Adaptive wavelet methods for solving operator equations: An overview

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Abstract In [*Math. Comp*, 70 (2001), 27-75] and [*Found. Comput. Math.*, 2(3) (2002), 203-245], Cohen, Dahmen and DeVore introduced adaptive wavelet methods for solving operator equations. These papers meant a break-through in the field, because their adaptive methods were not only proven to converge, but also with a rate better than that of their non-adaptive counterparts in cases where the latter methods converge with a reduced rate due a lacking regularity of the solution. Until then, adaptive methods were usually assumed to converge via a saturation assumption. An exception was given by the work of Dörfler in [*SIAM J. Numer. Anal.*, 33 (1996), 1106–1124], where an adaptive finite element method was proven to converge, with no rate though.

This work contains a complete analysis of the methods from the aforementioned two papers of Cohen, Dahmen and DeVore. Furthermore, we give an overview over the subsequent developments in the field of adaptive wavelet methods. This includes a precise analysis of the near-sparsity of an operator in wavelet coordinates needed to obtain optimal computational complexity; the avoidance of coarsening; quantitative improvements of the algorithms; their generalization to frames; and their application with tensor product wavelet bases which give dimension independent rates.

1 Introduction

1.1 Non-adaptive methods

In this survey, we discuss optimally converging adaptive wavelet methods for solving well-posed linear operator equations

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$$Bu = f.$$

Such methods were introduced by Cohen, Dahmen and DeVore in [CDD01, CDD02]. For wavelet methods in general for solving operator equations, we refer to [Dah97, Coh03, Urb09].

We assume that B is a *boundedly invertible linear operator* between $\mathscr{X}$ and $\mathscr{Y}'$, where $\mathscr{X}$ and $\mathscr{Y}$ are Hilbert spaces. As typical examples, we have in mind linear *partial differential* or *singular integral* equations, in which case $\mathscr{X}$ and $\mathscr{Y}$ are Sobolev spaces or, for non-scalar equations, products of such spaces. We assume that we have Riesz bases $\Psi^{\mathscr{X}} = \{\psi_\lambda^{\mathscr{X}} : \lambda \in \nabla\}$ and $\Psi^{\mathscr{Y}} = \{\psi_\lambda^{\mathscr{Y}} : \lambda \in \nabla\}$ for $\mathscr{X}$ and $\mathscr{Y}$ available, which are of *wavelet type*. In most applications, $\mathscr{X}$ and $\mathscr{Y}$ and $\Psi^{\mathscr{X}}$ and $\Psi^{\mathscr{Y}}$ will be equal.

The adaptive wavelet methodology has been extended to *non-linear* problems, see [CDD03a, DSX00a, CDD03b, CU05, BDS07, BU08]. Such problems, however, will not be discussed in this paper.

A standard *non-adaptive* numerical (wavelet) method for solving $Bu = f$ consists of selecting a Λ from a fixed sequence $\Lambda_0 \subset \Lambda_1 \subset \cdots$ with $\cup_i \Lambda_i = \nabla$, and computing a (quasi-) best approximation u_Λ to u from $\mathrm{span}\{\psi_\lambda^{\mathscr{X}} : \lambda \in \Lambda\}$. The standard choice for Λ_i is the set of all wavelet indices λ with "level" up to i, so that $\mathrm{span}\{\psi_\lambda^{\mathscr{X}} : \lambda \in \Lambda_i\}$ is equal to the span of all "scaling functions" on level i. The counterpart of this wavelet method in a *finite element setting* is the computation of the finite element approximation with respect to an i times *uniformly refined initial mesh*.

Associated to $\mathscr{X}$ and $\Psi^{\mathscr{X}}$, there exists a parameter

$$s_{\max} > 0$$

such that for a suitable choice of $(\Lambda_i)_i$, for *all* $u \in \mathscr{X}$ that are *sufficiently smooth*

$$\|u - u_{\Lambda_i}\|_{\mathscr{X}} \lesssim (\#\Lambda_i)^{-s_{\max}},$$

where this rate $s_{\max}$ *cannot be improved by imposing additional smoothness conditions or by another selection of* $(\Lambda_i)_i$.

For completeness, here and in the remainder of this work, with $C \lesssim D$ we will mean that C can be bounded by a multiple of D, independently of parameters on which C and D may depend. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

Remark 1. There exist $u \in \mathscr{X}$ for which a rate better than $s_{\max}$ can be realized. Indeed, if u happens to have a finite representation in $\Psi^{\mathscr{X}}$, or if it is exceptionally close to such a function, then with a suitable choice of $(\Lambda_i)_i$ any rate can be realized. Since such cases are exceptional, we may ignore them in the further considerations.

Typically, the parameter $s_{\max}$ is a function of the *order* of the wavelet basis $\Psi^{\mathscr{X}}$, the order of smoothness that is measured in the (Sobolev) space $\mathscr{X}$, and the dimension of the underlying domain.

Example 1. Let $\mathscr{X} = H^m(\Omega)$ where Ω is a bounded domain in $\mathbb{R}^n$, and let $\Psi^{\mathscr{X}}$ be a standard wavelet basis of order $d > m$. Then

$$s_{\max} = \frac{d-m}{n},$$

and with Λ_i being the set of all wavelet indices λ with levels up to i, this rate $s_{\max}$ is realized for $u \in H^d(\Omega)$. More generally, for $s \in (0, s_{\max}]$ and $u \in H^{sn+m}(\Omega)$, a rate s is realized. This result is sharp in the sense that for $\varepsilon > 0$, there exists no choice $(\Lambda_i)_i$ such that the rate s is realized for all $u \in H^{sn+m-\varepsilon}(\Omega)$.

1.2 Adaptive methods

Even for smooth right-hand sides f, in many applications the smoothness conditions on u to realize the optimal rate $s_{\max}$ with the standard choice of $(\Lambda_i)_i$ are not fulfilled. Typical examples are boundary value problems on non-smooth domains, where corners, edges etc. induce singularities in the solution. For simple model examples, the precise knowledge of these singularities enables one to select a sequence $(\Lambda_i)_i$ such that the optimal rate $s_{\max}$ is retrieved, assuming f is sufficiently smooth. Such a sequence $(\Lambda_i)_i$ involves local refinements towards the boundary, i.e., the addition of extra wavelets with supports near the boundary. For more general problems, however, such an a priori selection of $(\Lambda_i)_i$ is not feasible.

The topic of this work are *adaptive* (wavelet) methods. With these methods, the expansion of Λ_i to Λ_{i+1} is made based on information provided by u_{Λ_i}. In this way, the sequences $(\Lambda_i)_i$ and $(u_{\Lambda_i})_i$ depend (non-linearly) on u.

The method from [CDD01] is similar to an adaptive finite element method in the sense that information from an a posteriori error estimator is used to guide the expansion of Λ_i to Λ_{i+1} such that the error is reduced, at the expense of a (quasi-) minimal increase in the cardinality.

