About this manuscript

The following is a scanned version of:

Tom Koornwinder, *Self-duality for q-ultraspherical polynomials associated with root system A_n*, handwritten manuscript, February 2, 1988, 17 pp.

I wrote this while I was employed at CWI, Amsterdam. After I finished this work, I sent copies to I. G. Macdonald and to a few other interested people, but I never published it. The ideas of this manuscript were used by Macdonald in Chapter 6, §6 (on Pieri formulas) in his book *Symmetric functions and Hall polynomials*, Second Edition, Oxford University Press, 1995. He acknowledges the present manuscript (reference [K15] in the book) on p.343.

The Pieri formula for Macdonald polynomials (root system A_n) was first obtained by Macdonald in his unpublished manuscript of 1987 entitled *The symmetric functions $P_\lambda(x; q, t)$: facts and conjectures.* His proof was analogous to the proof by R. P. Stanley of the Pieri formula for Jack polynomials, which Stanley had communicated to Macdonald and which he later published in:

R. P. Stanley, *Some combinatorial properties of Jack symmetric functions*, Advances in Math. 77 (1989), 76–115.

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Self-duality for q-ultraspherical polynomials associated with root system A_n

(by Tom Koornwinder)

Let V be a real vector space of dimension n with inner product $(.,.)$. Let R be a root system in V. For convenience we assume that R is reduced, irreducible and with only one root length which we take as

$$(\alpha,\alpha) = 2 \qquad \forall \alpha \in R.$$

Hence $\check{\alpha} = \alpha \quad \forall \alpha \in R$. Some further notation:

$W :=$ Weyl group of R,

$R^+ :=$ choice of positive roots in R

$\{\alpha_1, \ldots, \alpha_n\}$: simple roots in R^+

$\{\lambda_1, \ldots, \lambda_n\}$: fundamental weights, i.e.

$$(\lambda_i, \alpha_j) = \delta_{i,j}, \quad i,j = 1,\ldots,n.$$

$$P := \{\lambda \in V \mid (\lambda,\alpha) \in \mathbb{Z} \ \ \forall \alpha \in R\}$$

$$= \mathbb{Z}\text{-span}\{\lambda_1, \ldots, \lambda_n\} = \text{weight lattice}$$

$$\mathbb{Z}^+ := \{0,1,2,\ldots\}$$

$$P^+ := \{\lambda \in V \mid (\lambda,\alpha) \in \mathbb{Z}^+ \ \ \forall \alpha \in R^+\}$$

$$= \mathbb{Z}^+\text{-span}\{\lambda_1,\ldots,\lambda_n\} = \{\text{dominant weights}\}$$

partial ordering on P:

$$\mu \le \lambda \iff \lambda - \mu \in \mathbb{Z}^+\text{-span}\{\alpha_1,\ldots,\alpha_n\}.$$

$$\rho := \tfrac{1}{2} \sum_{\alpha \in R^+} \alpha = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

Extend the inner product on V to a complex bilinear form $(.,.)$ on the complexification $V + iV$ of V. For $\lambda \in P$ let

e^λ be the function on $V+iV$ defined by

$$e^\lambda(z) := e^{(\lambda,z)}.$$

For $\lambda \in P^+$ let

$$S_\lambda := \sum_{\nu \in W\lambda} e^\nu .$$

Let $\varepsilon(w) := \det(w)$ $(w \in W)$ and

$$\delta := \sum_{w \in W} \varepsilon(w)\, e^{w\rho} = e^\rho \prod_{\alpha \in R^+} (1 - e^{-\alpha})$$

Let $q, t \in (0,1)$,

$$(a;q)_n := \prod_{k=1}^{n} (1 - a q^{k-1}) ,$$

where $a \in \mathbb{C}$ and $n \in \mathbb{Z}^+$ or $n = \infty$.

Let $C := \{ x \in V \mid 0 < \alpha(x) < 2\pi \quad \forall \alpha \in R^+ \}$,
dx Lebesgue measure on V,

$$\Delta^{(q,t)}(ix) := \prod_{\alpha \in R} \left(\frac{(e^{i(\alpha,x)};q)_\infty}{(t e^{i(\alpha,x)};q)_\infty} \right), \quad x \in C.$$

Definition 1. For $\lambda \in P^+$ define $p_\lambda^{(q,t)}$ by the two properties:

(i) $$p_\lambda^{(q,t)} = S_\lambda + \sum_{\substack{\mu \in P^+ \\ \mu < \lambda}} c_{\lambda,\mu}^{(q,t)} S_\mu$$

for certain coefficients $c_{\lambda,\mu}^{(q,t)}$;

(ii) $$\int_C p_\lambda^{(q,t)}(ix)\, S_\mu^{(q,t)}(ix)\, \Delta^{(q,t)}(ix)\, dx = 0$$

if $\mu \in P^+$ and $\mu < \lambda$.

