The q-Jacobi transform

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1. Some functional analytic preliminaries

Let $\mathcal{H}$ be a separable Hilbert space. We denote unbounded linear operators on $\mathcal{H}$ by $(L, \mathcal{D})$, where $\mathcal{D}$ is a linear subspace of $\mathcal{H}$ and $L: \mathcal{D} \to \mathcal{H}$ is a linear mapping. We write $(L_1, \mathcal{D}_1) \subset (L_2, \mathcal{D}_2)$ if $\mathcal{D}_1 \subset \mathcal{D}_2$ and $L_1 = L_2|_{\mathcal{D}_1}$. If $\mathcal{D}$ is dense in $\mathcal{H}$ then we write the adjoint $(L, \mathcal{D})^*$ of $(L, \mathcal{D})$ as $(L^*, \mathcal{D}^*)$. If $\mathcal{D}^*$ is again dense in $\mathcal{H}$ then we write the adjoint $(L, \mathcal{D})^{**}$ of $(L, \mathcal{D})^*$ as $(L^{**}, \mathcal{D}^{**})$.

Let $\mathcal{D}$ be dense in $\mathcal{H}$ and let $(L, \mathcal{D})$ be symmetric. Then $(L, \mathcal{D})^*$ is a closed extension of $(L, \mathcal{D})$ and every symmetric extension of $(L, \mathcal{D})$ is contained in $(L, \mathcal{D})^*$ (cf. [1, Lemma XII.4.1]). Furthermore, $(L, \mathcal{D})^{**}$ is the minimal closed extension of $(L, \mathcal{D})$ and it is symmetric. We have the inclusions

$$(L, \mathcal{D}) \subset (L, \mathcal{D})^{**} \subset (L, \mathcal{D})^*.$$

Put

$$B(v, w) := (L^* v, w) - (v, L^* w), \quad v, w \in \mathcal{D}^*. \tag{1.1}$$

Then

$$\mathcal{D}^{**} = \{v \in \mathcal{D}^* \mid B(v, w) = 0 \quad \forall w \in \mathcal{D}^*\}.$$

Let $\lambda \in \mathbb{C}$. Put

$$N_\lambda := \{v \in \mathcal{D}^* \mid L^* v = \lambda v\}$$

and

$$\begin{aligned} n_+ &:= \dim N_i, \\ n_- &:= \dim N_{-i}. \end{aligned}$$

The cardinalities n_+ and n_- are called the *deficiency indices* of the symmetric operator $(L, \mathcal{D})$. We will always assume that these are finite. We will keep the assumptions and notations of this paragraph in the remainder of this section.

Proposition 1.1. (a) There is the orthogonal direct sum

$$\mathcal{D}^* = \mathcal{D}^{**} + N_i + N_{-i}.$$

(b) If $\operatorname{Im} \lambda > 0$ then $\dim N_\lambda = n_+$ and if $\operatorname{Im} \lambda < 0$ then $\dim N_\lambda = n_-$.
(c) For $\lambda \in \mathbb{C} \backslash \mathbb{R}$ there is the (generally non-orthogonal) direct sum

$$\mathcal{D}^* = \mathcal{D}^{**} + N_\lambda + N_{\overline{\lambda}}.$$

Proof. See [1, Lemma XII.4.10] for (a) and [1, Theorem XII.4.19] for (b). Part (c) follows from (b), the linear independence of eigenvectors for distinct eigenvalues and the fact that the domain $\mathcal{D}^{**}$ of a symmetric operator L^{**} cannot contain eigenvectors for non-real eigenvalues. □

If $(L,\mathcal{D})$ has a self-adjoint extension $(\widetilde{L},\widetilde{\mathcal{D}})$ then

$$(L,\mathcal{D})^{**}\subset(\widetilde{L},\widetilde{\mathcal{D}})\subset(L,\mathcal{D})^{*}.$$

Conversely, we have:

Proposition 1.2. Let $\mathcal{D}^{**}\subset\widetilde{\mathcal{D}}\subset\mathcal{D}^{*}$. Let $\widetilde{L}:=L^{*}|_{\widetilde{\mathcal{D}}}$. Equivalent statements are:
(a) $(\widetilde{L},\widetilde{\mathcal{D}})$ is self-adjoint;
(b) $\widetilde{\mathcal{D}}=\{v\in\mathcal{D}^{*}\mid B(v,w)=0\quad\forall w\in\widetilde{\mathcal{D}}\}$;
(c) $\exists$ isometric bijection $U\colon N_i\to N_{-i}$ such that $\widetilde{\mathcal{D}}=\{v+w+Uw\mid v\in\mathcal{D}^{**},\ w\in N_i\}$.
Proof. The equivalence of (a) and (b) is by the definition of self-adjoint operator. For the equivalence of (a) and (c) see [1, Theorem XII.4.12]. □

Note that, by (c) of Proposition 1.2, the equality $n_+=n_-$ is a necessary and sufficient condition for the existence of a self-adjoint extension of $(L,\mathcal{D})$.

Let $C(\mathbb{Z})$, $\mathcal{H}:=L^2(\mathbb{Z})$ and $\mathcal{D}:=C_c(\mathbb{Z})$ respectively consist of all complex-valued functions on $\mathbb{Z}$, of all square integrable functions on $\mathbb{Z}$ and of all functions u on $\mathbb{Z}$ such that $u(n)\neq0$ for only finitely many n. Then $\mathcal{D}$ is dense in $\mathcal{H}$. Define the linear operator L on $C(\mathbb{Z})$ by

$$(Lu)(n):=a(n)\,u(n+1)+b(n)\,u(n)+a(n-1)\,u(n-1), \tag{1.2}$$

where $a(n)$ and $b(n)$ are real and $a(n)>0$. Then

$$\sum_{n=-\infty}^{\infty}(Lu)(n)\,\overline{v(n)}=\sum_{n=-\infty}^{\infty}u(n)\,\overline{(Lv)(n)},\quad u\in C_c(\mathbb{Z}),\ v\in C(\mathbb{Z}).$$

Hence the operator $(L,\mathcal{D})$ is symmetric and L^* is the restriction of L to $\mathcal{D}^*$, so we can write $(L,\mathcal{D})^*=(L,\mathcal{D}^*)$. Note that $\mathcal{D}^*$ can be characterized as

$$\mathcal{D}^*=\{u\in\mathcal{H}\mid Lu\in\mathcal{H}\}.$$

Clearly, $\dim(N_\lambda)\leq2$. Also, $u\in N_\lambda$ iff $\bar{u}\in N_{\overline{\lambda}}$. Hence

$$n_+=n_-\leq2.$$

Thus $(L,\mathcal{D})$ has self-adjoint extensions.

