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Combinatorial aspects of Macdonald and related polynomials

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This is a update (May 11, 2021) of sections 10.9.3 and 10.10 in Chapter 10 in the book *Encyclopedia of special functions: The Askey–Bateman project, Vol. 2: Multivariable special functions*, T. H. Koornwinder and J. V. Stokman (eds.), Cambridge University Press, 2021.

10.9 Other directions

10.9.3 k-Schur functions

For any positive integer k, Lascoux, Lapointe & Morse [19] introduced a family of symmetric functions which depend on a parameter t and reduce to Schur functions when $k = \infty$. These symmetric functions form a basis for a certain subspace of Λ. During the period 2001 – 2015, several other conjecturally equivalent definitions of this intriguing family were introduced; they are now commonly called *k-Schur functions* as in [15], denoted $s_\lambda^{(k)}(X;t)$ ($\lambda \in \Lambda$, $\lambda_1 \leq k$). This and other related conjectures have sparked a large amount of research over the last twenty years; see for example [14, 16, 17].

In a landmark 2019 paper [5], Blasiak, Morse, Pun and Summers proved many of the conjectures about k-Schur functions, by studying a broader family of functions, *Catalan functions*, introduced by Chen and Haiman [8] in association with their study of k-Schur functions. One conjecture that is still open, the main conjecture in [19], is that when the modified Macdonald polynomial $\tilde{H}_\mu(X;q,t)$ is expanded into the k-Schur basis with parameter q, i.e., $\{s_\lambda^{(k)}(X;q)\}$, where $k \geq \mu_1'$, the coefficients are in $\mathbf{N}[q,t]$.

Let the *bandwidth* of a LLT polynomial be the number of dotted diagonal lines which intersect the diagonal of some square in one of the skew shapes in the LLT tuple. For example, for the LLT polynomial in Figure 10.6, the bandwidth is 3, and for the tuple on the right in Figure 10.7, it is 4. It has been suggested that when expanding an LLT polynomial of bandwidth k into the k-Schur function basis $\{s_\lambda^{(k)}(X;q)\}$, the coefficients are in $\mathbf{N}[q]$. By (10.5.6), this refines the conjecture from [19] discussed in the previous paragraph.

k-Schur functions also have other remarkable properties. For example, T. Lam [13] proved a conjecture of M. Shimozono, which says that when $t = 1$ the k-Schur form the Schubert basis for the homology of the loop Grassmannian, a conjecture which was based in part on results in [18].

10.10 Recent developments

The work of Carlsson and Mellit

In August 2015 Carlsson & Mellit posted a preprint on the arXiv which proved the Compositional Shuffle Conjecture of Remark 10.4.8, and which has appeared now in [7]. As corollaries they obtain the first proof of Conjecture 10.4.1 (the combinatorial formula for $\mathrm{Hilb}(\mathrm{DH}_n; q, t)$) and more generally the first proof of the Shuffle Conjecture.

To prove the Compositional Shuffle Conjecture they introduce a new algebraic object, the double Dyck path algebra $\mathbf{A}_{q,t}$, which is closely related to the double affine Hecke algebra. (See [24] for an detailed description of how $\mathbf{A}_{q,t}$ arises as a stable limit of the family of GL_n double affine Hecke algebras). A crucial role in $\mathbf{A}_{q,t}$ is played new operators d_- and d_+, which have combinatorial interpretations involving weighted Dyck paths and parking functions.

There is a wonderful action of $\mathbb{A}_{q,t}$ on elements of Λ with coefficients in $\mathbb{Q}(q,t)[y_1, \ldots, y_k]$. The T_i operators act on monomials in the y_i (as in (10.7.8) with x^λ replaced by y^λ), while the action of $d_+^{(k)}$ and $d_-^{(k)}$ is defined using plethysm. The operators d_+ and d_- are constructed such that, if you start at the end of a Dyck path π and create a sequence of operators $L(\pi)$ by tracing the path backward, prepending d_+ to $L(\pi)$ for E steps and d_- to $L(\pi)$ for N steps, then $L(\pi)$ operating on the constant 1 gives a certain LLT product of single cells. Moreover, if for each EN corner of π you replace the corresponding d_-d_+ contribution to $L(\pi)$ by a factor of $(d_-d_+ - d_+d_-)/(q-1)$, then the resulting sequence $M(\pi)$, acting on the constant 1, will yield $\mathcal{F}_{\zeta^{-1}(\pi)}(X; q)$, where $\mathcal{F}$ and ζ are as in (10.4.3) and Figure 10.16.

Say we have a path π which begins with k N steps followed by an E step. Then $M(\pi)$ will begin with k d_- terms. Letting $M'(\pi)$ denote $M(\pi)$ with these k d_- terms removed, then $M'(\pi)$ applied to 1 will be a sum of symmetric functions in X with coefficients in $\mathbb{Q}(q,t)[y_1, \ldots, y_k]$. Carlsson & Mellit show that certain sums of the $M'(\pi)1$, corresponding to elements π for which $\zeta^{-1}(\pi)$ has touch points $(a_1, a_1), (a_1 + a_2, a_1 + a_2), \ldots, (n, n)$, satisfy a nice recurrence. They also show that the operator ∇ can be expressed using elements of $\mathbb{A}_{q,t}$, and that they can then, by using their commutation relations, prove the Compositional Shuffle Conjecture in two lines.

