Homework #2

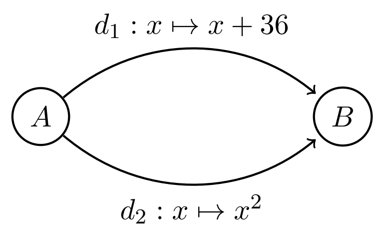
Deadline: Tuesday, 16 April 2024, 13:00

Submit your solutions for (up to) three of the following four exercises. If you solve all four, we will consult a random number generator to decide which three to look at and grade.

Exercise 1 (10 points)

In class, we had sketched the definition of the *price of anarchy*, PoA(G), of a given game $G = \langle N, \boldsymbol{A}, \boldsymbol{u} \rangle$ in the context of our discussion of congestion games. For a congestion game the PoA is defined as the ratio between the expected total cost for the worst equilibrium and the expected total cost for the best outcome. In normal-form games where all payoffs are non-negative (rather than non-positive, as for congestion games), the PoA instead is defined as the ratio of the expected *social welfare* (sum of payoffs) for the best outcome and the expected social welfare for the worst equilibrium. That is, the PoA is always a number greater than or equal to 1, and the best possible scenario is a PoA of 1, which corresponds to the case where the equilibrium is also socially optimal. In fact, the notion of PoA is considerably more general than defined here, as other objective functions and other solution concepts have also been considered. However, here we are going to restrict ourselves to sums of costs (or payoffs) and pure Nash equilibria. Answer the following questions:

(a) Calculate the PoA for the congestion game below, with $n = 10$ players, $m = 2$ resources, and action sets $A_1 = A_2 = \cdots = A_{10} = \{\{1\}, \{2\}\}$. Show your working.



(b) Define a normal-form game G for which PoA(G) = 1. This should be a nontrivial game, where not all action profiles have the same social welfare.

(c) Show that for any $K \in \mathbb{N}$ there exists a normal-form game G such that PoA(G) > K. That is, show that the PoA can become arbitrarily large. Show that this is possible even in case the utility functions of all players are strictly positive.

Exercise 2 (10 points)

We have seen that iterated elimination of strictly dominated strategies is order-independent. Show that the same is *not* true for the iterated elimination of weakly dominated strategies.

Exercise 3 (10 points)

We saw examples where a correlated equilibrium of a normal-form game intuitively seems 'better' than any of its Nash equilibria. For the Nash equilibria we had to choose between either an equal distribution of expected utility between the players or a high sum of expected utilities, while for the correlated equilibrium we could get both. In particular, the sum of expected utilities in the correlated equilibrium was *at least as high* as it was in any of the Nash equilibria, while at the same time also achieving perfect equity between the players.

This exercise is about showing that we can do even better. Find a game and a correlated equilibrium for that game where the sum of the expected utilities is *strictly higher* than for any of the Nash equilibria of the same game.

Exercise 4 (10 points)

Write a program to compute the set of all (mixed and pure) Nash equilibria of a given normal-form game with two players and two actions per player. The exact specification of the task (e.g., how to represent games in the input and how to report (possibly infinite) sets of equilibria in the output) is up to you, as is the choice of programming language.

Designing an algorithm that systematically covers every possible case is not straightforward. Full marks will be given for well-presented solutions that work correctly at least in those cases where the game in question has only finitely many equilibria, provided your report includes a brief discussion of what would be required to turn this into a general solution.