Define the *Wronskian* of two functions u and v on $\mathbb{Z}$ by

$$[u,v](n):=a(n)\big(u(n+1)\,v(n)-u(n)\,v(n+1)\big). \tag{1.3}$$

Then

$$(Lu)(n)\,v(n)-u(n)\,(Lv)(n)=[u,v](n)-[u,v](n-1). \tag{1.4}$$

Hence

$$[u,v](n)-[u,v](m)=\sum_{k=m+1}^{n}\big((Lu)(k)\,v(k)-u(k)\,(Lv)(k)\big).$$

It follows (on letting n tend to ∞ respectively $-\infty$) that the limits

$$[u,v](\pm\infty):=\lim_{n\to\pm\infty}[u,v](n)$$

exist for $+\infty$ respectively $-\infty$ if u, Lu, v, Lv are L^2 at $+\infty$ respectively $-\infty$. In particular, if $u, v \in \mathcal{D}^*$ then these limits exist and, by (1.1) and (1.4),

$$B(u,v) = \lim_{n\to\infty, m\to-\infty} ([u,\bar{v}](n) - [u,\bar{v}](m)).$$

Hence

$$B(u,v) = [u,\bar{v}](\infty) - [u,\bar{v}](-\infty), \quad u, v \in \mathcal{D}^*.$$

If u, v are functions on $\mathbb{Z}$ satisfying $Lu = \lambda u$, $Lv = \lambda v$ for some $\lambda \in \mathbb{C}$, then it follows from (1.4) that $[u,v] := [u,v](n)$ is independent of n.

We make now the following three assumptions.

1) $a(n)$ is bounded as $n \to -\infty$. Then, for $u, v \in \mathcal{D}^*$, we have $[u,\bar{v}](-\infty) = 0$, hence

$$B(u,v) = [u,\bar{v}](\infty).$$

2) For $\lambda \in \mathbb{C}\backslash\mathbb{R}$ the space $\{u \in C(\mathbb{Z}) \mid Lu = \lambda u,\ u \text{ is } L^2 \text{ at } -\infty\}$ has dimension one. Say that it is spanned by F_λ.
3) For $\lambda \in \mathbb{C}\backslash\mathbb{R}$ there exists $f_\lambda \in C(\mathbb{Z})$ such that $Lf_\lambda = \lambda f_\lambda$, f_λ is L^2 at ∞ and F_λ, f_λ are linearly independent.

Now there are two possibilities for given $\lambda \in \mathbb{C}\backslash\mathbb{R}$:

(i) F_λ is not L^2 at ∞. Then $(L, \mathcal{D}^*)$ is self-adjoint.
(ii) F_λ is L^2 at ∞. Then the possible self-adjoint extensions $(L, \widetilde{\mathcal{D}})$ of $(L, \mathcal{D})$ are given by

$$\widetilde{\mathcal{D}} := \{u \in \mathcal{D}^* \mid [u, e^{i\theta} F_i + e^{-i\theta} F_{-i}](\infty) = 0\}, \tag{1.5}$$

where $\theta \in [0, 2\pi)$.

In case (ii) we can assume that f_λ moreover satisfies $[f_\lambda, e^{i\theta} F_i + e^{-i\theta} F_{-i}](\infty) = 0$. Indeed, replace f_λ in (ii) by another solution $\tilde{f}_\lambda := f_\lambda + cF_\lambda$ of $Lu = \lambda u$ and find c such that $[\tilde{f}_\lambda, e^{i\theta} F_i + e^{-i\theta} F_{-i}](\infty) = 0$. Such c exists because $[F_\lambda, e^{i\theta} F_i + e^{-i\theta} F_{-i}](\infty) \neq 0$. This last inequality holds because otherwise F_λ would be in $\widetilde{\mathcal{D}}$ by (1.5), while an eigenvector with nonreal eigenvalue cannot be in the domain of a self-adjoint operator.

Let $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Define the *Green kernel* by

$$G_\lambda(m,n) := \begin{cases} \dfrac{F_\lambda(n)\, f_\lambda(m)}{[F_\lambda, f_\lambda]} & , n \le m, \\ \dfrac{f_\lambda(n)\, F_\lambda(m)}{[F_\lambda, f_\lambda]} & , n > m. \end{cases} \tag{1.6}$$

Fix a self-adjoint extension $(L, \widetilde{\mathcal{D}})$ of $(L, \mathcal{D})$.

Proposition 1.3. Let $\lambda \in \mathbb{C}\backslash\mathbb{R}$. The *resolvent* $(\lambda - L)^{-1}$ of $(L, \widetilde{\mathcal{D}})$ satisfies

$$((\lambda - L)^{-1} f)(m) = \sum_{n=-\infty}^{\infty} G_\lambda(m,n)\, f(n), \quad f \in \mathcal{H}. \tag{1.7}$$

Proof. Since $(L, \widetilde{\mathcal{D}})$ is self-adjoint, $(\lambda - L)^{-1}$ is a continuous one-to-one mapping from $\mathcal{H}$ onto $\widetilde{\mathcal{D}}$. Hence, for each $m \in \mathbb{Z}$, $f \mapsto ((\lambda - L)^{-1} f)(m)$ is a continuous linear functional on $\mathcal{H}$. Thus (1.7) holds for a certain kernel $G_\lambda(m,n)$ such that $G_\lambda(m, .) \in \mathcal{H}$. Thus, the proposition will follow if we can prove that (1.7) holds, with G_λ given by (1.6), for all f in the dense subspace $\mathcal{D}$ of $\mathcal{H}$.

