# On $q^{-1}$-Al-Salam-Chihara polynomials

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In [2, (3.3)] and [4] the Al-Salam-Chihara polynomials $Q_n^{ASC}(x; q; a, b, c)$ are considered, which can be defined as solutions of the recurrence relation (with usual starting values)

$$(1 - q^{n+1})Q_{n+1}(x) = (x - aq^n)Q_n(x) - (c - bq^{n-1})Q_{n-1}(x). \tag{1}$$

Note that the three parameters involve one scale parameter:

$$\lambda^{-n} Q_n^{ASC}(\lambda x; q; \lambda a, \lambda^2 b, \lambda^2 c) = Q_n^{ASC}(x; q; a, b, c). \tag{2}$$

In particular:

$$i^{-n} Q_n^{ASC}(ix; q; ia, -b, -c) = Q_n^{ASC}(x; q; a, b, c). \tag{3}$$

We suppose that $c \neq 0$ (for $c = 0$ we obtain Al-Salam-Carlitz polynomials, see [2, §3.7]). So, it is sufficient to consider for the parameter triples $(a, b, c)$ with $c \neq 0$ and $x \in \mathbb{R}$ the cases $(a, b, 1)$ and $(a, b, -1)$. In [8, §3.8] the Al-Salam-Chihara polynomials are notated in a different way; in terms of the Askey-Wilson polynomials $p_n(x; a, b, c, d \mid q)$:

$$Q_n^{KS}(x; a, b \mid q) = p_n(x; a, b, 0, 0 \mid q). \tag{4}$$

In this notation the recurrence relation (see [8, (3.8.4)]) becomes

$$2xQ_n(x) = Q_{n+1}(x) + (a + b)q^n Q_n(x) + (1 - q^n)(1 - abq^{n-1})Q_{n-1}(x), \tag{5}$$

I also introduce the polynomials

$$\widetilde{Q}_n^{KS}(x; a, b \mid q) := i^{-n}\, Q_n^{KS}(ix; ia, ib \mid q). \tag{6}$$

for which the recurrence relation becomes

$$2x\widetilde{Q}_n(x) = \widetilde{Q}_{n+1}(x) + (a + b)q^n \widetilde{Q}_n(x) - (1 - q^n)(1 + abq^{n-1})\widetilde{Q}_{n-1}(x). \tag{7}$$

The relationship between $Q_n^{ASC}$ and $Q_n^{KS}$ (and $\widetilde{Q}_n^{KS}$) is:

$$Q_n^{ASC}(x; q; a, b, c) = \frac{(\sqrt{c}\,)^n}{(q; q)_n}\, Q_n^{KS}\left(\tfrac{x}{2\sqrt{c}}; \tfrac{a+\sqrt{a^2-4b}}{2\sqrt{c}}, \tfrac{a-\sqrt{a^2-4b}}{2\sqrt{c}} \mid q\right), \tag{8}$$

or the other way around:

$$\begin{aligned} Q_n^{KS}(x; a, b \mid q) &= (q; q)_n\, Q_n^{ASC}(2x; q; a + b, ab, 1), & (9)\\ \widetilde{Q}_n^{KS}(x; a, b \mid q) &= (q; q)_n\, Q_n^{ASC}(2x; q; a + b, ab, -1). & (10) \end{aligned}$$

If we consider the polynomials $Q_n^{ASC}(x;q;a,b,\pm 1)$ for $q>1$ then the necessary and sufficient conditions for orthogonality of the polynomials with respect to a positive measure on $\mathbb{R}$ become $a\in\mathbb{R}$ together with $b>c>0$ or $b\ge 0>c$ (see [2, (3.69)]). For the polynomials $Q_n^{KS}(x;a,b\mid q)$ with $q>1$ these conditions are $ab>1$ together with $a,b\in\mathbb{R}$ or $b=\overline{a}$. For the polynomials $\widetilde{Q}_n^{KS}(x;a,b\mid q)$ with $q>1$ these conditions are $ab\ge 0$ together with $a,b\in\mathbb{R}$ or $b=\overline{a}$. Note that the $a,b\to 0$ limit to continuous $q$-Hermite polynomials with $q>1$ while remaining in the parameter domain allowing a positive orthogonality measure, is possible for the polynomials $\widetilde{Q}_n^{KS}(x;a,b\mid q)$ but not for the polynomials $Q_n^{KS}(x;a,b\mid q)$. Continuous $q$-Hermite polynomials with $q>1$ were studied in [1] and [7].

If the conditions of the previous paragraph are satisfied, then the necessary and sufficient conditions for the determinacy of the moment problem associated with the polynomials $Q_n^{ASC}(x;q;a,b,c)$ are

$$a^2>4b \quad\text{and}\quad \frac{4b}{\left(a+\sqrt{a^2-4b}\right)^2}\le q^{-1}. \tag{11}$$

See [2, (3.77)]. For the polynomials $Q_n^{KS}(x;a,b\mid q)$ and $\widetilde{Q}_n^{KS}(x;a,b\mid q)$ these conditions become:

$$a,b\in\mathbb{R} \quad\text{and}\quad ba^{-1}\le q^{-1}. \tag{12}$$

The indeterminate case $q>1$, $a^2\le 4b$ of the polynomials $Q_n^{ASC}(x;q;a,b,c)$ is studied in [4]. I assume that the authors intended this under the additional condition $b>c>0$ or $b\ge 0>c$ (not explicitly mentioned there) for existence of a positive orthogonality measure.