The Dyck path algebra method has already found other substantial applications. In a sequel to the Carlsson–Mellit paper, Mellit [22] proves the Compositional Rational Shuffle Conjecture from [4], which contains the Rational Shuffle Conjecture and Compositional Shuffle Conjecture, and hence all the conjectures from §10.4, as special cases. His proof starts by assuming the properties of the double Dyck path algebra developed in [7], then introduces some new ideas. In particular he relates actions of toric braids with parking functions, and exploits the known fact that the DAHA_n can be viewed as a quotient of the surface braid group of a torus. One question which the work of Carlsson and Mellit hasn't as of yet shed any light on is the problem of finding a combinatorial expression for the Schur coefficients of the $\mathcal{F}_\pi(X;q,t)$ of (10.4.3).

Another conjecture which has recently been proved by the Dyck path algebra method is the *Delta Conjecture* of Haglund, Remmel & Wilson [11]. For $f \in \Lambda$, let Δ_f be the linear operator defined on the $\tilde{H}_\mu$ basis of modified Macdonald polynomials via $\Delta_f \tilde{H}_\mu(X;q,t) = f[B_\mu(q,t)]\tilde{H}_\mu(X;q,t)$, with $B_\mu(q,t)$ as in (10.3.4). Furthermore let $\Delta'_f \tilde{H}_\mu(X;q,t) = f[B_\mu(q,t) - 1]\tilde{H}_\mu(X;q,t)$. The Delta Conjecture gives an elegant combinatorial formula, in terms of parking functions, for $\Delta'_{e_k} e_n$, for any $0 \leq k \leq n-1$. For $k = n-1$ it reduces to the Shuffle Conjecture. (There are actually two different formulas for the combinatorial side of the Delta Conjecture, the rise version and the valley version. Both these versions are fairly similar, but it is still an open problem to show they are equivalent.) In [10], D'Adderio, Iraci and Wyngaerd introduce a new family of useful operators called Theta operators, and showcase these in a compositional refinement of the Delta Conjecture. Shortly after this D'Adderio and Mellit [9] prove this Compositional Delta Conjecture, which implies the rise version of the Delta Conjecture, by embedding the Theta operators in $\mathbb{A}_{q,t}$.

In another major development, Blasiak, Haiman, Morse, Pun and Seelinger [6] have proved the Extended Delta Conjecture, which generalizes the rise version of the Delta Conjecture in a different way, by giving a combinatorial interpretation for $\Delta_{h_m}\Delta'_{e_k} e_n$, for any $0 \leq k, l$. Their proof uses a completely new method involving the Schiffman algebra and a new view of LLT polynomials as the polynomial part of certain Laurent series. This preprint is the second in a series; in the first they give another proof of the Rational Shuffle Conjecture (in fact a generalization of it involving lines of irrational slope). They have also announced that they can use the method to prove a conjecture of Loehr and Warrington going back to 2008 [21], which gives a combinatorial interpretation, in terms of nested Dyck paths, for ∇ applied to any Schur function.

There are a number of intriguing results and conjectures linking the Delta Conjecture to coinvariant algebras. Haglund, Rhoades & Shimozono [12] introduced a quotient ring whose bigraded character equals the symmetric function described by the combinatorial side of the Delta Conjecture when $t = 0$. The combinatorics of this $t = 0$ case is controlled by ordered set partitions. Moreover, Zabrocki [25] has conjectured that the symmetric function

$$\sum_{k=0}^{n-1} z^{n-k} \Delta'_{e_k} e_n \tag{10.10.1}$$

gives the tri-graded Frobenius characteristic of the *Super Diagonal Coinvariant Ring* SDR_n, defined as

$$\mathrm{SDR}_n = \frac{\mathbf{C}[x_1,\ldots,x_n,y_1,\ldots,y_n,\theta_1,\ldots,\theta_n]}{\mathbf{C}[x_1,\ldots,x_n,y_1,\ldots,y_n,\theta_1,\ldots,\theta_n]^{S_n,+}}, \tag{10.10.2}$$

where the θ_k are so-called fermionic variables (i.e. $\theta_i\theta_j = -\theta_j\theta_i$ for all $1 \leq i,j \leq n$) which commute with the ordinary bosonic commuting x and y variables. The denominator is all functions of positive homogeneous degree which are fixed by the symmetric group under the diagonal action (which permutes the x, y, θ variables in the exact same way). The z-parameter in 10.10.1 corresponds to the degree in the θ variables, and the q, t the degree in the x, y variables as usual. Zabrocki's Conjecture seems to be very difficult, and even the case $t = 0$ of it is still open.

In another direction Sergel [23] has proved the Square Paths Conjecture of Loehr and Warrington [20], which gives a combinatorial interpretation for ∇p_n, by showing it follows from the Compositional Shuffle Conjecture. Her proof uses clever combinatorial manipulations of parking functions. There is also a family of conjectures connected to the combinatorics of the character of diagonal harmonics in several sets of variables under the diagonal action of S_n; see [1, 2, 3].

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