Thus let $f \in \mathcal{D}$ and G_λ given by (1.6). By the properties of f_λ and F_λ we see that $G_\lambda(m, .) \in \mathcal{H}$ for all m, and that $G_\lambda f \in \widetilde{\mathcal{D}}$, where $(G_\lambda f)(m)$ denotes the right hand side of (1.7). We will finish the proof by showing that $(\lambda - L)(G_\lambda f) = f$. Indeed,

$$
\begin{aligned}
&\big((\lambda - L)(G_\lambda f)\big)(m)\\
&= -a(m)(G_\lambda f)(m+1) + (\lambda - b(m))(G_\lambda f)(m) - a(m-1)(G_\lambda f)(m-1)\\
&= \Big(\sum_{n=-\infty}^{m-1} + \sum_{n=m+1}^{\infty} + \sum_{n=m}^{m}\Big)\big(-a(m)G_\lambda(m+1,n) + (\lambda - b(m))G_\lambda(m,n) - a(m-1)G_\lambda(m-1,n)\big)f(n)\\
&= \frac{((\lambda - L)f_\lambda)(m)}{[F_\lambda, f_\lambda]} \sum_{n=-\infty}^{m-1} F_\lambda(n)f(n) + \frac{((\lambda - L)F_\lambda)(m)}{[F_\lambda, f_\lambda]} \sum_{n=m+1}^{\infty} f_\lambda(n)f(n)\\
&\quad + \big(-a(m)F_\lambda(m)f_\lambda(m+1) + (\lambda - b(m))f_\lambda(m)F_\lambda(m) - a(m-1)f_\lambda(m)F_\lambda(m-1)\big)f(m)/[F_\lambda, f_\lambda]\\
&= 0 + 0 + \frac{f_\lambda(m)}{[F_\lambda, f_\lambda]}((\lambda - L)F_\lambda)(m)f(m) + \frac{a(m)}{[F_\lambda, f_\lambda]}(F_\lambda(m+1)\, f_\lambda(m) - F_\lambda(m)f_\lambda(m+1))f(m)\\
&= f(m). \qquad \square
\end{aligned}
$$

See Rudin [4, Def. 12.17] for the definition of a *resolution of the identity* on a σ-algebra. If E is a resolution of the identity and v, w are elements of the corresponding Hilbert space then put

$$E_{v,w}(\omega) := (E(\omega)v, w),$$

where ω is in the σ-algebra. We formulate the *spectral theorem* for a self-adjoint operator (cf. for instance [4, Theorem 13.30].

Theorem 1.4. To every self-adjoint operator $(L, \mathcal{D})$ in $\mathcal{H}$ corresponds a unique resolution E of the identity on the Borel subsets of the real line such that

$$(Lv, w) = \int_{-\infty}^{\infty} t\, dE_{v,w}(t), \quad v \in \mathcal{D},\ w \in \mathcal{H}.$$

The resolution of the identity E in the above theorem may be calculated explicitly in terms of the resolvent of L (cf. [1, Theorem XII.2.10]):

Theorem 1.5. Let $(L, \mathcal{D})$ and E be as in Theorem 1.4 and let (a, b) be an open interval in $\mathbb{R}$. Then

$$E_{v,w}((a,b)) = \lim_{\delta\downarrow 0} \lim_{\varepsilon\downarrow 0} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} [((s - i\varepsilon - L)^{-1}v, w) - ((s + i\varepsilon - L)^{-1}v, w)]\, ds, \quad v, w \in \mathcal{H}.$$

2. The q-hypergeometric q-difference equation and its solutions

Let $0 < q < 1$. For definitions and elementary facts about q-hypergeometric functions the reader is referred to Gasper & Rahman [2, Ch.1].

First form. Consider the equation

$$(c - abz)\,U(qz) + \big(-(c+q) + (a+b)z\big)\,U(z) + (q-z)\,U(q^{-1}z) = 0, \quad a,b,c \in \mathbb{C}\backslash\{0\}. \tag{2.1}$$

Any solution of (2.1) may be considered up to multiplication by a function f satisfying $f(z) = f(qz)$, (e.g. $f(z) = z^{2\pi i n\, {}^q\!\log e}$, $n \in \mathbb{Z}$). We look for solutions of the form $U(z) = \sum_{k=0}^{\infty} c_k\, z^{k+\lambda}$ ($c_0 = 1$, convergent around 0) and $U(z) = \sum_{k=0}^{\infty} c_k\, z^{-k+\lambda}$ ($c_0 = 1$, convergent around ∞). The two solutions of the first type are

$$\begin{aligned} U_1(z) &:= {}_2\phi_1(a,b;c;q,z), \quad c \neq q^{-n} \text{ for } n \in \mathbb{Z}_+,\\ U_2(z) &:= z^{1-{}^q\!\log c}\, {}_2\phi_1\left(\frac{qa}{c}, \frac{qb}{c}; \frac{q^2}{c}; q, z\right), \quad c \neq q^{n+2} \text{ for } n \in \mathbb{Z}_+, \end{aligned}$$

and the two solutions of the second type are

$$\begin{aligned} U_3(z) &:= z^{-{}^q\!\log a}\, {}_2\phi_1\left(a, \frac{qa}{c}; \frac{qa}{b}; q, \frac{qc}{abz}\right), \quad a \neq q^{-n-1}b \text{ for } n \in \mathbb{Z}_+,\\ U_4(z) &:= z^{-{}^q\!\log b}\, {}_2\phi_1\left(b, \frac{qb}{c}; \frac{qb}{a}; q, \frac{qc}{abz}\right), \quad a \neq q^{n+1}b \text{ for } n \in \mathbb{Z}_+. \end{aligned}$$

Thus, for $c \notin \{q^n \mid n \in \mathbb{Z}\}$ we have obtained two linearly independent solutions U_1 and U_2 around 0 and for $a/b \notin \{q^n \mid n \in \mathbb{Z}\}$ we have obtained two linearly independent solutions U_3 and U_4 around ∞. Because of Jackson's transformation formula

$${}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, z\right] = \frac{(az;q)_\infty}{(z;q)_\infty}\, {}_2\phi_2\left[\begin{matrix} a, c/b \\ c, az \end{matrix}; q, bz\right],$$

U_1, U_2, U_3, U_4 have one-valued analytic continuations to $\{z \in \mathbb{C}\backslash\{0\} \mid |\arg(-z)| < \pi\}$. We can express $U_1(z)$ as a linear combination of $U_3(z)$ and $U_4(z)$ over the ring of functions f satisfying $f(z) = f(qz)$ by

$$\begin{aligned} {}_2\phi_1(a,b;c;q,z) = & \frac{(ca^{-1}, b;q)_\infty}{(c, ba^{-1};q)_\infty}\, \frac{(az, qa^{-1}z^{-1};q)_\infty}{(z, qz^{-1};q)_\infty}\, {}_2\phi_1\left(a, \frac{qa}{c}; \frac{qa}{b}; q, \frac{qc}{abz}\right)\\ & + \frac{(cb^{-1}, a;q)_\infty}{(c, ab^{-1};q)_\infty}\, \frac{(bz, qb^{-1}z^{-1};q)_\infty}{(z, qz^{-1};q)_\infty}\, {}_2\phi_1\left(b, \frac{qb}{c}; \frac{qb}{a}; q, \frac{qc}{abz}\right), \end{aligned} \tag{2.2}$$

provided that $z \neq 0$, $|\arg(-z)| < \pi$, $c \neq q^{-n}$ for $n \in \mathbb{Z}_+$, $a/b \neq q^{-n}$ for $n \in \mathbb{Z}$ (see Gasper & Rahman, [2, 4.3.2]. Mimachi [3].