Proposition 2 (Macdonald [3]). Let π be some fundamental weight λ_j with the property that it is

minuscule, i.e. $(\pi, \alpha) = \pm 1$ or 0 for all $\alpha \in R$.
Then, for all $\mu \in P^+$ and $y \in V + iV$:

$$(1) \quad S_\pi((\log t)\rho + (\log q)\mu)\, p_\mu^{(q,t)}(y)$$
$$= \sum_{\nu \in W\pi} \frac{\delta(y + (\log t)\nu)}{\delta(y)}\, p_\mu^{(q,t)}(y + (\log q)\nu)$$

Thus each minuscule fundamental weight π yields a difference operator for which the $p_\mu^{(q,t)}$ are eigenfunctions.

By way of motivating example we consider the case that R is of type A_1. Let $V := \mathbb{R}$,
$(x, y) := \frac{1}{2} x y \quad (x, y \in V)$,
$R := \{\pm 2\}$, $R^+ := \{2\}$. Then
$\alpha_1 = 2$, $\lambda_1 = 1$, $P = \mathbb{Z}$, $P^+ = \mathbb{Z}^+$,
$\mu \leq \lambda \iff \lambda - \mu \in 2\mathbb{Z}^+ \quad (\lambda, \mu \in P)$, $\rho = 1$,
$e^\lambda(y) = e^{\frac{1}{2}\lambda y} \quad (\lambda \in P,\ y \in \mathbb{C})$,

$$S_\lambda(y) = \begin{cases} e^{\frac{1}{2}\lambda y} + e^{-\frac{1}{2}\lambda y}, & \lambda = 1, 2, \ldots \\ 1, & \lambda = 0. \end{cases}$$

$\delta(y) = e^{\frac{1}{2}y} - e^{-\frac{1}{2}y}$,
$C = (0, 2\pi)$,

$$\Delta^{(q,t)}(ix) = \frac{(e^{ix}; q)_\infty (e^{-ix}; q)_\infty}{(te^{ix}; q)_\infty (te^{-ix}; q)_\infty} \quad (0 < x < 2\pi)$$

For $n \in \mathbb{Z}^+$:

$$p_n^{(q,t)}(ix) = 2^{1-\delta_{n,0}} \cos(\tfrac{1}{2} n x) + \sum_{k=1}^{[\frac{1}{2}n]} c_k \cos((\tfrac{1}{2}n - k)x),$$

$$\int_0^{2\pi} p_n^{(q,t)}(ix) \cos(\tfrac{1}{2} m x)\, \Delta^{(q,t)}(ix)\, dx = 0$$

if $m = n - 2k$, $k = 1, 2, \ldots, [\frac{1}{2}n]$.

~~But~~ From Askey & Ismail [1, (3.10), (4.3), (4.5)] we conclude that

$$p_n^{(q,t)}(ix) = \text{const.}\, C_n(\cos\tfrac{1}{2}x\,;\, t\,|\,q),$$

a q-ultraspherical polynomial.
We can rewrite $\Delta^{(q,t)}(ix)$ as

$$\Delta^{(q,t)}(ix) = \left| \frac{(e^{ix};q^{\frac{1}{2}})_\infty}{(t^{\frac{1}{2}}e^{\frac{1}{2}ix}, -t^{\frac{1}{2}}e^{\frac{1}{2}ix}, q^{\frac{1}{4}}e^{\frac{1}{2}ix}, -q^{\frac{1}{4}}e^{\frac{1}{2}ix}; q^{\frac{1}{2}})_\infty} \right|^2,$$

where we have put

$$(a,b,c,d;q)_\infty := (a;q)_\infty (b;q)_\infty (c;q)_\infty (d;q)_\infty .$$

By Askey & Wilson [2, Theorem 2.2] we obtain

$$p_n^{(q,t)}(ix) \overset{\text{const.}}{=} p_n(\cos\tfrac{1}{2}x\,;\, t^{\frac{1}{2}}, -t^{\frac{1}{2}}, q^{\frac{1}{4}}, -q^{\frac{1}{4}}\,|\,q^{\frac{1}{2}}),$$

a q-Wilson polynomial.
Now put

$$P_n^{(q,t)}(ix) := \frac{p_n^{(q,t)}(ix)}{p_n^{(q,t)}(\log t)} .$$

Then, by Askey & Wilson [2, (1.15)] we obtain:

$$P_n^{(q,t)}(ix) = {}_4\phi_3\left(\begin{matrix} q^{-\frac{1}{2}n}, q^{\frac{1}{2}n}t, t^{\frac{1}{2}}e^{\frac{1}{2}ix}, t^{\frac{1}{2}}e^{-\frac{1}{2}ix} \\ -t, t^{\frac{1}{2}}q^{\frac{1}{4}}, -t^{\frac{1}{2}}q^{\frac{1}{4}} \end{matrix} ; q^{\frac{1}{2}}, q^{\frac{1}{2}} \right).$$

Hence

$$P_n^{(q,t)}(\log t + m\log q) = P_m^{(q,t)}(\log t + n\log q),$$

a kind of self-duality.