The idea behind the method from [CDD02] is the application of *some* iterative method to construct $(\Lambda_i)_i$ such that $(u_{\Lambda_i})_i$ converges (linearly) to u. Here a (quasi-) optimal balance between support sizes and accuracy is realized by, after each fixed number of iterations, removing small coefficients from the current approximation, a process known as *coarsening*.

The key to the development of adaptive wavelet methods is the fact that for a large class of operators B, its bi-infinite stiffness or system matrix with respect to suitable wavelet bases is close to a sparse matrix. Here suitable means that the wavelets are *sufficiently smooth* and have sufficiently many *vanishing moments*. Thanks to this near-sparsity, given an approximation $\tilde{\mathbf{u}} \in \ell_0$ to $\mathbf{u}$, its generally infinitely supported residual can be accurately approximated at relatively low cost. This fact allows to run an iterative scheme to the bi-infinite matrix vector equation, in which residuals are computed approximately, essentially being the scheme from [CDD02], or to use the approximate residual as an a posteriori error estimator as in the scheme proposed in [CDD01].

1.3 Best N-term approximation and approximation classes

As a benchmark for these adaptive methods, we consider a (quasi-) *best possible choice* of $(\Lambda_i)_i$ *depending on* u, where we assume to have full knowledge of this function, and thus of its expansion in the wavelet basis $\Psi^{\mathscr{X}}$. Given $u = \mathbf{u}^\top \Psi^{\mathscr{X}} := \sum_{\lambda\in\nabla} \mathbf{u}_\lambda \psi_\lambda^{\mathscr{X}}$ and an approximation $v = \mathbf{v}^\top \Psi^{\mathscr{X}}$, because $\Psi^{\mathscr{X}}$ is a Riesz basis, it holds that

$$\|u - v\|_{\mathscr{X}} \eqsim \|\mathbf{u} - \mathbf{v}\|, \tag{1}$$

where $\|\cdot\|$ is the norm on $\ell_2 = \ell_2(\nabla) := \{\mathbf{v} : \nabla \to \mathbb{R} : \sum_{\lambda\in\nabla} |v_\lambda|^2 < \infty\}$. The subspace of finitely supported $\mathbf{v} \in \ell_2$ will be denoted as ℓ_0. As a consequence of (1), given a budget $N \in \mathbb{N}$, a (quasi-) best choice for an approximation $v = \mathbf{v}^\top \Psi^{\mathscr{X}} \in \mathscr{X}$ with $\#\operatorname{supp}\mathbf{v} \le N$ is to take $\mathbf{v}$ to be a *best N-term approximation* to $\mathbf{u}$, i.e., a vector with at most N non-zero coefficients that has ℓ_2-distance to $\mathbf{u}$ not larger than any vector with support length $\le N$. Obviously, such a best N-term approximation to $\mathbf{u}$, denoted as $\mathbf{u}_N$, coincides with $\mathbf{u}$ on those N positions where $\mathbf{u}$ has its N largest coefficients in modulus, and is zero elsewhere. Note that $\mathbf{u}_N$ is not necessarily unique.

All $\mathbf{u}$ whose best N-term approximations converge with rate $s > 0$ are collected in the approximation class

$$\mathscr{A}^s (= \mathscr{A}^s_\infty) := \{\mathbf{u} \in \ell_2 : \|\mathbf{u}\|_{\mathscr{A}^s} := \sup_{\varepsilon>0} \varepsilon \times [\min\{N \in \mathbb{N}_0 : \|\mathbf{u} - \mathbf{u}_N\|_{\ell_2} \le \varepsilon\}]^s < \infty\}. \tag{2}$$

Indeed, one may verify that $\|\mathbf{u}\|_{\mathscr{A}^s} \eqsim \sup_{N\in\mathbb{N}_0} (N+1)^s \|\mathbf{u} - \mathbf{u}_N\|$, being the commonly used definition of the (quasi-) norm on $\mathscr{A}^s$. Given $\mathbf{u} \in \mathscr{A}^s$ and $\varepsilon > 0$, the smallest N such that $\|\mathbf{u} - \mathbf{u}_N\| \le \varepsilon$ satisfies

$$N \le \varepsilon^{-1/s} \|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}, \tag{3}$$

which bound is generally sharp. Since for $\varepsilon < \|\mathbf{u}\|$, the value of N in the definition of $\|\mathbf{u}\|_{\mathscr{A}^s}$ is positive, furthermore note that

$$\|\mathbf{u}\|_{\mathscr{A}^s} \ge \sup_{0<\varepsilon<\|\mathbf{u}\|} \varepsilon = \|\mathbf{u}\|.$$

As discussed in Remark 1, for $s > s_{\max}$, the class $\mathscr{A}^s$, although not empty, is not relevant. For any $s \in (0, s_{\max}]$, the class $\mathscr{A}^s$ is much larger than the class of (representations of) functions that can be approximated with rate s for any fixed choice of $(\Lambda_i)_i$.

Example 2. In the situation of Example 1, with wavelets that are sufficiently smooth, for $s \in (0, \frac{d-m}{n})$ and with $\tau := (\frac{1}{2} + s)^{-1}$, (representations of) all functions in the Besov space $B^{sn+t}_\tau(L_\tau(\Omega))$ are contained in $\mathscr{A}^s$. Coarsely speaking, $B^{sn+t}_\tau(L_\tau(\Omega))$ is the space of all functions having $sn+t$ orders of smoothness in $L_\tau(\Omega)$, which space, since $\tau < 2$, is thus (much) larger than $H^{sn+t}(\Omega)$ with an increasing difference with growing s. For details about the relation between approximation classes and Besov spaces we refer to [DeV98, Coh03]. For several boundary value problems, assuming

a sufficiently smooth right-hand side f, it has been proved that the solution u has a much higher regularity in this scale of Besov spaces than in the scale of Sobolev spaces $(H^{sn+m}(\Omega))_s$, see [DD97, Dah99].

In view of the definition of $\mathscr{A}^s$, in particular (3), we will call an *adaptive wavelet method to be (quasi-) optimal* if

> *whenever u has a representation $u = \mathbf{u}^\top \Psi^{\mathscr{X}}$ with $\mathbf{u} \in \mathscr{A}^s$ for some $s \in (0, s_{\max}]$, then given a tolerance $\varepsilon > 0$, it produces an approximation $\mathbf{v} \in \ell_0$ with $\|\mathbf{u} - \mathbf{v}\| \leq \varepsilon$ and $\#\operatorname{supp}\mathbf{v} \lesssim \varepsilon^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}$, at the cost of a number of arithmetic operations that can be bounded by some absolute multiple of the same expression.*

1.4 Structure of the paper

The remainder of this work is organized as follows: In Sect. 2, we reformulate well-posed operator equations as bi-infinite matrix vector equations, and give some typical examples of such operator equations.

In Sect. 3 and 4, we define the adaptive wavelet schemes from [CDD02] and [CDD01], respectively, and prove their (quasi-) optimality. Note that we reverse the order in which these schemes were proposed.