Second form. Substitute $a := q^{\frac{1}{2}(\alpha+\beta+1+i\lambda)}$, $b := q^{\frac{1}{2}(\alpha+\beta+1-i\lambda)}$, $c := q^{\alpha+1}$, and assume that $\alpha > -1$, $\beta \in \mathbb{R}$, $\lambda \in \mathbb{C}$. Then equation (2.1) becomes

$$(q^{\alpha+1} - q^{\alpha+\beta+1}z)\,U(qz) + \big(-(q^{\alpha+1}+q) + q^{\frac{1}{2}(\alpha+\beta+1)}(q^{\frac{1}{2}i\lambda} + q^{-\frac{1}{2}i\lambda})z\big)\,U(z) + (q-z)\,U(q^{-1}z) = 0. \tag{2.3}$$

The solutions take the form

$$U_1(z) = {}_2\phi_1\left[\begin{matrix} q^{\frac12(\alpha+\beta+1+i\lambda)}, q^{\frac12(\alpha+\beta+1-i\lambda)} \\ q^{\alpha+1}\end{matrix}; q, z\right],$$

$$U_2(z) = z^{-\alpha}\, {}_2\phi_1\left[\begin{matrix} q^{\frac12(-\alpha+\beta+1+i\lambda)}, q^{\frac12(-\alpha+\beta+1-i\lambda)} \\ q^{-\alpha+1}\end{matrix}; q, z\right], \quad \alpha\notin\mathbb{N},$$

$$U_3(z) = z^{-\frac12(\alpha+\beta+1+i\lambda)}\, {}_2\phi_1\left[\begin{matrix} q^{\frac12(\alpha+\beta+1+i\lambda)}, q^{\frac12(-\alpha+\beta+1+i\lambda)} \\ q^{i\lambda+1}\end{matrix}; q, q^{1-\beta}\, z^{-1}\right], \quad \lambda\notin i\mathbb{N}+2\pi\mathbb{Z}^q\log e,$$

$$U_4(z) = z^{-\frac12(\alpha+\beta+1-i\lambda)}\, {}_2\phi_1\left[\begin{matrix} q^{\frac12(\alpha+\beta+1-i\lambda)}, q^{\frac12(-\alpha+\beta+1-i\lambda)} \\ q^{-i\lambda+1}\end{matrix}; q, q^{1-\beta}\, z^{-1}\right], \quad \lambda\notin -i\mathbb{N}+2\pi\mathbb{Z}^q\log e.$$

Identity (2.2) becomes

$$\begin{aligned}
&{}_2\phi_1\left[\begin{matrix} q^{\frac12(\alpha+\beta+1+i\lambda)}, q^{\frac12(\alpha+\beta+1-i\lambda)} \\ q^{\alpha+1}\end{matrix}; q, z\right] \\
&\quad= \frac{(q^{\frac12(\alpha-\beta+1-i\lambda)}, q^{\frac12(\alpha+\beta+1-i\lambda)}; q)_\infty}{(q^{\alpha+1}, q^{-i\lambda}; q)_\infty}\, \frac{(q^{\frac12(\alpha+\beta+1+i\lambda)}z, q^{\frac12(-\alpha-\beta+1-i\lambda)}z^{-1}; q)_\infty}{(z, qz^{-1}; q)_\infty} \\
&\qquad\qquad \times {}_2\phi_1\left[\begin{matrix} q^{\frac12(\alpha+\beta+1+i\lambda)}, q^{\frac12(-\alpha+\beta+1+i\lambda)} \\ q^{i\lambda+1}\end{matrix}; q, q^{1-\beta}\, z^{-1}\right] \\
&\quad+ \frac{(q^{\frac12(\alpha-\beta+1+i\lambda)}, q^{\frac12(\alpha+\beta+1+i\lambda)}; q)_\infty}{(q^{\alpha+1}, q^{i\lambda}; q)_\infty}\, \frac{(q^{\frac12(\alpha+\beta+1-i\lambda)}z, q^{\frac12(-\alpha-\beta+1+i\lambda)}z^{-1}; q)_\infty}{(z, qz^{-1}; q)_\infty} \\
&\qquad\qquad \times {}_2\phi_1\left[\begin{matrix} q^{\frac12(\alpha+\beta+1-i\lambda)}, q^{\frac12(-\alpha+\beta+1-i\lambda)} \\ q^{-i\lambda+1}\end{matrix}; q, q^{1-\beta}\, z^{-1}\right],
\end{aligned} \tag{2.4}$$

provided that $z\neq 0$, $|\arg(-z)|<\pi$, $\lambda\notin i\mathbb{Z}+2\pi\mathbb{Z}^q\log e$.

Third form. Let $\gamma\in\mathbb{R}$, put $v(n) := U(-q^{n+\gamma})$ $(n\in\mathbb{Z})$ and multiply both sides of (2.3) by $q^{-\frac12(\alpha+\beta+1)-\gamma-n}$. Then equation (2.3) takes the form

$$\begin{aligned}
&(q^{\frac12(\alpha-\beta+1)-\gamma-n} + q^{\frac12(\alpha+\beta+1)})\, v(n+1) - (q^{\alpha+1}+q)\, q^{-\frac12(\alpha+\beta+1)-\gamma-n}\, v(n) \\
&\qquad + (q^{-\frac12(\alpha+\beta+1)-\gamma-n+1} + q^{-\frac12(\alpha+\beta+1)})\, v(n-1) = (q^{\frac12 i\lambda} + q^{-\frac12 i\lambda})\, v(n).
\end{aligned} \tag{2.5}$$