Formula (1), with $n = \lambda_1 = 1$ becomes in the A_1 case:

$$\left(t^{\frac{1}{2}} q^{\frac{1}{2} m} + t^{-\frac{1}{2}} q^{-\frac{1}{2} m}\right) P_m^{(q,t)}(y)$$

$$= \frac{t^{\frac{1}{2}} e^{\frac{1}{2} y} - t^{-\frac{1}{2}} e^{-\frac{1}{2} y}}{e^{\frac{1}{2} y} - e^{-\frac{1}{2} y}} P_m^{(q,t)}(y + \log q)$$

$$+ \frac{t^{-\frac{1}{2}} e^{\frac{1}{2} y} - t^{\frac{1}{2}} e^{-\frac{1}{2} y}}{e^{\frac{1}{2} y} - e^{-\frac{1}{2} y}} P_m^{(q,t)}(y - \log q).$$

The same difference equation is contained in Askey & Wilson [2, (3.7), (5.8), (5.9)]. On the other hand, from Askey & Wilson [2, (1.24) – (1.27)] we can obtain the recurrence formula

$$\left(e^{\frac{1}{2} z} + e^{-\frac{1}{2} z}\right) P_n^{(q,t)}(z)$$

$$= \frac{t q^{\frac{1}{2} n} - t^{-1} q^{-\frac{1}{2} n}}{t^{\frac{1}{2}} q^{\frac{1}{2} n} - t^{-\frac{1}{2}} q^{-\frac{1}{2} n}} P_{n+1}^{(q,t)}(z)$$

$$+ \frac{q^{\frac{1}{2} n} - q^{-\frac{1}{2} n}}{t^{\frac{1}{2}} q^{\frac{1}{2} n} - t^{-\frac{1}{2}} q^{-\frac{1}{2} n}} P_{n-1}^{(q,t)}(z).$$

If we substitute $y = \log t + n \log q$ in the difference equation and $z = \log t + m \log q$ in the recurrence formula then the two formulas pass into each other by the self-duality.

We now continue with the difference equation (1). Substitute $y = (\log t)\rho + (\log q)\lambda$ with $\lambda \in P^+$. Then

$$(2)\quad S_\pi\big((\log t)\rho + (\log q)\mu\big)\, p_\mu^{(q,t)}\big((\log t)\rho + (\log q)\lambda\big)$$
$$= \sum_{v \in W\pi} \frac{\delta\big((\log t)(\rho+v) + (\log q)\lambda\big)}{\delta\big((\log t)\rho + (\log q)\lambda\big)}$$
$$\cdot\, p_\mu^{(q,t)}\big((\log t)\rho + (\log q)(\lambda+v)\big)$$

<u>Lemma 2</u> Let $t, q \in (0,1)$, $\lambda \in P^+$, π a minuscule fundamental weight and $v \in W\pi$. Then

(a) $\delta\big((\log t)\rho + (\log q)\lambda\big) \neq 0$,
(b) $\delta\big((\log t)(\rho+\pi) + (\log q)\lambda\big) \neq 0$ and
(c) $\delta\big((\log t)(\rho+v) + (\log q)\lambda\big) \neq 0 \Longrightarrow \lambda + v \in P^+$.

<u>Proof</u>.

(a) $\delta\big((\log t)\rho + (\log q)\lambda\big) =$

$$= t^{(\rho,\rho)} q^{(\rho,\lambda)} \prod_{\alpha \in R^+} \big(1 - t^{-(\rho,\alpha)} q^{(\lambda,\alpha)}\big) \neq 0$$

since $(\lambda,\alpha) \geq 0$ and $(\rho,\alpha) > 0$ for all $\alpha \in R^+$.

(b) $\delta\big((\log t)(\rho + \pi) + (\log q)\lambda\big)$

$$= t^{(\rho,\rho+\pi)} q^{(\rho,\lambda)} \prod_{\alpha \in R^+} \big(1 - t^{(\rho+\pi,\alpha)} q^{(\lambda,\alpha)}\big) \neq 0$$

for the same reasons (also $(\pi,\alpha) \geq 0$ if $\alpha \in R^+$).