The analysis from Sect. 3 and 4 applies under the assumption that the operator B in wavelet coordinates can be sufficiently well approximated by sparse matrices that are computable in linear complexity. In Sect. 5, we verify this assumption for a class of partial differential operators.

In Sect. 6, we discuss the generalization of the adaptive wavelet approach to the case that instead of Riesz bases we only have frames available. Our motivation will be that on general non-product domains, the construction of (wavelet) frames is easier than that of (wavelet) Riesz bases.

Finally, in Sect. 7, the application of tensor product wavelet bases is discussed. Approximation using such bases does not suffer from the so-called curse of dimensionality. An application of those bases is given by the (quasi-) optimal simultaneously space-time adaptive solution of parabolic initial boundary value problems.

1.5 Some properties of the (quasi-) norms $\|\cdot\|_{\mathscr{A}^s}$

We end this section by recalling two known properties of the $\|\cdot\|_{\mathscr{A}^s}$-norm (cf. e.g. [DeV98]) that will be used in Sect. 3 and 4. In order to keep the presentation in these sections self-contained, we include their proofs.

Lemma 1. *For $\mathbf{v} \in \mathscr{A}^s$ and $\mathbf{w} \in \ell_0$,*

$$\|\mathbf{w}\|_{\mathscr{A}^s} \leq 2\max(\|\mathbf{v}\|_{\mathscr{A}^s}, (\#\operatorname{supp}\mathbf{w})^s\|\mathbf{v} - \mathbf{w}\|).$$

Proof. For $\varepsilon \in (0, 2\|\mathbf{v}-\mathbf{w}\|]$, the approximation of $\mathbf{w}$ by itself shows that the expression εN^s in the definition of $\|\mathbf{w}\|_{\mathscr{A}^s}$ is bounded by $2\|\mathbf{v}-\mathbf{w}\|(\#\mathrm{supp}\mathbf{w})^s$.

For $\varepsilon \geq 2\|\mathbf{v}-\mathbf{w}\|$, let N be the smallest integer such that $\|\mathbf{v}-\mathbf{v}_N\| \leq \frac{\varepsilon}{2}$. Then $\|\mathbf{w}-\mathbf{v}_N\| \leq \varepsilon$ and $\varepsilon N^s = 2\frac{\varepsilon}{2}N^s \leq 2\|\mathbf{v}\|_{\mathscr{A}^s}$. □

Lemma 2. *For $s > 0$ and with $\tau := (\frac{1}{2}+s)^{-1}$,*

$$\|\mathbf{v}\|_{\mathscr{A}^s} \eqsim \sup_{\eta>0} \eta \times \#\{\lambda \in \nabla : |v_\lambda| > \eta\}^{1/\tau}, \quad (\mathbf{v} \in \mathscr{A}^s). \tag{4}$$

Proof. Let us denote the expression at the right-hand side of (4) as $|||\mathbf{v}|||_{\mathscr{A}^s}$. Let N be the smallest integer such that the entries of $\mathbf{v}-\mathbf{v}_N$ are in modulus not larger than η. Then $N \leq (|||\mathbf{v}|||_{\mathscr{A}^s}\eta^{-1})^\tau$, and

$$\begin{aligned}\|\mathbf{v}-\mathbf{v}_N\| &\leq \sum_{k=0}^{\infty} 2^{-k}\eta\sqrt{\#\{\lambda \in \nabla : |v_\lambda| \in (2^{-(k+1)}\eta, 2^{-k}\eta]\}} \\ &\leq \sum_{k=0}^{\infty} 2^{-k}\eta(|||\mathbf{v}|||_{\mathscr{A}^s}2^{k+1}\eta^{-1})^{\tau/2} \lesssim \eta^{1-\tau/2}|||\mathbf{v}|||_{\mathscr{A}^s}^{\tau/2},\end{aligned}$$

and so $\|\mathbf{v}\|_{\mathscr{A}^s} \lesssim \sup_{\eta>0} \eta^{1-\tau/2}|||\mathbf{v}|||_{\mathscr{A}^s}^{\tau/2} \times ((|||\mathbf{v}|||_{\mathscr{A}^s}\eta^{-1})^\tau)^s = |||\mathbf{v}|||_{\mathscr{A}^s}$.

To show the other direction, first we note that

$$\|\mathbf{v}-\mathbf{v}_N\| \leq N^{-s}\|\mathbf{v}\|_{\mathscr{A}^s} \quad (\mathbf{v} \in \mathscr{A}^s, N \geq 1). \tag{5}$$

Indeed, if $\|\mathbf{v}-\mathbf{v}_N\| = \|\mathbf{v}-\mathbf{v}_{N-1}\|$, then $\mathbf{v}_N = \mathbf{v}_{N-1} = \mathbf{v}$ and (5) is valid. Otherwise, i.e., when $\|\mathbf{v}-\mathbf{v}_N\| < \|\mathbf{v}-\mathbf{v}_{N-1}\|$, by putting $\varepsilon := \|\mathbf{v}-\mathbf{v}_N\|$, the definition of $\|\cdot\|_{\mathscr{A}^s}$ shows (5).

With $(\gamma_N(\mathbf{v}))_{N\in\mathbb{N}}$ denoting a non-decreasing re-arrangement of $\mathbf{v}$ in modulus, secondly we note that

$$\sup_{N\in\mathbb{N}} N^{1/\tau}|\gamma_N(\mathbf{v})| \lesssim \|\mathbf{v}\|_{\mathscr{A}^s} \quad (\mathbf{v} \in \mathscr{A}^s). \tag{6}$$

Indeed, $|\gamma_1(\mathbf{v})| \leq \|\mathbf{v}\| \leq \|\mathbf{v}\|_{\mathscr{A}^s}$. For $1 \leq k < N$,

$$(N-k)|\gamma_N(\mathbf{v})|^2 \leq \sum_{k<j\leq N} |\gamma_j(\mathbf{v})|^2 \leq \|\mathbf{v}-\mathbf{v}_k\|^2 \leq k^{-2s}\|\mathbf{v}\|_{\mathscr{A}^s}^2,$$

or $|\gamma_N(\mathbf{v})| \leq \min_{1\leq k<N} \frac{k^{-s}}{(N-k)^{\frac{1}{2}}}\|\mathbf{v}\|_{\mathscr{A}^s} \eqsim N^{-1/\tau}\|\mathbf{v}\|_{\mathscr{A}^s}$.

