Special functions and Lie theory (**week 1**)

Notes by Tom H. Koornwinder for the MSc course *Special functions and Lie theory*, University of Amsterdam, February–May 2008

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1 Representations of $SU(2)$ and Jacobi polynomials

Literature: Sugiura [3, Ch. 2] and Vilenkin & Klimyk [4, Ch. 6]; furthermore Askey, Andrews & Roy [1] on special functions.

1.1 Preliminaries about representation theory

Let G be a group. Representations of G can be defined on any vector space (possibly infinite dimensional) over any field, but we will only consider representations on finite dimensional complex vector spaces. Let V be a finite dimensional complex vector space. Let $GL(V)$ be the set of all invertible linear transformations of V. This is a group under composition. If V has dimension n and if we choose a basis $e_1, \ldots, e_n$ of V then the map $x = x_1e_1 + \cdots + x_ne_n \mapsto (x_1, \ldots, x_n) \colon V \to \mathbb{C}_n$ is an isomorphism of vector spaces. There is a corresponding group isomorphism $GL(V) \to GL(\mathbb{C}^n)$ which sends each invertible linear transformation of V to the corresponding invertible matrix with respect to this basis. We denote $GL(\mathbb{C}^n)$ by $GL(n, \mathbb{C})$: the group of all invertible complex $n \times n$ matrices. Here the group multiplication is by multiplication of matrices.

Definition 1.1. A *representation* of a group G on a finite dimensional complex vector space V is a group homomorphism $\pi \colon G \to GL(V)$. A linear subspace W of V is called *invariant* (with respect to the representation π) if $\pi(g)\, W \subset W$ for all $g \in G$. The representation π on V is called *irreducible* if V and $\{0\}$ are the only invariant subspaces of V.

Definition 1.2. Let π be a representation of a group G on a finite dimensional complex vector space V. Choose a basis $e_1, \ldots, e_n$ of V. Then, for $g \in G$, the linear map $\pi(g)$ has a matrix $(\pi_{i,j}(g))_{i,j=1,\ldots,n}$ with respect to this basis, which is determined by the formula

$$\pi(g)\, e_j = \sum_{i=1}^{n} \pi_{i,j}(g)\, e_i.$$

The $\pi_{i,j}$ are complex-valued functions on G which are called the *matrix elements* of the representation π with respect to the basis $e_1, \ldots, e_n$.

Remark 1.3. Let $\mathrm{End}(V)$ be the space of all linear transformations $A \colon V \to V$. If π is a map of the group G into $\mathrm{End}(V)$ such that $\pi(g_1g_2) = \pi(g_1)\pi(g_2)$ for all g_1, g_2 and $\pi(e) = \mathrm{id}$, then π maps into $GL(V)$ and π is a representation of G on V.

Definition 1.4. A *topological group* is a set G which is both a group and a topological space such that the maps $(g_1, g_2) \mapsto g_1 g_2 \colon G \times G \to G$ and $g \mapsto g^{-1} \colon G \to G$ are continuous.

Example 1.5. $GL(n, \mathbb{C})$ can be considered as an open subset of $\mathbb{C}^{n^2}$ by associating with the element $T = (t_{i,j})_{i,j=1,\dots,n} \in GL(n, \mathbb{C})$ ($\det T \neq 0$) the n^2 complex coordinates $t_{i,j}$. Then the group $GL(n, \mathbb{C})$, with the topology inherited from $\mathbb{C}^{n^2}$, is a topological group.

Let V be an n-dimensional complex vector space. With any basis of V the group $GL(V)$ is isomorphic with $GL(n, \mathbb{C})$. Give a topology to $GL(V)$ such that this isomorphism is also a homeomorphism. Then $GL(V)$ is a topological group and the topology is independent of the choice of the basis.

Definition 1.6. A *representation* of a topological group G on a finite dimensional complex vector space V is a continuous group homomorphism $\pi \colon G \to GL(V)$.

Remark 1.7. Let G be a topological group, V a finite dimensional complex vector space and $\pi \colon G \to GL(V)$ a group homomorphism. Let $e_1, \dots, e_n$ a basis for V. Then the following five properties are equivalent:

a) π is continuous;

b) for all $v \in V$ the map $g \mapsto \pi(g)\, v \colon G \to V$ is continuous;

c) for all j the map $g \mapsto \pi(g)\, e_j \colon G \to V$ is continuous;

d) for all $v \in V$ and for all complex linear functionals f on V the map $g \mapsto f(\pi(g)v) \colon G \to \mathbb{C}$ is continuous.

e) The matrix elements $\pi_{i,j}$ of π with respect to the basis $e_1, \dots, e_n$ are continuous functions on G.

Be aware that these equivalences are not necessarily true if V is an infinite dimensional topological vector space.

Remark 1.8. If π is a representation of G on V and if H is a subgroup of G then the restriction of the group homomorphism $\pi \colon G \to GL(V)$ to H is a group homomorphism $\pi \colon H \to GL(V)$, so it is a representation of H on V.

If G is moreover a topological group then H with the topology inherited from G becomes a topological group.

If, furthermore, π is a representation of G as a topological group on V then the restriction of π to H is a representation of H as a topological group on V.

Definition 1.9. Let V be a finite dimensional complex vector space with hermitian inner product $\langle\ ,\ \rangle$. A representation π of a group G on V is called *unitary* is $\pi(g)$ is a unitary operator on V for all $g \in G$, i.e., if

$$\langle \pi(g)\, v, \pi(g)\, w \rangle = \langle v, w \rangle \quad \text{for all } v, w \in V \text{ and for all } g \in G.$$

Remark 1.10. Let V and G be as in Definition 1.9 and let π be a representation of G on V. Let $e_1, \dots, e_n$ be an orthonormal basis of V and let $\pi(g)$ have matrix $(\pi_{i,j}(g))$ with respect to this basis. Then the representation π is unitary iff the matrix $(\pi_{i,j}(g))$ is unitary for each $g \in G$. One of the ways to characterize unitarity of the matrix $(\pi_{i,j}(g))$ is that

$$\overline{\pi_{i,j}(g)} = \pi_{j,i}(g^{-1}) \quad (i,j = 1,\ldots,n).$$

Proposition 1.11 (Complete reducibility of unitary representations)**.**
Let V and G be as in Definition 1.9 and let π be a unitary representation of G on V. Then:

a) If W is an invariant subspace of V then the orthoplement $W^\perp$ of W is also an invariant subspace.

b) V can be written as an orthogonal direct sum of subspaces V_i such that the representation π, when restricted to V_i, is irreducible.

1.2 A class of representations of $SU(2)$

Fix $l \in \{0, \frac{1}{2}, 1, \ldots\}$. Let $\mathcal{H}_l$ be the space of *homogeneous polynomials* of degree $2l$ in two complex variables z_1, z_2, i.e., consisting of polynomials $f(z_1, z_2)$ with complex coefficients such that $f(cz_1, cz_2) = c^{2l} f(z_1, z_2)$ for all $c, z_1, z_2 \in \mathbb{C}$. Then the monomials $z_1^{l-n} z_2^{l+n}$ $(n = -l, -l+1, \ldots, l)$ form a basis of $\mathcal{H}_l$, and $\mathcal{H}_l$ has dimension $2l+1$. For reasons which will become clear later, we will work with a renormalized basis

$$\psi_n^l(z_1, z_2) := \binom{2l}{l-n}^{\frac{1}{2}} z_1^{l-n} z_2^{l+n} \qquad (n = -l, -l+1, \ldots, l). \tag{1.1}$$

For $A \in GL(2, \mathbb{C})$ and $f \in \mathcal{H}_l$ define the function $t^l(A)f$ on $\mathbb{C}^2$ by

$$(t^l(A)f)(z) := f(A'z) \qquad (z = (z_1, z_2) \in \mathbb{C}^2), \tag{1.2}$$

where A' is the transpose of the matrix A. So

$$\left(t^l \begin{pmatrix} a & b \\ c & d \end{pmatrix} f\right)(z_1, z_2) = f(az_1 + cz_2, bz_1 + dz_2), \quad \text{where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C}).$$

From this it is clear that $(t^l(A)f)(z_1, z_2)$ is again a homogeneous polynomial of degree $2l$ in z_1, z_2. Moreover, t^l is a representation of $GL(2, \mathbb{C})$ on $\mathcal{H}_l$, since $t^l(I)f = f$ and

$$\begin{aligned}(t^l(AB)f)(z) = f((AB)'z) = f(B'A'z) = (t^l(B)f)(A'z) \\ = \big(t^l(A)(t^l(B)f)\big)(z) = \Big(\big(t^l(A)t^l(B)\big)f\Big)(z).\end{aligned}$$

The matrix elements $t_{m,n}^l$ $(m, n = -l, -l+1, \ldots, l)$ of t^l with respect to the basis (1.1) are determined by

$$t^l(g)\,\psi_n^l = \sum_{m=-l}^{l} t_{m,n}^l(g)\,\psi_m^l \qquad (g \in GL(2, \mathbb{C})), \tag{1.3}$$

Since

$$\left(t^l\begin{pmatrix} a & b\\ c & d\end{pmatrix}\psi_n^l\right)(z_1,z_2)=\begin{pmatrix} 2l\\ l-n\end{pmatrix}^{\frac12}(az_1+cz_2)^{l-n}\,(bz_1+dz_2)^{l+n}, \tag{1.4}$$

(1.3) can be written more explicitly as

$$\begin{pmatrix} 2l\\ l-n\end{pmatrix}^{\frac12}(az_1+cz_2)^{l-n}\,(bz_1+dz_2)^{l+n}=\sum_{m=-l}^{l}\begin{pmatrix} 2l\\ l-m\end{pmatrix}^{\frac12}t^l_{m,n}\begin{pmatrix} a & b\\ c & d\end{pmatrix}z_1^{l-m}z_2^{l+m},$$
$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in GL(2,\mathbb{C}). \tag{1.5}$$

From (1.5) we see that $t^l_{m,n}\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ is a homogeneous polynomial of degree $2l$ in a,b,c,d, so $t^l_{m,n}$ is continuous on $GL(2,\mathbb{C})$. By Remark 1.7 t^l is then also a representation of $GL(2,\mathbb{C})$ considered as a topological group.

For fixed n we can consider (1.5) as a *generating function* for the matrix elements $t^l_{m,n}$ with $m=-l,\ldots,l$: the matrix elements are obtained as the coefficients in the power series expansion of the elementary function in z_1,z_2 on the left-hand side.

From (1.5) for $n=l$ elementary expressions for the matrix elements $t^l_{n,l}$ can be obtained (exercise):

$$t^l_{m,l}\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} 2l\\ l-m\end{pmatrix}^{\frac12}b^{l-m}d^{l+m}. \tag{1.6}$$

From (1.5) we can derive a double generating function for the matrix elements $t^l_{m,n}$: Multiply both sides of (1.5) with

$$\begin{pmatrix} 2l\\ l-n\end{pmatrix}^{\frac12}w_1^{l-n}w_2^{l+n},$$

and sum over n. Then we obtain

$$(az_1w_1+bz_1w_2+cz_2w_1+dz_2w_2)^{2l}=\sum_{m,n=-l}^{l}\begin{pmatrix} 2l\\ l-m\end{pmatrix}^{\frac12}\begin{pmatrix} 2l\\ l-n\end{pmatrix}^{\frac12}t^l_{m,n}\begin{pmatrix} a & b\\ c & d\end{pmatrix}$$
$$\times\, z_1^{l-m}z_2^{l+m}\,w_1^{l-n}w_2^{l+n},\qquad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in GL(2,\mathbb{C}). \tag{1.7}$$

Formula (1.7) implies the symmetry

$$t^l_{m,n}\begin{pmatrix} a & b\\ c & d\end{pmatrix}=t^l_{n,m}\begin{pmatrix} a & c\\ b & d\end{pmatrix}, \tag{1.8}$$

while (1.5) implies that

$$t^l_{m,n}\begin{pmatrix} a & b\\ c & d\end{pmatrix}=t^l_{-m,-n}\begin{pmatrix} d & c\\ b & a\end{pmatrix}. \tag{1.9}$$

From (1.8) and (1.9) we obtain a third symmetry

$$t^l_{m,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = t^l_{-n,-m}\begin{pmatrix} d & b \\ c & a \end{pmatrix}. \tag{1.10}$$

Let $SU(2)$ denote the set of all 2×2 unitary matrices of determinant 1. This is clearly a subgroup of $GL(2,\mathbb{C})$. Note that $SU(2)$ consists of all matrices

$$\begin{pmatrix} a & -\overline{c} \\ c & \overline{a} \end{pmatrix} \quad \text{with } a, c \in \mathbb{C} \text{ and } |a|^2 + |c|^2 = 1. \tag{1.11}$$

Hence, as a topological space, $SU(2)$ is homeomorphic with $\{(a,c) \in \mathbb{C}^2 \mid |a|^2 + |c|^2 = 1\}$, which is the unit sphere in $\mathbb{C}^2$, i.e., the sphere S^3. In particular, $SU(2)$ is compact.

The representation t^l of $GL(2,\mathbb{C})$ given by (1.2), becomes by restriction a representation of $SU(2)$. Put a hermitian inner product on $\mathcal{H}_l$ such that the basis of functions ψ^l_n ($n = -l, -l+1, \ldots, l$) is orthonormal.

Proposition 1.12. *The representation t^l of $SU(2)$ is unitary.*

Proof The inverse of $\begin{pmatrix} a & -\overline{c} \\ c & \overline{a} \end{pmatrix} \in SU(2)$ is $\begin{pmatrix} \overline{a} & \overline{c} \\ -c & a \end{pmatrix}$. In view of Remark 1.10 we have to show that

$$\overline{t^l_{m,n}\begin{pmatrix} a & -\overline{c} \\ c & \overline{a} \end{pmatrix}} = t^l_{n,m}\begin{pmatrix} \overline{a} & \overline{c} \\ -c & a \end{pmatrix}.$$

Since, by (1.5), $t^l_{m,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a polynomial with real coefficients in a, b, c, d, we have

$$\overline{t^l_{m,n}\begin{pmatrix} a & -\overline{c} \\ c & \overline{a} \end{pmatrix}} = t^l_{m,n}\begin{pmatrix} \overline{a} & -c \\ \overline{c} & a \end{pmatrix}.$$

Hence we have to show that

$$t^l_{m,n}\begin{pmatrix} \overline{a} & -c \\ \overline{c} & a \end{pmatrix} = t^l_{n,m}\begin{pmatrix} \overline{a} & \overline{c} \\ -c & a \end{pmatrix}.$$

This last identity follows from (1.8). □

Exercise 1.13. Give variants of the representation t^l of $GL(2,\mathbb{C})$ (see (1.2)) which also define representations of $GL(2,\mathbb{C})$ (for instance using the entry-wise complex conjugate of A and/or powers of the determinant of A, or replacing A' by A^{-1}). What happens with these representations if you restrict them to $SU(2)$?

1.3 Computation of matrix elements of representations of $SU(2)$

We can use the generating function (1.5) in order to compute the matrix elements $t^l_{m,n}$. First we expand the two powers on the left-hand side of (1.5) by the binomial formula:

$$(az_1 + cz_2)^{l-n} = \sum_{j=0}^{l-n} \binom{l-n}{j} a^j z_1^j c^{l-n-j} z_2^{l-n-j},$$

$$(bz_1 + dz_2)^{l+n} = \sum_{k=0}^{l+n} \binom{l+n}{k} b^k z_1^k d^{l+n-k} z_2^{l+n-k}.$$

Hence the left-hand side of (1.5) can be rewritten as

$$\binom{2l}{l-n}^{\frac{1}{2}} \sum_{j=0}^{l-n} \sum_{k=0}^{l+n} \binom{l-n}{j} \binom{l+n}{k} a^j b^k c^{l-n-j} d^{l+n-k} z_1^{j+k} z_2^{2l-j-k}. \tag{1.12}$$

In this double sum we make a change of summation variables $(j,k) \mapsto (m,j)$, where $j+k = l-m$. Hence

$$(j,k) \mapsto (l-k-j, j) \text{ with inverse map } (m,j) \mapsto (j, l-m-j). \tag{1.13}$$

Now we have

$$0 \le j \le l-n \text{ and } 0 \le k \le l+n \iff \\ -l \le m \le l \text{ and } 0 \le j \le l-n \text{ and } -m-n \le j \le l-m. \tag{1.14}$$

Indeed, the inequalities to the left of the equivalence sign in (1.14) imply that $0 \le j+k \le 2l$, hence $0 \le l-m \le 2l$, hence $-l \le m \le l$. Also, $0 \le k \le l+n$ implies $0 \le l-m-j \le l+n$, hence $-m-n \le j \le l-m$. Conversely, $-m-n \le j \le l-m$ implies (substitute $m = l-k-j$) that $-l-n+k+j \le j \le k+j$, hence $0 \le k \le l+n$. (Note that $-l \le m \le l$ to the right of the equivalence sign in (1.14) is not strictly needed because it is implied by the other inequaltities on the right.)

We conclude that the double sum (1.12) can be rewritten by the substitution $j+k = l-m$ as follows:

$$\binom{2l}{l-n}^{\frac{1}{2}} \sum_{m=-l}^{l} \sum_{j=0\vee(-m-n)}^{(l-m)\wedge(l-n)} \binom{l-n}{j} \binom{l+n}{l-m-j} a^j b^{l-m-j} c^{l-n-j} d^{n+m+j} z_1^{l-m} z_2^{l+m}. \tag{1.15}$$

Here the first summation is by convention over all $m \in \{-l, -l+1, \ldots, l\}$. In the second summation the symbol $\vee$ means maximum and the symbol $\wedge$ means minimum. The range of the double summation in (1.15) is justified by the equivalence (1.14). Note that the second summation is an inner summation since its summation bounds depend on m, which is the summation variable for the outer summation. The summand in (1.15) is obtained from the summand in (1.12) by the substitution $k = l-m-j$.

Since (1.15) is a rewritten form of the left-hand side of (1.5), it must be equal to the right-hand side of (1.5). Both (1.15) and the right-hand side of (1.5) are polynomials in z_1, z_2 with explicit coefficients. Hence the corresponding coefficients must be equal. We conclude:

Proposition 1.14.

$$t^l_{m,n}\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \binom{2l}{l-m}^{-\frac12}\binom{2l}{l-n}^{\frac12}\sum_{j=0\vee(-m-n)}^{(l-m)\wedge(l-n)}\binom{l-n}{j}\binom{l+n}{l-m-j}a^j b^{l-m-j}c^{l-n-j}d^{n+m+j}. \tag{1.16}$$

Note that the summation bounds in (1.12) reduce to one of four alternatives depending on the signs of $m+n$ and $m-n$:

$$\begin{aligned}
0\le j\le l-m \quad &\text{if}\quad m+n\ge 0\ \text{ and }\ m-n\ge 0;\\
0\le j\le l-n \quad &\text{if}\quad m+n\ge 0\ \text{ and }\ m-n\le 0;\\
-m-n\le j\le l-m \quad &\text{if}\quad m+n\le 0\ \text{ and }\ m-n\ge 0;\\
-m-n\le j\le l-n \quad &\text{if}\quad m+n\le 0\ \text{ and }\ m-n\le 0.
\end{aligned}$$

These four alternatives correspond two four subsets of the set $\{(m,n)\mid m,n\in\{-l,-l+1,\dots,l\}\}$, which have triangular shape, overlapping boundaries, and together span the whole set. These four subsets are mapped onto each other by the symmetries (1.8)–(1.10).

Hence it is sufficient to compute $t^l_{m,n}$ if $m+n\ge 0$, $m-n\ge 0$. For a while we only assume $m+n\ge 0$ and not yet $m-n\ge 0$ Then (1.16) takes the form

$$t^l_{m,n}\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \binom{2l}{l-m}^{-\frac12}\binom{2l}{l-n}^{\frac12}\sum_{j\ge 0}\binom{l-n}{j}\binom{l+n}{l-m-j}a^j b^{l-m-j}c^{l-n-j}d^{n+m+j}. \tag{1.17}$$

We will rewrite the right-hand side of (1.17) first as a Gauss hypergeometric function (with some elementary factors in front) and next as a Jacobi polynomial. For this derivation remember the *Pochhammer symbol*

$$(a)_0 := 1,\qquad (a)_k := a(a+1)\dots(a+k-1)\quad (k\in\mathbb{Z}_{>0}).$$

In particular, note that

$$\frac{(n+k)!}{n!} = (n+1)_k\,,\qquad \frac{n!}{(n-k)!} = (-1)^k(-n)_k\,.$$

Now we have for $m+n\ge 0$:

$$
\begin{aligned}
&t^l_{m,n}\begin{pmatrix} a & b\\ c & d\end{pmatrix}\\
&=\left(\frac{(l+m)!\,(l-m)!}{(l+n)!\,(l-n)!}\right)^{\frac12}\sum_{j\ge0}\frac{(l-n)!}{j!\,(l-n-j)!}\,\frac{(l+n)!}{(l-m-j)!\,(n+m+j)!}\,a^j b^{l-m-j}c^{l-n-j}d^{n+m+j}\\
&=\left(\frac{(l+m)!\,(l-m)!}{(l+n)!\,(l-n)!}\right)^{\frac12}\frac{(l+n)!\,b^{l-m}c^{l-n}d^{m+n}}{(l-m)!\,(m+n)!}\\
&\qquad\qquad\times\sum_{j\ge0}\frac{(l-m)!}{(l-m-j)!}\,\frac{(l-n)!}{(l-n-j)!}\,\frac{(m+n)!}{(m+n+j)!\,j!}\left(\frac{ad}{bc}\right)^j\\
&=\left(\frac{(l+m)!\,(l+n)!}{(l-m)!\,(l-n)!}\right)^{\frac12}\frac{b^{l-m}c^{l-n}d^{m+n}}{(m+n)!}\sum_{j\ge0}\frac{(-l+m)_j\,(-l+n)_j}{(m+n+1)_j\,j!}\left(\frac{ad}{bc}\right)^j. \qquad (1.18)
\end{aligned}
$$

1.4 Hypergeometric series

An infinite series $\sum_{k=0}^\infty c_k$ is called a *hypergeometric series* if $c_0=1$ and there are complex-valued polynomials P,Q in one complex variable such that $Q(k)\ne0$ for $k\in\mathbb{Z}_{\ge0}$ and

$$c_{k+1}=c_k\,\frac{P(k)}{Q(k)}\qquad (k\in\mathbb{Z}_{\ge0}). \qquad (1.19)$$

This property is satisfied if

$$c_k:=\frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k\,k!}\,z^k \qquad (1.20)$$

for certain complex $a_1,\ldots,a_p$, $b_1,\ldots,b_q$, z with $b_1,\ldots,b_q\notin\mathbb{Z}_{\le0}$. Then

$$c_{k+1}=c_k\,\frac{(a_1+k)\ldots(a_p+k)\,z}{(b_1+k)\ldots(b_q+k)(1+k)}\,.$$

Conversely, we see that for any hypergeometric series we can write c_k in the form (1.20). (If the polynomial Q in (1.19) has no root -1, then multiply $P(k)$ and $Q(k)$ by $1+k$.) The reason for including the factor $k!$ in the denominator of (1.20) is to let the series $\sum_k c_k$ naturally start with $k=0$. Indeed, if we consider (1.19) also for $k\in\mathbb{Z}_{<0}$ and write it in the form $c_k=c_{k+1}Q(k)/P(k)$, then we get $c_k=0$ for $k\in\mathbb{Z}_{<0}$ if $Q(-1)=0$ and $P(k)\ne0$ for $k\in\mathbb{Z}_{<0}$.

For c_k given by (1.20), denote the hypergeometric series $\sum_{k=0}^\infty c_k$ by

$${}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};z\right)={}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z):=\sum_{k=0}^\infty\frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k\,k!}\,z^k. \qquad (1.21)$$

If for some i and for some $n\in\mathbb{Z}_{\ge0}$ we have $a_i=-n$, while $a_j\notin\{-n+1,\ldots,-1,0\}$ for $j\ne i$, then the series (1.21) terminates after the term for $k=n$. Then the series is a polynomial of degree n in z. In that case, if we replace in (1.21) $\sum_{k=0}^{\infty}$ by $\sum_{k=0}^{n}$, the sum will remain meaningful if one or more of the parameters b_j take integer values $\le -n$.

In the nonterminating case we can apply the ratio test for convergence of the series (1.21). We see that the series has radius of convergence ∞ if $p \le q$, 1 if $p = q+1$, and 0 if $p > q+1$. Then, for $p \le q+1$, the hypergeometric series is certainly an analytic function in the complex variable z on the open disk with that radius and with center 0. We call this function a *hypergeometric function*.

The hypergeometric series generalizes the geometric series

$$ {}_1F_0\left({1 \atop -};z\right) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \qquad (|z|<1). $$

The general case of the ${}_1F_0$ series is the infinite binomial series

$$ {}_1F_0\left({a \atop -};z\right) = \sum_{k=0}^{\infty} \frac{(a)_k}{k!}\, z^k = (1-z)^{-a} \qquad (|z|<1). \tag{1.22} $$

The ${}_0F_0$ series is the exponential series:

$$ {}_0F_0\left({- \atop -};z\right) = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z \qquad (z \in \mathbb{C}). $$

The general ${}_2F_1$ series is called *Gauß hypergeometric series*:

$$ {}_2F_1\left({a,b \atop c};z\right) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!}\, z^k \qquad (|z|<1). \tag{1.23} $$

See Dutka [2] for the early history of hypergeometric series. He attributes the introduction of the ${}_2F_1$ series to Euler in 1778 (see [2, (11)]), which is much earlier than Gauß' first memoir on this hypergeometric series in 1813.

The *Euler integral representation*

$$ {}_2F_1\left({a,b \atop c};z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt \qquad (\operatorname{Re} c > \operatorname{Re} b > 0) \tag{1.24} $$

is also misnamed according to [2]. Its first occurrence is in the thesis by Vorsselman de Heer (Utrecht, 1833), see [2, (27)]. However, according to Dutka, Euler obtained a tranformed version of the right-hand side of (1.24) of which Euler showed that it satisfies a transformed version of the differential equation (1.32). Formula (1.24) can be proved for $|z|<1$ by expanding $(1-tz)^{-a}$ as a power series (see (1.22)), next interchanging summation and integration (justify this), and finally using the beta integral. From (1.24) one can see that the right-hand side, as a function of z, has an analytic continuation to $\mathbb{C}\backslash[1,\infty)$. This shows that the Gauß hypergeometric series (1.23) extends to a one-valued analytic function on $\mathbb{C}\backslash[1,\infty)$. For the moment we have shown this analytic continuation only for $\operatorname{Re} c > \operatorname{Re} b > 0$, but the property can be extended to general parameter values (for instance by using that (1.23) is a solution of the differential equation (1.32) below).