(c) Suppose $w \in W$ and $\lambda + w\pi \notin P^+$. ~~for some~~ We will show that there is an $\alpha \in R^+$ such that $(\lambda,\alpha) = 0$, $(\rho,\alpha) = 1$ and $(w\pi,\alpha) = -1$. Indeed, $(\lambda + w\pi, \alpha_j) < 0$ for some j, but

$(\lambda, \alpha_j) \geq 0$. Hence $(w\pi, \alpha_j) < 0$, ~~so~~ $(w\pi, \alpha_j) = -1$. This forces $(\lambda, \alpha_j) = 0$. Clearly $(\rho, \alpha_j) = 1$. So $\alpha = \alpha_j$ will do what we want. It follows that, for $\nu = w\pi$,

$$\delta((\log t)(\rho+\nu) + (\log q)\lambda)$$
$$= t^{(\rho,\, \rho+\nu)} q^{(\rho,\lambda)} \prod_{\alpha\in R^+} \left(1 - t^{-(\rho+\nu,\alpha)} q^{-(\lambda,\alpha)}\right)$$

will have a zero factor. $\square$

Thus, the summation in (2) can be taken over all $\nu \in W\pi$ such that $\lambda + \nu \in P^+$.

We now specialise to R of ~~and~~ type ~~system~~ A_n. Then all fundamental weights $\lambda_1, \ldots, \lambda_n$ are minuscule. Put

$$(3) \quad A^{(q,t)}_{\lambda,\nu} := \frac{\delta((\log t)(\rho+\nu) + (\log q)\lambda)}{\delta((\log t)\rho + (\log q)\lambda)}$$

for the coefficients in (2). Then $A^{(q,t)}_{\lambda,\lambda_j} \neq 0$ and $A^{(q,t)}_{\lambda,\nu} = 0$ if $\nu \in W\lambda_j$ and $\lambda+\nu \notin P^+$. We also replace $p^{(q,t)}_\mu$ by some constant multiple $P^{(q,t)}_\mu$. Now we can rewrite (2) in the case of root system A_n as

$$(4) \quad S_{\lambda_j}((\log t)\rho + (\log q)\mu)\, P^{(q,t)}_\mu((\log t)\rho + (\log q)\lambda)$$
$$= \sum_{\substack{\nu\in W\lambda_j \\ \lambda+\nu\in P^+}} A^{(q,t)}_{\lambda,\nu}\, P^{(q,t)}_\mu((\log t)\rho + (\log q)(\lambda+\nu)),$$

where $\lambda, \mu \in P^+$ and $j = 1, \ldots, n$.

<u>Proposition ~~4~~.</u> $P^{(q,t)}_\mu((\log t)\rho) \neq 0$.

<u>Proof</u>. Suppose $P^{(q,t)}_\mu((\log t)\rho) = 0$.

Then $P^{(q,t)}_\mu((\log t)\rho + (\log q)\lambda) = 0 \quad \forall \lambda \in P^+$.
For let $\sigma \in P^+$ and let
$P^{(q,t)}_\mu((\log t)\rho + (\log q)\lambda) = 0$ for all $\lambda \in P^+$
with $(\lambda, \rho) < (\sigma, \rho)$. Then
$P^{(q,t)}_\mu((\log t)\rho + (\log q)\lambda)$ follows from
(4) with a suitable j with λ replaced
by $\lambda - \lambda_j$ and by using that $A^{(q,t)}_{\lambda-\lambda_j, \lambda_j} \neq 0$.

Now let w_0 be the Weyl group element which sends P^+ to $-P^+$. Then

$$e^{-(w_0\mu, y)} P^{(q,t)}_\mu(y) = F(e^{(y,\lambda_1)}, \ldots, e^{(y,\lambda_n)}),$$

where F is a polynomial in n complex variables which vanishes for $y = (\log t)\rho + (\log q)\lambda$, $\lambda \in P^+$.
Hence

$$F\left(t^{(\rho,\lambda_1)} q^{\sum_j m_j(\lambda_j,\lambda_1)}, \ldots, t^{(\rho,\lambda_n)} q^{\sum_j m_j(\lambda_j,\lambda_n)}\right)$$

$= 0$ for $m_1, \ldots, m_n \in \mathbb{Z}^+$.
The left hand side is a polynomial in $q^{\frac{m_1}{(n+1)2}}, \ldots, q^{\frac{m_n}{(n+1)2}}$ which vanishes for $m_1, \ldots, m_n \in \mathbb{Z}^+$. Hence the polynomial is zero. So $P_\mu \equiv 0$, which is absurd. $\square$

From now on we normalize $P^{(q,t)}_\mu$ such that

(5) $\quad P^{(q,t)}_\mu((\log t)\rho) = 1$.

