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Ch. 5 Gamma function

§5.2(i) Gamma and Psi functions (Definitions);
§5.9(i) Gamma function (Integral representations);
§5.13 Integrals

From Euler's integral (5.2.1) we obtain

$$\Gamma(x+iy)=\int_{-\infty}^{\infty}\exp(-e^t+xt)\,e^{iyt}\,dt\quad(x>0,\ y\in\mathbb{R}).$$

By Fourier inversion,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(x+iy)\,e^{-iyt}\,dy=\exp(-e^t+xt)\quad(x>0,\ t\in\mathbb{R}).$$

§5.4(i) Gamma function (Special values and extrema);
§5.5(ii) Reflection (Functional relations)

Additionally to (5.5.3),

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}\,,$$

we have

$$\begin{aligned}\Gamma(z)\Gamma(-z)&=-\,\frac{\pi}{z\sin(\pi z)}\,,\\ \Gamma\big(\tfrac12+z\big)\Gamma\big(\tfrac12-z\big)&=\frac{\pi}{\cos(\pi z)}\,,\\ \Gamma(1+z)\Gamma(1-z)&=\frac{\pi z}{\sin(\pi z)}\,.\end{aligned}$$

Additionally to (5.4.3) (rewritten) and (5.4.4),

$$\begin{aligned}\Gamma(iy)\Gamma(-iy)&=\frac{\pi}{y\sinh(\pi y)}\,,\\ \Gamma\big(\tfrac12+iy\big)\Gamma\big(\tfrac12-iy\big)&=\frac{\pi}{\cosh(\pi y)}\,,\end{aligned}$$

we have

$$\Gamma(1+iy)\Gamma(1-iy)=\frac{\pi y}{\sinh(\pi y)}\,.$$

Ch. 9 Airy and related functions

§9.6 Relations to other functions

Add a paragraph "Airy functions as ${}_0F_1$ hypergeometric functions":

$$\operatorname{Ai}(z) = \frac{1}{3^{2/3}\Gamma(2/3)}\, {}_0F_1\left({- \atop 2/3};z^3/9\right) - \frac{1}{3^{1/3}\Gamma(1/3)}\, {}_0F_1\left({- \atop 4/3};z^3/9\right),$$
$$\operatorname{Bi}(z) = \frac{1}{3^{1/6}\Gamma(2/3)}\, {}_0F_1\left({- \atop 2/3};z^3/9\right) + \frac{3^{1/6}}{\Gamma(1/3)}\, {}_0F_1\left({- \atop 4/3};z^3/9\right).$$

For the proof use (9.6.2) respectively (9.6.4) together with (10.39.9) or use (9.4.1) respectively (9.4.3) together with §9.1(ii).

Ch. 10 Bessel functions

§10.13 Other differential equations

Note the special case of (10.13.5) with $r = -\nu$ and $q = 1$:

$$w''(z) + \frac{2\nu+1}{z}\, w'(z) + \lambda^2 w(z) = 0, \qquad w(z) = z^{-\nu}\mathcal{C}_\nu(\lambda z).$$

§10.23(ii) Addition theorems

Note the special case of Gegenbauer's addition theorem (10.23.8) when $\mathcal{C} = J$, $\alpha = 0$ an $u = v$. Then

$$1 = 2^{2\nu}\,\Gamma(\nu)^2 \sum_{k=0}^{\infty} \frac{\nu+k}{\nu}\,\frac{(2\nu)_k}{k!}\,\frac{J_{\nu+k}(u)^2}{u^{2\nu}}.$$

See Watson (1944, §11.41 (14)), a formula going back to Gegenbauer.

Ch. 16 Generalized Hypergeometric Functions and Meijer G-Function

§16.4(ii) Examples

Watson's Sum

For $a = -n$ we get

$${}_3F_2\left({-n, b, c \atop \frac12(-n+b+1), 2c};1\right) = \begin{cases} \dfrac{(\frac12)_m(-\frac12 b + c + \frac12)_m}{(-\frac12 b + \frac12)_m(c+\frac12)_m}, & n = 2m, \\ 0, & n = 2m+1. \end{cases}$$

It is only for this terminating case that Watson gave (16.4.6) in:

G. N. Watson **(1925)**. A note on generalized hypergeometric series. Proc. London Math. Soc. (2) **23**, pp. xiii–xv.

The general case of (16.4.6) was first given by Whipple on p.113 of:

F. J. W. Whipple **(1925)**. A group of generalized hypergeometric series: relations between 120 allied series of the type $F[a, b, c; e, f]$. Proc. London Math. Soc. (2) **23**, pp. 104–114.