Gauss' summation formula evaluates

$$ {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad (\mathrm{Re}\,(c-a-b)>0), \tag{1.25} $$

where

$$ {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};1\right) := \lim_{x\uparrow 1} {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};x\right) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!}\,. \tag{1.26} $$

Here the infinite series converges absolutely converges if $\mathrm{Re}\,(c-a-b)>0$ and the limit then exists because summability implies Abel summability. If moreover $\mathrm{Re}\,c > \mathrm{Re}\,b > 0$ then (1.25) follows by letting $z \uparrow 1$ in (1.24). (Gauß obtained it in a different way.) For the general case that $\mathrm{Re}\,(c-a-b)>0$ one has to use analytic continuation of (1.25) with respect to a, b, c.

Another important corollary of (1.24) is *Pfaff's transformation formula*

$$ {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};z\right) = (1-z)^{-a}\ {}_2F_1\left(\begin{matrix} a,c-b \\ c \end{matrix};\frac{z}{z-1}\right). \tag{1.27} $$

If $\mathrm{Re}\,c > \mathrm{Re}\,b > 0$ then (1.27) indeed follows from (1.24) by substitution of $t = 1-s$ for the integration variable. For the general case one has to justify analytic continuation of both sides of (1.27) with respect to the parameters b and c. (Pfaff obtained (1.27) in a different way.) Note that we can expand the right-hand side of (1.27) as a power series in $z/(z-1)$ if $\mathrm{Re}\,z < \frac{1}{2}$. Hence (1.27) gives an analytic continuation of the left-hand side from the open unit disc to its union with the half plane $\{z \in \mathbb{C} \mid \mathrm{Re}\,z < \frac{1}{2}\}$.

As observed by Vorsselman de Heer in 1833, combination of two versions of (1.27) yields *Euler's tranformation formula*

$$ {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};z\right) = (1-z)^{c-a-b}\ {}_2F_1\left(\begin{matrix} c-a,c-b \\ c \end{matrix};z\right). \tag{1.28} $$

Euler obtained this in 1778 in a different way.

Termwise differentiation in (1.23) yields the differentiation formula

$$ \frac{d}{dz}\ {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};z\right) = \frac{ab}{c}\ {}_2F_1\left(\begin{matrix} a+1,b+1 \\ c+1 \end{matrix};z\right). \tag{1.29} $$

Note that all parameters are raised by 1. Similarly one obtains

$$ \frac{d}{dz}\left(z^c\ {}_2F_1\left(\begin{matrix} c-a,c-b \\ c+1 \end{matrix};z\right)\right) = cz^{c-1}\ {}_2F_1\left(\begin{matrix} c-a,c-b \\ c \end{matrix};z\right), \tag{1.30} $$

which, combined with (1.28), yields

$$ \frac{d}{dz}\left(z^c(1-z)^{a+b-c+1}\ {}_2F_1\left(\begin{matrix} a+1,b+1 \\ c+1 \end{matrix};z\right)\right) = cz^{c-1}(1-z)^{a+b-c}\ {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix};z\right). \tag{1.31} $$

Note that here all parameters are lowered by 1. Combination of (1.29) and (1.31) yields *Euler's differential equation*

$$\left(z(1-z)\frac{d^2}{dz^2}+\big(c-(a+b+1)z\big)\frac{d}{dz}-ab\right)\,{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};z\right)=0. \tag{1.32}$$

Exercise 1.15. Derive from (1.24) that

$$\begin{aligned}{}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};z\right)&=-\frac{e^{-i\pi c}\,\Gamma(c)}{4\sin(\pi b)\sin(\pi(c-b))\,\Gamma(b)\Gamma(c-b)}\\&\quad\times\int_{(1^+,0^+,1^-,0^-)}t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt\qquad(z\notin[0,\infty),\ b,1-c,c-b\notin\mathbb{Z}_{>0}).\end{aligned}$$

Here the closed integration contour goes successively around 1 in positive sense, around 0 in positive sense, around 1 in negative sense, and around 0 in negative sense. The factors t^{b-1} and $(1-t)^{c-b-1}$ are taken such that they are equal to their principal values at a point $t\in(0,1)$ where the path of integration starts. The factor $(1-tz)^{-a}$ is defined such that it equals 1 if $z\to 0$.

1.5 Jacobi polynomials

Let (a,b) be an open interval and let two systems of monic orthogonal polynomials $\{p_n\}_{n=0}^\infty$ and $\{q_n\}_{n=0}^\infty$ be defined with respect to strictly positive weight functions w respectively w_1 on (a,b). Suppose that w is continuous and w_1 is continuously differentiable on (a,b). Under suitable boundary assumptions on w and w_1, integration by parts yields

$$\int_a^b p_n'(x)\,q_{m-1}(x)\,w_1(x)\,dx=-\int_a^b p_n(x)\,w(x)^{-1}\,\frac{d}{dx}\big(w_1(x)\,q_{m-1}(x)\big)\,w(x)\,dx \tag{1.33}$$

without stock terms. Suppose that

$$w(x)^{-1}\,\frac{d}{dx}\big(w_1(x)\,x^{n-1}\big)=a_n\,x^n+\text{polynomial of degree}<n \tag{1.34}$$

for certain $a_n\neq 0$. Then (1.32) and (1.33) together with the orthogonality properties of $\{p_n\}_{n=0}^\infty$ and $\{q_n\}_{n=0}^\infty$ yield that

$$p_n'(x)=n\,q_{n-1}(x), \tag{1.35}$$

$$w(x)^{-1}\,\frac{d}{dx}\big(w_1(x)\,q_{n-1}(x)\big)=a_n\,p_n(x). \tag{1.36}$$

The pair of operators D_- and D_+ defined by

$$(D_-f)(x):=f'(x), \tag{1.37}$$

$$(D_+f)(x):=\left(w(x)^{-1}\,\frac{d}{dx}\circ w_1(x)\right)f(x)=\frac{w_1(x)}{w(x)}\,f'(x)+\frac{w_1'(x)}{w(x)}\,f(x) \tag{1.38}$$

will be called a pair of *shift operators*. So we have:
adjointness:

$$\int_a^b (D_- f)(x)\, g(x)\, w_1(x)\, dx = -\int_a^b f(x)\, (D_+ g)(x)\, w(x)\, dx \quad (f,\, g \text{ polynomials}), \tag{1.39}$$

shift formulas:

$$D_- p_n = n\, q_{n-1}, \quad D_+ q_{n-1} = a_n\, p_n, \tag{1.40}$$

second order differential equation:

$$(D_+ \circ D_-) p_n = n a_n\, p_n, \tag{1.41}$$

relation between squared L^2-norms:

$$n \int_a^b (q_{n-1}(x))^2\, w_1(x)\, dx = -a_n \int_a^b (p_n(x))^2\, w(x)\, dx. \tag{1.42}$$

From (1.42) we see that $a_n < 0$.

The following cases are examples for which the above considerations are valid:

i) *Jacobi*: $(a,b) = (-1,1)$, $w(x) = (1-x)^\alpha (1+x)^\beta$, $w_1(x) = (1-x^2)\, w(x)$, $\alpha, \beta > -1$.

ii) *Laguerre*: $(a,b) = (0,\infty)$, $w(x) = x^\alpha e^{-x}$, $w_1(x) = x\, w(x)$, $\alpha > -1$.

iii) *Hermite*: $(a,b) = (-\infty,\infty)$, $w(x) = w_1(x) = e^{-x^2}$.

These three cases are essentially the only cases which satisfy the above conditions.

Let $\alpha, \beta > -1$. Let a, b, w and w_1 be as in case (i) above. Note that the integral of w over $(-1,1)$ is a variant of the beta integral:

$$\int_{-1}^1 (1-x)^\alpha\, (1+x)^\beta\, dx = \frac{2^{\alpha+\beta+1}\, \Gamma(\alpha+1)\, \Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\,. \tag{1.43}$$

Denote the monic orthogonal polynomials on $(-1,1)$ with respect to the weight function w by $p_n^{(\alpha,\beta)}$ (monic *Jacobi polynomials*). Then the raising shift operator $D_+ = D_+^{(\alpha,\beta)}$ is given by

$$\begin{aligned}(D_+^{(\alpha,\beta)} f)(x) &= \left((1-x)^{-\alpha} (1+x)^{-\beta}\, \frac{d}{dx} \circ (1-x)^{\alpha+1} (1+x)^{\beta+1} \right) f(x) \\ &= (1-x^2)\, f'(x) + (\beta - \alpha - (\alpha+\beta+2)x)\, f(x).\end{aligned}$$

Hence $a_n = -(n+\alpha+\beta+1)$ and the shift relations become:

$$\frac{d}{dx}\, p_n^{(\alpha,\beta)}(x) = n\, p_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{1.44}$$

$$\begin{aligned}&\left((1-x)^{-\alpha} (1+x)^{-\beta}\, \frac{d}{dx} \circ (1-x)^{\alpha+1} (1+x)^{\beta+1} \right) p_{n-1}^{(\alpha+1,\beta+1)}(x) \\ &\qquad = \left((1-x^2)\, \frac{d}{dx} + (\beta - \alpha - (\alpha+\beta+2)x) \right) p_{n-1}^{(\alpha+1,\beta+1)}(x) = -(n+\alpha+\beta+1)\, p_n^{(\alpha,\beta)}(x).\end{aligned} \tag{1.45}$$

The second order differential equation becomes

$$\left((1-x^2)\,\frac{d^2}{dx^2}+(\beta-\alpha-(\alpha+\beta+2)x)\,\frac{d}{dx}\right)p_n^{(\alpha,\beta)}(x)=-n(n+\alpha+\beta+1)\,p_n^{(\alpha,\beta)}(x). \tag{1.46}$$

Note that the hypergeometric differential equation (1.32) with a,b,c equal to $-n,n+\alpha+\beta+1,\alpha+1$, respectively, can be obtained from (1.46) by the substitution $x=1-2z$. Since solutions of (1.32) are uniquely determined by a nonzero initial value at $z=0$, we can prove (1.51) (given below) already in this way.

Iteration of (1.45) yields the *Rodrigues formula*

$$p_n^{(\alpha,\beta)}(x)=\frac{(-1)^n}{(n+\alpha+\beta+1)_n}\,(1-x)^{-\alpha}(1+x)^{-\beta}\,\frac{d^n}{dx^n}\left((1-x)^{\alpha+n}\,(1+x)^{\beta+n}\right). \tag{1.47}$$

Formula (1.42) yields the recurrence

$$\int_{-1}^{1}\left(p_n^{(\alpha,\beta)}(x)\right)^2(1-x)^\alpha(1+x)^\beta\,dx$$
$$=\frac{n}{n+\alpha+\beta+1}\int_{-1}^{1}\left(p_{n-1}^{(\alpha+1,\beta+1)}(x)\right)^2(1-x)^{\alpha+1}(1+x)^{\beta+1}\,dx.$$

Iteration of this recurrence and combination with (1.43) yields

$$\frac{\int_{-1}^{1}\left(p_n^{(\alpha,\beta)}(x)\right)^2(1-x)^\alpha(1+x)^\beta\,dx}{\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta\,dx}=\frac{2^{2n}\,n!\,(\alpha+1)_n\,(\beta+1)_n}{(\alpha+\beta+2)_{2n}\,(n+\alpha+\beta+1)_n}. \tag{1.48}$$

Consider (1.45) for $x=1$. This yields the recurrence

$$-2(\alpha+1)\,p_{n-1}^{(\alpha+1,\beta+1)}(1)=-(n+\alpha+\beta+1)\,p_n^{(\alpha,\beta)}(1).$$

By iteration we obtain

$$p_n^{(\alpha,\beta)}(1)=\frac{2^n\,(\alpha+1)_n}{(n+\alpha+\beta+1)_n}. \tag{1.49}$$

By Taylor expansion and by use of (1.44) and (1.49) we obtain

$$\begin{aligned}
p_n^{(\alpha,\beta)}(x)&=\sum_{k=0}^{n}\frac{(x-1)^k}{k!}\left(\frac{d}{dx}\right)^k p_n^{(\alpha,\beta)}(x)\Big|_{x=1}\\
&=\sum_{k=0}^{n}\frac{(x-1)^k}{k!}\,\frac{n!}{(n-k)!}\,p_{n-k}^{(\alpha+k,\beta+k)}(1)\\
&=\sum_{k=0}^{n}\frac{n!\,2^{n-k}\,(\alpha+k+1)_{n-k}\,(x-1)^k}{k!\,(n-k)!\,(n+\alpha+\beta+k+1)_{n-k}}\\
&=\frac{2^n\,(\alpha+1)_n}{(n+\alpha+\beta+1)_n}\sum_{k=0}^{n}\frac{(-n)_k\,(n+\alpha+\beta+1)_k}{(\alpha+1)_k\,k!}\left(\frac{1-x}{2}\right)^k.
\end{aligned} \tag{1.50}$$

From (1.50) and (1.49) we obtain

$$\frac{p_n^{(\alpha,\beta)}(x)}{p_n^{(\alpha,\beta)}(1)} = \sum_{k=0}^{n} \frac{(-n)_k\,(n+\alpha+\beta+1)_k}{(\alpha+1)_k\,k!} \left(\frac{1-x}{2}\right)^k = {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{1-x}{2}\right). \quad (1.51)$$

The following general result can be proved in an immediate way. Let $\{p_n\}_{n=0}^{\infty}$ be a system of monic orthogonal polynomials with respect to a weight function w on (a,b). Put $v(x) := w(-x)$ and let $\{q_n\}_{n=0}^{\infty}$ be a system of monic orthogonal polynomials with respect to the weight function v on $(-b,-a)$. Then $p_n(-x) = (-1)^n q_n(x)$. In particular,

$$p_n^{(\alpha,\beta)}(-x) = (-1)^n p_n^{(\beta,\alpha)}(x). \quad (1.52)$$

The standard normalization of Jacobi polynomials is different from the monic normalization. Write $P_n^{(\alpha,\beta)}$ for the constant multiple of $p_n^{(\alpha,\beta)}$ such that

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!}\,. \quad (1.53)$$

Then, by (1.49) and (1.53),

$$P_n^{(\alpha,\beta)}(x) = \frac{(n+\alpha+\beta+1)_n}{2^n\,n!}\,p_n^{(\alpha,\beta)}(x) = \frac{(n+\alpha+\beta+1)_n}{2^n\,n!}\,x^n + \text{lower degree terms.}$$

From (1.53) and (1.51) we obtain:

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\;{}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{1-x}{2}\right). \quad (1.54)$$

One may now also rewrite the other previous formulas in terms of these renormalized Jacobi polynomials. For instance, (1.46) and (1.52) remain valid with p_n replaced by P_n, and (1.48) (more generally the orthogonality relations) now takes the form

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x)\,P_n^{(\alpha,\beta)}(x)\,(1-x)^\alpha(1+x)^\beta\,dx = h_n^{(\alpha,\beta)}\,\delta_{m,n} \quad (1.55)$$

with

$$h_n^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}(n+\alpha+\beta+1)_n\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{n!\,\Gamma(2n+\alpha+\beta+2)}\,. \quad (1.56)$$

Theorem 1.16. *For fixed $\alpha,\beta > -1$ the Jacobi polynomials $P_n^{(\alpha,\beta)}$ form a complete orthogonal system in $L^2((-1,1),(1-x)^\alpha(1+x)^\beta\,dx)$.*

For the proof use that we have L^2 with respect to a finite measure on a bounded interval. Therefore the continuous functions on $[-1,1]$ are dense in $L^2((-1,1),(1-x)^\alpha(1+x)^\beta\,dx)$, and the polynomials are dense in sup-norm (and hence also in L^2-norm) in $C([-1,1])$ by Weierstrass' approximation theorem.

1.6 Orthogonality of matrix elements

We now recognize the formula (1.18) for the matrix element $t^l_{m,n}$ ($m+n \geq 0$) of the representation t^l of $GL(2,\mathbb{C})$ as a hypergeometric series (1.23):

$$t^l_{m,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{(l+m)!\,(l+n)!}{(l-m)!\,(l-n)!}\right)^{\frac{1}{2}} \frac{b^{l-m}c^{l-n}d^{m+n}}{(m+n)!}\, {}_2F_1\left(\begin{matrix} -l+m, -l+n \\ m+n+1 \end{matrix}; \frac{ad}{bc}\right). \tag{1.57}$$

Note that the two upper parameters $-l+m, -l+n$ of the hypergeometric function in (1.57) are both non-positive, and that the series will terminate after the term with $j = (l-m) \wedge (l-n)$.

Pfaff's transformation (1.27) implies for the hypergeometric function in (1.57) that

$${}_2F_1\left(\begin{matrix} -l+m, -l+n \\ m+n+1 \end{matrix}; \frac{ad}{bc}\right) = b^{m-l}c^{m-l}(bc-ad)^{l-m}\, {}_2F_1\left(\begin{matrix} -l+m, l+m+1 \\ m+n+1 \end{matrix}; \frac{ad}{ad-bc}\right).$$

Hence we arrive at the following rewritten form of (1.57) (from now on assume $m+n \geq 0$, $m-n \geq 0$):

$$t^l_{m,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{(l+m)!\,(l+n)!}{(l-m)!\,(l-n)!}\right)^{\frac{1}{2}} \frac{c^{m-n}d^{m+n}(bc-ad)^{l-m}}{(m+n)!}\, {}_2F_1\left(\begin{matrix} -l+m, l+m+1 \\ m+n+1 \end{matrix}; \frac{ad}{ad-bc}\right). \tag{1.58}$$

Now use the expression (1.54) of Jacobi polynomials in terms of the Gauss hypergeometric function. This implies for the hypergeometric function in (1.58) that

$${}_2F_1\left(\begin{matrix} -l+m, l+m+1 \\ m+n+1 \end{matrix}; \frac{ad}{ad-bc}\right) = \frac{(l-m)!\,(m+n)!}{(l+n)!}\, P^{(m+n,m-n)}_{l-m}\left(\frac{bc+ad}{bc-ad}\right).$$

Hence we can further rewrite (1.58) (if $m \pm n \geq 0$) as follows:

$$t^l_{m,n}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{(l+m)!\,(l-m)!}{(l+n)!\,(l-n)!}\right)^{\frac{1}{2}} c^{m-n}d^{m+n}(bc-ad)^{l-m}\, P^{(m+n,m-n)}_{l-m}\left(\frac{bc+ad}{bc-ad}\right). \tag{1.59}$$

We are in particular interested in (1.59) if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$. Note that by (1.11) a general element of $SU(2)$ can be written as

$$\begin{pmatrix} \sin\theta\, e^{i\phi} & -\cos\theta\, e^{-i\psi} \\ \cos\theta\, e^{i\psi} & \sin\theta\, e^{-i\phi} \end{pmatrix} \qquad \text{with } 0 \leq \theta \leq \pi/2 \text{ and } \phi, \psi \in [0, 2\pi).$$

Hence we obtain:

Theorem 1.17. *If $m \pm n \geq 0$ then*

$$t^l_{m,n}\begin{pmatrix} \sin\theta\, e^{i\phi} & -\cos\theta\, e^{-i\psi} \\ \cos\theta\, e^{i\psi} & \sin\theta\, e^{-i\phi} \end{pmatrix} = (-1)^{l-m}\left(\frac{(l+m)!\,(l-m)!}{(l+n)!\,(l-n)!}\right)^{\frac{1}{2}} \\ \times e^{-i(m+n)\phi}\, e^{i(m-n)\psi}\, (\sin\theta)^{m+n}(\cos\theta)^{m-n}\, P^{(m+n,m-n)}_{l-m}(\cos 2\theta). \tag{1.60}$$

We introduce a special Borel measure μ on $SU(2)$ such that

$$\int_{SU(2)} f d\mu = \frac{1}{2\pi^2} \int_{\phi=0}^{2\pi} \int_{\psi=0}^{2\pi} \int_{\theta=0}^{\pi/2} f \begin{pmatrix} \sin\theta\, e^{i\phi} & -\cos\theta\, e^{-i\psi} \\ \cos\theta\, e^{i\psi} & \sin\theta\, e^{-i\phi} \end{pmatrix} \sin\theta\, \cos\theta\, d\theta\, d\psi\, d\phi. \tag{1.61}$$

for all continuous funtions f on $SU(2)$. Note that

$$\int_{SU(2)} d\mu = 1. \tag{1.62}$$

The matrix elements $t^l_{m,n}$ satisfy a remarkable orthogonality relation with respect to this measure:

$$\int_{SU(2)} t^l_{m,n}\, \overline{t^{l'}_{m',n'}}\, d\mu = \frac{1}{2l+1}\, \delta_{l,l'}\, \delta_{m,m'}\, \delta_{n,n'}\,. \tag{1.63}$$

For $(m,n) \neq (m',n')$ this follows immediately from (1.60), (1.61) and the symmetries (1.8)–(1.10). For $(m,n) = (m',n')$ with $m+n, m-n \geq 0$ we have to show that

$$\frac{(l+m)!\,(l-m)!}{(l+n)!\,(l-n)!} \int_0^{\pi/2} P^{(m+n,m-n)}_{l-m}(\cos 2\theta)\, P^{(m+n,m-n)}_{l'-m}(\cos 2\theta)\, (\sin\theta)^{2m+2n+1} (\cos\theta)^{2m-2n+1}\, d\theta \\ = \frac{1}{2l+1}\, \delta_{l,l'}\,.$$

By the substitution $x = \cos 2\theta$ this can be rewritten as

$$\frac{(l+m)!\,(l-m)!}{(l+n)!\,(l-n)!\, 2^{2m+1}} \int_{-1}^{1} P^{(m+n,m-n)}_{l-m}(x)\, P^{(m+n,m-n)}_{l'-m}(x)\, (1-x)^{m+n} (1+x)^{m-n}\, dx = \frac{\delta_{l,l'}}{2l+1}\,. \tag{1.64}$$

In order to show this identity we remember the orthogonality relations (1.55), (1.56) for Jacobi polynomials. Now observe that

$$h^{(m+n,m-n)}_{l-m} = \frac{2^{2m+1}(l+n)!\,(l-n)!}{(2l+1)(l+m)!\,(l-m)!}\,.$$

This settles (1.64) and hence (1.63).

Proposition 1.18. *The matrix elements $t^l_{m,n}$ ($l \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, $m,n \in \{-l,-l+1,\ldots,l\}$) form a complete orthogonal system in $L^2(SU(2), d\mu)$.*

The proof uses that the span of the matrix elements $t^l_{m,n}$, as functions of $\begin{pmatrix} a & -\overline{c} \\ c & \overline{a} \end{pmatrix} \in SU(2)$ ($|a|^2 + |c|^2 = 1$), equals the space of polynomials in the four real variables $\mathrm{Re}\, a, \mathrm{Im}\, a, \mathrm{Re}\, b, \mathrm{Im}\, b$, restricted to $|a|^2 + |c|^2 = 1$. Hence, by Weierstrass' approximation theorem, the $t^l_{m,n}$ span a dense subspace of $C(SU(2))$ with respect to the sup-norm, and thus also span a dense subspace of $L^2(SU(2), d\mu)$.

1.7 Schur's orthogonality relations

The orthogonality relation (1.63) is a special case of *Schur's orthogonality relations* for the matrix elements of the irreducible unitary representations of a compact group. For the formulation of this theorem we need the concept of the *Haar measure* on a compact group (see for instance Rudin [6, §5.12–5.14]).

Theorem 1.19. *Let G be a compact group. There is a unique Borel measure μ on G, called* Haar measure*, such that $\mu(G) = 1$ and, for all Borel sets $E \subset G$ and for all $g \in G$, $\mu(gE) = \mu(E)$. Then this measure also satisfies $\mu(E) = \mu(Eg)$ for all Borel sets $E \subset G$ and for all $g \in G$.*

The left and right invariance of the measure μ can be equivalently phrased as follows: For each continuous function f on G we have

$$\int_G f(hg)\,d\mu(g) = \int_G f(g)\,d\mu(g) = \int_G f(gh)\,d\mu(g) \qquad (h \in G). \tag{1.65}$$

When we write $L^2(G)$ (or $L^1(G)$, etc.) we will mean the L^2-space on G with respect to the Haar measure.

We will now show that the measure μ on $SU(2)$ as defined by (1.61) is equal to the Haar measure on $SU(2)$. By (1.11) the group $SU(2)$ is homeomorphic with the unit sphere $S^3 = \{(a,c) \in \mathbb{C}^2 \mid |a|^2 + |c|^2 = 1\}$. Let $S \in SU(2)$. A left multiplication $T \mapsto ST\colon SU(2) \to SU(2)$ corresponds to some rotation of S^3. Thus a rotation invariant measure on S^3 will provide, after suitable normalization, the Haar measure on $SU(2)$. There exists, up to a constant factor, a unique rotation invariant measure ω on S^3. This measure is such that, for all continuous functions on $\mathbb{R}^4$ of compact support and with λ Lebesgue measure on $\mathbb{R}^4$,

$$\int_{\mathbb{R}^4} f\,d\lambda = \int_{r=0}^{\infty} \int_{\xi \in S^3} f(r\xi)\,r^3\,d\omega(\xi)\,dr. \tag{1.66}$$

Now take coordinates

$$x = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta\cos\psi, r\cos\theta\sin\psi)$$

on $\mathbb{R}^4$, which means for $r = 1$ that $x_1 + ix_2 = \sin\theta\,e^{i\phi}$, $x_3 + ix_4 = \cos\theta\,e^{i\psi}$. These are just the coordinates chosen in (1.61) for $(a,c) \in \mathbb{C}^2$ with $|a|^2 + |c|^2 = 1$. A straightforward computation of the Jacobian yields:

$$\begin{aligned}\int_{\mathbb{R}^4} & f(x_1,x_2,x_3,x_4)\,dx_1\,dx_2\,dx_3\,dx_4 \\ &= \int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}\int_{\psi=0}^{2\pi} f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta\cos\psi, r\cos\theta\sin\psi) \\ &\qquad\qquad \times r^3\sin\theta\cos\theta\,dr\,d\theta\,d\phi\,d\psi. \end{aligned} \tag{1.67}$$

Comparison of (1.66) and (1.67) gives, for continuous functions F on $S^3 \subset \mathbb{C}^2$, that

$$\int_{S^3} F\,d\omega = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}\int_{\psi=0}^{2\pi} F(\sin\theta\,e^{i\phi}, \cos\theta\,e^{i\psi})\,\sin\theta\cos\theta\,d\theta\,d\phi\,d\psi. \tag{1.68}$$

In view of the previous observations we have thus shown that the Haar measure on $SU(2)$ is given by (1.61).

As a preliminary to Schur's orthogonality relations we recall Schur's Lemma:

Lemma 1.20 (Schur)**.** *Let G be a group, and let π, ρ be irreducible representations of G on finite dimensional complex linear spaces V, W, respectively.*

*a) Let $A\colon V \to W$ be a linear map which is G-*intertwining*, i.e.,*

$$A\,\pi(g) = \rho(g)\,A \quad \textit{for all } g \in G. \tag{1.69}$$

Then A is bijective or $A = 0$.

b) Let $A\colon V \to V$ be linear such that $A\,\pi(g) = \pi(g)\,A$ for all $g \in G$. Then $A = \lambda I$ for some $\lambda \in \mathbb{C}$.