Before stating and proving our main results we need two more lemmas.

<u>Lemma 5.</u> $P^{(q,t)}_\lambda = \sum_{\substack{\mu \in P^+ \\ \mu \leq \lambda}} C_{\lambda,\mu}(q,t)\, S_\mu$
with $C_{\lambda,\mu} \in \mathbb{Q}(q^{1/n+1}, t^{1/2})$.

This follows from Macdonald [3] and the normalization (5).

<u>Lemma 6.</u> Let X be a finite subset of $\mathbb{R}^n$. Then

$$\det\left(q^{(x,y)}\right)_{x,y\in X} \neq 0$$

for q sufficiently large.

<u>Proof.</u> The determinant is a linear combination with plus and minus signs of terms

$$q^{\sum_{x\in X}(x,f(x))},$$

where f runs through the permutations of X. Now

$$\sum_{x\in X}(x,f(x)) \leq \sum_{x\in X}\|x\|\,\|f(x)\| \leq$$

$$\leq \left(\sum_{x\in X}\|x\|^2\right)^{\frac{1}{2}}\left(\sum_{x\in X}\|f(x)\|^2\right)^{\frac{1}{2}} = \sum_{x\in X}\|x\|^2$$

with equality signs iff $\{f(x)=c_x\,x,\ c_x>0\ \forall x$ and $\|f(x)\|^2 = c\|x\|^2\ \ \forall x\in X\}$ iff $f=\mathrm{id}$. Hence the determinant has dominant term $q^{\sum_{x\in X}\|x\|^2}$ for q large, so it is nonzero. $\square$

<u>Theorem 7.</u> We have

$$(5)\quad S_{\lambda_j}P_\lambda^{(q,t)} = \sum_{\substack{r\in W\lambda_j\\ \lambda+r\in P^+}} A_{\lambda,r}^{(q,t)}\,P_{\lambda+r}^{(q,t)},\qquad \lambda\in P^+,\ j=1,\dots,n;$$

$$(6)\quad P_\mu^{(q,t)}((\log t)\rho+(\log q)\lambda)$$

$$= P_\lambda^{(q,t)}((\log t)\rho+(\log q)\mu),\ \cancel{\lambda,\mu\in P^+}$$

for all $\lambda,\mu\in P^+$.

<u>Proof</u>. By complete induction with respect to μ we will prove:

(6) is true for all $\lambda \in P^+$,

(7) $$S_{\lambda_j}((\log t)e + (\log q)\mu)\, P_\lambda^{(q,t)}((\log t)e + (\log q)\mu) = \sum_{\substack{\nu \in W\lambda_j \\ \lambda+\nu \in P^+}} A_{\lambda,\nu}^{(q,t)}\, P_{\lambda+\nu}^{(q,t)}((\log t)e + (\log q)\mu)$$
for all $\lambda \in P^+$ and all $j = 1, \dots, n$

(8) $$S_{\lambda_j}\, P_{\mu-\lambda_j}^{(q,t)} = \sum_{\substack{\nu \in W\lambda_j \\ \mu-\lambda_j+\nu \in P^+}} A_{\mu-\lambda_j,\nu}^{(q,t)}\, P_{\mu-\lambda_j+\nu}^{(q,t)}$$
for ~~all $\lambda \in P^+$ and~~ all $j = 1, \dots, n$ such that $\mu - \lambda_j \in P^+$.

For $\mu = 0$ both sides of (6) equal 1, (7) reduces to

$$S_{\lambda_j}((\log t)e) = \sum_{\substack{\nu \in W\lambda_j \\ \lambda+\nu \in P^+}} A_{\lambda,\nu}^{(q,t)},$$

which is obtained from (4) with $\mu = 0$, and (8) is empty for $\mu = 0$.
~~Suppose that~~ Let now $0 \neq \sigma \in P^+$ and suppose that the induction hypothesis is satisfied for all $\mu \in P^+$ such that $0 \neq \sigma - \mu = \sum_{j=1}^n c_j \alpha_j$ with all $c_j \geq 0$ and real. Take any j such that $\sigma - \lambda_j \in P^+$ (since $\sigma \neq 0$ such j exist). First we will prove (8) for $\mu = \sigma$ and for such j. Clearly,

(9) $$S_{\lambda_j}\, P_{\sigma-\lambda_j}^{(q,t)} = \sum_{\substack{\tau \in P^+ \\ \tau \leq \sigma}} B_\tau^{(q,t)}\, P_\tau^{(q,t)}$$
for certain coefficients $B_\tau^{(q,t)}$.

