A fractional generalisation of an operational formula and the Gauss hypergeometric function

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The n-th derivative formula

$$D^n(z^{a+n-1}F(a,b;c;z)) = (a)_n z^{a-1}F(a+n,b;c;z) \tag{1}$$

(see [2, 2.8 (21)]) can be understood as an iteration of its case $n=1$

$$D(z^a F(a,b;c;z)) = a z^{a-1}F(a+1,b;c;z) \tag{2}$$

by using the identity

$$z^n D^n z^n = (zDz)^n. \tag{3}$$

This identity (see also [3, (13)] and [4, (15.5.10)]) can be shown by induction. First observe

$$z^n D^n z^n = \sum_{k=0}^{n} \frac{n!^2}{k!^2(n-k)!} z^{n+k} D^k. \tag{4}$$

Denote the right-hand side of the last identity by S_n. Then show that

$$(zDz)S_n = S_{n+1}.$$

We can rewrite (3) as

$$z^n \circ D^n \circ z^n = (z\circ D\circ z)^n = z^{-1}\circ (z^2D)^n \circ z = \iota\circ z\circ(-D)^n\circ z^{-1}\circ\iota, \tag{5}$$

where $(\iota f)(z) := f(z^{-1})$. Thus (1) can be rewritten as

$$(zDz)^n(z^{a-1}F(a,b;c;z)) = (a)_n z^{a+n-1}F(a+n,b;c;z), \tag{6}$$

(see also [4, (15.5.3)]) and as

$$(-D)^n(z^{-a}F(a,b;c;z^{-1}) = (a)_n z^{-a-n}F(a+n,b;c;z^{-1}). \tag{7}$$

Recall the Riemann-Liouville type fractional integral

$$(I_\mu f)(x) := \frac{1}{\Gamma(\mu)}\int_0^x f(y)\,(x-y)^{\mu-1}\,dy, \tag{8}$$

and the Weyl type fractional integral

$$(W_\mu f)(x) := \frac{1}{\Gamma(\mu)}\int_x^\infty f(y)\,(y-x)^{\mu-1}\,dy. \tag{9}$$

The fractional integral formula (see [1, (2.10)])

$$\frac{1}{\Gamma(\mu)}\int_0^x \Gamma(a+\mu)\,y^{a-1}\,F(a+\mu,b;c;y)\,(x-y)^{\mu-1}\,dy \\ = \Gamma(a)\,x^{a+\mu-1}\,F(a,b;c;x) \qquad (\operatorname{Re} a>0,\ \operatorname{Re}\mu>0) \tag{10}$$

can be seen as a fractional iteration of (2) by observing that

$$x^{-\mu}\circ I_\mu\circ x^{-\mu}=\iota\circ x\circ W_\mu\circ x^{-1}\circ\iota. \tag{11}$$

Formula (11) follows because

$$g(x)=x^{-\mu}\,\frac{1}{\Gamma(\mu)}\int_0^x y^{-\mu}f(y)\,(x-y)^{\mu-1}\,dy$$

implies that

$$g(x^{-1})=x\,\frac{1}{\Gamma(\mu)}\int_x^\infty y^{-1}\,f(y^{-1})\,(y-x)^{\mu-1}\,dy.$$

Formula (11) also implies the equality of the first and last part of (5). We can rewrite (10) by use of (11) as

$$\frac{1}{\Gamma(\mu)}\int_x^\infty \Gamma(a+\mu)\,y^{-a-\mu}\,F(a+\mu,b;c;y^{-1})\,(y-x)^{\mu-1}\,dy=\Gamma(a)\,x^{-a}\,F(a,b;c;x^{-1}). \tag{12}$$

References

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