Now given $N \in \mathbb{N}_0$, let $\eta := \gamma_{N+1}(\mathbf{v})$, then $\#\{\lambda \in \nabla : |v_\lambda| > \eta\} = N$. From $\eta \lesssim (N+1)^{-1/\tau}\|\mathbf{v}\|_{\mathscr{A}^s}$, we arrive at $|||\mathbf{v}|||_{\mathscr{A}^s} \lesssim \sup_N (N+1)^{-1/\tau}\|\mathbf{v}\|_{\mathscr{A}^s} \times N^{1/\tau} \leq \|\mathbf{v}\|_{\mathscr{A}^s}$. □

2 Well-posed linear operator equations

2.1 Reformulation as a bi-infinite matrix vector equation

Let $\mathscr{X}, \mathscr{Y}$ be separable (infinite dimensional) Hilbert spaces over $\mathbb{R}$ (the complex case does not impose additional difficulties apart from requiring somewhat more complicated notations). Let us assume that we have available a *Riesz basis* $\boldsymbol{\Psi}^{\mathscr{X}} = \{\psi_\lambda^{\mathscr{X}} : \lambda \in \nabla\}$ for $\mathscr{X}$, meaning that the *analysis operator*

$$\mathscr{F}_{\mathscr{X}} : \mathscr{X}' \to \ell_2 : g \mapsto [g(\psi_\lambda^{\mathscr{X}})]_{\lambda\in\nabla},$$

is boundedly invertible. By identifying ℓ_2 with its dual, its adjoint $\mathscr{F}'_{\mathscr{X}}$, known as the *synthesis operator*, and defined by $g(\mathscr{F}'_{\mathscr{X}}\mathbf{c}) = \langle \mathscr{F}_{\mathscr{X}} g, \mathbf{c}\rangle_{\ell_2\times\ell_2}$ $(g \in \mathscr{X}', \mathbf{c} \in \ell_2)$, reads as

$$\mathscr{F}'_{\mathscr{X}} : \ell_2 \to \mathscr{X} : \mathbf{c} \mapsto \mathbf{c}^\top \boldsymbol{\Psi}^{\mathscr{X}} := \sum_{\lambda\in\nabla} c_\lambda \psi_\lambda^{\mathscr{X}}.$$

Similarly, let $\boldsymbol{\Psi}^{\mathscr{Y}} = \{\psi_\lambda^{\mathscr{Y}} : \lambda \in \nabla\}$ be a Riesz basis for $\mathscr{Y}$, with analysis operator $\mathscr{F}_{\mathscr{Y}}$ and adjoint $\mathscr{F}'_{\mathscr{Y}}$. For both $\boldsymbol{\Psi}^{\mathscr{X}}$ and $\boldsymbol{\Psi}^{\mathscr{Y}}$, we have suitable *wavelet* bases in mind. Note that w.l.o.g. we could assume that the index set ∇ is the same for $\boldsymbol{\Psi}^{\mathscr{X}}$ and $\boldsymbol{\Psi}^{\mathscr{Y}}$.

Now given an $f \in \mathscr{Y}'$, and a boundedly invertible $B \in \mathscr{L}(\mathscr{X}, \mathscr{Y}')$, we are interested in solving the *linear operator equation* of finding $u \in \mathscr{X}$ such that

$$Bu = f.$$

Writing $u = s_{\mathscr{X}}\mathbf{u}$, and applying $\mathscr{F}_{\mathscr{Y}}$ to both sides of the equation, we infer that the problem can equivalently be written as the bi-infinite matrix vector problem

$$\mathbf{Bu} = \mathbf{f}, \tag{7}$$

where $\mathbf{f} := \mathscr{F}_{\mathscr{Y}} f = [f(\psi_\lambda^{\mathscr{Y}})]_{\lambda\in\nabla} \in \ell_2$, and the *"stiffness"* or *system matrix*

$$\mathbf{B} := \mathscr{F}_{\mathscr{Y}} B \mathscr{F}'_{\mathscr{X}} = [(B\psi_\mu^{\mathscr{X}})(\psi_\lambda^{\mathscr{Y}})]_{\lambda,\mu\in\nabla} \in \mathscr{L}(\ell_2, \ell_2)$$

is boundedly invertible. With $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{\ell_2\times\ell_2}$, for any $\mathbf{v}, \mathbf{w} \in \ell_2$,

$$\langle \mathbf{Bv}, \mathbf{w}\rangle = \langle \mathscr{F}_{\mathscr{Y}} B \mathscr{F}'_{\mathscr{X}} \mathbf{v}, \mathbf{w}\rangle = (Bv)(w), \tag{8}$$

where $v = \mathscr{F}'_{\mathscr{X}}\mathbf{v}$ and $w = \mathscr{F}'_{\mathscr{Y}}\mathbf{w}$.

With the Riesz constants

$$\Lambda_{\boldsymbol{\Psi}^{\mathscr{X}}} := \|\mathscr{F}_{\mathscr{X}}\|_{\mathscr{X}'\to\ell_2} = \sup_{0\neq g\in\mathscr{X}} \frac{\|\mathscr{F}_{\mathscr{X}} g\|_{\ell_2}}{\|g\|_{\mathscr{X}'}},$$

$$\lambda_{\boldsymbol{\Psi}^{\mathscr{X}}} := \|(\mathscr{F}_{\mathscr{X}})^{-1}\|^{-1}_{\ell_2\to\mathscr{X}'} = \inf_{0\neq g\in\mathscr{X}} \frac{\|\mathscr{F}_{\mathscr{X}} g\|_{\ell_2}}{\|g\|_{\mathscr{X}'}},$$

and $\Lambda_{\mathscr{Y}}$ and $\lambda_{\mathscr{Y}}$ defined analogously, and with $\|\cdot\| := \|\cdot\|_{\ell_2\to\ell_2}$, obviously it holds that

$$\|\mathbf{B}\| \leq \|B\|_{\mathscr{X}\to\mathscr{Y}'}\Lambda_{\Psi^{\mathscr{X}}}\Lambda_{\mathscr{Y}}, \tag{9}$$

$$\|\mathbf{B}^{-1}\| \leq \frac{\|B^{-1}\|_{\mathscr{Y}'\to\mathscr{X}}}{\lambda_{\Psi^{\mathscr{X}}}\lambda_{\mathscr{Y}}}. \tag{10}$$

Remark 2. Although not strictly necessary for the remainder of this paper, we make a few comments about *dual bases*. The collection

$$\Psi^{\mathscr{X}'} = (\mathscr{F}'_{\mathscr{X}}\mathscr{F}_{\mathscr{X}})^{-1}\,\Psi^{\mathscr{X}}$$

is the Riesz basis for $\mathscr{X}'$ that is dual to $\Psi^{\mathscr{X}}$. Indeed, since the corresponding analysis operator $\mathscr{F}_{\mathscr{X}'}$ reads as $(\mathscr{F}'_{\mathscr{X}})^{-1}$, it is boundedly invertible, and so $\Psi^{\mathscr{X}'}$ is a Riesz basis for $\mathscr{X}'$. It holds that $\mathscr{F}'_{\mathscr{X}'}\mathscr{F}_{\mathscr{X}} = I$, which, since $\Psi^{\mathscr{X}'}$ is a basis, implies that $\psi_\lambda^{\mathscr{X}'}(\psi_\mu^{\mathscr{X}}) = \delta_{\lambda\mu}$.