Ch. 17 q-Hypergeometric and Related Functions

§17.2(iv) Derivatives
Note in particular (17.2.40) for $n=2$:

$$\mathcal{D}_q^2 f(z) = \frac{qf(z)-(1+q)f(qz)+f(q^2z)}{q(1-q)^2z^2}, \quad z\neq 0.$$

§17.4(i) ${}_r\phi_s$ functions
In connection with (17.4.1) note the confluence relations

$$\begin{aligned}\lim_{a_0\to\infty} {}_{r+1}\phi_s\left(\begin{matrix}a_0,a_1,a_2,\ldots,a_r\\ b_1,b_2,\ldots,b_s\end{matrix};q,a_0^{-1}z\right) &= {}_r\phi_s\left(\begin{matrix}a_1,a_2,\ldots,a_r\\ b_1,b_2,\ldots,b_s\end{matrix};q,z\right), \quad r\le s,\\ \lim_{b_s\to\infty} {}_{r+1}\phi_s\left(\begin{matrix}a_0,a_1,a_2,\ldots,a_r\\ b_1,b_2,\ldots,b_s\end{matrix};q,b_sz\right) &= {}_{r+1}\phi_{s-1}\left(\begin{matrix}a_0,a_1,a_2,\ldots,a_r\\ b_1,b_2,\ldots,b_{s-1}\end{matrix};q,z\right), \quad r\le s-1.\end{aligned}$$

§17.5 ${}_0\phi_0$, ${}_1\phi_0$, ${}_1\phi_1$ Functions
From (17.6.27) or its equivalent form we can get by confluence a q-differential equation for ${}_1\phi_1$. With $u(z)={}_1\phi_1(a;c;q,z)$ we have

$$(c-aqz)\big(qu(z)-(1+q)u(qz)+u(q^2z)\big)+q\big(1-c+(a(1+q)-1)z\big)(u(z)-u(qz))+q(1-a)zu(z)=0.$$

Note also the q-difference recurrence relations

$$\begin{aligned}{}_1\phi_1\left(\begin{matrix}a\\ c\end{matrix};q,z\right)-{}_1\phi_1\left(\begin{matrix}a\\ c\end{matrix};q,qz\right) &= -\frac{1-a}{1-c}\,z\,{}_1\phi_1\left(\begin{matrix}qa\\ qc\end{matrix};q,qz\right),\\ {}_1\phi_1\left(\begin{matrix}qa\\ qc\end{matrix};q,z\right)-(c-az)\,{}_1\phi_1\left(\begin{matrix}qa\\ qc\end{matrix};q,qz\right) &= (1-c)\,{}_1\phi_1\left(\begin{matrix}a\\ c\end{matrix};q,z\right),\\ (z-c)\,{}_1\phi_1\left(\begin{matrix}a\\ c\end{matrix};q,z\right)-(az-c)\,{}_1\phi_1\left(\begin{matrix}a\\ c\end{matrix};q,qz\right) &= \frac{1-a}{1-c}\,z\,{}_1\phi_1\left(\begin{matrix}qa\\ qc\end{matrix};q,z\right).\end{aligned}$$

The second and the third equation follow by substituting the first equation twice or once in (17.6.27).

§17.6(iv) Differential Equations
With $u(z)={}_2\phi_1(a,b;c;q,z)$ we can write (17.6.27) equivalently as

$$\begin{aligned}(c-abqz)\big(qu(z)-(1+q)u(qz)+u(q^2z)\big)+q\big(1-c+(ab(1+q)-a-b)z\big)(u(z)-u(qz))\\ -q(1-a)(1-b)zu(z)=0.\end{aligned}$$