For the proof use in a) that $A(V)$ and $A^{-1}\{0\}$ are G-invariant subspaces, and in b) that A must have at least one (complex) eigenvalue and that eigenspaces of A are G-invariant.

In the case of bijective A in a) of Schur's lemma we call the representations π and ρ equivalent:

Definition 1.21. Let G be a group, and let π, ρ be representations of G on finite dimensional complex linear spaces V, W, respectively. Then π and ρ are called *equivalent* if there is a bijective linear map $A\colon V \to W$ satisfying (1.69).

Equivalence is an equivalence relation on the collection of finite dimensional representations of G, and also on the collection of finite dimensional irreducible representations of G.

Theorem 1.22. (Schur's orthogonality relations)
Let G be a compact group with Haar measure μ. Let π and ρ be finite dimensional complex irreducible unitary representations of G which are inequivalent to each other. Let π resp. ρ have matrix elements $(\pi_{i,j})_{i,j=1,\ldots,d_\pi}$ and $(\rho_{k,l})_{k,l=1,\ldots,d_\rho}$ with respect to certain orthonormal bases of their representation spaces. Then

$$\int_G \pi_{i,j}(g)\,\overline{\rho_{k,l}(g)}\,d\mu(g) = 0$$

and

$$\int_G \pi_{i,j}(g)\,\overline{\pi_{k,l}(g)}\,d\mu(g) = \frac{1}{d_\pi}\,\delta_{i,k}\,\delta_{j,l}.$$

For the proof fix j and l and put

$$A_{ik} := \int_G \pi_{ij}(g)\,\overline{\rho_{k,l}(g)}\,d\mu(g).$$

Then

$$\sum_k A_{ik}\rho_{k,r}(h) = \sum_m \pi_{im}(h) A_{m,r}.$$

Hence $A\rho(h) = \pi(h)A$ for all $h \in G$. Then $A = 0$ by Schur's lemma. But if we put $\rho := \pi$ then we conclude that $A = \lambda I$ for some $\lambda_{j,l} \in \mathbb{C}$. Then

$$\int_G \pi_{i,j}(g)\, \overline{\pi_{k,l}(g)}\, d\mu(g) = \lambda_{j,l}\, \delta_{i,k}.$$

We can compute $\lambda_{j,l} = (d_\pi)^{-1}$ by putting $k = i$ and summing over i.

Let $(\pi^\alpha)_{\alpha\in A}$ be a maximal set of mutually inequivalent finite dimensional complex irreducible unitary representations of G, and put $d_\alpha := d_{\pi^\alpha}$. Then the orthogonality relations in Theorem 1.22 can be written as

$$\int_G \pi^\alpha_{i,j}(g)\, \overline{\pi^\beta_{k,l}(g)}\, d\mu(g) = \frac{1}{d_\alpha}\, \delta_{\alpha,\beta}\, \delta_{i,k}\, \delta_{j,l}. \tag{1.70}$$

The functions $\pi_{i,j}$ are continuous on G, so they are certainly in $L^2(G)$. By (1.70) the functions $d_\alpha^{\frac{1}{2}}\, \pi^\alpha_{i,j}$ ($\alpha \in A$, $i,j = 1,\ldots,d_\alpha$) form an orthonormal system in $L^2(G)$. It can be shown in general that this orthonormal system is complete, but in special cases (like $G = SU(2)$) the completeness will already be obvious (it will follow from Proposition 1.18).

1.8 Irreducibility of representations

Comparison of (1.63) with (1.70) strongly suggests that (1.63) is the specialization of (1.70) to $SU(2)$. But we still have to show that the representations t^l are irreducible.

Put $a_\phi := \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi}\end{pmatrix}$. Then $a_\phi a_\psi = a_{\phi+\psi}$ and $a_{\phi+2\pi} = a_\phi$. The group $A := \{a_\phi \mid 0 \le \phi < 2\pi\}$ is a closed abelian subgroup of $SU(2)$. It is isomorphic and homeomorphic with the group $U(1)$ of complex numbers of absolute value 1, which has multiplication of complex numbers as the group multiplication. It follows from (1.4) that

$$t^l(a_\phi)\, \psi^l_n = e^{-2in\phi}\, \psi^l_n. \tag{1.71}$$

Lemma 1.23. *Let V be an invariant subspace of $\mathcal{H}_l$ with respect to the representation π^l of $SU(2)$. If $v \in V$ and $\langle v, \psi^l_m\rangle \neq 0$ then $\psi^l_m \in V$.*

Proof We have

$$v = \sum_{n=-l}^{l} \langle v, \psi^l_n\rangle\, \psi^l_n,$$

$$t^l(a_\phi)\, v = \sum_{n=-l}^{l} \langle v, \psi^l_n\rangle\, t^l(a_\phi)\, \psi^l_n = \sum_{n=-l}^{l} \langle v, \psi^l_n\rangle\, e^{-2in\phi}\, \psi^l_n.$$

Hence

$$\int_0^{2\pi} e^{2im\phi}\, t^l(a_\phi)\, v\, d\phi = \sum_{n=-l}^{l} \langle v, \psi^l_n\rangle \left(\int_0^{2\pi} e^{2im\phi} e^{-2in\phi}\, d\phi\right) \psi^l_n = 2\pi \langle v, \psi^l_m\rangle \psi^l_m.$$

The integral on the left should be interpreted as a Riemann integral of vectors, which can be approximated by Riemann sums of vectors. Since $v \in V$, each approximating Riemann sum is in V, and hence also their limit, the Riemann integral, is in V. Hence $2\pi\langle v, \psi_m^l\rangle\psi_m^l \in V$. So $\psi_m^l \in V$ if $\langle v, \psi_m^l\rangle \neq 0$. $\square$

This Lemma implies the following Proposition.

Proposition 1.24. *Let V be an invariant subspace of $\mathcal{H}_l$ with respect to the representation π^l of $SU(2)$. Then there is a subset $\mathcal{A}$ of $\{-l, \ldots, l\}$ such that $V = \mathrm{Span}\{\psi_n^l \mid n \in \mathcal{A}\}$. Let W be the orthoplement of V and $\mathcal{B}$ the complement of $\mathcal{A}$. Then W is also an invariant subspace and $W = \mathrm{Span}\{\psi_n^l \mid n \in \mathcal{B}\}$.*

Theorem 1.25. *The representation t^l of $SU(2)$ is irreducible.*

Proof Suppose t^l is not irreducible. By Proposition 1.24 $\mathcal{H}_l$ is the orthogonal direct sum of invariant subspaces $V = \mathrm{Span}\{\psi_n^l \mid n \in \mathcal{A}\}$ and $W = \mathrm{Span}\{\psi_n^l \mid n \in \mathcal{B}\}$, where $\{-l, \ldots, l\}$ is the disjoint union of certain nonempty subsets $\mathcal{A}$ and $\mathcal{B}$. One of these subsets, say $\mathcal{A}$, will contain l. Then some m will be in $\mathcal{B}$. Then $t^l(T)\,\psi_l^l$ will be in V for all $T \in SU(2)$, and therefore orthogonal to ψ_m^l. Hence $t_{m,l}^l(T) = 0$ for all $T \in SU(2)$. In particular, also using (1.6), we obtain

$$0 = t_{m,l}^l \begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} = (-1)^{l-m}\,(\cos\theta)^{l-m}\,(\sin\theta)^{l+m},$$

which gives a contradiction. $\square$

Exercise 1.26. (An interpretation of Krawtchouk polynomials as matrix elements of irreducible representations of $SU(2)$)

a) Prove that

$$ {}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}; x\right) = \frac{(c-b)_n}{(c)_n}\,{}_2F_1\left(\begin{matrix} -n, b \\ b-c-n+1 \end{matrix}; 1-x\right) \qquad (n = 0, 1, 2, \ldots). \tag{1.72}$$

(Use (1.54) and (1.52).)

b) Prove that

$$ {}_2F_1\left(\begin{matrix} -n, -m \\ c \end{matrix}; x\right) = \frac{(c)_{m+n}}{(c)_n (c)_m}\,{}_2F_1\left(\begin{matrix} -n, -m \\ -c-n-m+1 \end{matrix}; 1-x\right) \qquad (n, m = 0, 1, 2, \ldots). \tag{1.73}$$

(Use (1.72).)

c) Prove that, for $m + n \geq 0$,

$$ t_{m,n}^l \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \binom{2l}{l-m}^{\frac{1}{2}} \binom{2l}{l-n}^{\frac{1}{2}} b^{l-m} c^{l-n} d^{m+n}\,{}_2F_1\left(\begin{matrix} -l+m, -l+n \\ -2l \end{matrix}; \frac{bc-ad}{bc}\right). \tag{1.74}$$

(Use (1.73) and (1.57).)

d) Prove that, for $m+n \ge 0$,

$$t^l_{m,n}\begin{pmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{pmatrix} = \binom{2l}{l-m}^{\frac12}\binom{2l}{l-n}^{\frac12}(-1)^{l-m}(\cos\theta)^{2l-m-n}(\sin\theta)^{m+n}$$
$$\times K_{l-m}(l-n;\cos^2\theta,2l), \quad (1.75)$$

where the *Krawtchouk polynomials* are given by

$$K_n(x;p,N) := {}_2F_1\left(\begin{matrix}-n,-x\\ -N\end{matrix};p^{-1}\right) \qquad (n=0,1,\ldots,N). \quad (1.76)$$

e) Prove that

$$K_n(x;p,N) = (1-p^{-1})^{x+n-N}\,K_{N-n}(N-x;p,N). \quad (1.77)$$

(Use (1.76) and (1.28).)

f) Prove that (1.75) remains valid for all m,n.
(Use (1.9) and (1.77).)

g) Show that

$$\sum_{n=-l}^{l} t^l_{m,n}\begin{pmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{pmatrix} t^l_{m',n}\begin{pmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{pmatrix} = \delta_{m,m'}\,, \quad (1.78)$$

and rewrite this as an orthogonality relation for the Krawtchouk polynomials occurring on the right-hand side of (1.75).

Exercise 1.27. (Addition formula and product formula for Legendre polynomials)
Let $l=0,1,2,\ldots$.

a) Prove that, for $ad-bc=1$,

$$t^l_{0,0}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = P_l(2ad-1), \quad (1.79)$$

where $P_l := P^{(0,0)}_l$ is the *Legendre polynomial.*

b) Prove that

$$t^l_{0,0}(T) = P_l(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\phi)$$

if

$$T = \begin{pmatrix}\sin\frac12\theta_1 & -\cos\frac12\theta_1\\ \cos\frac12\theta_1 & \sin\frac12\theta_1\end{pmatrix}\begin{pmatrix}e^{\frac12 i\phi} & 0\\ 0 & e^{-\frac12 i\phi}\end{pmatrix}\begin{pmatrix}\sin\frac12\theta_2 & \cos\frac12\theta_2\\ -\cos\frac12\theta_2 & \sin\frac12\theta_2\end{pmatrix}.$$

c) Prove that

$$P_l(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\phi) = P_l(\cos\theta_1)\,P_l(\cos\theta_2)$$
$$+\sum_{0<|k|\le l} t^l_{0,k}\begin{pmatrix}\sin\frac12\theta_1 & -\cos\frac12\theta_1\\ \cos\frac12\theta_1 & \sin\frac12\theta_1\end{pmatrix} t^l_{k,0}\begin{pmatrix}\sin\frac12\theta_2 & \cos\frac12\theta_2\\ -\cos\frac12\theta_2 & \sin\frac12\theta_2\end{pmatrix} e^{-ik\phi}. \quad (1.80)$$

d) Prove that

$$P_l(\cos\theta_1)\,P_l(\cos\theta_2)=\frac{1}{2\pi}\int_0^{2\pi}P_l(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\phi)\,d\phi. \tag{1.81}$$

1.9 On the proof of Proposition 1.18

I give a little more detail about the proof of Proposition 1.18. First observe, by (1.59), by the symmetries (1.8)–(1.10), and by (1.52) (also valid if we replace p_n by P_n) that, for $m,n\in\{-l,-l+1,\ldots,l\}$ such that $m\pm n\ge 0$ and for $a,c\in\mathbb{C}$:

$$\left.\begin{aligned}
t^l_{m,n}\begin{pmatrix}a&-\overline{c}\\ c&\overline{a}\end{pmatrix}&=c^{m-n}\overline{a}^{m+n}\\
t^l_{n,m}\begin{pmatrix}a&-\overline{c}\\ c&\overline{a}\end{pmatrix}&=(-\overline{c})^{m-n}\overline{a}^{m+n}\\
t^l_{-m,-n}\begin{pmatrix}a&-\overline{c}\\ c&\overline{a}\end{pmatrix}&=(-\overline{c})^{m-n}a^{m+n}\\
t^l_{-n,-m}\begin{pmatrix}a&-\overline{c}\\ c&\overline{a}\end{pmatrix}&=c^{m-n}a^{m+n}
\end{aligned}\right\}$$

$$\times\left(\frac{(l+m)!\,(l-m)!}{(l+n)!\,(l-n)!}\right)^{\frac12}(|a|^2+|c|^2)^{l-m}\,P_{l-m}^{(m-n,m+n)}\left(\frac{|a|^2-|c|^2}{|a|^2+|c|^2}\right). \tag{1.82}$$

These exhaust all matrix elements of t^l. They are homogeneous of degree $2l$ in $a,\overline{a},c,\overline{c}$. We get their restriction to $SU(2)$ if we take $|a|^2+|c|^2=1$. For the restriction it makes no difference if we replace in (1.82) the factor $(|a|^2+|c|^2)^{l-m}$ by $(|a|^2+|c|^2)^{r-m}$ with $r-l\in\mathbb{Z}_{\ge0}$. Then we obtain homogeneous polynomials of degree $2r$ in $a,\overline{a},c,\overline{c}$. By a little thought we see that all functions thus obtained, i.e., $(|a|^2+|c|^2)^{l-m}$ replaced by $(|a|^2+|c|^2)^{r-m}$, r fixed, $0\le l\in\{r,r-1,\ldots\}$, $m,n\in\{-l,-l+1,\ldots,l\}$, $m\pm n\ge0$ yield a basis for all homogeneous polynomials of degree $2r$ in $a,\overline{a},c,\overline{c}$. Then there follows our claim after the statement of Proposition 1.18 that the span of the matrix elements $t^l_{m,n}$, as functions of $\begin{pmatrix}a&-\overline{c}\\ c&\overline{a}\end{pmatrix}\in SU(2)$ ($|a|^2+|c|^2=1$), equals the space of polynomials in the four real variables $\operatorname{Re}a,\operatorname{Im}a,\operatorname{Re}c,\operatorname{Im}c$, restricted to $|a|^2+|c|^2=1$. It is a consequence of the Stone-Weierstrass theorem (see for instance [11, §36]) that, for a compact subset X of $\mathbb{R}^n$, the space of polynomials on $\mathbb{R}^n$ restricted to X lies dense in $C(X)$ with respect to the subnorm. Then, if μ is a finite Borel measure on X, this space of polynomials is also dense in $C(X)$ with respect to the norm of $L^2(X,\mu)$. Finally, $C(X,\mu)$ is then dense in $L^2(X,\mu)$, see for instance [10, Theorem 3.14].

1.10 Lie groups and Lie algebra

References for the general theory of Lie groups and Lie algebras are [7], [8], [9], [12].

Definition 1.28. A *Lie group* is a group G which is also a C^∞ manifold such that the maps $(g,h)\mapsto gh\colon G\times G\to G$ and $g\mapsto g^{-1}\colon G\to G$ are C^∞.

The fact that G is a C^∞ manifold implies that G is a topological space (Hausdorff and satisfying the second axiom of countability) and the fact that the group operations for G are C^∞ implies that they are continuous. So a Lie group is in particular a topological group.

The fact that the multiplication on a Lie group G is C^∞ can be described as follows. Let $g_0, h_0 \in H$. Take an open neighbourhood W of $g_0 h_0$ on which there are local coordinates $(z_1, \ldots, z_n)$. Then there are open neigbourhoods U of g_0 and V of h_0 such that $UV \subset W$ and such that we have local coordinates $(x_1, \ldots, x_n)$ on U and local coordinates $(y_1, \ldots, y_n)$ on V. Then we can write the multiplication on $U \times V$ as

$$(x_1, \ldots, x_n)(y_1, \ldots, y_n) = (z_1, \ldots, z_n),$$

which gives z_j as a function $z_j(x_1, \ldots, x_n, y_1, \ldots, y_n)$ of $2n$ real variables. Then we require that these functions are C^∞.

Example 1.29. Consider $G = GL(n, \mathbb{C})$. This is an open part of $M_n(\mathbb{C})$ (the space of all complex $n \times n$ matrices), so it is an open part of an n^2-dimensional complex linear space, and we can take on the whole group one system of local coordinates, namely the real and imaginary parts of the matrix entries. Even better, we can take the matrix entries themselves as complex coordinates. Thus $GL(n, \mathbb{C})$ is a complex analytic manifold, and the group operations are complex analytic (even stronger: multiplication is a polynomial map and taking the inverse is a rational map). We say that $GL(n, \mathbb{C})$ is a *complex analytic group*. Of course, $GL(n, \mathbb{C})$ is then also a *real analytic group* (real analytic manifold and group operations are real analytic) and a C^∞ group (Lie group as defined in Definition 1.28).

It can be shown that every Lie group is in fact a real analytic group, but complex analytic groups are much more special.

Suppose that M is a C^∞ manifold. Let $p \in M$ and let U be an open neighbourhood of p on which there are local coordinates $(x_1, \ldots, x_n)$. Let p have coordinates $x_0 = (x_{01}, \ldots, x_{0n})$. With respect to this system of coordinates the *tangent space* T_pM to M at p is just the linear space $\mathbb{R}^n$. With a tangent vector $a \in \mathbb{R}^n$ we associate on the one hand an equivalence class of C^∞ curves through p and on the other hand a linear functional on the space $C^\infty(U)$ of real-valued C^∞ functions on U. The equivalence class of curves consists of all C^∞ maps $t \mapsto x(t) = (x_1(t), \ldots, x_n(t))$ such that $x(0) = x_0$, i.e. p, and $x'(0) = (x_1'(0), \ldots, x_n'(0)) = (a_1, \ldots, a_n) = a$. The linear functional A is given by

$$Af := \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j} f(x_1, \ldots, x_n)\Big|_{x=x_0} \qquad (f \in C^\infty(U)). \tag{1.83}$$

Then we can also connect $x(t)$ and A, if they are both assiciated with a as above:

$$Af = \frac{d}{dt} f(x(t))\Big|_{t=0} \qquad (f \in C^\infty(U)). \tag{1.84}$$

A *vector field* X on M is an assignment of a tangent vector $X_p \in T_pM$ to each $p \in M$ such that, on any open set U in M with local coordinates $(x_1, \ldots, x_n)$, we have

$$X_x f = \sum_{j=1}^{n} c_j(x_1, \ldots, x_n) \frac{\partial}{\partial x_j} f(x_1, \ldots, x_n) \qquad (f \in C^\infty(U)) \tag{1.85}$$

with the c_j being C^∞ functions in $x_1, \ldots, x_n$. Here X_x means the tangent vector attributed by X to the point p with coordinates $x = (x_1, \ldots, x_n)$. Thus we can consider vector fields as first order linear operators on M with C^∞ coefficients.

Let X and Y be vector fields on M. Define the *commutator* $[X, Y]$ of X and Y by

$$[X,Y]f := X(Yf) - Y(Xf) \qquad (f \in C^\infty(M)).$$

Then $[X, Y]$ is again a vector field. Clearly $[X, Y]$ is linear in X and Y and we have *anticommutativity*

$$[X,Y] = -[Y,X]$$

and the *Jacobi identity*

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0.$$

Thus the real linear space of vector fields on X is a real *Lie algebra*.

Now let G be a Lie group and let V be an open neighbourhood of e with local coordinates $(y_1, \ldots, y_n)$ such that e has coordinates $(0, \ldots, 0)$. Consider the tangent space T_eG as the space of linear operators A associated with $a \in \mathbb{R}^n$ and given by

$$Af := \sum_{j=1}^n a_j \frac{\partial}{\partial y_j} f(y_1, \ldots, y_n)\Big|_{y=0} \qquad (f \in C^\infty(V)). \tag{1.86}$$

Define in terms of A the vector field X_A on G by

$$(X_A f)(g) := A(h \mapsto f(gh)) \qquad (f \in C^\infty(G),\ g \in G). \tag{1.87}$$

Indeed, for given $g \in G$ and $f \in C^\infty(G)$ the function $h \mapsto f(gh)$ restricted to V is a C^∞ function on V, by which $A(h \mapsto f(gh))$ is well defined. Also, for g in an open set U with local coordinates $(x_1, \ldots, x_n)$ we can write the right-hand side of (1.87) in the form of the right-hand side of (1.85) with C^∞ coefficients. Thus X_A is a vector field and it is *left invariant* in the sense that

$$(X_A f)(g_1 g) = X_A(g \mapsto f(g_1 g)) \qquad (f \in C^\infty(G),\ g, g_1 \in G). \tag{1.88}$$

It is not difficult to show that all left invariant vector fields on G are of the form X_A for some $A \in T_eG$, and that the map $A \mapsto X_A$ is a linear bijection from T_eG onto the linear space of left invariant vector fields on G. Note that we can recover A from X_A by

$$Af = (X_A f)(e). \tag{1.89}$$

By this linear bijection the Lie algebra structure of the space of left invariant vector fields can be transfered to T_eG. Thus T_eG becomes a Lie algebra with *Lie bracket* $[A, B]$ defined by

$$X_{[A,B]} := [X_A, X_B] \qquad (A, B \in T_eG). \tag{1.90}$$

Note that we cannot a priori consider this Lie bracket $[A, B]$ as a commutator $AB - BA$ because it is not clear how to define the product AB for $A, B \in T_eG$. The tangent space T_eG considered as a Lie algebra is also denoted by $\mathfrak{g}$ or $\mathrm{Lie}(G)$. It is called the *Lie algebra of the Lie group G*.

1.11 Linear Lie groups

Definition 1.30. Let G be a Lie group with subgroup H. We call H a *regularly embedded Lie subgroup* of G if, for each $h \in H$, there is an open neighbourhood U of h in G on which there are local coordinates $(x_1, \ldots, x_n)$ $(x_1, \ldots, x_n \in (-a, a))$ such that $H \cap U$ consists of all elements of U with coordinates $(x_1, \ldots, x_m, 0, \ldots, 0)$ $(x_1, \ldots, x_m \in (-a, a))$. Then H becomes a Lie group itself with the relative topology from G and with the structure of C^∞ manifold given by the local coordinates $(x_1, \ldots, x_m)$ on the sets $H \cap U$. Then H is also a closed subset of G.

If a subgroup H of G would satisfy the requirements in the above definition except for the rule that in the local coordinate neighbourhood U the intersection $H \cap U$ should consist of only one slice, then we still speak of a Lie subgroup H but it may not be regularly embedded and it may not be a closed subset of G. Its topology, compatible with the structure of C^∞ manifold, may be different from the relative topology. An example is the abelian Lie group $G := \mathbb{R}^2/\mathbb{Z}^2$ with subgroup $H := \{(x, cx) \pmod{\mathbb{Z}^2} \mid x \in \mathbb{R}\}$, where c is irrational. Usually we will only consider regularly embedded Lie subgroups.

Definition 1.31. A *regularly embedded linear Lie group* is a regularly embedded Lie subgroup of $GL(n, \mathbb{C})$ for some $n \in \mathbb{Z}_{>0}$.

There is a theorem stating that every closed subgroup of $GL(n, \mathbb{C})$ has a unique structure of C^∞ manifold by which it becomes a regularly embedded linear Lie group. For an example of a subgroup of $GL(1, \mathbb{C})$ which is not a Lie group, consider $G := \{e^x \mid x \in \mathbb{Q}\}$ with the relative topology as a subset of $GL(1, \mathbb{C})$. It is not locally Euclidean, so it cannot be a Lie group.

Let $G \subset GL(n, \mathbb{C})$ be a regularly embedded linear Lie group. Then one way of obtaining the tangent space $T_I G$ to G at the identity element I is as the set of all matrices $T'(0)$ such that $t \mapsto T(t)$ is a C^∞ curve in $GL(n, \mathbb{C})$ completely lying in G and with $T(0) = I$. This is indeed a real linear subspace of $M_n(\mathbb{C})$. Moreover, the Lie algebra structure of $T_I G$, which is by definition induced by the Lie algebra structure of the left invariant vector fields on G, is also obtained as a commutator product:

Theorem 1.32. *Let $G \subset GL(n, \mathbb{C})$ be a regularly embedded linear Lie group with Lie algebra $\mathfrak{g} = T_I G \subset M_n(\mathbb{C})$. Then the Lie bracket on $\mathfrak{g}$ equals*

$$[A, B] = AB - BA \qquad (A, B \in \mathfrak{g}),$$

where AB and BA mean matrix multiplication.

An important tool in connecting the Lie algebra $\mathfrak{g}$ of $G \subset GL(n, \mathbb{C})$ with G is the *exponential map*

$$\exp(A) := \sum_{k=0}^{\infty} \frac{1}{k!} A^k \qquad (A \in M_n(\mathbb{C})). \tag{1.91}$$

Some properties of exp:

a) $\exp(A + B) = \exp(A) \exp(B)$ if A and B commute.

b) $\exp(A) \in GL(n,\mathbb{C})$ for all $A \in M_n(\mathbb{C})$.

c) $\exp(-A) = (\exp(A))^{-1}$.

d) $\exp(TAT^{-1}) = T\exp(A)\,T^{-1} \quad (A \in M_n(\mathbb{C}),\, T \in GL(n,\mathbb{C}))$.

e) $\det(\exp A) = e^{\operatorname{tr} A}$.

f) If $T(t) := \exp(tA)$ then $T'(t) = AT(t) = T(t)A$. In particular, $T'(0) = A$.

g) There are open neighbourhoods U of 0 in $M_n(\mathbb{C})$ and V of I in $GL(n,\mathbb{C})$ such that $\exp\colon U \to V$ is bijective and a C^∞ diffeomorphism (even complex analytic).