Given a $\tilde{\nabla} \subset \nabla$, $Q_{\tilde{\nabla}} : \mathscr{X} \to \mathscr{X} : v \mapsto \sum_{\lambda\in\tilde{\nabla}} \psi_\lambda^{\mathscr{X}'}(v)\psi_\lambda^{\mathscr{X}}$ is the *biorthogonal projector* onto $\mathrm{span}\{\psi_\lambda^{\mathscr{X}} : \lambda \in \tilde{\nabla}\}$, i.e., $Q_{\tilde{\nabla}}^2 = Q_{\tilde{\nabla}}$ and $\psi_\lambda^{\mathscr{X}'}$ vanishes on $\mathrm{Im}(\mathrm{Id} - Q_{\tilde{\nabla}})$ for all $\lambda \in \tilde{\nabla}$. We have $\|Q_{\tilde{\nabla}}\|_{\mathscr{X}\to\mathscr{X}} \leq \Lambda_{\Psi^{\mathscr{X}}}/\lambda_{\Psi^{\mathscr{X}}}$, and so

$$\|v - Q_{\tilde{\nabla}}v\|_{\mathscr{X}} \leq (1 + \Lambda_{\Psi^{\mathscr{X}}}/\lambda_{\Psi^{\mathscr{X}}}) \inf_{w\in\mathrm{span}\{\psi_\lambda^{\mathscr{X}}:\lambda\in\tilde{\nabla}\}} \|v - w\|_{\mathscr{X}}.$$

The dual projector $Q'_{\tilde{\nabla}} : \mathscr{X}' \to \mathscr{X}'$, that reads as $Q'_{\tilde{\nabla}}(g) = \sum_{\lambda\in\tilde{\nabla}} g(\psi_\lambda^{\mathscr{X}})\psi_\lambda^{\mathscr{X}'}$, has analogous properties.

If we identify $\mathscr{X}'$ with $\mathscr{X}$ using the Riesz map, then if, using this identification, $\Psi^{\mathscr{X}}$ and $\Psi^{\mathscr{X}'}$ are equal, then $Q_{\tilde{\nabla}}$, being equal to its adjoint, is the orthogonal projector onto $\mathrm{span}\{\psi_\lambda^{\mathscr{X}} : \lambda \in \tilde{\nabla}\}$.

Obviously, similar observations can be made for the collections $\Psi^{\mathscr{Y}}$ and its dual $\Psi^{\mathscr{Y}'}$.

2.2 Some model examples

We give some examples of partial differential equations or singular integral equations that are of the form $Bu = f$ with $B \in \mathscr{L}(\mathscr{X}, \mathscr{Y}')$ boundedly invertible. More examples can be found in [CDD02, DK05].

2.2.1 Second order elliptic boundary value problems

The variational formulation of a second order elliptic boundary value problem on a domain $\Omega \subset \mathbb{R}^n$ with homogeneous Dirichlet boundary conditions reads as $Bu = f$,

where

$$(Bu)(v) := \int_\Omega \mathbf{A}\nabla u \cdot \nabla v + (\mathbf{b}\cdot\nabla u)v + cuv\,dx.$$

If $\mathbf{A} \in L_\infty(\Omega)^{n\times n}$, $\mathbf{b} \in L_\infty(\Omega)^n$, $c \in L_\infty(\Omega)$, $c \geq 0$ (*a.e.*), $\nabla\cdot\mathbf{b} = 0$ (*a.e.*) and, for some $\delta > 0$, $\mathbf{A} \geq \delta > 0$ (*a.e.*), then $(Bv)(v) \geq \delta |v|^2_{H^1(\Omega)} \gtrsim \|v\|^2_{H^1(\Omega)}$ ($v \in H^1_0(\Omega)$), i.e., B is *coercive*. The Lax-Milgram lemma now shows that with $\mathscr{X} := H^1_0(\Omega)$, $B : \mathscr{X} \to \mathscr{X}'$ is boundedly invertible.

Remark 3. If $\partial\Omega \in C^2$ or Ω is convex, and the coefficients of the differential operator satisfy some mild smoothness conditions, then $B : H^2(\Omega)\cap H^1_0(\Omega) \to L_2(\Omega)$ is boundedly invertible, e.g., see [Hac92] + references cited there. Since the same is valid for the adjoint B', defined by $(B'v)(u) = (Bu)(v)$, we also have that $B : L_2(\Omega) \to (H^2(\Omega)\cap H^1_0(\Omega))'$ is boundedly invertible. In view of the possibility to take $\mathscr{X} \neq \mathscr{Y}$, we infer that the adaptive wavelet method can be used also to realize the best possible convergence rate in $L_2(\Omega)$.

2.2.2 Boundary integral equations

For Ω being some domain in $\mathbb{R}^3$, let $\Gamma := \partial\Omega$. The Laplace equation on Ω or on $\mathbb{R}^3\backslash\Omega$, with either Dirichlet or Neumann boundary conditions can be reformulated as a boundary integral equation of type $(Bu)(v) := \int_\Gamma Lu(x)v(x)ds_x = f(v)$ ($v \in \mathscr{X}$), where either

$$Lu(x) := \int_\Gamma \frac{u(y)}{4\pi|x-y|}ds_y, \quad \mathscr{X} := H^{-\frac{1}{2}}(\Gamma), \tag{11}$$

or

$$Lu(x) := \pm\frac{1}{2}u(x) + \int_\Gamma \frac{(x-y)^\top \mathbf{n}_y v(y)}{4\pi|x-y|^3}ds_y, \quad \mathscr{X} := L_2(\Gamma), \tag{12}$$

or

$$Lu(x) := -\partial_{\mathbf{n}_x}\int_\Gamma \frac{(x-y)^\top \mathbf{n}_y v(y)}{4\pi|x-y|^3}ds_y, \quad \mathscr{X} := H^{\frac{1}{2}}(\Gamma)/\mathbb{R}. \tag{13}$$

In all three cases, $B : \mathscr{X} \to \mathscr{X}'$ is known to be boundedly invertible.

2.2.3 Stokes equations

The variational formulation of the Stokes equations on a domain $\Omega \subset \mathbb{R}^n$ with homogeneous Dirichlet boundary conditions reads as

$$(B(\vec{u},p))(\vec{v},q) := \int_\Omega \nabla\vec{u} : \nabla\vec{v}\,dx + \int_\Omega p\,\mathrm{div}\,\vec{v}\,dx + \int_\Omega q\,\mathrm{div}\,\vec{u}\,dx = \vec{f}(\vec{v})$$

($\vec{v} \in H^1_0(\Omega)^n$, $q \in L_{2,0}(\Omega)$). With $\mathscr{X} := H^1_0(\Omega)^n \times L_{2,0}(\Omega)$, it is well-known that $B : \mathscr{X} \to \mathscr{X}'$ is boundedly invertible.