We write the four solutions of (2.5) corresponding to U_1, U_2, U_3, U_4 as $\phi_\lambda^{(\alpha,\beta,\gamma)}(n)$, $\psi_\lambda^{(\alpha,\beta,\gamma)}(n)$, $\Phi_\lambda^{(\alpha,\beta,\gamma)}(n)$ and $\Phi_{-\lambda}^{(\alpha,\beta,\gamma)}(n)$. Their definitions are

$$\phi_\lambda^{(\alpha,\beta,\gamma)}(n) := {}_2\phi_1\left[\begin{matrix} q^{\frac12(\alpha+\beta+1+i\lambda)}, q^{\frac12(\alpha+\beta+1-i\lambda)} \\ q^{\alpha+1}\end{matrix}; q, -q^{n+\gamma}\right],$$

$$\psi_\lambda^{(\alpha,\beta,\gamma)}(n) := q^{-\alpha n}\, {}_2\phi_1\left[\begin{matrix} q^{\frac12(-\alpha+\beta+1+i\lambda)}, q^{\frac12(-\alpha+\beta+1-i\lambda)} \\ q^{-\alpha+1}\end{matrix}; q, -q^{n+\gamma}\right], \quad \alpha\notin\mathbb{N},$$

$$\Phi_\lambda^{(\alpha,\beta,\gamma)}(n) := q^{-\frac12(\alpha+\beta+1+i\lambda)n}\, {}_2\phi_1\left[\begin{matrix} q^{\frac12(\alpha+\beta+1+i\lambda)}, q^{\frac12(-\alpha+\beta+1+i\lambda)} \\ q^{i\lambda+1}\end{matrix}; q, -q^{-n-\gamma-\beta+1}\right],$$

$$\lambda\notin -i\mathbb{N}+2\pi\mathbb{Z}^q\log e.$$

Identity (2.4) takes the form

$$\phi_\lambda^{(\alpha,\beta,\gamma)}(n) = c_{\alpha,\beta,\gamma}(\lambda)\,\Phi_\lambda^{(\alpha,\beta,\gamma)}(n) + c_{\alpha,\beta,\gamma}(-\lambda)\,\Phi_{-\lambda}^{(\alpha,\beta,\gamma)}(n),$$

where

$$c_{\alpha,\beta,\gamma}(\lambda) := \frac{(q^{\frac12(\alpha-\beta+1-i\lambda)}, q^{\frac12(\alpha+\beta+1-i\lambda)}; q)_\infty}{(q^{\alpha+1}, q^{-i\lambda}; q)_\infty}\,\frac{(-q^{\frac12(\alpha+\beta+1+i\lambda)+\gamma}, -q^{1-\frac12(\alpha+\beta+1+i\lambda)-\gamma}; q)_\infty}{(-q^\gamma, -q^{1-\gamma}; q)_\infty}$$

and $\lambda \notin i\mathbb{Z} + 2\pi\mathbb{Z}^q \log e$.

Fourth form. Put $u(n) := w(n)v(n)$ where $w(n) > 0$ and

$$\begin{aligned} w(n)^2 :=& q^{n(\alpha+1)}\,\frac{(-q^{\gamma+n}, -q^{1-\beta-\gamma}, -q^{\beta+\gamma}; q)_\infty}{(-q^{\beta+\gamma+n}, -q^{1-\gamma}, -q^\gamma; q)_\infty} \\ =& q^{n(\alpha+\beta+1)}\,\frac{(-q^{-n+1-\beta-\gamma}; q)_\infty}{(-q^{-n+1-\gamma}; q)_\infty}\,. \end{aligned}$$

Then

$$w(n) \sim \begin{cases} \text{const.}\, q^{\frac12 n(\alpha+1)} & \text{as } n \to \infty, \\ q^{\frac12 n(\alpha+\beta+1)} & \text{as } n \to -\infty. \end{cases}$$

Now equation (2.5) takes the form

$$(Lu)(n) := a(n)\,u(n+1) + b(n)\,u(n) + a(n-1)\,u(n-1) = (q^{\frac12 i\lambda} + q^{-\frac12 i\lambda})\,u(n), \tag{2.6}$$

where

$$\begin{aligned} a(n) &:= (1 + q^{-\beta-\gamma-n})^{\frac12}\,(1 + q^{-\gamma-n})^{\frac12}, \\ b(n) &:= -q^{-\frac12(\alpha+\beta+1)-\gamma+1}\,(1 + q^\alpha)\,q^{-n}. \end{aligned}$$

Thus the operator L in (2.6) has the form of L in (1.2) and assumption 1) of the previous section is satisfied.

The solutions of (2.6) determined by their asymptotic behaviour as $n \to \infty$ are

$$\begin{aligned} u_1(n) &:= w(n)\,\phi_\lambda^{(\alpha,\beta,\gamma)}(n) \sim \text{const.}\, q^{\frac12 n(\alpha+1)} \quad \text{as } n \to \infty, \\ u_2(n) &:= w(n)\,\psi_\lambda^{(\alpha,\beta,\gamma)}(n) \sim \text{const.}\, q^{\frac12 n(-\alpha+1)} \quad \text{as } n \to \infty \quad (\alpha \notin \mathbb{N}), \end{aligned}$$

while the solutions determined by their asymptotic behaviour as $n \to -\infty$ are

$$\begin{aligned} u_3(n) &:= w(n)\,\Phi_\lambda^{(\alpha,\beta,\gamma)}(n) \sim q^{-\frac12 in\lambda} \quad \text{as } n \to -\infty \quad (\lambda \notin -i\mathbb{N} + 2\pi\mathbb{Z}^q \log e), \\ u_4(n) &:= w(n)\,\Phi_{-\lambda}^{(\alpha,\beta,\gamma)}(n) \sim q^{\frac12 in\lambda} \quad \text{as } n \to -\infty \quad (\lambda \notin i\mathbb{N} + 2\pi\mathbb{Z}^q \log e). \end{aligned}$$

We conclude ($\alpha > -1$, $\beta \in \mathbb{R}$, $\lambda \in \mathbb{C}$):

$$\begin{aligned} & w\,\phi_\lambda^{(\alpha,\beta,\gamma)} \text{ is } L^2 \text{ at } \infty\,, \\ \text{for } \alpha \notin \mathbb{N}\text{:} \quad & w\,\psi_\lambda^{(\alpha,\beta,\gamma)} \text{ is } L^2 \text{ at } \infty \iff \alpha < 1, \\ \text{for } \lambda \notin -i\mathbb{N} + 2\pi\mathbb{Z}^q \log e\text{:} \quad & w\,\Phi_\lambda^{(\alpha,\beta,\gamma)} \text{ is } L^2 \text{ at } -\infty \iff \operatorname{Im}\lambda < 0. \end{aligned}$$

Thus, in the generic case, assumptions 2) and 3) of the previous section are also satisfied. For the self-adjoint extensions of L we have possibility (i) of the previous section, except if $-1<\alpha<1$ for which possibility (ii) holds.