Let $G \subset GL(n,\mathbb{C})$ be a regularly embedded linear Lie group with Lie algebra $\mathfrak{g} \subset M_n(\mathbb{C})$. Then

$$\exp(\mathfrak{g}) \subset G. \tag{1.92}$$

Furthermore, after possibly shrinking U and V in item g) above while keeping the diffeomorphism property of exp, we have

$$\exp(\mathfrak{g} \cap U) = G \cap V. \tag{1.93}$$

The converse also holds:

Theorem 1.33. *Let G be a subgroup of $GL(n,\mathbb{C})$, $\mathfrak{g}$ a real linear subspace of $M_n(\mathbb{C})$, let U and V be as in item g) above, and suppose that* (1.93) *holds. Then G is a regularly embedded linear Lie group with Lie algebra $\mathfrak{g}$. For any basis $A_1,\ldots,A_m$ of $\mathfrak{g}$ the map $(x_1,\ldots,x_m) \mapsto \exp(x_1A_1 + \cdots + x_mA_m)$ gives a system of local coordinates on $G \cap V$.*

Example 1.34. Let $U(n)$ be the subgroup of $GL(n,\mathbb{C})$ consisting all *unitary* matrices (matrices T such that $TT^* = I$). Let $u(n)$ be the real linear subspace of $M_n(\mathbb{C})$ consisting of all *skew hermitian* matrices (matrices A such that $A+A^* = 0$). Then we can use Theorem 1.33 in order to show that $U(n)$ is a regularly embedded linear Lie group with Lie algebra $u(n)$. Indeed, let U and V be as in item g). Replace U by $U_0 := U \cap U^* \cap (-U) \cap (-U^*)$ and V by $V_0 := \exp(U_0)$. Then U_0 is still an open neighbourhood of 0 in $M_n(\mathbb{C})$ and it is closed under taking opposites or adjoints. If $A \in u(n) \cap U_0$ then $(\exp A)^* = \exp(A^*) = \exp(-A) = (\exp A)^{-1}$. Hence $\exp A \in U(n) \cap V_0$. Conversely, if $T \in U(n) \cap V_0$ then $T = \exp A$ for some $A \in U_0$ and $(\exp A)^* = (\exp A)^{-1}$. Hence $\exp(A^*) = \exp(-A)$. Since A^* and $-A$ are in U_0 and exp is injective on U_0, we conclude that $A^* = -A$. Hence $A \in u(n) \cap U_0$. By Theorem 1.33, $U(n)$ is a regularly embedded linear Lie group with Lie algebra $u(n)$.

Let next $SU(n)$ be the subgroup of $GL(n,\mathbb{C})$ consisting of all unitary matrices of determinant 1, and let $su(n)$ be the real linear subspace of $M_n(\mathbb{C})$ consisting of all skew hermitian matrices of trace 0. Then we can show that $SU(n)$ is a regularly embedded linear Lie group with Lie algebra $su(n)$ by the following refinement of the above reasoning. Start with U and V such that moreover $|\operatorname{tr} A| < \pi$ if $A \in U$. Then construct U_0 and V_0 as above. Then use that $\det(\exp A) = e^{\operatorname{tr} A} = 1$ if $\operatorname{tr} A = 0$. Conversely, if $\det(\exp A) = 1$ and $|\operatorname{tr} A| < \pi$ then $\operatorname{tr} A = 0$.

Let $G \subset GL(n,\mathbb{C})$ be a regularly embedded linear Lie group with Lie algebra $\mathfrak{g}$. By (1.84) and item f) we can write A and X_A given by (1.86) and (1.87) for $f \in C^\infty(G)$ and $A \in \mathfrak{g}$ as:

$$Af = \frac{d}{dt} f(\exp(tA))\Big|_{t=0}, \tag{1.94}$$

$$X_A f(T) = \frac{d}{dt} f(T \exp(tA))\Big|_{t=0} \qquad (T \in G). \tag{1.95}$$

1.12 Representations of Lie groups

Let G be a Lie group. Let π be a representation of G on $\mathbb{C}^n$, while G is only considered as an abstract group, i.e., we have a group homomorphism $\pi\colon G \to GL(n, \mathbb{C})$. Since G is in particular a topological group, we also require that π is continuous. In order to let π also respect the structure of C^∞ manifold of G, we want to require moreover that π is a C^∞ map, i.e., dat all matrix elements π_{ij} are C^∞ functions $\pi_{ij}\colon G \to \mathbb{C}$. However, there is a theorem that for a Lie group G continuity of the homomorphism π already implies that it is C^∞.

This obviously can also be phrased for a representation of a Lie group G on a finite dimensional complex vector space V. Then $\pi\colon G \to GL(V)$ is a group homomorphism, and for any choice of basis $e_1, \ldots, e_n$ of V we must have that the matrix elements of $\pi(g)$ wiht respect to this basis are C^∞ functions of $g \in G$.

If $\mathfrak{g}$ is a (real) Lie algebra then a *representation* of $\mathfrak{g}$ on a finite dimensional complex vector space V is a real linear map $\phi\colon \mathfrak{g} \to \mathrm{End}(V)$ such that $\phi([A, B]) = \phi(A)\phi(B) - \phi(B)\phi(A)$. A representation of a Lie group G implies a representation of its Lie algebra:

Theorem 1.35. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\pi\colon G \to GL(n, \mathbb{C})$ be a representation of G on $\mathbb{C}^n$ (i.e., π is a C^∞ homomorphism). Define the real linear map $d\pi\colon \mathfrak{g} \to M_n(\mathbb{C})$ by*

$$d\pi(A) := A(x \mapsto \pi(x)), \quad \text{i.e.,} \quad (d\pi(A))_{i,j} := A(x \mapsto \pi_{i,j}(x)) \qquad (A \in \mathfrak{g}),$$

where A is considered as in (1.86). Then $d\pi$ is a Lie algebra representation of $\mathfrak{g}$.

If moreover $G \subset GL(m, \mathbb{C})$ is a regularly embedded linear Lie group then, for $A \in \mathfrak{g}$:

$$\exp(d\pi(A)) = \pi(\exp A), \tag{1.96}$$

$$d\pi(A) = \frac{d}{dt} \pi(\exp(tA))\Big|_{t=0}. \tag{1.97}$$

This theorem can also be phrased for a representation $\pi\colon G \to GL(V)$. Then $d\pi\colon \mathfrak{g} \to \mathrm{End}(V)$. In particular, we still have (1.97).

Corollary 1.36. *Under the assumptions of the first part of Theorem 1.35 and with X_A the left invariant vector field on G given by (1.87) we have:*

$$X_A\, \pi_{i,j} = \sum_{k=1}^{n} (d\pi(A))_{k,j}\, \pi_{i,k} \qquad (A \in \mathfrak{g}). \tag{1.98}$$

Proof For $g, h \in G$ and $A \in \mathfrak{g}$ we successively have:

$$\pi_{i,j}(gh) = \sum_{k=1}^{n} \pi_{k,j}(h)\, \pi_{i,k}(g),$$

$$A(h \mapsto \pi_{i,j}(gh)) = \sum_{k=1}^{n} A(h \mapsto \pi_{k,j}(h))\, \pi_{i,k}(g),$$

$$(X_A \pi_{i,j})(g) = \sum_{k=1}^{n} (d\pi(A))_{k,j}\, \pi_{i,k}(g). \qquad \square$$

1.13 Representations of $su(2)$

The Lie algebra $su(2)$ consists of all 2×2 matrices A such that $A + A^* = 0$ and $\operatorname{tr} A = 0$. These are precisely the matrices $\begin{pmatrix} it & -\bar{c} \\ c & -it \end{pmatrix}$ with $t \in \mathbb{R}$ and $c \in \mathbb{C}$. It has real dimension 3. Choose a basis A, B, C of $su(2)$ given by

$$A := \tfrac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B := \tfrac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad C := \tfrac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Then

$$[A, B] = C, \quad [B, C] = A, \quad [C, A] = B. \tag{1.99}$$

By exponentiation A, B, C generate three one-parameter subgroups of $SU(2)$:

$$a_\theta := \exp(\theta A) = \begin{pmatrix} \cos \frac{1}{2}\theta & -\sin \frac{1}{2}\theta \\ \sin \frac{1}{2}\theta & \cos \frac{1}{2}\theta \end{pmatrix}, \tag{1.100}$$

$$b_\phi := \exp(\phi B) = \begin{pmatrix} e^{\frac{1}{2}i\phi} & 0 \\ 0 & e^{-\frac{1}{2}i\phi} \end{pmatrix}, \tag{1.101}$$

$$c_\psi := \exp(\psi C) = \begin{pmatrix} \cos \frac{1}{2}\psi & i\sin \frac{1}{2}\psi \\ i\sin \frac{1}{2}\psi & \cos \frac{1}{2}\psi \end{pmatrix}. \tag{1.102}$$

We compute the representations dt^l of $su(2)$, where the representations t^l of $SU(2)$ on $\mathcal{H}_l$ are defined by (1.2). Then

$$(t^l(a_\theta)f)(z_1, z_2) = f(z_1 \cos \tfrac{1}{2}\theta + z_2 \sin \tfrac{1}{2}\theta, -z_1 \sin \tfrac{1}{2}\theta + z_2 \cos \tfrac{1}{2}\theta) \qquad (f \in \mathcal{H}_l).$$

By (1.100) and (1.97) we compute for $f \in \mathcal{H}_l$:

$$(dt^l(A)f)(z_1, z_2) = \tfrac{1}{2}\Big(z_2 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_2}\Big) f(z_1, z_2). \tag{1.103}$$

Similarly, (1.97) combined with (1.101) or (1.102) yield

$$(dt^l(B)f)(z_1,z_2) = \tfrac12 i\Big(z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2}\Big)f(z_1,z_2), \tag{1.104}$$

$$(dt^l(C)f)(z_1,z_2) = \tfrac12 i\Big(z_2\frac{\partial}{\partial z_1} + z_1\frac{\partial}{\partial z_2}\Big)f(z_1,z_2). \tag{1.105}$$

Note that the partial differential operators on the right-hand sides of (1.103)–(1.105) are independent of l and act on the space of all polynomials in z_1, z_2. This is no surprise since the original definition of t^l by (1.2) is also independent of l. As such they also satisfy the commutator relations (1.99): they span a Lie algebra of differential operators isomorphic to $su(2)$. This follows either by the homomorphism property of t^l or by direct computation of the commutators.

Let us see how $dt^l(A)$, $dt^l(B)$, $dt^l(C)$ act on the orthonormal basis vectors ψ_n^l of $\mathcal{H}^l$ given by (1.1). By substitution of $f := \psi_n^l$ in (1.103)–(1.105) we obtain:

$$dt^l(A)\psi_n^l = \tfrac12(l-n)^{\frac12}(l+n+1)^{\frac12}\psi_{n+1}^l - \tfrac12(l+n)^{\frac12}(l-n+1)^{\frac12}\psi_{n-1}^l, \tag{1.106}$$

$$dt^l(B)\psi_n^l = -in\psi_n^l, \tag{1.107}$$

$$dt^l(C)\psi_n^l = \tfrac12 i(l-n)^{\frac12}(l+n+1)^{\frac12}\psi_{n+1}^l + \tfrac12 i(l+n)^{\frac12}(l-n+1)^{\frac12}\psi_{n-1}^l, \tag{1.108}$$

where terms involving ψ_{l+1}^l or ψ_{-l-1}^l disappear. We see that $dt^l(B)$ acts diagonally on this basis and that $dt^l(\pm A - iC)$ act as *ladder operators*:

$$dt^l(A-iC)\psi_n^l = (l-n)^{\frac12}(l+n+1)^{\frac12}\psi_{n+1}^l, \tag{1.109}$$

$$dt^l(-A-iC)\psi_n^l = (l+n)^{\frac12}(l-n+1)^{\frac12}\psi_{n-1}^l. \tag{1.110}$$

The elements $\pm A - iC$ are no longer in the Lie algebra $su(2)$ but they are certainly in its complexification $sl(2,\mathbb{C})$, the Lie algebra of complex 2×2 matrices of trace 0. Furthermore, $sl(2,\mathbb{C})$ is the Lie algebra of $SL(2,\mathbb{C})$, the linear Lie group of complex 2×2 matrices of determinant 1. We defined the representations t^l originally for $GL(2,\mathbb{C})$, so certainly for $SL(2,\mathbb{C})$. Therefore, we can also use Theorem (1.97) for $\pi = t^l$, $G = SL(2,\mathbb{C})$, $\mathfrak{g} = sl(2,\mathbb{C})$. Note that

$$\exp(t(A-iC)) = \begin{pmatrix}1&0\\t&1\end{pmatrix}, \qquad \exp(t(-A-iC)) = \begin{pmatrix}1&t\\0&1\end{pmatrix}. \tag{1.111}$$

From (1.98) combined with (1.109), (1.110) we conclude:

$$X_{A-iC}\, t_{m,n}^l = (l-n)^{\frac12}(l+n+1)^{\frac12}\, t_{m,n+1}^l, \tag{1.112}$$

$$X_{-A-iC}\, t_{m,n}^l = (l+n)^{\frac12}(l-n+1)^{\frac12}\, t_{m,n-1}^l. \tag{1.113}$$

From (1.112), (1.113) we want to derive differential relations for Jacobi polynomials which shift the degree and the parameters.

Let $n \pm m \ge 0$, $ad - bc = 1$. From (1.59), (1.8) and (1.52) we obtain

$$t_{m,n}^l\begin{pmatrix}a&b\\c&d\end{pmatrix} = \left(\frac{(l+n)!\,(l-n)!}{(l+m)!\,(l-m)!}\right)^{\frac12} b^{n-m}d^{n+m}\,P_{l-n}^{(n-m,n+m)}(2ad-1). \tag{1.114}$$

By (1.95) and (1.111) we have

$$X_{A-iC}\, t^l_{m,n} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{d}{dt}\Big|_{t=0} t^l_{m,n} \begin{pmatrix} a+bt & b \\ c+dt & d \end{pmatrix}.$$

In combination with (1.114) and (1.112) this yields

$$\begin{aligned} b^{n-m} d^{n+m} \frac{d}{dt}\Big|_{t=0} & P^{(n-m,n+m)}_{l-n}(2(a+bt)d-1) \\ &= (l+n+1)\, b^{n-m+1} d^{n+m+1}\, P^{(n-m+1,n+m+1)}_{l-n-1}(2ad-1). \end{aligned}$$

Finally this is reduced to

$$\frac{d}{dx}\Big|_{x=2ad-1} P^{(n-m,n+m)}_{l-n}(x) = \tfrac{1}{2}(l+n+1)\, P^{(n-m+1,n+m+1)}_{l-n-1}(2ad-1).$$

Thus we have given a proof by representation theory of the differentiation formula

$$\frac{d}{dx} P^{(\alpha,\beta)}_n(x) = \tfrac{1}{2}(n+\alpha+\beta+1)\, P^{(\alpha+1,\beta+1)}_{n-1}(x) \tag{1.115}$$

in the case $\alpha, \beta \in \mathbb{Z}_{\geq 0}$. Formula (1.115) is essentially the same as (1.44).

Exercise 1.37. Use (1.95), (1.111), (1.114) and (1.113) in order to derive

$$\frac{d}{dx}\Big((1-x)^\alpha (1+x)^\beta P^{(\alpha,\beta)}_n(x)\Big) = -2(n+1)(1-x)^{\alpha-1}(1+x)^{\beta-1} P^{(\alpha-1,\beta-1)}_{n+1}(x) \tag{1.116}$$

for $\alpha, \beta \in \mathbb{Z}_{>0}$. Formula (1.116) is essentially the same as (1.45).

Exercise 1.38. Let G be a compact group and let π be an irreducible unitary representation of G on a finite dimensional complex linear space V. Define the *character* χ_π of the representation π by

$$\chi_\pi(g) := \mathrm{tr}\,(\pi(g)) \qquad (g \in G). \tag{1.117}$$

a) Prove that χ_π is a *central function* on G:

$$\chi_\pi(hgh^{-1}) = \chi_\pi(g) \qquad (g, h \in G).$$

b) Let μ be the Haar measure on G. Show that $\int_G |\chi_\pi(g)|^2\, d\mu(g) = 1$ and that, for ρ another irreducible unitary representation of G which is not equivalent to π, $\int_G \chi_\pi(g)\, \overline{\chi_\rho(g)}\, d\mu(g) = 0$.

c) Let π have matrix elements π_{ij} $(i, j = 1, \ldots, d_\pi)$ with respect to some orthonormal basis of V. Let $f \in \mathrm{Span}\{\pi_{ij}\}_{i,j=1,\ldots,d_\pi}$ be a central function on G. Show that $f = \mathrm{const.}\, \chi_\pi$.

d) Define the *Chebyshev polynomial of the second kind* by

$$U_n(\cos\theta) := \frac{\sin((n+1)\theta)}{\sin\theta}. \tag{1.118}$$

Show that $U_n(x)$ is a polynomial of degree n in x and that (by the orthogonality properties)

$$U_n(x) = \frac{n+1}{P_n^{(\frac12,\frac12)}(1)}\, P_n^{(\frac12,\frac12)}(x). \tag{1.119}$$

e) Let χ_l be the character of the representation t^l of $SU(2)$. Show that

$$\chi_l \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = U_{2l}(\cos\theta), \qquad \chi_l \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} = U_{2l}(\operatorname{Re} a) \quad (|a|^2 + |c|^2 = 1). \tag{1.120}$$

1.14 The Casimir operator

Let $\mathfrak{g}$ be a Lie algebra over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then the *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ is defined as the quotient algebra $\mathcal{T}(\mathfrak{g})/J$, where $\mathcal{T}(\mathfrak{g})$ is the tensor algebra $\mathbb{F} + \mathfrak{g} + \mathfrak{g}^2 + \cdots$ of $\mathfrak{g}$ ($\mathfrak{g}$ considered as a linear space) and J is the two-sided ideal generated by all elements $AB - BA - [A, B]$ ($A, B \in \mathfrak{g}$). So $\mathcal{U}(\mathfrak{g})$ is an associative algebra with unit element. Suppose that $\mathfrak{g}$ is finite dimensional with basis $X_1, \ldots, X_n$. Then the *PBW theorem* (theorem of Poincaré-Birkhoff-Witt, see for instance [14, §17.3]) says that the elements $X_1^{k_1} \ldots X_n^{k_n}$ ($k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}$) form a basis of $\mathcal{U}(\mathfrak{g})$. In particular this implies that $\mathfrak{g}$ is injectively embedded in $\mathcal{U}(\mathfrak{g})$ as the span of $X_1, \ldots, X_n$.

If ϕ is a representation of the Lie algebra $\mathfrak{g}$ on the linear space V then ϕ extends to an algebra representation of $\mathcal{U}(\mathfrak{g})$ on V by

$$\phi(A_1 \ldots A_m)v := \phi(A_1) \ldots \phi(A_m)v \qquad (A_1, \ldots, A_m \in \mathfrak{g},\ v \in V). \tag{1.121}$$

Suppose that $\mathfrak{g} = \mathrm{Lie}(G)$ for some Lie group G. Then the action $A \to X_A$ of $\mathfrak{g}$ on $C^\infty(G)$ by left invariant differential operators of order 1 extends to an action of $\mathcal{U}(\mathfrak{g})$ on $C^\infty(G)$ by left invariant differential operators:

$$(X_{A_1 \ldots A_m} f)(g) := \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_m} f\big(g \exp(t_1 A_1) \ldots \exp(t_m A_m)\big)\Big|_{t_1, \ldots, t_m = 0}$$
$$(f \in C^\infty(G),\ A_1, \ldots, A_m \in \mathfrak{g},\ g \in G). \tag{1.122}$$

It can be shown that (1.122) defines an algebra isomorphism $Y \mapsto X_Y$ of $\mathcal{U}(\mathfrak{g})$ onto the algebra of left invariant differential operators on G (see [13, Ch. II, Proposition 1.9]). If π is a representation of G on a finite dimensional complex vector space with matrix elements $\pi_{k,l}$ with respect to some basis of V then it follows from (1.121) and (1.122) that

$$(X_Y \pi_{k,l})(g) = \sum_m \pi_{k,m}(g)\, d\pi_{m,l}(Y) \qquad (g \in G,\ Y \in \mathcal{U}(\mathfrak{g})). \tag{1.123}$$

Let $\mathfrak{g}$ be a Lie algebra. We call an element Ω of $\mathcal{U}(\mathfrak{g})$ a *Caimir element* if Ω is in the *center* of $\mathcal{U}(\mathfrak{g})$, i.e., if it commutes with all elements of $\mathcal{U}(\mathfrak{g})$, or equivalently, if $A\Omega = \Omega A$ for all $A \in \mathfrak{g}$.

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let π be a representation of G on a finite dimensional complex linear space V. Let Ω be a Casimir element of $\mathfrak{g}$. Then

$$d\pi(A) d\pi(\Omega) - d\pi(\Omega) d\pi(A) = 0 \qquad (A \in \mathfrak{g}).$$

It can then be shown by exponentiation that

$$\pi(\exp(A))\, d\pi(\Omega)\, \big(\pi(\exp(A))\big)^{-1} = d\pi(\Omega) \qquad (A \in \mathfrak{g}).$$

Here exp is the matrix exponential if G is a linear Lie group and a generalized exponential map $\exp\colon \mathfrak{g} \to G$ otherwise. If the Lie group G is moreover connected then it follows that

$$\pi(g)d\pi(\Omega) = d\pi(\Omega)\pi(g)$$

for all $g \in G$. If moreover the representation π is irreducible then it follows by Schur's lemma that $d\pi(\Omega) = \text{const.}\, I$. So we have:

Proposition 1.39. *If π is an irreducible finite dimensional complex representation of a connected Lie group G with Lie algebra $\mathfrak{g}$ and if Ω is a Casimir element in $\mathcal{U}(\mathfrak{g})$ then there exists $\omega \in \mathbb{C}$ such that $d\pi(\Omega) = \omega I$.*

Now consider the Lie group $G := SU(2)$ with Lie algebra $\mathfrak{g} := su(2)$. Since $SU(2)$ is homeomorphic with S^3, it is connected. Let A, B, C be the basis of $su(2)$ introduced in the beginning of §1.13. Then it is immediately verified that

$$\Omega := A^2 + B^2 + C^2 \tag{1.124}$$

commutes with A, B, C. Hence Ω is a Casimir element in $\mathcal{U}(su(2))$. By Proposition 1.39 and the irreducibility of the representations t^l of $SU(2)$ we already know that $dt^l(\Omega) = \omega_l I$ for some $\omega_l \in \mathbb{C}$, and we might compute ω_l from $dt^l(\Omega)f = \omega_l f$ for just one suitable nonzero $f \in \mathcal{H}_l$. However, let us compute $dt^l(\Omega)\psi_n^l$ for all basis elements ψ_n^l of $\mathcal{H}_l$. Rewrite (1.124) as

$$\Omega = -(A - iC)(-A - iC) + B^2 - iB$$

and use (1.109), (1.110), (1.107). Then we obtain

$$\begin{aligned} dt^l(\Omega)\psi_n^l &= -dt^l(A - iC)dt^l(-A - iC)\psi_n^l + (dt^l(B))^2\psi_n^l - idt^l(B)\psi_n^l \\ &= -(l-n)(l+n+1)\psi_n^l - n^2\psi_n^l - n\psi_n^l = -l(l+1)\psi_n^l. \end{aligned}$$

Hence

$$dt^l(\Omega) = -l(l+1)I. \tag{1.125}$$

Now it follows by (1.123) that

$$X_\Omega t_{m,n}^l = -l(l+1)t_{m,n}^l. \tag{1.126}$$

Let $n \pm m \geq 0$, $ad - bc = 1$. Then we can write $t_{m,n}^l$ as in (1.114), and we can reduce (1.126) to

$$\begin{aligned} (1-x)^{-n+m}(1+x)^{-n-m}\frac{d}{dx}\left((1-x)^{n-m+1}(1+x)^{n+m+1}\frac{d}{dx}P_{l-n}^{(n-m,n+m)}(x)\right)& \\ - n(n+1)P_{l-n}^{(n-m,n+m)}(x) = -l(l+1)P_{l-n}^{(n-m,n+m)}(x).& \end{aligned} \tag{1.127}$$

Thus we have obtained the second order differential equation (1.48) for Jacobi polynomials $P_n^{(\alpha,\beta)}$ ($\alpha, \beta \in \mathbb{Z}_{\geq 0}$) in a new conceptual way, from the representation theory of $SU(2)$.

2 Spherical harmonics

Reference for this Chapter is for instance Stein & Weiss [15, Ch. IV, §2].

2.1 Definition of spherical harmonics

Let $O(d)$ be the group of real orthogonal $d \times d$ matrices, i.e., $O(d) := \{T \in M_d(\mathbb{R}) \mid T'T = I\}$, where $M_d(\mathbb{R})$ is the space of all real $d \times d$ matrices and T' is the adjoint of T. Another way to describe $O(d)$ is as the set of all real $d \times d$ matrices for which the column vectors form an orthonormal system. It can easily be seen that $O(d)$ is a linear Lie group, but for the moment we will only need that it is a compact group (note that $O(d)$ is a closed and bounded subset of $M_d(\mathbb{R})$).

We say that a group G *acts* on a space X if we have a map $(g, x) \mapsto g \,.\, x \colon G \times X \to X$ such that $(gh) \,.\, x = g \,.\, (h \,.\, x)$ and $e \,.\, x = x$. If G is a topological group and X is a topological space then we call an action of G on X *continuous* if the map $(g, x) \mapsto g \,.\, x \colon G \times X \to X$ is continuous.

We call an action of G on X *transitive* if for all $x, y \in X$ there is $g \in G$ such that $g \,.\, x = y$. If we fix some element x_0 of X then we see that the action of G on X is already transitive if for each $x \in X$ there exists $g \in G$ such that $g \,.\, x_0 = x$.

For an action of G on X and $x_0 \in X$ we call the subgroup $H := \{g \in G \mid g \cdot x_0 = x_0\}$ the *stabilizer* of x_0 in G. If H is any subgroup of G and G/H denotes the space of right cosets gH ($g \in G$) then we have a transitive action of G on G/H by

$$g_1 \,.\, (g_2 H) := (g_1 g_2) H,$$

and then the stabilizer of eH in G is H. The space G/H is called a *homogeneous space.*

For a transitive action of G on X and $x_0 \in X$ and H the stabilizer of x_0 in G we have a bijective map

$$gH \mapsto g \,.\, x_0 \colon G/H \to X$$

which commutes with the G-actions: if gH is sent to x then $g_1 \,.\, (gH)$ is sent to $g_1 \,.\, x$.

Clearly, the natural action of $O(d)$ on $\mathbb{R}^d$, i.e., $(T, x) \mapsto Tx \colon O(d) \times \mathbb{R}^d \to \mathbb{R}^d$, is a continuous group action. Let S^{d-1} be the unit sphere in $\mathbb{R}^d$ given by $\{x \in \mathbb{R}^d \mid |x| = 1\}$, where $|x| := \sqrt{x_1^2 + \cdots + x_d^2}$. Note that S^{d-1} is compact. By restriction to S^{d-1} of the $O(d)$-action on $\mathbb{R}^d$, the group $O(d)$ acts continuously on S^{d-1}. Moreover, this action is transitive. For a proof, let $x \in S^{d-1}$. Apply Gram-Schmidt orthonormalization to $x, e_1, \ldots, e_d$ in order to find an orthonomal system of d vectors of which x is the first. The orthogonal transformation T having these vectors as columns sends e_1 to x.