2.2.4 Parabolic evolution equations

For some domain $\Omega \subset \mathbb{R}^n$ and $T > 0$, we consider the parabolic problem

$$\begin{cases} (\partial_t u + \nabla_x \cdot \mathbf{A}\nabla_x u + \mathbf{b} \cdot \nabla_x u + cu)(t,x) = g(t,x) & (t \in (0,T), x \in \Omega), \\ u(t,x) = 0 & (t \in (0,T), x \in \partial\Omega), \\ u(0,x) = h(x) & (x \in \Omega), \end{cases}$$

where $\mathbf{A} \in L_\infty((0,T) \times \Omega)^{n \times n}$, $\mathbf{b} \in L_\infty((0,T) \times \Omega)^n$, $c \in L_\infty((0,T) \times \Omega)$, and, for some $\delta > 0$, $\mathbf{A} \geq \delta > 0$ (a.e.). With

$$\mathscr{X} := L_2(0,T) \otimes H^1_0(\Omega) \cap H^1(0,T) \otimes H^{-1}(\Omega)$$

i.e., $\mathscr{X}$ is an intersection of Bochner spaces, and

$$\mathscr{Y} := (L_2(0,T) \otimes H^1_0(\Omega)) \times L_2(\Omega),$$

and assuming that $g \in L_2((0,T); H^1_0(\Omega))'$ and $h \in L_2(\Omega)$, a variational formulation of this problem reads as: Find $u \in \mathscr{X}$ such that

$$\begin{aligned} (Bu)(v_1,v_2) &:= \int_0^T \int_\Omega (\partial_t u)v_1 + \mathbf{A}\nabla_x u \cdot \nabla_x v_1 + (\mathbf{b} \cdot \nabla_x u)v_1 + cuv_1 \, dx\, dt + \int_\Omega u(0,\cdot)v_2 \, dx \\ &= \int_0^T \int_\Omega g v_1 dx dt + \int_\Omega h v_2 \, dx \quad ((v_1,v_2) \in \mathscr{Y}). \end{aligned}$$

The operator $B : \mathscr{X} \to \mathscr{Y}'$ is boundedly invertible (cf. [SS09], [DL92, Ch.XVIII, §3], [Wlo82, Ch.IV,§26]).

3 Adaptive wavelet schemes I: Inexact Richardson iteration ([CDD02])

3.1 Richardson iteration

Throughout this section, until Sect. 3.4, we will *assume* that there exists an $\alpha \in \mathbb{R}$ such that

$$\|\mathrm{Id} - \alpha\mathbf{B}\| < 1, \tag{14}$$

i.e., we will assume that a properly damped Richardson iteration

$$\mathbf{u}^{(i+1)} = \mathbf{u}^{(i)} + \alpha(\mathbf{f} - \mathbf{B}\mathbf{u}^{(i)})$$

applied to (7) converges linearly.

Lemma 3. *In addition to being boundedly invertible, let* $\mathbf{B}$ *satisfy* $\mathbf{B} = \mathbf{B}^\top > 0$. *Then for* $\alpha \in (0, 2/\|\mathbf{B}\|)$,

$$\|\mathrm{Id} - \alpha\mathbf{B}\| = \max(\alpha\|\mathbf{B}\| - 1, 1 - \alpha\|\mathbf{B}^{-1}\|^{-1}) < 1,$$

with minimum $\frac{\kappa(\mathbf{B})-1}{\kappa(\mathbf{B})+1}$ *when* $\alpha = 2/(\|\mathbf{B}\| + \|\mathbf{B}^{-1}\|^{-1})$, *where* $\kappa(\mathbf{B}) := \|\mathbf{B}\|\|\mathbf{B}^{-1}\|$.

Proof. Since $\mathbf{B} = \mathbf{B}^\top$, $\|\mathrm{Id} - \alpha\mathbf{B}\| = \max_{\lambda\in\sigma(\mathrm{Id}-\alpha\mathbf{B})} |\lambda| = \max_{\mu\in\sigma(\mathbf{B})} |1 - \alpha\mu|$, and from $\mathbf{B} > 0$, we have $\sigma(\mathbf{B}) \subset [\|\mathbf{B}^{-1}\|^{-1}, \|\mathbf{B}\|]$. Elementary calculations now complete the proof. □

If, apart from being boundedly invertible between $\mathscr{X}$ and $\mathscr{Y}'$, B is *symmetric*, i.e., $\mathscr{X} = \mathscr{Y}$ and $(Bv)(w) = (Bw)(v)$ $(v, w \in \mathscr{X})$, and *positive definite*, i.e., $(Bv)(v) > 0$ $(v \in \mathscr{X})$, then, because of (8), so is $\mathbf{B}$ and Lemma 3 applies. The example from Sect.2.2.1 when $\mathbf{b} = 0$, as well as the example from Sect. 2.2.2 in the cases (11) and (13) fall into this category.

If $\mathscr{X} = \mathscr{Y}$ and B is bounded and *coercive*, i.e., for some $\delta > 0$, $(Bv)(v) \geq \delta\|v\|^2_{\mathscr{X}}$ $(v \in \mathscr{X})$, then (8) and the next lemma show that the properly damped Richardson iteration is again convergent. An application is given by the example from Sect.2.2.1 for general $\mathbf{b} \in L_\infty(\Omega)^n$ with $\nabla \cdot \mathbf{b} = 0$ (*a.e.*).

Lemma 4. *If, in addition to* $\mathbf{B}$ *being bounded,* $\mathbf{B}_S := \frac{1}{2}(\mathbf{B} + \mathbf{B}^\top) > 0$ *and has a bounded inverse, then for* $\alpha \in (0, 1/(\|\mathbf{B}_S\| + \|\mathbf{B}_S^{-1}\|^{-1})]$ *with* $\alpha < 2/(\|\mathbf{B}_S^{-1}\|\|\mathbf{B}\|)$,

$$\|\mathrm{Id} - \alpha\mathbf{B}\| \leq \sqrt{1 - 2\alpha\|\mathbf{B}_S^{-1}\|^{-1} + \alpha^2\|\mathbf{B}\|^2} < 1.$$

Proof. As shown in Lemma 3, for $\alpha \in (0, 1/(\|\mathbf{B}_S\| + \|\mathbf{B}_S^{-1}\|^{-1})]$, $\|\mathrm{Id} - 2\alpha\mathbf{B}_s\| \leq 1 - 2\alpha\|\mathbf{B}_S^{-1}\|^{-1}$. This shows that

$$\begin{aligned}\|\mathrm{Id} - \alpha\mathbf{B}\|^2 &= \|(\mathrm{Id} - \alpha\mathbf{B})(\mathrm{Id} - \alpha\mathbf{B}^\top)\| \\ &= \|\mathrm{Id} - 2\alpha\mathbf{B}_S + \alpha^2\mathbf{B}\mathbf{B}^\top\| \leq 1 - 2\alpha\|\mathbf{B}_S^{-1}\|^{-1} + \alpha^2\|\mathbf{B}\|^2 < 1,\end{aligned}$$

when $\alpha < 2/(\|\mathbf{B}_S^{-1}\|\|\mathbf{B}\|)$.

