Special functions and Lie theory

Exercises, week 10 (last modified July 20, 2008)

Exercise 1 Let the Bessel function $\mathcal{J}_\alpha$ be defined by

$$\mathcal{J}_\alpha(z) := \sum_{k=0}^{\infty} \frac{1}{(\alpha+1)_k\, k!} \left(-\frac{z^2}{4}\right)^k .$$

We derived that

$$e^{ir\cos\psi} = \sum_{m=-\infty}^{\infty} \frac{(ir/2)^{|m|}}{|m|!} \mathcal{J}_{|m|}(r)\, e^{im\psi} \qquad (r>0,\ \psi\in\mathbb{R})$$

with absolute convergent series. Put

$$J_{-m}(r) = (-1)^m\, J_m(r) := \frac{(r/2)^m}{m!}\, \mathcal{J}_m(r) \qquad (m\in\mathbb{Z}_{\geq 0},\ r>0).$$

Then we see that

$$e^{ir\sin\psi} = \sum_{m\in\mathbb{Z}} J_m(r)\, e^{im\psi} \qquad (r>0,\ \psi\in\mathbb{R}).$$

Conclude that

$$\sum_{m\in\mathbb{Z}} J_m(r)\, J_{m+k}(r) = \delta_{k,0} \qquad (k\in\mathbb{Z},\ r>0)$$

with absolute convergent series. Also show that $|J_m(r)| \leq 1$ ($m\in\mathbb{Z}$, $r>0$). Finally show that the vectors $v^{(k)} = (v_m^{(k)})_{m\in\mathbb{Z}}$ ($k\in\mathbb{Z}$) with $v_m^{(k)} := J_{m+k}(r)$ form an orthonormal basis of the Hilbert space $\ell^2(\mathbb{Z})$.

Exercise 2 Let $G := (\mathbb{Z}_2)^3 \ltimes S_3$, a semidirect product of the finite abelian group $(\mathbb{Z}_2)^3$ (where $\mathbb{Z}_2 := \mathbb{Z}/(2\mathbb{Z})$) and the symmetric group S_3 in 3 letters, where the action of S_3 on $(\mathbb{Z}_2)^3$ by means of automorphisms is given by $\alpha_\sigma(j_1,j_2,j_3) := (j_{\sigma^{-1}(1)}, j_{\sigma^{-1}(2)}, j_{\sigma^{-1}(3)})$. Use Mackey's theorem in order to find all irreducible representations of G, up to equivalence. Check your result by verifying that the sum of the squared degrees of the irreducible representations must be equal to the order of G.