The special indeterminate case $q>1$, $c<0=a\le b$ of the polynomials $Q_n^{ASC}(x;q;a,b,c)$, i.e., the special indeterminate case $q>1$, $b=-a\in i\mathbb{R}$ of the polynomials $\widetilde{Q}_n^{KS}(x;a,b\mid q)$, is studied in [5].

**Dual little $q$-Jacobi polynomials**

Rosengren [9] (somewhat implicitly), Groenevelt [6, Remark 3.1], and Atakishiyev & Klimyk [3] observed that $q^{-1}$-Al-Salam-Chihara polynomials are duals of little $q$-Jacobi polynomials:

$$\begin{aligned}&\frac{(-1)^n q^{\frac12 n(n-1)} b^{-n}}{((ab)^{-1};q)_n}\, Q_n^{KS}\left(\tfrac12(aq^{-k}+a^{-1}q^k);a,b\mid q^{-1}\right)\\ &\qquad=\frac{(-ab^{-1})^k (qa^{-1}b;q)_k}{q^{\frac12 k(k+1)}((ab)^{-1};q)_k}\, p_k(q^n;a^{-1}b,(qab)^{-1};q),\qquad q>1,\ n,k=0,1,2,\ldots\,,\end{aligned} \tag{13}$$

$$\begin{aligned}&\frac{(-1)^n q^{\frac12 n(n-1)} b^{-n}}{(-(ab)^{-1};q)_n}\, \widetilde{Q}_n^{KS}\left(\tfrac12(aq^{-k}-a^{-1}q^k);a,b\mid q^{-1}\right)\\ &\qquad=\frac{(-ab^{-1})^k (qa^{-1}b;q)_k}{q^{\frac12 k(k+1)}(-(ab)^{-1};q)_k}\, p_k(q^n;a^{-1}b,-(qab)^{-1};q),\qquad q>1,\ n,k=0,1,2,\ldots\,.\end{aligned} \tag{14}$$

Then, the orthogonality relations for little $q$-Jacobi polynomials give dually the orthogonality relations for $q^{-1}$-Al-Salam-Chihara polynomials:

$$\sum_{y=0}^{\infty} \frac{1-q^{2y}a^{-2}}{1-a^{-2}} \frac{(a^{-2},(ab)^{-1};q)_y}{(q,a^{-1}bq;q)_y} (a^{-1}b)^y q^{y^2} (Q_n^{KS}Q_m^{KS}) \left(\tfrac{1}{2}(aq^{-y}+a^{-1}q^y);a,b \mid q^{-1}\right)$$
$$= \frac{(qa^{-2};q)_\infty}{(a^{-1}bq;q)_\infty} (q,(ab)^{-1};q)_n (ab)^n q^{-n^2} \delta_{n,m}, \qquad 0<q<1,\ ab>1,\ a^{-1}b<q^{-1}, \qquad (15)$$

$$\sum_{y=0}^{\infty} \frac{1+q^{2y}a^{-2}}{1+a^{-2}} \frac{(-a^{-2},-(ab)^{-1};q)_y}{(q,a^{-1}bq;q)_y} (a^{-1}b)^y q^{y^2} (\widetilde{Q}_n^{KS}\widetilde{Q}_m^{KS}) \left(\tfrac{1}{2}(aq^{-y}-a^{-1}q^y);a,b \mid q^{-1}\right)$$
$$= \frac{(-qa^{-2};q)_\infty}{(a^{-1}bq;q)_\infty} (q,-(ab)^{-1};q)_n (ab)^n q^{-n^2} \delta_{n,m}, \qquad 0<q<1,\ ab>0,\ a^{-1}b<q^{-1}. \quad (16)$$

If the constraint $a^{-1}b<q^{-1}$ is narrowed to $a^{-1}b \le q$ then these are the orthogonality relations obtained in [2, (3.82)]. For $q<a^{-1}b<q^{-1}$ the orthogonality relations (15) and (16) remain valid if $a$ and $b$ are interchanged. Thus for $q<a^{-1}b<q^{-1}$ and $a \ne b$ we have an explicit example of two distinct orthogonality measures.

Note that it is allowed above to take the duals of the orthogonality relations for little $q$-Jacobi polynomials, because the little $q$-Jacobi polynomials form a complete orthogonal system in the $L^2$-space corresponding to their orthogonality measure, since this measure has bounded support. Then we can use the characterization of unitary operators on a Hilbert space as surjective isometric operators (see for instance [10, Theorem 12.13]).

## References


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