□

3.2 Practical scheme

Of course the Richardson iteration cannot performed exactly. Generally the right-hand side $\mathbf{f}$ is infinitely supported, and although $\mathbf{B}$ is close to being sparse, generally so is any column of $\mathbf{B}$. The idea proposed in [CDD02] is to apply Richardson iteration with inexact evaluations of the matrix-vector product and of the right-hand

side $\mathbf{f}$. It is easily seen that with a proper decay of the tolerances for these inexact evaluations as the iteration proceeds, the perturbed iteration is still linearly convergent. The issues at stake are whether the support lengths of the iterands are, up to a constant multiple, equal to the generally best possible bounds on the lengths of the best N-term approximations that give rise to the same error, and whether the computational costs to produce such iterands are bounded by the same expressions. To ensure these properties, i.e., to ensure (quasi-) *optimality* of the algorithm, assumptions are needed on the cost of the inexact application of $\mathbf{B}$ and that of the inexact evaluation of the right-hand side as a function of the prescribed tolerance.

Definition 1. For $\bar{s} > 0$, $\mathbf{B}$ will be called to be *$\bar{s}$-admissible* when we have available an approximate matrix times vector routine

$$\mathbf{APPLY}[\mathbf{w}, \varepsilon] \to \mathbf{z}_\varepsilon$$

that, for any $\varepsilon > 0$ and $\mathbf{w} \in \ell_0$, yields a $\mathbf{z}_\varepsilon \in \ell_0$ with

$$\|\mathbf{B}\mathbf{w} - \mathbf{z}_\varepsilon\| \leq \varepsilon,$$

and, for any $s \in (0, \bar{s}]$,

$$\#\operatorname{supp} \mathbf{z}_\varepsilon \lesssim \varepsilon^{-1/s} \|\mathbf{w}\|_{\mathscr{A}^s}^{1/s}, \tag{15}$$

where the number of operations used by the call $\mathbf{APPLY}[\mathbf{w}, \varepsilon]$ is bounded by some absolute multiple of

$$\varepsilon^{-1/s} \|\mathbf{w}\|_{\mathscr{A}^s}^{1/s} + \#\operatorname{supp} \mathbf{w} + 1. \tag{16}$$

As we will see, in order to guarantee optimality of the inexact Richardson iteration, as well as of the alternative Adaptive Wavelet-Galerkin Method discussed in Sect. 4, it will be needed that $\bar{s}$ not less than that s for which the solution $\mathbf{u}$ happens to be in $\mathscr{A}^s$. That is, with the best possible rate $s_{\max}$ as introduced in Sect. 1.1, *it will be sufficient, and generally necessary,* when

$$\bar{s} \geq s_{\max}, \tag{17}$$

an issue that was somewhat ignored in the early publications on adaptive wavelet methods. In Sect. 5, we will see that for partial differential operators with sufficiently smooth coefficients and for wavelets that are sufficiently smooth and have sufficiently many vanishing moments (or, more generally, cancellation properties) indeed (17) is valid. We include pointers to the literature where it is shown that the same holds for classes of singular integral operators,

In view of the definition of $\mathscr{A}^s$, a consequence of (15) is that $\mathbf{B}$, restricted to ℓ_0, is a bounded mapping from $\mathscr{A}^s$ to $\mathscr{A}^s$ for $s \in (0, \bar{s}]$. As shown in [CDD01, Prop. 3.8], we even have:

Proposition 1. *Let $\mathbf{B}$ be $\bar{s}$-admissible. Then for $s \in (0, \bar{s}]$, $\mathbf{B} : \mathscr{A}^s \to \mathscr{A}^s$ is bounded, and for $\mathbf{z}_\varepsilon := \mathbf{APPLY}[\mathbf{w}, \varepsilon]$, we have $\|\mathbf{z}_\varepsilon\|_{\mathscr{A}^s} \lesssim \|\mathbf{w}\|_{\mathscr{A}^s}$, uniformly in ε.*

Proof. For $s \in (0,\bar{s}]$, $\mathbf{w} \in \mathscr{A}^s$ and $\varepsilon > 0$, let $N \in \mathbb{N}$ be such that $\|\mathbf{B}\|\|\mathbf{w}-\mathbf{w}_N\| \le \varepsilon/2$, and let $\mathbf{z}_{\varepsilon/2} := \mathbf{APPLY}[\mathbf{w}_N, \varepsilon/2]$. Then $\#\mathrm{supp}\,\mathbf{z}_{\varepsilon/2} \lesssim \varepsilon^{-1/s}\|\mathbf{w}_N\|_{\mathscr{A}^s}^{1/s} \le \varepsilon^{-1/s}\|\mathbf{w}\|_{\mathscr{A}^s}^{1/s}$, and $\|\mathbf{B}\mathbf{w}-\mathbf{z}_{\varepsilon/2}\| \le \varepsilon$, showing the first statement.

Lemma 1 and (15) show that $\|\mathbf{z}_\varepsilon\|_{\mathscr{A}^s} \lesssim \max(\|\mathbf{B}\mathbf{w}\|_{\mathscr{A}^s}, \|\mathbf{w}\|_{\mathscr{A}^s}) \lesssim \|\mathbf{w}\|_{\mathscr{A}^s}$. □

The requirement (16) basically means that the cost of producing $\mathbf{z}_\varepsilon$ is proportional to its length plus that of $\mathbf{w}$.

Concerning the inexact evaluation of the right-hand side, *throughout this paper we assume availability of the following routine:*

$\mathbf{RHS}[\varepsilon] \to \mathbf{f}_\varepsilon$:
% Input: $\varepsilon > 0$.
% Output: $\mathbf{f}_\varepsilon \in \ell_0$ *with*

$$\|\mathbf{f}-\mathbf{f}_\varepsilon\| \le \varepsilon \quad \textit{and} \quad \#\mathrm{supp}\,\mathbf{f}_\varepsilon \lesssim \min\{N : \|\mathbf{f}-\mathbf{f}_N\| \le \varepsilon\},$$

% taking a number of operations that is bounded by some absolute multiple of
% $\#\mathrm{supp}\,\mathbf{f}_\varepsilon + 1$.

A realization of **RHS** generally has to depend on the right-hand side f at hand, that, however, in contrast to the solution u, is known to the user. Noting that for $\tilde{\nabla} \subset \nabla$,

$$\|\mathbf{f}-\mathbf{f}|_{\tilde{\nabla}}\| \eqsim \|f - \sum_{\lambda\in\tilde{\nabla}} f(\psi_\lambda^{\mathscr{Y}})\psi_\lambda^{\mathscr{Y}'}\|_{\mathscr{Y}'} \eqsim \inf_{\tilde{f}\in\mathrm{span}\{\psi_\lambda^{\mathscr{Y}'}:\lambda\in\tilde{\nabla}\}} \|f-\tilde{f}\|_{\mathscr{Y}'}$$

(cf. Remark 2), we see that for sufficiently smooth f, **RHS** is realized by collecting, or more precisely, by approximating using suitable quadrature, the wavelet coefficients of f up to some suitable level.