Let $[u,v](n)$ be the Wronskian of two functions u and v on $\mathbb{Z}$, as defined by (1.3). This is independent of n if u and v are both solutions of (2.6). Hence

$$\begin{aligned}
&[w\,\Phi_\lambda^{(\alpha,\beta,\gamma)}, w\,\phi_\lambda^{(\alpha,\beta,\gamma)}]\\
&\quad= c_{\alpha,\beta,\gamma}(-\lambda)\,[w\,\Phi_\lambda^{(\alpha,\beta,\gamma)}, w\,\Phi_{-\lambda}^{(\alpha,\beta,\gamma)}]\\
&\quad= c_{\alpha,\beta,\gamma}(-\lambda)\,\lim_{n\to-\infty}[w\,\Phi_\lambda^{(\alpha,\beta,\gamma)}, w\,\Phi_{-\lambda}^{(\alpha,\beta,\gamma)}](n)\\
&\quad= c_{\alpha,\beta,\gamma}(-\lambda)\,\lim_{n\to-\infty} a(n)\,\big((w\Phi_\lambda^{(\alpha,\beta,\gamma)})(n+1)\,(w\Phi_{-\lambda}^{(\alpha,\beta,\gamma)})(n)-(w\Phi_\lambda^{(\alpha,\beta,\gamma)})(n)\,(w\Phi_{-\lambda}^{(\alpha,\beta,\gamma)})(n+1)\big)\\
&\quad= c_{\alpha,\beta,\gamma}(-\lambda)\,\lim_{n\to-\infty}(q^{-\frac12(n+1)i\lambda}\,q^{\frac12 ni\lambda}-q^{-\frac12 ni\lambda}\,q^{\frac12(n+1)i\lambda})\\
&\quad= c_{\alpha,\beta,\gamma}(-\lambda)\,(q^{-\frac12 i\lambda}-q^{\frac12 i\lambda}).
\end{aligned}$$

Thus we have proved:

$$[w\,\Phi_\lambda^{(\alpha,\beta,\gamma)}, w\,\phi_\lambda^{(\alpha,\beta,\gamma)}]=c_{\alpha,\beta,\gamma}(-\lambda)\,(q^{-\frac12 i\lambda}-q^{\frac12 i\lambda}). \tag{2.7}$$

3. The Plancherel formula for the q-Jacobi transform

Let μ run from $-\infty$ to $+\infty$ over the real axis. With the parametrization

$$\mu=q^{\frac12 i\lambda}+q^{-\frac12 i\lambda} \tag{3.1}$$

this trajectory can be put in one-to-one correspondence with a broken line in the complex λ-plane falling apart into three parts as given below. Put

$$a:=\frac{2\pi}{\log q^{-1}}\,.$$

1) μ runs from $-\infty$ to -2 $\iff$ λ runs from $a+i\infty$ to a over the line $\operatorname{Re}\lambda=a$.
2) μ runs from -2 to 2 $\iff$ λ runs from a to 0 over the line $\operatorname{Im}\lambda=0$.
3) μ runs from 2 to $+\infty$ $\iff$ λ runs from 0 to $+i\infty$ over the line $\operatorname{Re}\lambda=0$.

Note that $\operatorname{Im}\mu<0$ in the infinite rectangle $\{\lambda\in\mathbb{C}\mid \operatorname{Im}\lambda>0,\ 0<\operatorname{Re}\lambda<a\}$ (to the right of the contour in the λ-plane) and that $\operatorname{Im}\mu>0$ in the three infinite rectangles $\{\lambda\in\mathbb{C}\mid \operatorname{Im}\lambda>0,\ a<\operatorname{Re}\lambda<2a\}$, $\{\lambda\in\mathbb{C}\mid \operatorname{Im}\lambda<0,\ 0<\operatorname{Re}\lambda<a\}$ and $\{\lambda\in\mathbb{C}\mid \operatorname{Im}\lambda>0,\ -a<\operatorname{Re}\lambda<0\}$ (to the left of the contour in the λ-plane).

We consider first part 2) of the contour. Write $\lambda[\mu]$ for λ corresponding to μ on this contour, also holomorphically extended to a neighbourhood of (μ_1,μ_2), where $-2<\mu_1<\mu_2<2$.

Let $f,g\in C_c(\mathbb{Z})$, $\mu_1\le\mu\le\mu_2$, $\varepsilon>0$. We suppose α,β,γ fixed and suppress these indices. Then

$$\begin{aligned}
&((\mu\pm i\varepsilon-L)^{-1}f,g)\\
&\quad=\sum_{n\le m}\frac{w(n)\,\Phi_{\pm\lambda[\mu\pm i\varepsilon]}(n)\,w(m)\,\phi_{\lambda[\mu\pm i\varepsilon]}(m)\,f(n)\,\overline{g(m)}}{[w\Phi_{\pm\lambda[\mu\pm i\varepsilon]},w\phi_{\lambda[\mu\pm i\varepsilon]}]}
\end{aligned}$$

$$+\sum_{n>m} \frac{w(n)\,\phi_{\lambda[\mu\pm i\varepsilon]}(n)\,w(m)\,\Phi_{\pm\lambda[\mu\pm i\varepsilon]}(m)\,f(n)\,\overline{g(m)}}{[w\Phi_{\pm\lambda[\mu\pm i\varepsilon]}, w\phi_{\lambda[\mu\pm i\varepsilon]}]}$$

$$= \sum_{n\le m} \frac{\Phi_{\pm\lambda[\mu\pm i\varepsilon]}(n)\,\phi_{\lambda[\mu\pm i\varepsilon]}(m)}{[w\Phi_{\pm\lambda[\mu\pm i\varepsilon]}, w\phi_{\lambda[\mu\pm i\varepsilon]}]}\,w(n)\,w(m)\,\frac{f(n)\,\overline{g(m)}+f(m)\,\overline{g(n)}}{2}\,.$$

$$= \sum_{n\le m} \frac{\Phi_{\pm\lambda[\mu\pm i\varepsilon]}(n)\,\phi_{\lambda[\mu\pm i\varepsilon]}(m)}{c(\mp\lambda[\mu\pm i\varepsilon])\,(q^{\mp\frac{1}{2}i\lambda[\mu\pm i\varepsilon]}-q^{\pm\frac{1}{2}i\lambda[\mu\pm i\varepsilon]})}\,w(n)\,w(m)\,\frac{f(n)\,\overline{g(m)}+f(m)\,\overline{g(n)}}{2}\,.$$