On the other hand, by the induction hypothesis, (7) yields

$$(10)\quad S_{\lambda_j}\big((\log t)\rho + (\log q)\mu\big)\, P^{(q,t)}_{\sigma-\lambda_j}\big((\log t)\rho + (\log q)\mu\big)$$
$$= \sum_{\substack{\nu\in W\lambda_j \\ \sigma-\lambda_j+\nu\in P^+}} A^{(q,t)}_{\sigma-\lambda_j,\nu}\, P^{(q,t)}_{\sigma-\lambda_j+\nu}\big((\log t)\rho + (\log q)\mu\big)$$

for $\mu \in P^+$ such that $\mu < \sigma$.

From (9) we obtain

$$(11)\quad S_{\lambda_j}\big((\log t)\rho + (\log q)\sigma\big)\, P^{(q,t)}_{\sigma}\big((\log t)\rho + (\log q)(\sigma-\lambda_j)\big)$$
$$= \sum_{\substack{\nu\in W\lambda_j \\ \sigma-\lambda_j+\nu\in P^+}} A^{(q,t)}_{\sigma-\lambda_j,\nu}\, P^{(q,t)}_{\sigma}\big((\log t)\rho + (\log q)(\sigma-\lambda_j+\nu)\big)$$

If we apply (6) and the induction hypothesis to (11) and if we observe that (6) is trivially satisfied for $\mu=\lambda=\sigma$ then it follows that (10) is also valid for $\mu=\sigma$, hence for all $\mu\in P^+$ such that $\mu\le\sigma$.

On comparing (9) with (10) we obtain that

$$(12)\quad \sum_{\substack{\tau\in P^+ \\ \tau\le\sigma}} B^{(q,t)}_{\tau}\, P^{(q,t)}_{\tau}\big((\log t)\rho + (\log q)\mu\big)$$
$$= \sum_{\substack{\nu\in W\lambda_j \\ \sigma-\lambda_j+\nu\in P^+}} A^{(q,t)}_{\sigma-\lambda_j,\nu}\, P^{(q,t)}_{\sigma-\lambda_j+\nu}\big((\log t)\rho + (\log q)\mu\big)$$

Now we want to show that

(13) $$\det\left(P_\tau^{(q,t)}\big((\log t)\rho+(\log q)\mu\big)\right)_{\substack{\tau,\mu\in P^+\\ \tau,\mu\le\sigma}}$$

is nonzero as element of $\mathbb{Q}(q^{\frac{1}{n+1}},t^{\frac{1}{2}})$.
Equivalently we ~~have~~ have to show that

$$\det\left(S_\tau\big((\log t)\rho+(\log q)\mu\big)\right)_{\substack{\tau,\mu\in P^+\\ \tau,\mu\le\sigma}}$$

is nonzero as element of $\mathbb{Q}(q^{\frac{1}{n+1}},t^{\frac{1}{2}})$.
Since

$$S_\tau\big((\log t)\rho+(\log q)\mu\big)$$
$$=\sum_{\nu\in W\tau} t^{(\nu,\rho)}\,q^{(\nu,\mu)},$$

the dominant term of the above determinant for $t\to\infty$ equals

$$t^{\left(\sum_{\tau\in P^+,\,\tau\le\sigma}\tau,\,\rho\right)}\det\left(q^{(\tau,\mu)}\right)_{\substack{\tau,\mu\in P^+\\ \tau,\mu\le\sigma}}$$

By Lemma 6, $\det\left(q^{(\tau,\mu)}\right)_{\substack{\tau,\mu\in P^+\\ \tau,\mu\le\sigma}}\neq 0$
for big q. We conclude that (13) is indeed a nonzero element of $\mathbb{Q}(q^{\frac{1}{n+1}},t^{\frac{1}{2}})$.
Since $B_\tau^{(q,t)}$ is in $\mathbb{Q}(q^{\frac{1}{n+1}},t^{\frac{1}{2}})$ by (9) and Lemma 5 and
$A_{\sigma-\lambda_j,\nu}^{(q,t)}$ is in $\mathbb{Q}(q^{\frac{1}{n+1}},t^{\frac{1}{2}})$ by (3),
we conclude from (12) that

$$B_\tau^{(q,t)}=\begin{cases}A_{\sigma-\lambda_j,\nu}^{(q,t)} & \text{is } \tau=\sigma-\lambda_j+\nu,\ \nu\in W\lambda_j\\ 0 & \text{otherwise.}\end{cases}$$

Hence we have proved (8) for $\mu=\sigma$. This implies that

$$(14)\quad S_{\lambda_j}((\log t)\rho+(\log q)\lambda)\, P^{(q,t)}_{\sigma-\lambda_j}((\log t)\rho+(\log q)\lambda)$$
$$= \sum_{\substack{\nu\in W\lambda_j\\ \sigma-\lambda_j+\nu\in P^+}} A^{(q,t)}_{\sigma-\lambda_j,\nu}\, P^{(q,t)}_{\sigma-\lambda_j+\nu}((\log t)\rho+(\log q)\lambda)$$

for all $\lambda\in P^+$.