The stabilizer of e_1 in $O(d)$ is easily be seen to consist of all block matrices $\begin{pmatrix} 1 & 0 \\ 0 & T_1 \end{pmatrix}$ such that $T_1 \in O(d-1)$. This stabilizer subgroup, which is clearly isomorphic to $O(d-1)$, is also denoted by $O(d-1)$. So S^{d-1} with the transitive action of $O(d)$ is isomorphic to the homogeneous space $O(d)/O(d-1)$.

If we have an action of a group G on a space X then we have a (usually infinite dimensional) representation of G on the complex (resp. real) linear space of complex-valued (resp. real-valued) functions f on X by

$$(g\,.\,f)(x) := f(g^{-1}\,.\,x).$$

If the group action is continuous then this also defines a representation of G on the space of continuous functions on X. For the action of $O(d)$ on $\mathbb{R}^d$ we can further restrict to the space of real-valued polynomials on $\mathbb{R}^d$.

Let $\mathcal{P}_n$ denote the real linear space of real-valued homogeneous polynomials of degree n on $\mathbb{R}^d$. It is clearly finite dimensional and it has a basis consisting of the monomials

$$x_1^{n_1}\ldots x_d^{n_d} \qquad (n_1+\cdots+n_d=n). \tag{2.1}$$

Then we have a representation of $O(d)$ on $\mathcal{P}_n$:

$$(T\,.\,f)(x) := f(T^{-1}x) \qquad (f\in\mathcal{P}_n), \tag{2.2}$$

and this representation is continuous. The map which sends $f\in\mathcal{P}_n$ to its restriction $f\big|_{S^{d-1}}$ is a linear bijection since we can reconstruct f from its restriction:

$$f(rx) = r^n\,f(x) \qquad (x\in S^{d-1},\ r\in[0,\infty),\ f\in\mathcal{P}_n).$$

Thus we can consider (2.2) also as a representation of $O(d)$ on the space of restrictions to S^{d-1} of homogeneoous polynomials of degree n on $\mathbb{R}^d$.

We have

$$\dim(\mathcal{P}_n) = \binom{n+d-1}{d-1}. \tag{2.3}$$

For the proof we count the number of monomials (2.1). This equals the number of ways to write $n=n_1+\cdots+n_d$ with $n_1,\ldots,n_d\in\mathbb{Z}_{\ge 0}$, and this equals the number of subsets of size $d-1$ in a set of $n+d-1$ elements.

We will often use that

$$\sum_{j=1}^d x_j\,\frac{\partial f(x)}{\partial x_j} = nf(x) \qquad (f\in\mathcal{P}_n). \tag{2.4}$$

For the proof note that

$$n\,f(x) = \frac{d}{dt}(t^n)\Big|_{t=1}\,f(x) = \frac{d}{dt}f(tx)\Big|_{t=1} = \sum_{j=1}^d x_j\,\frac{\partial f(x)}{\partial x_j}\,.$$

Let Δ denote the *Laplace operator* in d variables:

$$\Delta := \frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_d^2}\,.$$

If $y=T^{-1}x$ $(T\in O(d))$ then

$$\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_d^2} = \frac{\partial^2}{\partial y_1^2}+\cdots+\frac{\partial^2}{\partial y_d^2}\,.$$

Indeed,

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^d \frac{\partial y_j}{\partial x_i}\frac{\partial}{\partial y_j} = \sum_{j=1}^d (T^{-1})_{j,i}\frac{\partial}{\partial y_j} = \sum_{j=1}^d T_{i,j}\frac{\partial}{\partial y_j}\,.$$

Hence

$$\sum_{i=1}^d \left(\frac{\partial}{\partial x_i}\right)^2 = \sum_{i,j,k=1}^d T_{i,j}T_{i,k}\frac{\partial}{\partial y_j}\frac{\partial}{\partial y_k} = \sum_{j,k}\delta_{j,k}\frac{\partial}{\partial y_j}\frac{\partial}{\partial y_k} = \sum_{j=1}^d \left(\frac{\partial}{\partial y_j}\right)^2.$$

It follows that

$$\Delta(T\,.\,f) = T\,.\,(\Delta f) \qquad (T \in O(d),\ f \text{ a } C^2 \text{ function}).$$

We call a C^2 function f on $\mathbb{R}^d$ *harmonic* if $\Delta f = 0$. Thus we see that if f is harmonic and $T \in O(d)$ then $T\,.\,f$ is harmonic.

Let $\mathcal{H}_n$ denote the real linear space of real harmonic homogeneous polynomials on $\mathbb{R}^d$ of degree n:

$$\mathcal{H}_n := \{f \in \mathcal{P}_n \mid \Delta f = 0\}.$$

Then, under the representation of $O(d)$ on $\mathcal{P}_n$, the subspace $\mathcal{H}_n$ is invariant, so we have also a (continuous) representation of $O(d)$ on $\mathcal{H}_n$.

The restriction map $f \mapsto f\big|_{S^{d-1}}$ is a linear bijection of $\mathcal{H}_n$ on the space of its restrictions to S^{d-1}. This last space is called the space of *spherical harmonics* of degree n on S^{d-1}. We can consider the representation of $O(d)$ on $\mathcal{H}_n$ equivalently as a representation on the space of spherical harmonics of degree n.

Note that, for $d \ge 2$, $f(x) := (x_1 + ix_2)^n$ yields a function $f \in \mathcal{H}_n$. So $\mathcal{H}_n$ has certainly nonzero dimension if $d \ge 2$. On the other hand, for $n \ge 2$ there are many $f \in \mathcal{P}_n$ outside $\mathcal{H}_n$:

Lemma 2.1. *Let $n \ge 2$, $f \in \mathcal{P}_{n-2}$ and $F(x) := |x|^2 f(x)$ (so certainly $F \in \mathcal{P}_n$). If $F \in \mathcal{H}_n$ then $F = 0$.*

Proof Suppose $F \ne 0$. Then there is a maximal k ($1 \le k \le \frac{1}{2}n$) such that $F(x) = |x|^{2k} g(x)$ for some $g \in \mathcal{P}_{n-2k}$. Write $r := |x|$. We use that Δ acting on a function only depending on r acts as $\frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr}$, and that $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$, and (2.4). Then

$$\begin{aligned}
0 = \Delta\Big(r^{2k} g(x)\Big) &= \Delta(r^{2k})\, g(x) + 2\sum_{i=1}^d \frac{\partial}{\partial x_i}(r^{2k})\,\frac{\partial}{\partial x_i} g(x) + r^{2k}\,\Delta g(x)\\
&= \Big(\frac{d^2}{dr^2} + \frac{d-1}{r}\,\frac{d}{dr}\Big)(r^{2k})\, g(x) + 4kr^{2k-2}\sum_{i=1}^d x_i \frac{\partial}{\partial x_i} g(x) + r^{2k}\,\Delta g(x)\\
&= 2k(2k+d-2)r^{2k-2} g(x) + 4k(n-2k)r^{2k-2} g(x) + r^{2k}\,\Delta g(x)\\
&= 2k(2n-2k+d-2)r^{2k-2} g(x) + r^{2k}\,\Delta g(x).
\end{aligned}$$

Hence $g(x) = \text{const.}\, r^2 \Delta g(x)$, so $F(x) = \text{const.}\, r^{2k+2}\,\Delta g(x)$. This contradicts the maximality of k. $\square$

Proposition 2.2. *We have*

$$\mathcal{P}_n = \mathcal{H}_n \oplus |x|^2\,\mathcal{P}_{n-2} \quad (n \geq 2) \quad \text{and} \quad \mathcal{P}_n = \mathcal{H}_n \quad (n = 0, 1). \tag{2.5}$$

Proof The case $n = 0, 1$ is seen immediately. Let $n \geq 2$. $\mathcal{H}_n$ and $|x|^2\,\mathcal{P}_{n-2}$ are linear subspaces of $\mathcal{P}_n$. By Lemma 2.1 they have intersection $\{0\}$. So it is sufficient to prove that $\dim(\mathcal{P}_n) \leq \dim(\mathcal{H}_n) + \dim(|x|^2\,\mathcal{P}_{n-2})$. But this inequality holds because

$$\dim(\mathcal{P}_n) = \dim(\mathcal{H}_n) + \dim(\Delta(\mathcal{P}_n)) \leq \dim(\mathcal{H}_n) + \dim(\mathcal{P}_{n-2}) = \dim(\mathcal{H}_n) + \dim(|x|^2\,\mathcal{P}_{n-2}),$$

Here the first identity is the fact from linear algebra that, for a linear map $A \colon V \to W$, we have $\dim V = \dim A^{-1}(0) + \dim A(V)$. The second inequality follows because $\Delta(\mathcal{P}_n) \subset \mathcal{P}_{n-2}$. $\square$

As a corollary, we see that

$$\dim \mathcal{H}_n = \dim \mathcal{P}_n - \dim \mathcal{P}_{n-2} \quad (n \geq 2) \quad \text{and} \quad \dim \mathcal{H}_n = \dim \mathcal{P}_n \quad (n = 0, 1).$$

By (2.1) we compute for $d \geq 2$ that

$$\dim \mathcal{H}_n = \frac{(2n + d - 2)(n + d - 3)!}{n!\,(d - 2)!} \quad (n \geq 1) \quad \text{and} \quad \dim \mathcal{H}_0 = 1. \tag{2.6}$$

For $d = 1$ we have trivially $\dim \mathcal{H}_n = 1$ if $n = 0, 1$ and $\dim \mathcal{H}_n = 0$ if $n \geq 2$.

Exercise 2.3. Let $SO(d) := \{T \in O(d) \mid \det T = 1\}$. This is again a compact group.

a) Show that $SO(d)$ acts transitively on S^{d-1} if $d \geq 2$ (but not if $d = 1$) and that then the stabilizer of e_1 in $SO(d)$ equals the subgroup of block matrices $\begin{pmatrix} 1 & 0 \\ 0 & T_1 \end{pmatrix}$ with $T_1 \in SO(d-1)$. (This subgroup is isomorphic to $SO(d-1)$ and it is also denoted by $SO(d-1)$.)

b) Let $\langle x, y \rangle := \sum_{j=1}^{d} x_j y_j$ denote the inner product on $\mathbb{R}^d$. Show that $O(d)$ acts on S^{d-1} in a *doubly transitive* way, i.e., if $x, y, x', y' \in S^{d-1}$ such that $\langle x, y \rangle = \langle x', y' \rangle$ then there exists $T \in O(d)$ such that $Tx = x'$, $Ty = y'$.

c) Show that $SO(d)$ acts on S^{d-1} in a doubly transitive way if $d \geq 3$ (but not for $d = 1, 2$).

d) Let F be a function on $S^{d-1} \times S^{d-1}$ such that $F(Tx, Ty) = F(x, y)$ for all $T \in O(d)$. Show that $F(x, y) = f(\langle x, y \rangle)$ for some function f on $[-1, 1]$. For $d \geq 3$ show a similar conclusion for F if $F(Tx, Ty) = F(x, y)$ for all $T \in SO(d)$.

Exercise 2.4. Let $\mathcal{P}_n$ be the space of real-valued homogeneous polynomials of degree n on $\mathbb{R}^d$. Write ∂_j for $\frac{\partial}{\partial x_j}$. Define on $\mathcal{P}_n$ the bilinear form

$$\langle f, g \rangle_n := \big(f(\partial_1, \ldots, \partial_d) g\big)(0) \qquad (f, g \in \mathcal{P}_n). \tag{2.7}$$

Here $f(\partial_1, \ldots, \partial_d)$ is the partial differential operator obtained by replacing $x_1, \ldots, x_d$ by $\partial_1, \ldots, \partial_d$ in $f(x_1, \ldots, x_d) = \sum_{n_1 + \cdots + n_d = n} c_{n_1, \ldots, n_d} x_1^{n_1} \ldots x_d^{n_d}$.

a) Show that (2.7) defines an inner product on $\mathcal{P}_n$.

b) Show that

$$\langle T\,.\,f, T\,.\,g\rangle_n = \langle f, g\rangle_n \qquad (T \in O(d)). \tag{2.8}$$

c) Show for $n \geq 2$ that

$$\langle |x|^2 f, g\rangle_n = \langle f, \Delta g\rangle_{n-2} \qquad (f \in \mathcal{P}_{n-2},\ g \in \mathcal{P}_n). \tag{2.9}$$

d) Show for $n \geq 2$ that $g \in \mathcal{H}_n$ iff $\langle |x|^2 f, g\rangle_n = 0$ for all $f \in \mathcal{P}_{n-2}$. (This also gives an alternative proof of Proposition 2.2.)

Hint In a) compute $\langle f, g\rangle_n$ when f and g are monomials. In b) the left hand side equals $f\big(T^{-1}(\partial/\partial x_1, \ldots, \partial/\partial x_d)\big)\, g\big(T^{-1}(x_1, \ldots, x_d)\big)\big|_{x=0}$. Put $y = T^{-1}x$ and compute the j-th coordinate of $T^{-1}(\partial/\partial x_1, \ldots, \partial/\partial x_d)$ in terms of $\partial/\partial y_1, \ldots, \partial/\partial y_d$.

2.2 Zonal spherical harmonics

We keep the notation of §2.1, but we suppose from now on that $d \geq 2$.

We call a function f on S^{d-1} *zonal* if $T\,.\,f = f$ for all $T \in O(d-1)$. We want to find the zonal functions in $\mathcal{P}_n\big|_{S^{d-1}}$, and next the zonal spherical harmonics. Since $\mathcal{P}_n$ and $\mathcal{P}_n\big|_{S^{d-1}}$ are isomorphic as $O(d)$-modules, and similarly for $\mathcal{H}_n$ and $\mathcal{H}_n\big|_{S^{d-1}}$, it is sufficient for this purpose to search for $O(d-1)$-invariant functions in $\mathcal{P}_n$ and in $\mathcal{H}_n$.

Lemma 2.5. *Let $f \in \mathcal{P}_n$. Then f is zonal iff, for certain coefficients c_j,*

$$f(x) = \sum_{j=0}^{[\frac{1}{2}n]} c_j\, x_1^{n-2j}\, (x_2^2 + \cdots + x_d^2)^j. \tag{2.10}$$

Proof Clearly, every f of the form (2.10) is in $\mathcal{P}_n$ and zonal. Let conversely $f \in \mathcal{P}_n$ be zonal. Then, for certain homogeneous polynomials f_k of degree k in $x_2, \ldots, x_d$ we have:

$$f(x) = \sum_{k=0}^{n} x_1^{n-k} f_k(x_2, \ldots, x_d) = \sum_{k=0}^{n} x_1^{n-k} f_k(-x_2, \ldots, -x_d) = \sum_{k=0}^{n} x_1^{n-k} (-1)^k f_k(x_2, \ldots, x_d),$$

where we used $O(d-1)$-invariance and homogeneity. Hence, the terms for odd k vanish. Then, again by $O(d-1)$-invariance and homogeneity, we have for certain coefficients c_j:

$$\begin{aligned} f(x) = \sum_{j=0}^{[n/2]} x_1^{n-2j} f_{2j}(x_2, \ldots, x_d) &= \sum_{j=0}^{[n/2]} x_1^{n-2j} f_{2j}\Big(\sqrt{x_2^2 + \cdots + x_d^2}, 0, \ldots, 0\Big) \\ &= \sum_{j=0}^{[\frac{1}{2}n]} c_j\, x_1^{n-2j}\, (x_2^2 + \cdots + x_d^2)^j. \qquad \square \end{aligned}$$

Proposition 2.6. *The linear space of zonal functions in $\mathcal{H}_n$ has dimension 1.*

Proof A function $f \in \mathcal{P}_n$ will be a zonal function in $\mathcal{H}_n$ iff it has the form (2.10) and satisfies $\Delta f = 0$. Put $\rho := \sqrt{x_2^2 + \ldots + x_d^2}$. Then, similarly as in the proof of Lemma 2.1, observe that $\frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$ acts on a function only depending on ρ as $\frac{d^2}{d\rho^2} + \frac{d-2}{\rho}\frac{d}{d\rho}$. Now split Δ, acting on the terms in (2.1), as the sum of $\frac{\partial^2}{\partial x_1^2}$ and $\frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$, where the first part acts only on the factor x_1^{n-2j} and the second part acts only on the factor ρ^{2j}. Thus $\Delta f = 0$ is equivalent with

$$\sum_{j=0}^{[\frac{1}{2}n]} c_j \Big((n-2j)(n-2j-1) x_1^{n-2j-2} \rho^{2j} + 2j(2j+d-3) x_1^{n-2j} \rho^{2j-2} \Big) = 0. \tag{2.11}$$

Note that $2j(2j+d-3)x_1^{n-2j}\rho^{2j-2}$ vanishes if $j = 0$ and $(n-2j)(n-2j-1)x_1^{n-2j-2}\rho^{2j}$ vanishes if $j = [\frac{1}{2}n]$. Hence (2.11) can be equivalently written as

$$\sum_{j=1}^{[\frac{1}{2}n]} \Big((n-2j+2)(n-2j+1) c_{j-1} + 2j(2j+d-3) c_j \Big) x_1^{n-2j} \rho^{2j-2} = 0,$$

and this is equivalent with

$$c_j = -\frac{(n-2j+2)(n-2j+1)}{2j(2j+d-3)}\, c_{j-1} \qquad (j = 1, \ldots, [\tfrac{1}{2}n]). \qquad \square$$

2.3 Compact homogeneous spaces and reproducing kernels

Let G be a topological group which is compact Hausdorff as a topological space. Let X be a compact Hausdorff space on which G acts in a continuous and transitive way. Fix $x_0 \in X$. Let K be the stabilizer of x_0 in G. Then K is a closed subgroup of G, hence a compact group itself. Let μ be the Haar measure on G. It implies a G-invariant normalized Borel measure ω on X by

$$\omega(E) := \mu(\{g \in G \mid g \,.\, x_0 \in E\}) \qquad (E \text{ Borel set in } X),$$

or equivalently,

$$\int_X f \, d\omega := \int_G f(g \,.\, x_0) \, d\mu(g) \qquad (f \in C(X)).$$

It can be shown that this measure ω is the unique G-invariant normalized measure on X. Let $L^2(X) := L^2(X, \omega)$. Write the inner product on $L^2(X)$ as

$$\langle f_1, f_2 \rangle := \int_X f_1 \, \overline{f_2} \, d\omega \qquad (f_1, f_2 \in L^2(X)). \tag{2.12}$$

Then, by G-invariance of ω, this inner product is G-invariant:

$$\langle g \,.\, f_1, g \,.\, f_2 \rangle = \langle f_1, f_2 \rangle \qquad (f_1, f_2 \in L^2(X),\ g \in G).$$

We may consider the action of G on $L^2(X)$ as a unitary representation of G on the (ususally infinite dimensional) Hilbert space $L^2(X)$. The continuity of the representation will now hold in the form:

$$g \mapsto \langle g \,.\, f_1, f_2 \rangle \colon G \to \mathbb{C} \quad \text{is continuous for all } f_1, f_2 \in L^2(X). \tag{2.13}$$

This can first be proved for $f_1, f_2 \in C(X)$, and next for $f_1, f_2 \in L^2(X)$ by using the density of $C(X)$ in $L^2(X)$. If we write $\lambda(g)\, f$ for $g \,.\, f$ ($f \in L^2(X)$, $g \in G$) then $\lambda(g)$ is a unitary operator on $L^2(X)$ for each $g \in G$, so certainly $\lambda(g) \in \mathcal{B}(L^2(X))$, the space of bounded linear operators on $L^2(X)$. In general, $\lambda \colon G \mapsto \mathcal{B}(L^2(X))$ will not be continuous with respect to the operator norm topology of $\mathcal{B}(L^2(X))$. The property (2.13) is called *weak continuity* of λ. *Strong continuity* for λ can also be shown. This means that

$$g \mapsto g \,.\, f \colon G \to L^2(X) \quad \text{is continuous for all } f \in L^2(X).$$

It is only in the case of infinite dimensional representations that these distinctions between various types of continuity have to be made.

Let now $V \subset C(X) \subset L^2(X)$ be a linear space of finite nonzero dimension N which is G-invariant. Then we have a finite dimensional unitary representation of G on V. Choose an orthonormal basis $f_1, \ldots, f_N$ of V. Put

$$\Phi(x,y) := \sum_{j=1}^{N} f_j(x)\, \overline{f_j(y)} \qquad (x, y \in X). \tag{2.14}$$

This definition is independent of the choice of the orthonormal basis of V (for which the G-invariance of V is not yet needed). This can either be proved by working with the unitary matrix which connects two orthonormal bases of V or, more conceptually, by observing that the operator P, defined by

$$(Pf)(x) := \int_X \Phi(x,y)\, f(y)\, d\omega(y) \qquad (f \in L^2(X)),$$

is the orthogonal projection of $L^2(X)$ onto V. In particular,

$$\int_X \Phi(x,y)\, f(y)\, d\omega(y) = f(x) \qquad (f \in V),$$

and for that reason $\Phi(x,y)$ is called the *reproducing kernel* of V.

By G-invariance of ω the functions $g \,.\, f_1, \ldots, g \,.\, f_N$ form an orthonormal basis of V for each $g \in G$. Hence Φ is G-invariant:

$$\Phi(g \,.\, x, g \,.\, y) = \Phi(x,y) \qquad (x, y \in X,\ g \in G). \tag{2.15}$$

In particular,

$$\Phi(k \,.\, x, x_0) = \Phi(k \,.\, x, k \,.\, x_0) = \Phi(x, x_0) \qquad (x \in X,\ k \in K).$$

Put

$$\phi(x) := N^{-1} \Phi(x, x_0) \qquad (x \in X). \tag{2.16}$$

Proposition 2.7. *The function ϕ, defined by* (2.16)*, is a zonal (i.e. K-invariant) function in V. Furthermore,*

$$\phi(x_0) = 1 \quad \text{and} \quad \langle \phi, \phi \rangle = N^{-1}. \tag{2.17}$$

Proof We already showed that ϕ is zonal. By (2.16) and (2.14) we see that $\phi \in V$. For all $g \in G$ we have

$$N\phi(x_0) = \Phi(x_0, x_0) = \Phi(g \,.\, x_0, g \,.\, x_0).$$

Hence, for all $x \in X$ we have

$$N\phi(x_0) = \Phi(x, x) = \sum_{j=1}^{N} |f_j(x)|^2.$$

Integration of both sides over X yields

$$N\phi(x_0) = \sum_{j=1}^{N} \langle f_j, f_j \rangle = \sum_{j=1}^{N} 1 = N, \quad \text{hence} \quad \phi(x_0) = 1.$$

Next,

$$\begin{aligned}
\langle \phi, \phi \rangle = N^{-2} \int_X \Phi(x, x_0)\, \overline{\Phi(x, x_0)}\, d\omega(x) &= N^{-2} \sum_{i,j=1}^{N} \int_X f_i(x)\, \overline{f_i(x_0)}\, \overline{f_j(x)}\, f_j(x_0)\, d\omega(x) \\
&= N^{-2} \sum_{i,j=1}^{N} f_j(x_0)\, \overline{f_i(x_0)}\, \delta_{i,j} = N^{-2} \sum_{j=1}^{N} f_j(x_0)\, \overline{f_j(x_0)} \\
&= N^{-2} \Phi(x_0, x_0) = N^{-1} \phi(x_0) = N^{-1}. \qquad \square
\end{aligned}$$

Theorem 2.8. *Let V be a G-invariant linear subspace of $C(X)$ of nonzero finite dimension. Let V_K be the subspace of zonal functions in V. Then:*

a) $0 \neq \phi \in V_K$ *and* $\dim V_K \geq 1$.

b) If $\dim V_K = 1$ *then the representation of G on V is irreducible and V_K is spanned by ϕ.*

Proof For the proof of b) suppose that the representation is not irreducible. Then V is the orthogonal direct sum of two invariant subspaces V_1 and V_2 of nonzero dimension. Then $\dim V_K \geq 2$ by a). $\square$

2.4 Zonal spherical harmonics (continued)

We continue §2.2, where we keep the notation of §2.1. It follows from Proposition 2.6 and Theorem 2.8 that:

Theorem 2.9. *The representation of $O(d)$ on $\mathcal{H}_n$ is irreducible.*

The Lebesgue measure on $\mathbb{R}^d$ induces on every $(d-1)$-dimensional smooth submanifold a surface measure, certainly also on S^{d-1}. We denote the surface measure on S^{d-1} by σ. It is for instance determined by the property that

$$\int_{R^d} f(y)\,dy = \int_{r=0}^{\infty} \int_{x\in S^{d-1}} f(rx)\,r^{d-1}\,d\sigma(x)\,dr \tag{2.18}$$

for all continuous L^1 functions on $\mathbb{R}^d$. Clearly, the measure σ on S^{d-1} is $O(d)$-invariant. Therefore the normalized $O(d)$-invariant measure ω on S^{d-1} is given by

$$d\omega = \frac{1}{\sigma(S^{d-1})}\,d\sigma.$$

Let $L^2(S^{d-1}) := L^2(S^{d-1},\omega)$ and write the inner product in $L^2(S^{d-1})$ as in (2.12) with $X := S^{d-1}$.