Hence

$$\lim_{\varepsilon\downarrow 0}\big(((\mu+i\varepsilon-L)^{-1}f,g)-((\mu-i\varepsilon-L)^{-1}f,g)\big)$$

$$= \sum_{n\le m} \frac{\big(c(\lambda[\mu])\,\Phi_{\lambda[\mu]}(n)+c(-\lambda[\mu])\,\Phi_{-\lambda[\mu]}(n)\big)\,\phi_{\lambda[\mu]}(m)}{c(\lambda[\mu])\,c(-\lambda[\mu])\,(q^{-\frac{1}{2}i\lambda[\mu]}-q^{\frac{1}{2}i\lambda[\mu]})}\,w(n)\,w(m)\,\frac{f(n)\,\overline{g(m)}+f(m)\,\overline{g(n)}}{2}$$

$$= \sum_{n\le m} \frac{\phi_{\lambda[\mu]}(n)\,\phi_{\lambda[\mu]}(m)}{c(\lambda[\mu])\,c(-\lambda[\mu])\,(q^{-\frac{1}{2}i\lambda[\mu]}-q^{\frac{1}{2}i\lambda[\mu]})}\,w(n)\,w(m)\,\frac{f(n)\,\overline{g(m)}+f(m)\,\overline{g(n)}}{2}$$

$$= \frac{\Big(\sum_{n\in\mathbb{Z}} f(n)\,\phi_{\lambda[\mu]}(n)\,w(n)\Big)\,\overline{\Big(\sum_{m\in\mathbb{Z}} g(m)\,\phi_{\lambda[\mu]}(m)\,w(m)\Big)}}{c(\lambda[\mu])\,c(-\lambda[\mu])\,(q^{-\frac{1}{2}i\lambda[\mu]}-q^{\frac{1}{2}i\lambda[\mu]})}\,.$$

Since (3.1) implies

$$\frac{d\mu}{d\lambda} = \tfrac{1}{2}i\log(q^{-1})\,(q^{-\frac{1}{2}i\lambda}-q^{\frac{1}{2}i\lambda}),$$

it follows that

$$\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\mu_1}^{\mu_2}[((\mu-i\varepsilon-L)^{-1}f,g)-((\mu+i\varepsilon-L)^{-1}f,g)]\,d\mu$$

$$= \frac{\log q^{-1}}{4\pi}\int_{\lambda[\mu_2]}^{\lambda[\mu_1]}\Big(\sum_{n\in\mathbb{Z}} f(n)\,\phi_\lambda(n)\,w(n)\Big)\,\overline{\Big(\sum_{m\in\mathbb{Z}} g(m)\,\phi_\lambda(m)\,w(m)\Big)}\,\frac{d\lambda}{c(\lambda)\,c(-\lambda)}\,.$$

Hence, by Theorem 1.5, we have for $-2\le\mu_1<\mu_2\le 2$ that

$$E_{f,g}((\mu_1,\mu_2)) = \frac{\log q^{-1}}{4\pi}\int_{\lambda[\mu_2]}^{\lambda[\mu_1]}\Big(\sum_{n\in\mathbb{Z}} f(n)\,\phi_\lambda(n)\,w(n)\Big)\,\overline{\Big(\sum_{m\in\mathbb{Z}} g(m)\,\phi_\lambda(m)\,w(m)\Big)}\,\frac{d\lambda}{c(\lambda)\,c(-\lambda)}\,.$$

Let now $\mu<-2$ or $\mu>2$. Then

$$((\mu\pm i\varepsilon-L)^{-1}f,g)$$

$$= \sum_{n\le m} \frac{w(n)\,\Phi_{-\lambda[\mu\pm i\varepsilon]}(n)\,w(m)\,\phi_{\lambda[\mu\pm i\varepsilon]}(m)\,f(n)\,\overline{g(m)}}{[w\Phi_{-\lambda[\mu\pm i\varepsilon]}, w\phi_{\lambda[\mu\pm i\varepsilon]}]}$$

$$+\sum_{n>m} \frac{w(n)\,\phi_{\lambda[\mu\pm i\varepsilon]}(n)\,w(m)\,\Phi_{-\lambda[\mu\pm i\varepsilon]}(m)\,f(n)\,\overline{g(m)}}{[w\Phi_{-\lambda[\mu\pm i\varepsilon]}, w\phi_{\lambda[\mu\pm i\varepsilon]}]}$$

$$= \sum_{n\le m} \frac{\Phi_{-\lambda[\mu\pm i\varepsilon]}(n)\,\phi_{\lambda[\mu\pm i\varepsilon]}(m)}{[w\Phi_{-\lambda[\mu\pm i\varepsilon]}, w\phi_{\lambda[\mu\pm i\varepsilon]}]}\,w(n)\,w(m)\,\frac{f(n)\,\overline{g(m)}+f(m)\,\overline{g(n)}}{2}\,.$$

$$= \sum_{n\le m} \frac{\Phi_{-\lambda[\mu\pm i\varepsilon]}(n)\,\phi_{\lambda[\mu\pm i\varepsilon]}(m)}{c(\lambda[\mu\pm i\varepsilon])\,(q^{\frac{1}{2}i\lambda[\mu\pm i\varepsilon]}-q^{-\frac{1}{2}i\lambda[\mu\pm i\varepsilon]})}\,w(n)\,w(m)\,\frac{f(n)\,\overline{g(m)}+f(m)\,\overline{g(n)}}{2}\,.$$

Hence

$$\lim_{\varepsilon\downarrow 0}\big(((\mu+i\varepsilon-L)^{-1}f,g)-((\mu-i\varepsilon-L)^{-1}f,g)\big)=0 \quad \text{if } c(\lambda[\mu])\neq 0.$$

We have

$$c(it)=\frac{(q^{\frac12(\alpha-\beta+1+t)},q^{\frac12(\alpha+\beta+1+t)};q)_\infty}{(q^{\alpha+1},-q^\gamma,-q^{1-\gamma};q)_\infty}\,\frac{(-q^{\frac12(\alpha+\beta+1+2\gamma-t)},-q^{1-\frac12(\alpha+\beta+1+2\gamma-t)};q)_\infty}{(q^t;q)_\infty}\,.$$