On the other hand, (4) implies that

$$(15)\quad S_{\lambda_j}((\log t)\rho+(\log q)\lambda)\, P^{(q,t)}_{\lambda}((\log t)\rho+(\log q)\lambda)$$
$$= \sum_{\substack{\nu\in W\lambda_j\\ \sigma-\lambda_j+\nu\in P^+}} A^{(q,t)}_{\sigma-\lambda_j,\nu}\, P^{(q,t)}_{\lambda}((\log t)\rho+(\log q)(\sigma-\lambda_j+\nu))$$

for all $\lambda\in P^+$.

If we compare (14) with (15) and use the induction hypothesis for (6) then we obtain

$$A^{(q,t)}_{\sigma-\lambda_j,\lambda_j}\, P^{(q,t)}_{\sigma}((\log t)\rho+(\log q)\lambda)$$
$$= A^{(q,t)}_{\sigma-\lambda_j,\lambda_j}\, P^{(q,t)}_{\lambda}((\log t)\rho+(\log q)\sigma).$$

Since $A^{(q,t)}_{\sigma-\lambda_j,\lambda_j}\neq 0$, (6) follows for $\mu=\sigma$.

Finally, (7) ~~follow~~ for $\mu=\sigma$ follows from (4) together with (6) for $\mu=\sigma$. $\square$

There are several corollaries of (5) together with (3).

Corollary 8. Let $k_\lambda^{(q,t)}$ denote the coefficient of $S_\lambda^{(q,t)}$ in the expansion of $P_\lambda^{(q,t)}$ in terms of functions $S_\mu^{(q,t)}$. Then

$$(16)\quad k_\lambda^{(q,t)} = t^{(\rho,\lambda)} \prod_{\alpha\in R^+} \frac{(t^{(\alpha,\rho)};q)_{(\lambda,\alpha)}}{(t^{(\alpha,\rho)+1};q)_{(\lambda,\alpha)}}$$

Proof. The coefficients of $S_{\lambda+\lambda_j}$ have to be equal at both sides of (5). Hence
$k_\lambda^{(q,t)} = A_{\lambda,\lambda_j}^{(q,t)} k_{\lambda+\lambda_j}^{(q,t)}$. Now

$$A_{\lambda,\lambda_j}^{(q,t)} = \frac{\delta((\log t)(\rho+\lambda_j)+(\log q)\lambda)}{\delta((\log t)\rho+(\log q)\lambda)}$$

$$= \frac{t^{-(\rho,\rho+\lambda_j)}q^{(-\rho,\lambda)}}{t^{-(\rho,\rho)}q^{-(\rho,\lambda)}} \prod_{\alpha\in R^+} \frac{t^{(\alpha,\rho+\lambda_j)}q^{(\alpha,\lambda)}-1}{t^{(\alpha,\rho)}q^{(\alpha,\lambda)}-1}$$

$$= t^{-(\rho,\lambda_j)} \prod_{\substack{\alpha\in R^+\\ (\lambda_j,\alpha)=1}} \frac{1-t^{(\alpha,\rho)+1}q^{(\alpha,\lambda)}}{1-t^{(\alpha,\rho)}q^{(\alpha,\lambda)}}$$

$$= t^{-(\rho,\lambda+\lambda_j)}\, t^{(\rho,\lambda)}$$

$$\cdot \prod_{\alpha\in R^+}\left(\frac{(t^{(\alpha,\rho)+1};q)_{(\lambda+\lambda_j,\alpha)}}{(t^{(\alpha,\rho)+1};q)_{(\lambda,\alpha)}} \frac{(t^{(\alpha,\rho)};q)_{(\lambda,\alpha)}}{(t^{(\alpha,\rho)};q)_{(\lambda+\lambda_j,\alpha)}}\right)$$

Hence

$$k_\lambda^{(q,t)}\, t^{-(\rho,\lambda)} \prod_{\alpha\in R^+} \frac{(t^{(\alpha,\rho)+1};q)_{(\lambda,\alpha)}}{(t^{(\alpha,\rho)};q)_{(\lambda,\alpha)}}$$

is indendent of $\lambda\in P^+$. Substitute $\lambda=0$. $\square$

<u>Corollary 9</u>. Let

$$h_\lambda := \int_C \left| P_\lambda^{(q,t)}(ix) \right|^2 \Delta^{(q,t)}(ix)\, dx .$$