Theorem 2.10. *If $h_n \in \mathcal{H}_n$, $h_m \in \mathcal{H}_m$ and $n \neq m$ then $\langle h_n, h_m\rangle = 0$.*

Proof Without loss of generality we may assume f_n, f_m to be real-valued. Then

$$0 = \int_{|x|\le 1} (h_n\,\Delta h_m - h_m\,\Delta h_n)\,dx = \int_{S^{d-1}} \Big(h_n\,\frac{\partial h_m}{\partial\nu} - h_m\,\frac{\partial h_n}{\partial\nu}\Big)\,d\sigma = (m-n)\int_{S^{d-1}} h_n\,h_m\,d\sigma.$$

Division by $m-n$ yields the required orthogonality. Above we used a formula by Green in the second identity, where $\frac{\partial}{\partial\nu}$ denotes normal derivative. In the third identity we used that $\frac{\partial}{\partial\nu} f(x) = nf(x)$ $(x \in S^{d-1})$ if f is homogeneous of degree n on $\mathbb{R}^d$. □

Proposition 2.11. *Let $d \ge 3$. Consider $S^{d-2} := \{(x_1,\ldots,x_d) \in S^{d-1} \mid x_1 = 0\}$ as a subset of S^{d-1} and let σ' be the surface measure on S^{d-2}. Then, for all $f \in C(S^{d-1})$,*

$$\int_{S^{d-1}} f\,d\sigma = \int_{x'\in S^{d-2}} \int_{t=-1}^{1} f\Big(te_1 + \sqrt{1-t^2}\,x'\Big)\,(1-t^2)^{\frac12 d-\frac32}\,dt\,d\sigma'(x'). \tag{2.19}$$

Proof Let $f \in C_c(\mathbb{R}^d)$ (continuous with compact support). Then

$$\begin{aligned}
\int_{\mathbb{R}^d} f(y)\,dy &= \int_{y'\in\mathbb{R}^{d-1}} \int_{y_1=-\infty}^{\infty} f(y_1e_1 + y')\,dy_1\,dy' \\
&= \int_{x'\in S^{d-2}} \int_{\rho=0}^{\infty} \int_{y_1=-\infty}^{\infty} f(ye_1 + \rho x')\,\rho^{d-2}\,dy_1\,d\rho\,d\sigma'(x') \\
&= \int_{x'\in S^{d-2}} \int_{t=-1}^{1} \int_{r=0}^{\infty} f\Big(rte_1 + r\sqrt{1-t^2}\,x'\Big)\,(1-t^2)^{\frac12 d-\frac32}\,r^{d-1}\,dr\,dt\,d\sigma'(x').
\end{aligned}$$

Here we used (2.18) (with d replaced by $d-1$) in the second identity. In the third identity we passed from integration variables y_1,ρ to r,t by $y_1 = rt$, $\rho = r\sqrt{1-t^2}$. The Jacobian of this transformation can be computed to be $r(1-t^2)^{-\frac12}$. Now compare the above rewriting of the integral of f over $\mathbb{R}^d$ with the rewriting of this integral by (2.18). In particular we can make this comparison for f of the form $f(rx) = f_1(x) f_2(r)$ ($x \in S^{d-1}$, $r \in [0,\infty)$). This yields (2.19). $\square$

For $d = 2$ we trivially see that

$$\int_{S_1} f\,d\sigma = \int_0^{2\pi} f(\cos\theta\, e_1 + \sin\theta\, e_2)\,d\theta = \sum_{j=0,1} \int_{-1}^{1} f\Big(te_1 + (-1)^j \sqrt{1-t^2}\, e_2\Big)\,(1-t^2)^{-\frac12}\,dt. \tag{2.20}$$

Corollary 2.12. *let $d \ge 2$ and let $f \in C(S^{d-1})$ be zonal. Then*

$$\int_{S^{d-1}} f\,d\omega = \frac{\Gamma(\frac12 d)}{\Gamma(\frac12 d - \frac12)\Gamma(\frac12)} \int_{-1}^{1} f\Big(te_1 + \sqrt{1-t^2}\, e_2\Big)\,(1-t^2)^{\frac12 d - \frac32}\,dt. \tag{2.21}$$

Proof Use (2.19) and (2.20) and the fact that $f(te_1 + \sqrt{1-t^2}\,x')$ is independent of $x' \in S^{d-2}$ if f is zonal. The constant factor in front of the integral on the right in (2.21) is obtained from

$$\int_{-1}^{1} (1-t^2)^{\frac12 d - \frac32}\,dt = \frac{\Gamma(\frac12 d - \frac12)\Gamma(\frac12)}{\Gamma(\frac12 d)}\,. \qquad \square$$

Theorem 2.13. *Let $f \in \mathcal{H}_n$. Then f is zonal iff, for the restriction of f to S^{d-1},*

$$f\Big(te_1 + \sqrt{1-t^2}\,x'\Big) = \text{const.}\, P_n^{(\frac12 d - \frac32, \frac12 d - \frac32)}(t) \qquad (t \in [-1,1],\ x' \in S^{d-2}). \tag{2.22}$$

Here $P_n^{(\frac12 d - \frac32, \frac12 d - \frac32)}(t)$ is a Jacobi polynomial.

Proof For each n choose a nonzero real-valued zonal function $\phi_n \in \mathcal{H}_n$ By (2.10) there are constants c_j such that

$$\phi_n\Big(te_1 + \sqrt{1-t^2}\,x'\Big) = \sum_{j=0}^{[\frac12 n]} c_j\, t^{n-2j}\,(1-t^2)^j.$$

Hence ϕ_n only depends on t and is a polynomial of degree $\le n$ in t, which we denote by $p_n(t)$. By Theorem 2.10 and Corollary 2.12 we have

$$\int_{-1}^{1} p_n(t)\,p_m(t)\,(1-t^2)^{\frac12 d - \frac32}\,dt = 0 \qquad (n \ne m).$$

Hence p_n is equal to the orthogonal polynomial of degree n with respect to the weight function $(1-t^2)^{\frac12 d - \frac32}$ on $(-1,1)$. $\square$

Remark 2.14. By Theorems 2.8 and 2.13 and Proposition 2.7 the zonal function ϕ on S^{d-1} associated with $\mathcal{H}_n$ by (2.16) is given by

$$\phi\Big(te_1 + \sqrt{1-t^2}\,x'\Big) = \frac{P_n^{(\frac12 d - \frac32, \frac12 d - \frac32)}(t)}{P_n^{(\frac12 d - \frac32, \frac12 d - \frac32)}(1)}\,.$$

By (2.17) and (2.21) we must have

$$\frac{\Gamma(\frac{1}{2}d)}{\Gamma(\frac{1}{2}d-\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{1}\left(\frac{P_n^{(\frac{1}{2}d-\frac{3}{2},\frac{1}{2}d-\frac{3}{2})}(t)}{P_n^{(\frac{1}{2}d-\frac{3}{2},\frac{1}{2}d-\frac{3}{2})}(1)}\right)^2(1-t^2)^{\frac{1}{2}d-\frac{3}{2}}\,dt=\frac{1}{\dim\mathcal{H}_n}\,. \tag{2.23}$$

This can indeed be independently verified by (1.53), (1.55), (1.56) and (2.6).

Exercise 2.15. Show, by a variation of Lemma 2.5 and its proof, that $f\in\mathcal{H}_n$ is zonal iff

$$f(x)=\sum_{j=0}^{[\frac{1}{2}n]}a_j x_1^{n-2j}\,(x_1^2+\cdots+x_d^2)^j$$

with

$$a_j=-\,\frac{(n-2j+2)(n-2j+1)}{2j(2n-2j+d-2)}\,a_{j-1}\qquad (j=1,2,\ldots,[\tfrac{1}{2}n]).$$

Conclude that

$$P_n^{(\frac{1}{2}d-\frac{3}{2},\frac{1}{2}d-\frac{3}{2})}(t)=\text{const.}\sum_{j=0}^{[\frac{1}{2}n]}\frac{(-1)^j\,(\frac{1}{2}d-1)_{n-j}}{j!\,(n-2j)!}\,(2t)^{n-2j}.$$

Determine the constant factor by comparing the coefficients of t^n on the left and the right (use the formula after (1.53)).

Exercise 2.16. Let $d=d_1+d_2$ $(d_1,d_1\in\mathbb{Z}_{>0})$. Consider $O(d_1)\times O(d_2)$ as the subgroup of $O(d)$ consisting of the block matrices $\begin{pmatrix}T_1&0\\0&T_2\end{pmatrix}$, where T_1 is a $d_1\times d_1$ orthogonal matrix and T_2 is a $d_2\times d_2$ orthogonal matrix. Prove that the space of $(O(d_1)\times O(d_2))$-invariant functions in $\mathcal{H}_n$ is zero-dimensional if n is odd and one-dimensional if n is even.

For the persevering, identify these $(O(d_1)\times O(d_2))$-invariant functions in $\mathcal{H}_n$ with certain orthogonal polynomials.

2.5 Compact homogeneous spaces and reproducing kernels (continued)

As a preparation of the next theorem we prove the following lemma, which is related to Schur's lemma (Lemma 1.20) and to Definition 1.21 of equivalence of representations.

Lemma 2.17. *Let G be a group and let π and ρ be irreducible unitary representations of G on finite dimensional inner product spaces V and W, respectively. Let $A\colon V\to W$ be a linear bijection such that A is G-*intertwining*, i.e., $A\pi(g)=\rho(g)A$ for all $g\in G$. Then there exists $\lambda>0$ such that $\langle Av_1,Av_2\rangle=\lambda\langle v_1,v_2\rangle$ for all $v_1,v_2\in V$.*

Proof Let $A^*\colon W\to V$ be the *adjoint* of V, which is defined by the property that $\langle A^*w,v\rangle=\langle w,Av\rangle$ for all $v\in V$, $w\in W$. Then A^* is G-intertwining and hence $A^*A\colon V\to V$ is G-intertwining. By Schur's lemma $A^*A=\lambda I$ for some $\lambda\in\mathbb{C}$. Then

$$\langle Av_1,Av_2\rangle=\langle A^*Av_1,v_2\rangle=\lambda\langle v_1,v_2\rangle.$$

Because of bijectivity, $\lambda \neq 0$. By taking $v_1 = v_2$ we see that $\lambda > 0$. $\square$

In the remainder of this subsection let G, X, ω, x_0, K be as in §2.3.

Theorem 2.18. *Let V, W be G-invariant linear subspaces of $C(X)$ of finite nonzero dimension. Suppose that the representations of G on V and W are irrreducible, that $\dim V_K = 1$ (V_K being the space of zonal functions in V), and that $V \neq W$. Then:*

a) The representations of G on V and W are inequivalent.

b) V is orthogonal to W.

Proof For the proof of a) suppose that the conclusion is not true, so suppose that the representations of G on V and W are equivalent. Then there is a bijective G-intertwining linear map $A\colon V \to W$. By Lemma 2.17, this map A can be taken such that $\langle Ap, Aq \rangle = \langle p, q \rangle$ for all $p, q \in V$. Take an orthonormal basis $f_1, \ldots, f_N$ of V. Then $Af_1, \ldots, Af_N$ is an orthonormal basis of W. Put

$$\Phi(x, y) := \sum_{j=1}^{N} f_j(x)\, \overline{f_j(y)}, \qquad \Psi(x, y) := \sum_{j=1}^{N} f_j(x)\, \overline{Af_j(y)}.$$

We have seen in (2.15) that $\Phi(g\,.\,x, g\,.\,y) = \Phi(x, y)$ for all $g \in G$. Similarly $\Psi(g\,.\,x, g\,.\,y) = \Psi(x, y)$ for all $g \in G$.

The functions Φ and Ψ are linearly independent. Indeed, we have for all $f \in V$ that

$$f(y) = \int_X \overline{\Phi(x, y)}\, f(x)\, d\omega(x), \qquad (Af)(y) = \int_X \overline{\Psi(x, y)}\, f(x)\, d\omega(x)$$

(show it first for $f := f_j$). So $\Phi = c\Psi$ with $c \neq 0$ would imply that $V = W$.

From this, together with G-invariance of Φ and Ψ, It follows that the functions $x \mapsto \Phi(x, x_0)$ and $x \mapsto \Psi(x, x_0)$ are linearly independent. But these two functions are both in V and zonal. This is a contradiction.

For the proof of b) consider the orthogonal projection $A\colon V + W \to V$. This is a G-intertwining map. Then A restricted to W gives a G-intertwining map $A\colon W \to V$. By Schur's lemma A is bijective or $A = 0$. If A is bijective then the representations of G on V and W are equivalent, which cannot be the case by a). Hence $A = 0$, so all $w \in W$ are orthogonal to V. $\square$

Proposition 2.19. *Let V be a G-invariant linear subspace of $C(X)$ of finite nonzero dimension N and with $\dim V_K = 1$. Let ϕ be the zonal function in V as defined by (2.16). Let dk be the normalized Haar measure on K. Then*

$$\int_K f(k\,.\,x)\, dk = f(x_0)\, \phi(x) \qquad (f \in V,\ x \in X). \tag{2.24}$$

Proof Put $f_0(x) := \int_K f(k\,.\,x)\, dk$. Then $f_0 \in V$. Indeed, by Lebesgue's theorem f_0 is continuous on X. If $h \in L^2(X)$ is orthogonal to V then, by Fubini's theorem,

$$\langle f_0, h \rangle = \int_K \langle k^{-1}\,.\,f, h \rangle\, dk = 0,$$

where the last identity follows because $k^{-1} . f \in V$. Hence f_0 is orthogonal to the orthoplement of V, and therefore $f_0 \in V$.

Furthermore, f_0 is zonal by right invariance of the measure dk. Since $\dim V_K = 1$ we must have $f_0 = c\,\phi$ for some $c \in \mathbb{C}$. In particular, $f_0(x_0) = c\,\phi(x_0) = c$. So $c = f_0(x_0) = \int_K f(k . x_0)\,dk = f(x_0)$. □

Theorem 2.20. *Let V be a G-invariant linear subspace of $C(X)$ of finite nonzero dimension N and with* $\dim V_K = 1$. *Let Φ and ϕ be given by* (2.14), (2.16). *Then we have the* product formula

$$\frac{1}{N}\int_K \Phi(k . x, y)\,dk = \phi(x)\,\overline{\phi(y)} \qquad (x, y \in X), \tag{2.25}$$

another version of which reads as

$$\int_K \phi\big((g_1 k g_2) . x_0\big)\,dk = \phi(g_1 . x_0)\,\phi(g_2 . x_0) \qquad (g_1, g_2 \in G). \tag{2.26}$$

Proof For the proof of (2.25) consider (2.24) with $f(x) := N^{-1}\Phi(x, y)$. This function is indeed in V. We obtain from (2.24) that the left-hand side of (2.25) equals $N^{-1}\Phi(x_0, y)\,\phi(x)$. Now use that $\Phi(x_0, y) = \overline{\Phi(y, x_0)} = N\,\overline{\phi(y)}$.

For the proof of (2.26) first rewrite the left-hand side of (2.26) and then apply (2.24):

$$\frac{1}{N}\int_K \Phi\big(k . (g_2 . x_0), g_1^{-1} . x_0\big)\,dk = N^{-1}\Phi(x_0, g_1^{-1} . x_0)\,\phi(g_2 . x_0) = N^{-1}\Phi(g_1 . x_0, x_0)\,\phi(g_2 . x_0),$$

which is equal to the right-hand side of (2.26). □

2.6 Applications of the abstract theory to spherical harmonics

We continue §2.2 and §2.4, where we keep the notation of §2.1. Consider first the applications of Theorem 2.18 to the space $\mathcal{H}_n$. Since the space of zonal functions in $\mathcal{H}_n$ has diemnsion 1, Theorem 2.18a) would also prove the orthogonality of the spaces $\mathcal{H}_n$ (see Theorem 2.10) as soon as we have shown that $\mathcal{H}_n \neq \mathcal{H}_m$ if $n \neq m$. Anyhow, if we know that the spaces $\mathcal{H}_n$ are orthogonal then we know that $\mathcal{H}_n \neq \mathcal{H}_m$ if $n \neq m$, and then we can also use Theorem 2.18b) in order to conclude:

Theorem 2.21. *The representations of $O(d)$ on the spaces $\mathcal{H}_n$ are mutually inequivalent.*

There are various other ways to prove this last result. First, for $d > 2$ we can observe from (2.6) that $\dim \mathcal{H}_m < \dim \mathcal{H}_n$ if $m < n$. Indeed, for $d = 3$ we have $\dim \mathcal{H}_n = 2n+1$ and for $d > 3$ we have

$$\dim \mathcal{H}_n = \frac{(n+1)(n+2)\ldots(n+d-3)\,(2n+d-2)}{(d-2)!},$$

which is strictly increasing in n. Clearly, representations on spaces of different dimensions cannot be equivalent.

For another, more conceptual proof of the inequivalence of the representations recall that the polynomial $(x_1+ix_2)^n$ belongs to $\mathcal{H}_n$. Consider also the group $SO(2)$, embedded in $O(d)$ as the subgroup consisting of all matrices

$$A_\theta := \begin{pmatrix} \cos\theta & \sin\theta & 0 & \ldots & 0 \\ -\sin\theta & \cos\theta & 0 & \ldots & 0 \\ 0 & 0 & 1 & & \text{\Large 0} \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \text{\Large 0} & & 1 \end{pmatrix}.$$

For $\varepsilon_n(x) := (x_1+ix_2)^n$ we have $A_\theta\,.\,\varepsilon_n = e^{in\theta}\varepsilon_n$, so the restriction to $SO(2)$ of the representation of $O(d)$ on $\mathcal{H}_n$ contains the one-dimensional irreducible representation $A_\theta \mapsto e^{in\theta}$ as a subrepresentation. However, if $m<n$ and $0\ne f\in\mathcal{H}_m$ then we cannot have that $A_\theta f = e^{in\theta} f$ for all θ. Indeed, we can expand $f\in\mathcal{H}_m$ as

$$f(x) = \sum_{m_1+\cdots+m_d=m} c_{m_1,\ldots,m_d}(x_1+ix_2)^{m_1}(x_1-ix_2)^{m_2}x_3^{m_3}\ldots x_d^{m_d}.$$

Then

$$(A_\theta f)(x) = \sum_{m_1+\cdots+m_d=m} e^{i(m_1-m_2)\theta} c_{m_1,\ldots,m_d}(x_1+ix_2)^{m_1}(x_1-ix_2)^{m_2}x_3^{m_3}\ldots x_d^{m_d}.$$

If we would have $A_\theta f = e^{in\theta} f$ for all θ then

$$f(x) = \frac{1}{2\pi}\int_0^{2\pi}(A_\theta f)(x)\,e^{-in\theta}\,d\theta = 0$$

since $m_1-m_2\ne n$ for $m_1,m_2\ge 0$ and $m_1+m_2\le m<n$.

We conclude that the representations of $O(d)$ on $\mathcal{H}_n$ and $\mathcal{H}_m$ $(m<n)$ are inequivalent because their restrictions to $SO(2)$ are already inequivalent.

Put

$$R_n^{(\alpha,\beta)}(x) := \frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} \tag{2.27}$$

and

$$p_n(t) := R_n^{(\frac12 d-\frac32,\frac12 d-\frac32)}(t) = \phi\Big(te_1+\sqrt{1-t^2}\,x'\Big) \qquad (t\in[-1,1],\ x'\in S^{d-2}), \tag{2.28}$$

where the second equality was given in Remark 2.14. Put also

$$N_n := \dim\mathcal{H}_n = \frac{(2n+d-2)(n+d-3)!}{n!\,(d-2)!}, \tag{2.29}$$

where the second equality was given in (2.6). Then

$$\Phi(x,y) = N_n\,p_n(\langle x,y\rangle) \qquad (x,y\in S^{d-1}). \tag{2.30}$$

Indeed, we can write $y = Te_1$ for some $T \in O(d)$. Then

$$\Phi(x,y) = \Phi(x,Te_1) = \Phi(T^{-1}x,e_1) = N_n\, p_n(\langle T^{-1}x,e_1\rangle) = N_n\, p_n(\langle x,Te_1\rangle) = N_n\, p_n(\langle x,y\rangle).$$

Now we derive an integral representation for p_n by applying Proposition 2.19 to the case that $V = \mathcal{H}_n$ and $f(x) = \varepsilon_n(x) = (x_1 + ix_2)^n$. Note that $\varepsilon_n(e_1) = 1$. Then (2.24) yields

$$\int_{O(d-1)} \varepsilon_n(Tx)\, dT = \phi(x) = p_n(\langle x,e_1\rangle) \qquad (x \in S^{d-1}),$$

where dT is the normalized Haar measure on $O(d-1)$. Put $x := te_1 + \sqrt{1-t^2}\, e_2$ $(t \in [-1,1])$. Then $\langle x,e_1\rangle = t$ and $\varepsilon_n(Tx) = \big(t + i\sqrt{1-t^2}\,\langle Te_2,e_2\rangle\big)^n$. So

$$p_n(t) = \int_{O(d-1)} \big(t + i\sqrt{1-t^2}\,\langle Te_2,e_2\rangle\big)^n\, dT.$$

In general, for a continuous function F on $[-1,1]$ and with ω' the normalized $O(d-1)$-invariant measure on S^{d-2} we have

$$\int_{O(d-1)} F\big(\langle Te_2,e_2\rangle\big)\, dT = \int_{S^{d-2}} F(\langle x',e_2\rangle)\, d\omega'(x') = \frac{\Gamma(\frac12 d - \frac12)}{\Gamma(\frac12 d - 1)\Gamma(\frac12)} \int_{-1}^{1} F(u)\,(1-u^2)^{\frac12 d-2}\, du, \tag{2.31}$$

where we used the beginning of §2.3 in the first equality and (2.21) in the second equality. We conclude that

$$R_n^{(\frac12 d-\frac32,\frac12 d-\frac32)}(t) = \frac{\Gamma(\frac12 d - \frac12)}{\Gamma(\frac12 d - 1)\Gamma(\frac12)} \int_{-1}^{1} (t + i\sqrt{1-t^2}\,u)^n\,(1-u^2)^{\frac12 d-2}\, du. \tag{2.32}$$

Finally we derive a product formula for p_n by applying Theorem 2.20 to the case that $V = \mathcal{H}_n$. From (2.25) (note that p_n is real-valued) and (2.30) we get:

$$\int_{O(d-1)} p_n(\langle Tx,y\rangle)\, dT = p_n(\langle x,e_1\rangle)\, p_n(\langle y,e_1\rangle).$$

Put $x = se_1 + \sqrt{1-s^2}\, x'$ and $y = te_1 + \sqrt{1-t^2}\, y'$ $(s,t \in [-1,1],\ x',y' \in S^{d-2})$. Then

$$\int_{O(d-1)} p_n\Big(st + \sqrt{1-s^2}\,\sqrt{1-t^2}\,\langle Tx',y'\rangle\Big)\, dT = p_n(s)\, p_n(t).$$

There are $T_1, T_2 \in O(d-1)$ such that $x' = T_1e_2$, $y' = T_2e_2$. Then $\langle Tx',y'\rangle = \langle T_2^{-1}TT_1e_2,e_2\rangle$. By left and right invariance of dT we get

$$\int_{O(d-1)} p_n\Big(st + \sqrt{1-s^2}\,\sqrt{1-t^2}\,\langle Te_2,e_2\rangle\Big)\, dT = p_n(s)\, p_n(t).$$

By using (2.31) we conclude that

$$\begin{aligned} &R_n^{(\frac12 d-\frac32,\frac12 d-\frac32)}(s)\, R_n^{(\frac12 d-\frac32,\frac12 d-\frac32)}(t) \\ &\qquad = \frac{\Gamma(\frac12 d - \frac12)}{\Gamma(\frac12 d - 1)\Gamma(\frac12)} \int_{-1}^{1} R_n^{(\frac12 d-\frac32,\frac12 d-\frac32)}\Big(st + \sqrt{1-s^2}\,\sqrt{1-t^2}\,u\Big)\,(1-u^2)^{\frac12 d-2}\, du \end{aligned} \tag{2.33}$$

Exercise 2.22. Show that, for $d \geq 3$, there is a convergent expansion

$$\frac{1}{|e_1 - rx|^{d-2}} = \sum_{n=0}^{\infty} r^n f_n(x) \qquad (0 \leq r < 1,\ x \in \mathbb{R}^d,\ |x| \leq 1)$$

with f_n some homogeneous polynomial of degree n.
(*Hint* Use Taylor series on several variables.)
Show that $\Delta f_n = 0$ and that f_n is $O(d-1)$-invariant.
Compute $f_n(e_1)$. Conclude that

$$\frac{1}{(1-2rt+t^2)^{\frac{1}{2}d-1}} = \sum_{n=0}^{\infty} \frac{(d-2)_n}{(\frac{1}{2}d-\frac{1}{2})_n}\, P_n^{(\frac{1}{2}d-\frac{3}{2},\frac{1}{2}d-\frac{3}{2})}(t) \qquad (0 \leq r < 1,\ -1 \leq t \leq 1),$$

where $P_n^{(\frac{1}{2}d-\frac{3}{2},\frac{1}{2}d-\frac{3}{2})}(t)$ is a Jacobi polynomial.

Exercise 2.23. Let X, V, x_0, ω, ϕ be as in §2.3. Let $f \in V$. Show that $f(x_0) = 0$ iff $\langle f, \phi \rangle = 0$.

Exercise 2.24. Show that $\frac{\partial}{\partial x_1}$ sends an $O(d-1)$-invariant harmonic homogeneous polynomial of degree n on $\mathbb{R}^d$ to a similar polynomial of degree $n-1$. Use this to obtain a differentiation formula sending the Jacobi polynomial $P_n^{(\frac{1}{2}d-\frac{3}{2},\frac{1}{2}d-\frac{3}{2})}(t)$ to $P_{n-1}^{(\frac{1}{2}d-\frac{3}{2},\frac{1}{2}d-\frac{3}{2})}(t)$.

2.7 The addition formula

We continue §2.2, §2.4 and §2.6, where we keep the notation of §2.1.

Example 2.25. Let $d = 2$. If $n \in \mathbb{Z}_{>0}$ then $\dim \mathcal{H}_n = 2$ and $\mathcal{H}_n$ has basis $(x_1 \pm ix_2)^n$. The restrictions of these polynomials to S^1 are the functions

$$(\cos\theta, \sin\theta) \mapsto e^{\pm in\theta}.$$

They form an orthonormal basis of $\mathcal{H}_n$. Then (2.14), for $V := \mathcal{H}_n$ and the orthonormal basis just chosen, gives

$$\Phi\big((\cos\theta_1, \sin\theta_1), (\cos\theta_2, \sin\theta_2)\big) := e^{in\theta_1}e^{-in\theta_2} + e^{-in\theta_1}e^{in\theta_2} = 2\cos\big(n(\theta_1-\theta_2)\big).$$

Then (2.16) gives for the zonal spherical function:

$$\phi(\cos\theta, \sin\theta) = \cos(n\theta). \tag{2.34}$$

Now define the *Chebyshev polynomial of the first kind* by

$$T_n(\cos\theta) := \cos(n\theta). \tag{2.35}$$

From the trigonometric identity

$$\cos\theta\cos(n\theta) = \tfrac{1}{2}\cos\big((n+1)\theta\big) + \tfrac{1}{2}\cos\big((n-1)\theta\big) \qquad (n \in \mathbb{Z}_{>0}) \tag{2.36}$$

we see that $T_n(x)$ is a polynomial of degree n in x, and from the orthogonality

$$\int_0^\pi \cos(m\theta)\,\cos(n\theta)\,d\theta = 0 \qquad (m,n \in \mathbb{Z}_{\ge 0},\ m \ne n)$$

we see that

$$\int_{-1}^1 T_m(x)\,T_n(x)\,(1-x^2)^{-\frac12}\,dx = 0 \qquad (m \ne n). \tag{2.37}$$

So the polynomials T_n are special Jacobi polynomials:

$$T_n(x) = \frac{n!}{(\frac12)_n}\,P_n^{(-\frac12,-\frac12)}(x). \tag{2.38}$$

Now (2.30) gives

$$T_n\big(\cos(\theta_1-\theta_2)\big) = \tfrac12\big(e^{in\theta_1}e^{-in\theta_2} + e^{-in\theta_1}e^{in\theta_2}\big). \tag{2.39}$$

Another orthonormal basis of $\mathcal{H}_n$ can be chosen by starting with the zonal function $(\cos\theta,\sin\theta) \mapsto 2^{\frac12}\cos(n\theta)$ and complementing it with the function $(\cos\theta,\sin\theta) \mapsto 2^{\frac12}\sin(n\theta)$. Then (2.30) gives

$$T_n\big(\cos(\theta_1-\theta_2)\big) = \cos(n\theta_1)\,\cos(n\theta_2) + \sin(n\theta_1)\,\sin(n\theta_2), \tag{2.40}$$

which is a well-known trigonometric identity when we rewrite the left-hand side by means of (2.35). Formula (2.40) is a prototype of an *addition formula* because of the addition (or rather difference) of θ_1 and θ_2 in the argument on the left-hand side.