For $t>0$ this is regular and has no zeros if

$$\alpha-\beta+1>0,\quad \alpha+\beta+1>0.$$

Let us assume these inequalities for α,β. Then we conclude that

$$E_{f,g}((2,\infty))=0.$$

Put

$$\sigma:=\alpha+\beta+1+2\gamma.$$

We have

$$c(a+it)=\frac{(-q^{\frac12(\alpha-\beta+1+t)},-q^{\frac12(\alpha+\beta+1+t)};q)_\infty}{(q^{\alpha+1},-q^\gamma,-q^{1-\gamma};q)_\infty}\,\frac{(q^{\frac12(\sigma-t)},q^{1-\frac12(\sigma-t)};q)_\infty}{(q^t;q)_\infty}\,.$$

For $t>0$ this is regular and it vanishes iff $t\in\{\sigma+2k>0\mid k\in\mathbb{Z}\}$. All zeros are simple. We may add an integer to γ without essential changes. So choose γ such that

$$0<\alpha+\beta+1+2\gamma\le 2.$$

Then $c(a+it)$ vanishes for $t>0$ iff

$$t\in\sigma+2\mathbb{Z}_+.$$

Take $\mu_1<\mu_2\le -2$ such that $c(\lambda[\mu])\neq 0$ for $\mu\in[\mu_1,\mu_2]$ except for precisely one $\mu_0\in(\mu_1,\mu_2)$. Put

$$a+i(\sigma+2k_0)=\lambda_0:=\lambda[\mu_0].$$

Then

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\mu_1}^{\mu_2}[((\mu-i\varepsilon-L)^{-1}f,g)-((\mu+i\varepsilon-L)^{-1}f,g)]\,d\mu\\
&\quad=\frac{1}{2\pi i}\oint_{(\mu_0)}((\mu-L)^{-1}f,g)\,d\mu\\
&\quad=\sum_{n,m\in\mathbb{Z}}f(n)\,w(n)\,\overline{g(m)}\,w(m)\,\frac{1}{2\pi i}\oint_{(\mu_0)}\frac{\Phi_{-\lambda[\mu]}(n)\,\phi_{\lambda[\mu]}(m)}{c(\lambda[\mu])\,(q^{\frac12 i\lambda[\mu]}-q^{-\frac12 i\lambda[\mu]})}\,d\mu\\
&\quad=\sum_{n,m\in\mathbb{Z}}f(n)\,w(n)\,\overline{g(m)}\,w(m)\,\frac{-\log q^{-1}}{4\pi}\oint_{(\lambda_0)}\frac{\Phi_{-\lambda}(n)\,\phi_\lambda(m)}{c(\lambda)}\,d\lambda\\
&\quad=\sum_{n,m\in\mathbb{Z}}f(n)\,w(n)\,\overline{g(m)}\,w(m)\,\frac{-\log q^{-1}}{4\pi}\,\frac{\phi_{\lambda_0}(n)\,\phi_{\lambda_0}(m)}{c(-\lambda_0)}\,2\pi i\,\mathrm{Res}_{\lambda=\lambda_0}\frac{1}{c(\lambda)}\,.
\end{aligned}$$

Now

$$
\begin{aligned}
&\mathrm{Res}_{\lambda=\lambda_0} \frac{1}{c(\lambda)} \\
&\quad = i\,\mathrm{Res}_{t=\sigma+2k_0} \frac{1}{c(a+it)} \\
&\quad = \frac{i\,(q^{\alpha+1}, -q^{\gamma}, -q^{1-\gamma}, q^{\sigma+2k_0}; q)_\infty}{(-q^{\alpha+\gamma+k_0+1}, -q^{\alpha+\beta+\gamma+1+k_0}, q^{1+k_0}; q)_\infty\,(q^{-k_0}; q)_{k_0}\,(q;q)_\infty}\,\mathrm{Res}_{s=0} \frac{1}{1-q^{-\frac{1}{2}s}} \\
&\quad = \frac{-2i}{\log q^{-1}}\,\frac{(-1)^{k_0}\,q^{\frac{1}{2}k_0(k_0+1)}\,(q^{\alpha+1}, -q^{\gamma}, -q^{1-\gamma}, q^{\sigma+2k_0}; q)_\infty}{(-q^{\alpha+\gamma+k_0+1}, -q^{\alpha+\beta+\gamma+1+k_0}, q, q; q)_\infty}
\end{aligned}
$$

and

$$
\frac{1}{c(-\lambda_0)} = \frac{(q^{\alpha+1}, -q^{\gamma}, -q^{1-\gamma}, q^{-(\sigma+2k_0)}; q)_\infty}{(-q^{-\beta-\gamma-k_0}, -q^{-\gamma-k_0}, q^{\sigma+k_0}, q^{1-\sigma-k_0}; q)_\infty}\,.
$$

Hence

$$
\begin{aligned}
&E_{f,g}((\mu_1,\mu_2)) \\
&= \Big(\sum_{n\in\mathbb{Z}} f(n)\,\phi_{\lambda_0}(n)\,w(n)\Big)\,\overline{\Big(\sum_{m\in\mathbb{Z}} g(m)\,\phi_{\lambda_0}(m)\,w(m)\Big)}\,\frac{1}{c(-\lambda_0)} \\
&\quad \times \frac{(-1)^{k_0+1}\,q^{\frac{1}{2}k_0(k_0+1)}\,(q^{\alpha+1}, -q^{\gamma}, -q^{1-\gamma}, q^{\sigma+2k_0}; q)_\infty}{(-q^{\alpha+\gamma+k_0+1}, -q^{\alpha+\beta+\gamma+1+k_0}, q, q; q)_\infty} \\
&= \left(\frac{(q^{\alpha+1}, -q^{\gamma}, -q^{1-\gamma}; q)_\infty}{(q;q)_\infty}\right)^2\,\Big(\sum_{n\in\mathbb{Z}} f(n)\,\phi_{\lambda_0}(n)\,w(n)\Big)\,\overline{\Big(\sum_{m\in\mathbb{Z}} f(m)\,\phi_{\lambda_0}(m)\,w(m)\Big)} \\
&\quad \times \frac{q^{-(k_0+1)(k_0+\sigma)}\,(1-q^{\sigma+2k_0})}{(-q^{\alpha+\gamma+k_0+1}, -q^{\alpha+\beta+\gamma+1+k_0}, -q^{-\beta-\gamma-k_0}, -q^{-\gamma-k_0}; q)_\infty}\,.
\end{aligned}
$$

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