Then

$$(17)\quad \frac{h_\lambda}{h_0} = t^{2(\rho,\lambda)} \prod_{\alpha \in R^+} \frac{(t^{(\alpha,\rho)+1};q)_{(\lambda,\alpha)}}{(t^{(\alpha,\rho)-1}q;q)_{(\lambda,\alpha)}}$$

<u>Proof</u>. It follows from Theorem 7 that

$$h_{\lambda+\lambda_j}^{(q,t)} A_{\lambda,\lambda_j}^{(q,t)} = \int_C S_{\lambda_j}(ix) P_\lambda^{(q,t)}(ix) \overline{P_{\lambda+\lambda_j}^{(q,t)}(ix)} \cdot \Delta^{(q,t)}(ix)\, dx$$

$$= \int_C P_\lambda^{(q,t)}(ix) \overline{S_{-w_0\lambda_j}(ix) P_{\lambda+\lambda_j}^{(q,t)}(ix)} \cdot \Delta^{(q,t)}(ix)\, dx$$

$$= h_\lambda^{(q,t)} \overline{A_{\lambda+\lambda_j,-\lambda_j}^{(q,t)}}$$

Similarly as in the proof of Corollary 8 we compute that

$$\overline{A_{\lambda+\lambda_j,-\lambda_j}^{(q,t)}} = t^{(\rho,\lambda+\lambda_j)} t^{-(\rho,\lambda)} \cdot \prod_{\alpha \in R^+} \left(\frac{(t^{(\rho,\alpha)-1}q;q)_{(\lambda+\lambda_j,\alpha)}\,(t^{(\rho,\alpha)}q;q)_{(\lambda,\alpha)}}{(t^{(\rho,\alpha)-1}q;q)_{(\lambda,\alpha)}\,(t^{(\rho,\alpha)}q;q)_{(\lambda+\lambda_j,\alpha)}} \right)$$

If we combine this with the expression for $A_{\lambda,\lambda_j}^{(q,t)}$ in the previous proof and iterate, then (17) follows. $\square$

Observe that in the previous proof we only used the orthogonality for $P_\lambda^{(q,t)}$ and $P_\mu^{(q,t)}$ for the cases $\lambda < \mu$ and $\lambda > \mu$.

We now turn to the linearization of products. For convenience, omit the parameters q, t. We certainly have

$$(18) \quad P_\lambda P_\mu = \sum_{\substack{\tau \in P^+ - \lambda \\ \tau \leq \mu}} A_{\lambda,\mu,\tau} P_{\lambda+\tau}$$

for certain coefficients $A_{\lambda,\mu,\tau}$. Let, for $\mu \in P^+$

$$C(\mu) := \{\tau \in P \mid w\tau \leq \mu \quad \forall w \in W\}.$$

Clearly we have:

$$(19) \qquad A_{\lambda,0,\tau} = \begin{cases} 1 & \text{if } \tau = 0 \\ 0, & \text{otherwise} \end{cases}$$

It follows from (5) and (3) that

$$S_{\lambda_j} = A_{0,\lambda_j} P_{\lambda_j} = \frac{\delta((\log t)(\rho + \lambda_j))}{\delta((\log t)(\rho))} P_{\lambda_j}.$$

Hence (5) together with (18) and (3) yield that

$$(20) \quad A_{\lambda,\lambda_j,\nu} = \begin{cases} \dfrac{A_{\lambda,\nu}}{A_{0,\lambda_j}} = \dfrac{\delta((\log t)(\rho+\nu) + (\log q)\lambda)\, \delta((\log t)\rho)}{\delta((\log t)(\rho+\lambda_j))\, \delta((\log t)\rho + (\log q)\lambda)} & \text{if } \nu \in W\lambda_j \cap (P^+ - \lambda) \\ 0, & \text{otherwise.} \end{cases}$$

Hence, if $\mu = 0, \lambda_1, \ldots, \lambda_n$ then the sum in (18) runs over $C(\mu) \cap P^+ - \lambda$. We ~~will~~ can show that this holds generally. ~~but~~ The proof, by recurrence w.r.t. μ, is straightforward. In this way it is also possible to compute the numbers $A_{\lambda,\mu,w\mu}$ $(w \in W)$ explicitly.

Remark. Formula (5) together with (3) and its corollaries (16) and (17) were earlier obtained in Macdonald [4] by a combinatorial argument. The self-duality (6) seems to be new.

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