It is also interesting to see the explicit realization of the product formula for $d=2$. We start with the formula just above (2.33), which becomes for $d=2$ as follows:

$$\int_{O(1)} T_n\Big(st + \sqrt{1-s^2}\,\sqrt{1-t^2}\,\langle Te_2, e_2\rangle\Big)\,dT = T_n(s)\,T_n(t).$$

This can be rewritten as:

$$\tfrac12\Big(T_n\big(\cos(\theta_1-\theta_2)\big) + T_n\big(\cos(\theta_1+\theta_2)\big)\Big) = T_n(\cos\theta_1)\,T_n(\cos\theta_2), \tag{2.41}$$

again a well-known trigonometric identity after rewriting by means of (2.35). Note that the product formula (2.41) follows from the addition formula (2.40) by adding to (2.40) the identity obtained from it by the substitution $\theta_2 \to -\theta_2$.

Now we will work with $\mathcal{H}_n$ for general $d>2$. Consider (2.30) with p_n given by (2.28), N_n by (2.29) and Φ by (2.14). Choose the orthonormal basis $f_1,\dots,f_{N_n}$ of $\mathcal{H}_n$ such that $f_1 := \sqrt{N_n}\,\phi_n$. Then (2.30) can be written as:

$$p_n(\langle x,y\rangle) = p_n(\langle x,e_1\rangle)\,p_n(\langle y,e_1\rangle) + N_n^{-1}\sum_{j=2}^{N_n} f_j(x)\,\overline{f_j(y)} \qquad (x,y \in S^{d-1}). \tag{2.42}$$

Observe that the product formula for p_n, when written in the form of the first formula after (2.32), follows from (2.42). First replace x by Tx ($T \in O(d-1)$) in (2.42):

$$p_n(\langle Tx, y\rangle) = p_n(\langle x, e_1\rangle)\, p_n(\langle y, e_1\rangle) + N_n^{-1} \sum_{j=2}^{N_n} f_j(Tx)\, \overline{f_j(y)} \qquad (x, y \in S^{d-1},\ T \in O(d-1)). \tag{2.43}$$

Then integrate both sides of (2.43) over $O(d-1)$ with respect to dT. We will arrive at the mentioned product formula if we know that $\int_{O(d-1)} f_j(Tx)\, dT = 0$ for $j = 2, \ldots, N_n$. This follows from a slight extension of (2.24):

Proposition 2.26. *Under the assumptions of Proposition 2.19 we have*

$$\int_K f(k\,.\,x)\, dk = f(x_0)\, \phi(x) = N\langle f, \phi\rangle\, \phi(x) \qquad (f \in V,\ x \in X). \tag{2.44}$$

Proof Write $f_0(x)$ for the left part of (2.44), as in the proof of Proposition 2.19. Then, by (2.24), $f_0 = f(x_0)\, \phi$. Hence $\langle f_0, \phi\rangle = N^{-1} f(x_0)$. On the other hand, by Fubini's theorem, $\langle f_0, \phi\rangle = \int_K \langle k^{-1}\,.\,f, \phi\rangle\, dk = \int_K \langle f, k\,.\,\phi\rangle\, dk = \langle f, \phi\rangle$. □

So, indeed, we obtain from (2.43) again the product formula

$$\int_{O(d-1)} p_n(\langle Tx, y\rangle)\, dT = p_n(\langle x, e_1\rangle)\, p_n(\langle y, e_1\rangle).$$

As we saw in §2.6, this product formula can be rewritten as (2.33). The integrand in (2.33) can be seen, for fixed s, t, as a polynomial of degree n in u which is multiplied by $(1-u^2)^{\frac{1}{2}d-2}$ (the weight function for the Jacobi polynomials $R_j^{(\frac{1}{2}d-2, \frac{1}{2}d-2)}(u)$). This suggests to look for an expansion

$$R_n^{(\frac{1}{2}d-\frac{3}{2}, \frac{1}{2}d-\frac{3}{2})}\left(st + \sqrt{1-s^2}\,\sqrt{1-t^2}\, u\right) = \sum_{j=0}^{n} f_{n,j}(s,t)\, R_j^{(\frac{1}{2}d-2, \frac{1}{2}d-2)}(u).$$

From this expansion we see, by (2.33), that

$$f_{n,0}(s,t) = R_n^{(\frac{1}{2}d-\frac{3}{2}, \frac{1}{2}d-\frac{3}{2})}(s)\, R_n^{(\frac{1}{2}d-\frac{3}{2}, \frac{1}{2}d-\frac{3}{2})}(t).$$

Below we will find $f_{n,j}(s,t)$ explicitly for other values of j. This will be done by choosing the f_j in (2.43) more specially, adapted to a decomposition of $\mathcal{H}_n$ with respect to the subgroup $O(d-1)$.

Let $k \in \{0, 1, \ldots, n\}$. Let $\mathcal{H}'_k$ denote the space of harmonic homogeneous polynomials $h(x_2, \ldots, x_d)$ of degree k.

Lemma 2.27. *There exists a nonzero zonal homogeneous polynomial $\phi_n^k \in \mathcal{P}_{n-k}$ of degree k, unique up to a constant factor, with the following property:*
For all nonzero zonal $f \in \mathcal{P}_{n-k}$ and for all nonzero $h \in \mathcal{H}'_k$ we have that $\Delta(fh) = 0$ iff $f = c\,\phi_n^k$ for some nonzero c.

Note that Proposition 2.6 is the special case $k = 0$ of the above lemma.

Proof of Lemma 2.27.
Put $\rho := \sqrt{x_2^2 + \ldots + x_d^2}$. Let $f \in \mathcal{P}_{n-k}$ be nonzero and zonal. By Lemma 2.5 we can write

$$f(x) = \sum_{j=0}^{[\frac{1}{2}(n-k)]} c_j x_1^{n-k-2j} \rho^{2j} \qquad (x \in \mathbb{R}^d). \tag{2.45}$$

Let $h \in \mathcal{H}'_k$ be nonzero. Clearly, $fh \in \mathcal{P}_n$. We have

$$\begin{aligned}\Delta(x_1^{n-k-2j}\rho^{2j}h) &= \frac{\partial^2\, x_1^{n-k-2j}}{\partial x_1^2}\,\rho^{2j}h + x_1^{n-k-2j}\left(\left(\frac{\partial^2}{\partial\rho^2} + \frac{d-2}{\rho}\,\frac{d}{d\rho}\right)\rho^{2j}\right)h + 2x_1^{n-k-2j}\sum_{i=2}^{d}\frac{\partial\rho^{2j}}{\partial x_i}\,\frac{\partial h}{\partial x_i}\\ &= (n-k-2j)(n-k-2j-1)\,x_1^{n-k-2j-2}\rho^{2j}h + 2j(2j+2k+d-3)\,x_1^{n-k-2j}\rho^{2j-2}h,\end{aligned}$$

where we used that

$$\sum_{i=2}^{d}\frac{\partial\rho^{2j}}{\partial x_i} = 2j\rho^{2j-2}\sum_{i=2}^{d} x_i\,\frac{\partial h}{\partial x_i} = 2jk\rho^{2j-2}h$$

(since h is homogeneous of degree k in $x_2, \ldots, x_d$). Then we find for f given by (2.45) that $\Delta(fh) = 0$ iff

$$c_j = -\,\frac{(n-k-2j-2)(n-k-2j-3)}{2j(2j+2k+d-3)}\,c_{j-1}.$$

Hence there is a one-dimensional space of functions f as in (2.45) such that $\Delta(fh) = 0$. This space is independent of the choice of h, it is only dependent on k. $\square$

Proposition 2.28. *Let ϕ_n^k and $\mathcal{H}'_k$ be as in Lemma 2.27. Then:*

a) We have

$$\phi_n^k(te_1 + \sqrt{1-t^2}\,x') = \text{const.}\,(1-t^2)^{\frac{1}{2}k}\,P_{n-k}^{(\frac{1}{2}d-\frac{3}{2}+k,\frac{1}{2}d-\frac{3}{2}+k)}(t) \qquad (t \in [-1,1],\ x' \in S^{d-2}). \tag{2.46}$$

b) The subspaces $\phi_n^k\,\mathcal{H}'_k$ of $\mathcal{H}_n$ are mutually orthogonal.

c) $\dim\mathcal{H}_n = \sum_{k=0}^{n}\dim\mathcal{H}'_k$

d) There is the orthogonal direct sum decomposition $\mathcal{H}_n = \bigoplus_{k=0}^{n}\phi_n^k\,\mathcal{H}'_k$.

Proof For a) let ω be the $O(d)$-invariant measure on S^{d-1} and ω' be the $O(d-1)$-invariant measure on S^{d-2}. By (2.19) and (2.21) we have

$$d\omega(x) = \frac{\Gamma(\frac{1}{2}d-\frac{1}{2})}{\Gamma(\frac{1}{2}d-1)\Gamma(\frac{1}{2})}\,(1-t^2)^{\frac{1}{2}d-\frac{3}{2}}\,dt\,d\omega'(x') \qquad (x = te_1+\sqrt{1-t^2}\,x',\ t\in[-1,1],\ x'\in S^{d-2}).$$

Take $0\neq h\in\mathcal{H}'_k$ and $f_n := \phi_n^k\,h$. Then

$$\begin{aligned} f_n(te_1+\sqrt{1-t^2}\,x') &= \phi_n^k(te_1+\sqrt{1-t^2}\,x')\,h(\sqrt{1-t^2}\,x') \\ &= p_n^k(t)\,(1-t^2)^{\frac{1}{2}k}\,h(x') \qquad (t\in[-1,1],\ x'\in S^{d-2}) \end{aligned}$$

for some nonzero polynomial p_n^k of degree $\le n-k$. By orthogonality of $\mathcal{H}_n$ and $\mathcal{H}_m$ $(n\neq m)$ we get

$$\begin{aligned} 0 = \int_{S^{d-1}} f_n(x)\,\overline{f_m(x)}\,d\omega(x) &= \frac{\Gamma(\frac{1}{2}d-\frac{1}{2})}{\Gamma(\frac{1}{2}d-1)\Gamma(\frac{1}{2})} \\ &\times \int_{x'\in S^{d-2}}\int_{t=-1}^{1} p_n^k(t)\,(1-t^2)^{\frac{1}{2}k}\,h(x')\,p_m^k(t)\,(1-t^2)^{\frac{1}{2}k}\,\overline{h(x')}\,(1-t^2)^{\frac{1}{2}d-\frac{3}{2}}\,dt\,d\omega'(x') \\ &= \frac{\Gamma(\frac{1}{2}d-\frac{1}{2})}{\Gamma(\frac{1}{2}d-1)\Gamma(\frac{1}{2})}\int_{S^{d-2}}|h(x')|^2\,d\omega'(x')\int_{-1}^{1}p_n^k(t)\,p_m^k(t)\,(1-t^2)^{\frac{1}{2}d-\frac{3}{2}+k}\,dt. \end{aligned}$$

Hence

$$\int_{-1}^{1}p_n^k(t)\,p_m^k(t)\,(1-t^2)^{\frac{1}{2}d-\frac{3}{2}+k}\,dt = 0 \qquad (n\neq m)$$

and we conclude that

$$p_n^k(t) = \text{const.}\,P_{n-k}^{(\frac{1}{2}d-\frac{3}{2}+k,\frac{1}{2}d-\frac{3}{2}+k)}(t). \tag{2.47}$$

For b) let $h_k\in\mathcal{H}'_k$, $h_l\in\mathcal{H}'_l$ $(k\neq l)$. Then

$$\begin{aligned} \int_{S^{d-1}}\phi_n^k\,h_k\,\overline{\phi_n^l\,h_l}\,d\omega &= \frac{\Gamma(\frac{1}{2}d-\frac{1}{2})}{\Gamma(\frac{1}{2}d-1)\Gamma(\frac{1}{2})}\int_{-1}^{1}p_n^k(t)\,p_n^l(t)\,(1-t^2)^{\frac{1}{2}(d-3+k+l)}\,dt \\ &\times\int_{S^{d-2}}h_k(x')\,\overline{h_l(x')}\,d\omega'(x') = 0. \end{aligned}$$

For c) use (2.6):

$$\begin{aligned} \dim\mathcal{H}_n - \dim\mathcal{H}_{n-1} &= \frac{(2n+d-2)(n+d-3)!}{n!\,(d-2)!} - \frac{(2n+d-4)(n+d-4)!}{(n-1)!\,(d-2)!} \\ &= \frac{(2n+d-3)(n+d-4)!}{n!\,(d-3)!} = \dim\mathcal{H}'_n. \end{aligned}$$

Finally, d) follows from b) and c). $\square$

Let us refine (2.42) by a special choice of the orthonormal basis functions f_j. First put $x = se_1 + \sqrt{1-s^2}\,x'$ and $y = te_1 + \sqrt{1-t^2}\,y'$ ($s,t \in [-1,1]$, $x',y' \in S^{d-2}$). Then (2.42) becomes

$$p_n\Big(st+\sqrt{1-s^2}\,\sqrt{1-t^2}\,\langle x',y'\rangle\Big) = p_n(s)\,p_n(t)+N_n^{-1}\sum_{j=2}^{N_n} f_j(se_1+\sqrt{1-s^2}\,x')\,\overline{f_j(te_1+\sqrt{1-t^2}\,y')}. \tag{2.48}$$

Now choose the f_j corresponding to the orthogonal direct sum decomposition of $\mathcal{H}_n$ in Proposition 2.28 d). Choose an orthonormal basis $h_{k,j}$ ($j = 1,2,\ldots,N'_k := \dim \mathcal{H}'_k$) for $\mathcal{H}'_k$. Let $p_n^k(t)$ be as in (2.47). Then the functions

$$se_1 + \sqrt{1-s^2}\,x' \mapsto (c_n^k)^{\frac12}\,p_n^k(s)\,(1-s^2)^{\frac12 k}\,h_{k,j}(x') \qquad (k=0,1,\ldots,n,\ j=1,\ldots,N'_k)$$

with normalization constants c_n^k given by

$$(c_n^k)^{-1} := \frac{\Gamma(\frac12 d-\frac12)}{\Gamma(\frac12 d-1)\Gamma(\frac12)}\int_{-1}^1 (p_n^k(s))^2\,(1-s^2)^{\frac12 d-\frac32+k}\,ds, \tag{2.49}$$

form an orthonormal basis of $\mathcal{H}_n$. With this basis formula (2.48) takes the form

$$\begin{aligned} p_n\Big(st+\sqrt{1-s^2}\,\sqrt{1-t^2}\,\langle x',y'\rangle\Big) &= p_n(s)\,p_n(t) \\ &\quad + N_n^{-1}\sum_{k=1}^n c_n^k\,p_n^k(s)\,(1-s^2)^{\frac12 k}\,p_n^k(t)\,(1-t^2)^{\frac12 k}\sum_{j=1}^{N'_k} h_{k,j}(x')\,\overline{h_{k,j}(y')}. \end{aligned}$$

Now we apply (2.30), (2.28) with d replaced by $d-1$ to the inner sum on the right-hand side above. Then this inner sum will be equal to $N'_k\,R_k^{(\frac12 d-2,\frac12 d-2)}(\langle x',y'\rangle)$. Since in the resulting identity x' and y' only occur in the form $\langle x',y'\rangle$, we may put $u := \langle x',y'\rangle$ and we arrive at the addition formula

$$\begin{aligned} R_n^{(\frac12 d-\frac32,\frac12 d-\frac32)}\Big(st+\sqrt{1-s^2}\,\sqrt{1-t^2}\;u\Big) &= R_n^{(\frac12 d-\frac32,\frac12 d-\frac32)}(s)\,R_n^{(\frac12 d-\frac32,\frac12 d-\frac32)}(t) \\ &\quad + N_n^{-1}\sum_{k=1}^n c_n^k\,N'_k\,p_n^k(s)\,(1-s^2)^{\frac12 k}\,p_n^k(t)\,(1-t^2)^{\frac12 k}\,R_k^{(\frac12 d-2,\frac12 d-2)}(u). \end{aligned} \tag{2.50}$$

Now use (2.47), (2.49), (2.27) (1.53), (1.56) in (2.50). Then we have proved the case $\alpha = 0,\frac12,1,\ldots$ of the following:

Theorem 2.29 (addition formula for Jacobi polynomials $P_n^{(\alpha,\alpha)}$)**.** *For $x,y \in [-1,1]$ and $\alpha > -\frac12$ (and by suitable analytic continuation for more general complex values of x,y,α) we have:*

$$\begin{aligned} P_n^{(\alpha,\alpha)}\Big(xy+\sqrt{1-x^2}\,\sqrt{1-y^2}\;t\Big) &= \sum_{k=0}^n \frac{(\alpha+k)\,(n+2\alpha+1)_k\,(2\alpha+1)_k\,(n-k)!}{2^{2k}\,(\alpha+\frac12 k)\,(\alpha+\frac12)_k\,(\alpha+1)_n} \\ &\quad \times P_{n-k}^{(\alpha+k,\alpha+k)}(x)\,(1-x^2)^{\frac12 k}\,P_{n-k}^{(\alpha+k,\alpha+k)}(y)\,(1-y^2)^{\frac12 k}\,P_k^{(\alpha-\frac12,\alpha-\frac12)}(t). \end{aligned} \tag{2.51}$$

Proof Fix n, x, y, t. Both sides of (2.51) are rational functions of α (quotients of two polynomials in α). We know that the formula is true for $\alpha \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, this is for an infinite number of values of α. Hence the formula is valid for all $\alpha \in \mathbb{C}$ (outside the poles). $\square$

Exercise 2.30. Show that differentiation of both sides of (2.51) with respect to t yields the same formula with α, n replaced by $\alpha+1, n-1$, respectively. Conclude that (2.51) would already follow in general if we know its special case $\alpha = 0$ (i.e., the case obtained by an interpretation on S^2).

Exercise 2.31. Divide both sides of (2.51) by y^n and let $y \to \infty$. Show that the resulting formula is

$$\left(x + i\sqrt{1-x^2}\,t\right)^n = \sum_{k=0}^{n} \frac{i^k\,(\alpha+k)\,(2\alpha+1)_k\,n!}{2^k\,(\alpha+\frac{1}{2}k)\,(\alpha+\frac{1}{2})_k\,(\alpha+1)_n} \times P_{n-k}^{(\alpha+k,\alpha+k)}(x)\,(1-x^2)^{\frac{1}{2}k}\,P_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(t). \qquad (2.52)$$

Show that (2.32) (even for real $d > 2$) is also a consequence of (2.52).

2.8 Compact Gelfand pairs

In this subsection G will be a compact group with normalized Haar measure dg and K will be a closed subgroup of G with normalized Haar measure dk. Let $\widehat{G}$ be the set of all equivalence classes of finite dimensional irreducible unitary representations of G. For each $\pi \in \widehat{G}$ we choose in the equivalence class a concrete representation π on a linear space V_π. Let

$$d_\pi := \dim V_\pi, \qquad c_\pi := \dim\{v \in V_\pi \mid \forall k \in K\ \pi(k)v = v\}, \qquad (G/K)\widehat{\ }:= \{\pi \in \widehat{G} \mid c_\pi \neq 0\}.$$

So c_π is the dimension of the (evidently linear) space of K-fixed vectors in V_π. Also $\pi \in (G/K)\widehat{\ }$ iff V_π contains a nonzero K-fixed vector.

Definition 2.32. The pair (G, K) is called a *Gelfand pair* if $c_\pi \leq 1$ for each $\pi \in \widehat{G}$.
For $\pi \in \widehat{G}$ with $c_\pi = 1$ choose a K-fixed vector $v_K \in V_\pi$ with $\|v_K\| = 1$ and define the *spherical function* ϕ_π by

$$\phi_\pi(g) := \langle \pi(g)\,v_K, v_K\rangle \qquad (g \in G). \qquad (2.53)$$

Note that the definition of ϕ_π is independent of the choice of v_K.

The following properties of a spherical function ϕ_π are seen directly:

a) ϕ_π is a continuous function on G.

b) ϕ_π is K-biinvariant, i.e., $\phi_\pi(k_1 g k_2) = \phi_\pi(g)$ for all $k_1, k_2 \in K$, $g \in G$.

c) $\phi_\pi(e) = 1$.

For the proof of two further properties we need a lemma:

Lemma 2.33. *Let $c_\pi = 1$, $v \in V_\pi$. Then*

$$\int_K \pi(k)\, v\, dk = \langle v, v_K\rangle\, v_K. \tag{2.54}$$

Proof Put $v_0 := \int_K \pi(k)\, v\, dk$. Then $v_0 \in V$ and $\pi(k)\, v_0 = v_0$ for all $k \in K$ by left invariance of the Haar measure on K. Hence $v_0 = c\, v_K$ for some $c \in \mathbb{C}$. So

$$c = \langle v_0, v_K\rangle = \int_K \langle \pi(k)v, v_K\rangle\, dk = \int_K \langle v, \pi(k^{-1})v_K\rangle\, dk = \langle v, v_K\rangle \int_K dk = \langle v, v_K\rangle. \qquad \square$$

Proposition 2.34. *Let $c_\pi = 1$. Let f be a K-biinvariant function on G which is a linear combination of functions $g \mapsto \langle \pi(g)v, w\rangle$ $(v, w \in V_\pi)$. Then*

$$f = f(e)\, \phi_\pi. \tag{2.55}$$

Proof It is sufficient to show that $f = c\, \phi_\pi$ for some $c \in \mathbb{C}$. We can write $f(g)$ as a finite sum of terms $\langle \pi(g)\, v_j, w_j\rangle$ $(v_j, w_j \in V_\pi)$. Then

$$\begin{aligned} f(g) &= \int_K \int_K f(k_2^{-1} g k_1)\, dk_1\, dk_2 = \sum_j \int_K \int_K \langle \pi(k_2^{-1} g k_1)\, v_j, w_j\rangle\, dk_1\, dk_2 \\ &= \sum_j \langle \pi(g)\ \textstyle\int_K \pi(k_1)\, v_j\, dk_1, \int_K \pi(k_2)\, w_j\, dk_2\rangle = \sum_j \langle v_j, v_K\rangle\, \overline{\langle w_j, v_K\rangle}\, \langle \pi(g)v_K, v_K\rangle = c\, \phi_\pi(g). \end{aligned}$$

In the forelast equality we used (2.54). $\square$

Theorem 2.35 (product formula for spherical functions)**.** *Let $c_\pi = 1$. Then*

$$\int_K \phi_\pi(g_1 k g_2)\, dk = \phi_\pi(g_1)\, \phi_\pi(g_2) \qquad (g_1, g_2 \in G). \tag{2.56}$$

Proof For fixed g_1 the function $f\colon g_2 \mapsto \int_K \phi_\pi(g_1 k g_2)\, dk$ is K-biinvariant and $\int_K \phi_\pi(g_1 k g_2)\, dk = \langle \pi(g_2)\, v_K, \int_K \pi(k^{-1})\, \pi(g_1^{-1})\, v_K\, dk\rangle$. Hence we can apply Proposition 2.34. Then (2.55) yields:

$$\int_K \phi_\pi(g_1 k g_2)\, dk = f(g_2) = f(e)\, \phi_\pi(g_2) = \Big(\int_K \phi_\pi(g_1 k)\, dk\Big) \phi_\pi(g_2) = \phi_\pi(g_1)\, \phi_\pi(g_2). \qquad \square$$

For another characterization of Gelfand pairs we need the concepts of convolution and Fourier transform of functions on a compact group G. The *convolution product* $f_1 * f_2$ of $f_1, f_2 \in C(G)$ is defined by

$$(f_1 * f_2)(g) := \int_G f_1(g_1)\, f_2(g_1^{-1} g)\, dg_1 \qquad (g \in G). \tag{2.57}$$

Then $f_1 * f_2 \in C(G)$ and one can show *associativity*:

$$(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3).$$

However, if the group G is not commutative then the convolution product is usually not commutative.

For each $\pi \in \widehat{G}$ choose an orthonormal basis $e_1, \ldots, e_{d_\pi}$ of V_π with corresponding matrix elements $\pi_{i,j}$ $(i,j = 1, \ldots, d_\pi)$ of π. The functions $(d_\pi)^{\frac{1}{2}}\, \pi_{i,j}$ $(\pi \in \widehat{G},\ i,j = 1, \ldots, d_\pi)$ form an orthonormal system in $L^2(G)$ (see Theorem 1.22), and they form an orthonormal basis of the Hilbert space $L^2(G)$ by the *Peter-Weyl theorem*.

The *Fourier transform* of a function $f \in L^2(G)$ is defined as a "function" $\widehat{f}$ on $\widehat{G}$ such that $\widehat{f}(\pi)$, for $\pi \in \widehat{G}$, is a linear operator on V_π with matrix elements

$$\langle \widehat{f}(\pi)\, e_j, e_i \rangle = \big(\widehat{f}(\pi)\big)_{i,j} := \int_G f(g)\, \pi_{i,j}(g)\, dg. \tag{2.58}$$

By elementary theory of general Hilbert spaces we can recover f from $\widehat{f}$ by

$$\begin{aligned} f(g) &= \sum_{\pi \in \widehat{G}} \sum_{i,j=1}^{d_\pi} d_\pi \, \big(\widehat{f}(\pi)\big)_{i,j} \overline{\pi_{i,j}(g)} = \sum_{\pi \in \widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \big(\widehat{f}(\pi)\big)_{i,j}\, \pi_{j,i}(g^{-1}) = \sum_{\pi \in \widehat{G}} d_\pi \sum_{i=1}^{d_\pi} \big(\widehat{f}(\pi)\, \pi(g^{-1})\big)_{i,i} \\ &= \sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}\big(\widehat{f}(\pi)\, \pi(g^{-1})\big) \qquad (g \in G), \end{aligned} \tag{2.59}$$

where the sum is convergent with respect to the L^2 norm.

We also see for $f \in L^2(G)$ (in particular for $f \in C(G)$) that $f = 0$ iff $\widehat{f}(\pi) = 0$ for all $\pi \in \widehat{G}$.

Proposition 2.36.

$$(f_1 * f_2)\widehat{\ }(\pi) = \widehat{f_1}(\pi)\, \widehat{f_2}(\pi) \qquad (f_1, f_2 \in C(G),\ \pi \in \widehat{G}). \tag{2.60}$$

Proof

$$\begin{aligned} \big((f_1 * f_2)\widehat{\ }(\pi)\big)_{i,j} &= \int_{g \in G} \int_{g_1 \in G} f_1(g_1)\, f_2(g_1^{-1} g)\, \pi_{i,j}(g)\, dg_1\, dg \\ &= \int_{g_1 \in G} \int_{g \in G} f_1(g_1)\, f_2(g_1^{-1} g)\, \pi_{i,j}(g)\, dg\, dg_1 \\ &= \int_{g_1 \in G} \int_{g_2 \in G} f_1(g_1)\, f_2(g_2)\, \pi_{i,j}(g_1 g_2)\, dg_2\, dg_1 \\ &= \sum_{l=1}^{d_\pi} \int_{g_1 \in G} f_1(g_1)\, \pi_{i,l}(g_1)\, dg_1 \int_{g_2 \in G} f_2(g_2)\, \pi_{l,j}(g_2)\, dg_2 \\ &= \sum_{l=1}^{d_\pi} \big(\widehat{f_1}(\pi)\big)_{i,l} \big(\widehat{f_2}(\pi)\big)_{l,j} = \big(\widehat{f_1}(\pi)\, \widehat{f_2}(\pi)\big)_{i,j}, \end{aligned}$$

where we used Fubini's theorem in the second equality and made the substitution of inner integration variable $g_2 = g_1^{-1} g$ in the third equality, also using there the left invariance of Haar measure on G. $\square$

Let $C(K\backslash G/K)$ denote the space of K-biinvariant continuous functions on G. It follows from (2.57) together with the left invariance of Haar measure on G that

$$f_1, f_2 \in C(K\backslash G/K) \implies f_1 * f_2 \in C(K\backslash G/K).$$

Theorem 2.37. *(G,K) is a Gelfand pair iff $f_1 * f_2 = f_2 * f_1$ for all $f_1, f_2 \in C(K\backslash G/K)$.*

Proof For $\pi \in \widehat{G}$ we can choose the orthonormal basis $e_1, \ldots, e_{d_\pi}$ of V_π such that e_j is K-fixed if $j \le c_\pi$. Now consider e_j with $j > c_\pi$. Then $\pi(k)\, e_j$ is orthogonal to e_i $(i \le c_\pi)$ for all $k \in K$. Hence $\int_K \pi(k)\, e_j\, dk$ is orthogonal to e_i $(i \le c_\pi)$, but this vector is also K-fixed. Hence, because the dimension of the K-fixed vectors in V_π equals c_π, we must have $\int_K \pi(k)\, e_j\, dk = 0$ $(j > c_\pi)$.

Now let $f \in C(K\backslash G/K)$, $\pi \in \widehat{G}$. Then

$$\begin{aligned}
\big(\widehat{f}(\pi)\big)_{i,j} &= \int_G f(g)\, \pi_{i,j}(g)\, dg \\
&= \int_G \int_K \int_K f(k_2 g k_1^{-1})\, \pi_{i,j}(g)\, dk_1\, dk_2\, dg \\
&= \int_K \int_K \int_G f(k_2 g k_1^{-1})\, \pi_{i,j}(g)\, dg\, dk_1\, dk_2 \\
&= \int_K \int_K \int_G f(g)\, \pi_{i,j}(k_2^{-1} g k_1)\, dg\, dk_1\, dk_2 \\
&= \int_G \int_K \int_K f(g)\, \langle \pi(g)\, \pi(k_1)\, e_j, \pi(k_2)\, e_i \rangle\, dk_1\, dk_2\, dg \\
&= \int_G f(g)\, \langle \pi(g) \textstyle\int_K \pi(k_1)\, e_j\, dk_1, \int_K \pi(k_2)\, e_i\, dk_1 \rangle\, dg,
\end{aligned}$$

which is zero if $i > c_\pi$ or $j > c_\pi$.

Now assume that (G,K) is a Gelfand pair. Then $c_\pi = 0$ or 1. Let $f_1, f_2 \in C(K\backslash G/K)$. Then $\widehat{f_1}(\pi), \widehat{f_2}(\pi)$ are zero matrices if $c_\pi = 0$ and they are matrices with all entries except possibly the 1,1 entry equal to zero if $c_\pi = 1$. In both cases $\widehat{f_1}(\pi)$ and $\widehat{f_2}(\pi)$ commute. But then $(f_1 * f_2 - f_2 * f_1)\widehat{\ }(\pi) = 0$ for all $\pi \in \widehat{G}$. Hence $f_1 * f_2 - f_2 * f_1 = 0$.

Conversely, assume that (G,K) is not a Gelfand pair. Then there is a $\pi \in \widehat{G}$ such that $c_\pi > 1$. For this π consider $\pi_{1,2}$ and $\pi_{2,1}$, which are both in $C(K\backslash G/K)$. Then

$$\begin{aligned}
(\pi_{1,2} * \pi_{2,1})(g) &= \int_G \pi_{1,2}(g_1)\, \pi_{2,1}(g_1^{-1} g)\, dg_1 = \sum_{j=1}^{d_\pi} \Big(\int_G \pi_{1,2}(g_1)\, \pi_{2,j}(g_1^{-1})\, dg_1 \Big) \pi_{j,1}(g) \\
&= \sum_{j=1}^{d_\pi} \Big(\int_G \pi_{1,2}(g_1)\, \overline{\pi_{j,2}(g_1)}\, dg_1 \Big) \pi_{j,1}(g) = \sum_{j=1}^{d_\pi} (d_\pi)^{-1} \delta_{1,j}\, \pi_{j,1}(g) = (d_\pi)^{-1}\, \pi_{1,1}(g).
\end{aligned}$$

Similarly, $(\pi_{2,1} * \pi_{1,2})(g) = (d_\pi)^{-1}\, \pi_{2,2}(g)$. Hence $\pi_{1,2} * \pi_{2,1} \ne \pi_{2,1} * \pi_{1,2}$. So the convolution algebra $C(K\backslash G/K)$ is not commutative. □

The following sufficient condition for (G,K) in order to be a Gelfand pair is often useful.

Theorem 2.38. *If there exists a continuous group automorphism $\sigma\colon G \to G$ such that $\sigma(g) \in Kg^{-1}K$ for all $g \in G$, then (G, K) is a Gelfand pair.*

Proof By uniqueness of the normalized Haar measure on G, this measure is invariant under σ. Also, $f(\sigma(g)) = f(g^{-1})$ for $f \in C(K\backslash G/K)$, $g \in G$. Hence, for $f_1, f_2 \in C(K\backslash G/K)$ and $g \in G$ we have:

$$\begin{aligned}(f_1 * f_2)(g^{-1}) = (f_1 * f_2)(\sigma(g)) &= \int_G f_1(g_1)\, f_2(g_1^{-1}\sigma(g))\, dg_1 = \int_G f_1(\sigma(g_1))\, f_2(\sigma(g_1^{-1})\sigma(g))\, dg_1 \\ &= \int_G f_1(g_1^{-1})\, f_2((g_1^{-1}g)^{-1})\, dg_1 = \int_G f_1(g_2^{-1}g^{-1})\, f_2(g_2)\, dg_2 = (f_2 * f_1)(g^{-1}).\end{aligned}$$

So (G, K) is a Gelfand pair by Theorem 2.37. □

Example 2.39. Let $G := O(d)$, $K := O(d-1)$ and A the subgroup of G consisting of the matrices

$$A_\theta := \begin{pmatrix} \cos\theta & -\sin\theta & 0 & \dots & 0 \\ \sin\theta & \cos\theta & 0 & \dots & 0 \\ 0 & 0 & 1 & & \text{\Large 0} \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \text{\Large 0} & & 1 \end{pmatrix} \qquad (0 \le \theta < 2\pi).$$

Then any $x \in S^{d-1}$ can be written as $x = T_1 A_\theta e_1$ for some $T_1 \in O(d-1)$, $\theta \in [0, \pi]$. Hence any $T \in O(d)$ can be written $T = T_1 A_\theta T_2$ for some $T_1, T_2 \in O(d-1)$, $\theta \in [0, \pi]$. So $G = KAK$. Let J be the $d \times d$ diagonal matrix with diagonal elements $-1, 1, \dots, 1$, so $J \in O(d)$ and $J = J^{-1}$. Define a continuous automorphism σ of $O(d)$ by $\sigma(T) := JTJ$. Then $\sigma(T) = T$ if $T \in O(d-1)$ and $\sigma(A_\theta) = A_{-\theta} = (A_\theta)^{-1}$. Hence, for $T = T_1 A_\theta T_2$ with $T_1, T_2 \in O(d-1)$ we have

$$\sigma(T) = \sigma(T_1 A_\theta T_2) = T_1 (A_\theta)^{-1} T_2 = T_1 T_2 T^{-1} T_1 T_2 \in O(d-1)\, T^{-1}\, O(d-1).$$

So $(O(d), O(d-1))$ is a Gelfand pair by Theorem 2.38.

It can be similarly proved that $(SO(d), SO(d-1))$ is a Gelfand pair if $d \ge 3$. Of course, if (G, K) is a Gelfand pair and K_1 is a closed subgroup of G with $K_1 \supset K$, then (G, K_1) is also a Gelfand pair. For instance, $(O(d), O(1) \times O(d-1))$ is a Gelfand pair.

Exercise 2.40. Let G be a compact group. The group $G \times G$ acts continuously and transitively on G by $(g_1, g_2) \,.\, g := g_1 g g_2^{-1}$.

a) Show that the stabilizer subgroup of $e \in G$ in $G \times G$ equals the diagonal subgroup $G_1 := \{(g, g) \mid g \in G\}$ of $G \times G$.

b) Show that for each $\pi \in \widehat{G}$ the span W_π of the matrix elements π_{ij} is a $G \times G$ invariant subspace of $C(G)$ and that the subspace of G_1-fixed functions in W_π is one-dimensional and spanned by the character $\chi_\pi(g) := \operatorname{tr} \pi(g)$.

c) Find the product formula for the characters χ_π, as in (2.25) or (2.56).

d) Show that $(G \times G, G_1)$ is a Gelfand pair. (Use for instance Theorem 2.38.)

Exercise 2.41. Show that the following pairs (G, K) are Gelfand pairs.

a) $(O(d_1 + d_2), O(d_1) \times O(d_2))$

b) $(U(d_1 + d_2), U(d_1) \times U(d_2))$

c) $(S_{m+n}, S_m \times S_n)$, where S_n is the symmetric group in n letters.

2.9 Connecting the analysis on compact homogeneous spaces with the analysis on compact groups

We keep the notations of the beginning of §2.8. If $\pi \in \widehat{G}$ and $\pi = (\pi_{i,j})_{i,j=1,\ldots,d_\pi}$ with respect to an orthonormal basis of V_π then put

$$\pi^*_{i,j}(g) := \overline{\pi_{i,j}(g)} = \pi_{j,i}(g^{-1}) \qquad (g \in G), \tag{2.61}$$

where we used in the second equality that we are working with unitary matrices. Then π^* is again an irreducible unitary representation of G, called the *contragredient* or *dual* or *complex conjugate* of π.

Define the *left regular representation* λ and *right regular representation* ρ of G on $L^2(G)$ by

$$(\lambda(g)f)(g_1) := f(g^{-1}g_1), \quad (\rho(g)f)(g_1) := f(g_1 g) \qquad (f \in L^2(G),\ g, g_1 \in G). \tag{2.62}$$

These are indeed unitary representations of G and they are weakly continuous: the functions sending g to

$$\int_G f_1(g^{-1}g_1)\,\overline{f_2(g_1)}\,dg_1 \quad \text{and} \quad \int_G f_1(g_1 g)\,\overline{f_2(g_1)}\,dg_1$$

are continuous (prove this first for f_1, f_2 in the dense subspace of continuous functions with compact support).

Now let $\pi \in \widehat{G}$ and $\pi = (\pi_{i,j})_{i,j=1,\ldots,d_\pi}$ with respect to an orthonormal basis of V_π. Then

$$(\lambda(g)\pi_{i,j})(g_1) = \pi_{i,j}(g^{-1}g_1) = \sum_{l=1}^{d_\pi} \pi_{i,l}(g^{-1})\,\pi_{l,j}(g_1) = \sum_{l=1}^{d_\pi} \pi^*_{l.i}(g)\,\pi_{l,j}(g_1).$$

Hence, for fixed $j = 1, \ldots, d_\pi$ we have

$$\lambda(g)\pi_{i,j} = \sum_{l=1}^{d_\pi} \pi^*_{l,i}(g)\,\pi_{l,j} \qquad (i = 1, \ldots, d_\pi). \tag{2.63}$$

Note that the functions $d_\pi^{\frac{1}{2}}\pi_{i,j}$ $(i = 1, \ldots, d_\pi)$ form an orthonormal basis of the linear span $W_j(\pi)$ of the elements in the j-th column of the matrix $(\pi_{i,j})_{i,j=1,\ldots,d_\pi}$. Thus $W_j(\pi)$ is an invariant subspace of $L^2(G)$ under the representation λ of G and the restriction of λ to $W_j(\pi)$ is equivalent with π^*. We conclude that, corresponding to the orthogonal direct sum decomposition of Hilbert space

$$L^2(G) = \bigoplus_{\pi \in \widehat{G}} \Big(\bigoplus_{j=1}^{d_\pi} W_j(\pi) \Big)$$

we have a direct sum decomposition of unitary representation into irreducible representations:

$$\lambda = \bigoplus_{\pi \in \widehat{G}} \Big(\bigoplus_{j=1}^{d_\pi} \pi^* \Big) = \bigoplus_{\pi \in \widehat{G}} d_\pi \pi^*.$$

For $\pi \in \widehat{G}$ choose the orthonormal basis $e_1, \ldots, e_{d_\pi}$ of V_π such that e_j is K-fixed if $j \le c_\pi$. Let $W(\pi)$ be the linear span of the $\pi_{i,j}$ $(i, j = 1, \ldots, d_\pi)$. Then $f \in W(\pi)$ and right invariant under K iff $f \in \oplus_{j=1}^{c_\pi} W_j(\pi)$ (see the proof of Theorem 2.37). Let $L^2(G/K) := \{f \in L^2(G) \mid f \in L^2(G)$ and right-K-invariant$\}$. Then we have an orthogonal direct sum decomposition of $L^2(G/K)$ and a corresponding direct sum decomposition of λ restricted to $L^2(G/K)$:

$$L^2(G/K) = \bigoplus_{\pi \in (G/K)^\wedge} \Big(\bigoplus_{j=1}^{c_\pi} W_j(\pi) \Big), \tag{2.64}$$

$$\lambda|_{G/K} = \bigoplus_{\pi \in (G/K)^\wedge} \Big(\bigoplus_{j=1}^{c_\pi} \pi^* \Big) = \bigoplus_{\pi \in (G/K)^\wedge} c_\pi \pi^*. \tag{2.65}$$

We conclude that the following are equivalent:

a) (G, K) is a Gelfand pair.

b) $c_\pi = 1$ for all $\pi \in (G/K)^\wedge$.

c) Each $\pi \in \widehat{G}$ occurring in the direct sum decomposition of λ on $L^2(G/K)$ has multiplicity 1 in this decomposition.

Now let G, X, ω, x_0, K be as in §2.3. Then the map

$$f \mapsto \big(g \mapsto f(g \,.\, x_0)\big) \colon L^2(X) \to L^2(G/K)$$

is an isomorphism of Hilbert spaces and it is intertwining for the action of G on $L^2(X)$ and the representation λ on $L^2(G/K)$. Thus the decompositions (2.64), (2.65) can be rewritten as decompositons for $L^2(X)$ and the representation of G on $L^2(X)$. We conclude that the following are equivalent:

a) (G, K) is a Gelfand pair.

b) For all $\pi \in \widehat{G}$ occurring in the direct sum decomposition of $L^2(X)$ into irreducible subspaces under G we have $c_\pi = 1$.

c) Each $\pi \in \widehat{G}$ occurring in the direct sum decomposition of $L^2(X)$ into irreducible subspaces under G has multiplicity 1 in this decomposition.

An important class of compact Gelfand pairs is given by the *compact symmetric pairs* (G, K). Here G is a compact connected semisimple Lie group and K is a closed subgroup of G which is maximal in the following sense: if K_1 is a closed subgroup of G such that $K \subset K_1 \ne G$ then the Lie algebras of K and K_1 are isomorphic.

A compact Lie group G is called *semisimple* if its Lie algebra $\mathfrak{g}$ has the property that the bilinear form (*Killing form*)

$$B(X,Y) := \operatorname{tr}\big((\operatorname{ad} X)\circ(\operatorname{ad} Y)\big) \qquad (X,Y\in\mathfrak{g})$$

is negative definite. Here the linear map $\operatorname{ad} X\colon \mathfrak{g}\to\mathfrak{g}$ is defined by:

$$(\operatorname{ad} X)(Z) := [X,Z] \qquad (X,Z\in\mathfrak{g}).$$

For instance, the compact group $SU(n)$ with Lie algebra $su(n)$ is semisimple, but the compact group $U(n)$ with Lie algebra isomorphic to $su(n)\oplus\mathbb{R}$ is not semisimple.

For a compact symmetric pair (G,K) it can be shown that G has a closed connected abelian subgroup A such that G admits the *Cartan decomposition* $G = KAK$ and there is a continuous group automorphism θ of G (*Cartan involution*) such that $\theta^2 = \mathrm{id}$, $\theta|_K = \mathrm{id}$ and $\theta(a) = a^{-1}$ $(a\in A)$. The group A is isomorphic with a torus $\mathbb{T}^r$. Here r is called the *rank* of the symmetric pair.

The following table gives a classification of the compact symmetric pairs of rank 1. As already mentioned, compact symmetric pairs are special cases of compact Gelfand pairs. On all compact symmetric pairs of rank 1 the spherical functions turn out to be Jacobi polynomials $R_n^{(\alpha,\beta)}$. The table also lists α,β for the Jacobi polynomials occurring as spherical functions.

G	K	α	β
$SO(d)$	$SO(d-1)$	$\frac12 d-\frac32$	$\frac12 d-\frac32$
$SO(d)$	$S(O(1)\times O(d-1))$	$\frac12 d-\frac32$	$-\frac12$
$SU(d)$	$S(U(1)\times U(d-1))$	$d-2$	0
$Sp(d)$	$Sp(1)\times Sp(d-1)$	$2d-3$	1
F_4	$Spin(9)$	7	3

The first line gives for G/K the sphere S^{d-1}, which case we have extensively studied. For the second, third and fourth line G/K is a real, complex and quaternionic projective space, respectively. The case $(F_4, Spin(9))$ is exceptional; for this case there is not an infinite family. Here $F_4/Spin(9)$ is the *Cayley elliptic plane.*

2.10 Riemannian manifolds and invariant differential operators

A *Riemannian manifold* is a C^∞ manifold X on which a *line element* ds is given which takes on an open subset U of X with local coordinates $(x^1,\ldots,x^n)$ the form

$$ds^2 = \sum_{i,j=1}^n g_{i,j}(x)\,dx^i\,dx^j \qquad (x\in U), \tag{2.66}$$

where $g_{i,j}\in C^\infty(U)$ and the matrix $(g_{i,j}(x))_{i,j=1,\ldots,n}$ is real symmetric positive definite for each $x\in U$. Under a local change of coordinates $(x^1,\ldots,x^n)\mapsto(y^1,\ldots,y^n)$ the line element transforms as

$$ds^2 = \sum_{k,l=1}^n \sum_{i,j=1}^n g_{i,j}(x)\,\frac{\partial x^i}{\partial y^k}\,\frac{\partial x^j}{\partial y^l}\,dy^k\,dy^l. \tag{2.67}$$

We call $(g_{i,j})$ the *metric tensor*. A C^1 curve $\gamma\colon t \mapsto x(t)\colon [0,1] \to U$ has *length*

$$L(\gamma) := \int_0^1 \Big(\textstyle\sum_{i,j=1}^n g_{i,j}(x(t))\,\dfrac{dx^i(t)}{dt}\,\dfrac{dx^j(t)}{dt}\Big)^{\frac12}\,dt. \tag{2.68}$$

In an evident way, this definition can be extended to curves not included completely within one coordinate neighbourhood. A connected Riemannian manifold becomes a metric space by defining $d(a,b)$ as the infimum of all $L(\gamma)$ with γ a curve connecting a and b.

Still for local coordinates put

$$|g| := \det(g_{i,j}), \qquad (g^{i,j})_{i,j=1,\ldots,n} := \big((g_{i,j})_{i,j=1,\ldots,n}\big)^{-1}. \tag{2.69}$$

Then a measure on X is obtained from the *volume element*

$$d\sigma := |g|^{\frac12}\,dx^1\ldots dx^n. \tag{2.70}$$

Also a second order partial differential operator Δ on X, the *Laplace-Beltrami operator*, is defined by

$$\Delta := |g|^{-\frac12}\sum_{i,j=1}^n \frac{\partial}{\partial x^i}\circ |g|^{\frac12}\,g^{i,j}\,\frac{\partial}{\partial x^j}\,. \tag{2.71}$$

We will use these concepts in the observation that a diffeomorphism Φ of X which preserves the line element, also preserves the volume element, while the differential operator Δ will be invariant under Φ (i.e., $\Phi\,.\,(\Delta(f)) = \Delta(\Phi\,.\,f)$, where $(\Phi\,.\,f)(x) := f(\Phi^{-1}(x))$).

Example 2.42. Consider $\mathbb{R}^d$ with elements $x = (x_1,\ldots,x_d)$. Then $\mathbb{R}^d$ becomes a Riemannina manifold if we put for the line element $ds^2 = dx_1^2 + \cdots + dx_d^2$. Then clearly the volume element is $dx_1\ldots dx_d$ and the Laplace-Beltrami operator is $\Delta = \partial^2/\partial x_1^2 + \cdots \partial^2/\partial x_d^2$. Let ds' be the restriction of the line element to the submanifold S^{d-1}. For $0 \ne x \in \mathbb{R}^d$ write $x = rx'$ ($r > 0$, $x' \in S^{d-1}$. Take (unspecified) local coordinates $x'_1,\ldots,x'_{d-1}$ on S^{d-1}. Then use $r, x'_1,\ldots,x'_{d-1}$ as local coordinates on $\mathbb{R}^d$. Since ds is invariant under orthogonal transformations T and S^{d-1} is also invariant under such T, the line element ds' on S^{d-1} will be invariant under T. Hence also the volume element $d\sigma'$ and the Laplace-Beltrami operator Δ' associated with ds' will be invariant under T.

Now we have

$$ds^2 = dr^2 + r^2\,ds'^2, \qquad |g| = r^{2d-2}\,|g'|, \qquad d\mu(x) = r^{d-1}\,dr\,d\mu'(x'). \tag{2.72}$$

Furthermore,

$$(g_{i,j}) = \begin{pmatrix} 1 & 0\\ 0 & r^2\,(g'_{i,j})\end{pmatrix}, \qquad (g^{i,j}) = \begin{pmatrix} 1 & 0\\ 0 & r^{-2}\,(g'^{i,j})\end{pmatrix}.$$

Hence,

$$\begin{aligned}\Delta &= r^{-d+1}\,|g'|^{-\frac12}\Big(\frac{\partial}{\partial r}\circ r^{d-1}\,|g'|^{\frac12}\,\frac{\partial}{\partial r} + \textstyle\sum_{i,j=1}^{d-1}\dfrac{\partial}{\partial x'_i}\circ r^{d-1}\,|g'|^{\frac12}\,r^{-2}\,g'^{i,j}\,\dfrac{\partial}{\partial x'_j}\Big)\\ &= \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\,\frac{\partial}{\partial r} + \frac{1}{r^2}\,|g'|^{-\frac12}\sum_{i,j=1}^{d-1}\frac{\partial}{\partial x'_i}\circ |g'|^{\frac12}\,g'^{i,j}\,\frac{\partial}{\partial x'_j}\,.\end{aligned}$$

Hence

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta' \tag{2.73}$$

Now let Δ act on C^∞ function f which is homogeneous of degree n, i.e.,

$$f(rx') = r^n f(x') \qquad (r > 0,\ x' \in S^{d-1}).$$

Then

$$\begin{aligned}\Delta f(x) &= \Big(\frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta'\Big) r^n f(x') \\ &= r^{n-2}\big(\Delta' + n(n+d-2)\big) f(x').\end{aligned}$$

In particular,

$$\Delta f(x) = 0 \quad \Longleftrightarrow \quad \Delta' f(x') = -n(n+d-2) f(x'). \tag{2.74}$$

So a spherical harmonic of degree n, restricted to S^{d-1}, is an eigenfunction of Δ' for the eigenvalue $-n(n+d-2)$.

If (G,K) is a compact symmetric pair then it can be shown that $X = G/K$ is a Riemannian manifold with line element invariant under G (X is then called a compact Riemannian *symmetric space*). Then the Laplace-Beltrami operator Δ on X is G-invariant. Then it can be shown that the spherical functions, considered as functions on X, are C^∞ functions which are eigenfunctions of Δ. If (G,K) is a compact symmetric pair of rank one then the spherical functions are, up to a constant factor, characterized as the K-invariant eigenfunctions of Δ. If the rank is greater than 1 then it can be shown that the algebra of G-invariant differential operators on X (including Δ) is commutative and that the spherical functions are, up to a constant factor, characterized as the K-invariant joint eigenfunctions of the G-invariant differential operators on X.

References

[1] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.

[2] J. Dutka, *The early history of the hypergeometric function*, Arch. Hist. Exact Sci. 31 (1984), 15–34.

[3] M. Sugiura, *Unitary representations and harmonic analysis — an introduction*, Wiley, 1975.

[4] N. Ja. Vilenkin and A. U. Klimyk, *Representation of Lie groups and special functions, Vol. 1*, Kluwer, 1991.

[6] W. Rudin, *Functional analysis*, McGraw-Hill, 1973.

[7] A. Baker, *Matrix groups. An introduction to Lie group theory*, Springer-Verlag, 2002.

[8] B. C. Hall, *Lie groups, Lie algebras, and representations. An elementary introduction*, Springer-Verlag, 2003.

[9] W. Rossmann, *Lie groups. An introduction through linear groups*, Oxford University Press, 2002.

[10] W. Rudin, *Real and complex analysis*, McGraw-Hill, second ed., 1974.

[11] G. F. Simmons, *Introduction to topology and modern analysis*, McGraw-Hill, 1963.

[12] F. W. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer-Verlag, 1983.

[13] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, 1978.

[14] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, 1972.

[15] E. M. Stein and G. Weiss, *Introduction to Fourier analyis on Euclidean spaces*, Princeton University Press, 1971.