Corollary 1. *Let* $\mathbf{B}$ *be* $\bar{s}$*-admissible. If, for some* $s \in (0,\bar{s}]$, $\mathbf{u} \in \mathscr{A}^s$, *then* $\mathbf{f}_\varepsilon := \mathbf{RHS}[\varepsilon]$ *satisfies* $\#\mathrm{supp}\,\mathbf{f}_\varepsilon \lesssim \varepsilon^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}$, *where the number of operations used by the call* $\mathbf{RHS}[\varepsilon]$ *is bounded by some absolute multiple of*

$$\varepsilon^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s} + 1.$$

Proof. By the assumptions and Proposition 1, we have $\mathbf{f} \in \mathscr{A}^s$, with $\|\mathbf{f}\|_{\mathscr{A}^s} \lesssim \|\mathbf{u}\|_{\mathscr{A}^s}$. Now the proof is completed by the definition of $\mathscr{A}^s$ and the assumptions made on **RHS**. □

Remark 4. Recalling that $s_{\max}$ is the approximation order of $\Psi^{\mathscr{X}}$ in $\mathscr{X}$, let $\tilde{s}_{\max}$ denote the approximation order of $\Psi^{\mathscr{Y}'}$ in $\mathscr{Y}'$.

The property, shown in Corollary 1, that for any $\mathbf{u} \in \mathscr{A}^s$ with $s \le \bar{s}$, it holds that $\mathbf{f} \in \mathscr{A}^s$ can only be expected when $\tilde{s}_{\max} \ge \min(\bar{s}, s_{\max})$. This means that $\bar{s}$-admissibility of $\mathbf{B}$ with $\bar{s} \ge s_{\max}$ requires that $\tilde{s}_{\max} \ge s_{\max}$.

In the scalar model situation of $\mathscr{X} = \mathscr{Y} = H^m(\Omega)$ for some domain $\Omega \subset \mathbb{R}^n$, and $\Psi^{\mathscr{X}}$, $\Psi^{\mathscr{X}'}$ being wavelet collections of order d, $\tilde{d}$, normalized in $H^m(\Omega)$ or $(H^m(\Omega))'$, respectively, it holds that $s_{\max} = \frac{d-m}{n}$ and $\tilde{s}_{\max} = \frac{\tilde{d}+m}{n}$. In this case,

$\tilde{s}_{\max} \geq s_{\max}$ means that $\tilde{d} \geq d-2m$. For differential- and integral operators, in Sect. 5 we will see that the condition $\tilde{d} > d-2m$ suffices to demonstrate $\bar{s}$-admissibility of $\mathbf{B}$ for $\bar{s} \geq s_{\max}$.

Remark 5. The properties that $\|\mathbf{f}-\mathbf{f}_\varepsilon\| \leq \varepsilon$ and, when $\mathbf{u} \in \mathscr{A}^s$, that $\#\operatorname{supp}\mathbf{f}_\varepsilon \lesssim \varepsilon^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}$, with the cost of producing it being bounded by some absolute multiple of $\varepsilon^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}+1$ is all that will be needed about $\mathbf{f}_\varepsilon := \mathbf{RHS}[\varepsilon]$. Our assumptions on **RHS** together with Corollary 1 show that these properties hold when $\mathbf{B}$ is $\bar{s}$-admissible for some $\bar{s} \geq s_{\max}$. The assumption, formulated in the description of the **RHS** routine, that we can realize quasi-best N-term approximations for $\mathbf{f}$ in linear complexity is actually stronger than what is needed when $\tilde{s}_{\max} > s_{\max}$.

Besides **APPLY** and **RHS**, the inexact Richardson iteration requires another subroutine:

COARSE$[\mathbf{w},\varepsilon] \to \mathbf{w}_\varepsilon$*:*
% Input: $\mathbf{w} \in \ell_0$ *and* $\varepsilon > 0$.
% Output: $\mathbf{w}_\varepsilon \in \ell_0$ *with*

$$\|\mathbf{w}-\mathbf{w}_\varepsilon\| \leq \varepsilon \quad \textit{and} \quad \#\operatorname{supp}\mathbf{w}_\varepsilon \lesssim \min\{N : \|\mathbf{w}-\mathbf{w}_N\| \leq \varepsilon\}, \tag{18}$$

% taking a number of operations that is bounded by an absolute multiple of

$$\#\operatorname{supp}\mathbf{w} + \max\big(\log(\varepsilon^{-1}\|\mathbf{w}\|),1\big).$$

An implementation of a routine **COARSE** with these properties will be given in Sect. 3.3.

The routine **COARSE** will be applied after each fixed number of (inexact) Richardson steps. The idea is to remove small coefficients from the iterands, that, because they are small, little contribute to the approximation, but, because their possibly large number, may hamper an optimal balance between accuracy and support length. Although obviously an application of **COARSE** generally increases the error, the following proposition ([Coh03, Th. 4.9.1]) shows that indeed it creates the aforementioned optimal balance.

Proposition 2. *Let* $\zeta > 1$ *and* $s > 0$. *Then for any* $\varepsilon > 0$, $\mathbf{v} \in \mathscr{A}^s$ *and* $\mathbf{w} \in \ell_0$ *with*

$$\|\mathbf{v}-\mathbf{w}\| \leq \varepsilon,$$

for $\mathbf{w}_{\zeta\varepsilon} :=$ **COARSE**$[\zeta\varepsilon,\mathbf{w}]$ *we have that*

$$\#\operatorname{supp}\mathbf{w}_{\zeta\varepsilon} \lesssim \varepsilon^{-1/s}\|\mathbf{v}\|_{\mathscr{A}^s}^{1/s}, \quad \|\mathbf{w}_{\zeta\varepsilon}\|_{\mathscr{A}^s} \lesssim \|\mathbf{v}\|_{\mathscr{A}^s},$$

and $\|\mathbf{v}-\mathbf{w}_{\zeta\varepsilon}\| \leq (1+\zeta)\varepsilon$.

Proof. The smallest $N \in \mathbb{N}_0$ with $\|\mathbf{v}-\mathbf{v}_N\| \le (\zeta-1)\varepsilon$ satisfies $N \le ((\zeta-1)\varepsilon)^{-1/s}\|\mathbf{v}\|_{\mathscr{A}^s}^{1/s}$. From $\|\mathbf{w}-\mathbf{v}_N\| \le \|\mathbf{w}-\mathbf{v}\| + \|\mathbf{v}-\mathbf{v}_N\| \le \varepsilon + (\zeta-1)\varepsilon = \zeta\varepsilon$ and (18), it follows that $\#\operatorname{supp}\mathbf{w}_{\zeta\varepsilon} \lesssim N \lesssim \varepsilon^{-1/s}\|\mathbf{v}\|_{\mathscr{A}^s}^{1/s}$.

The second and last statement follow from Lemma 1 and an application of the triangle inequality, respectively. □

We are ready to give the inexact Richardson iteration:

Rich$[\varepsilon,\varepsilon_0] \to \mathbf{u}_\varepsilon$:
% Input: $\varepsilon > 0$ *and* $\varepsilon_0 \ge \|\mathbf{u}\|$.
% Parameters: $\theta < 1/2$, $K \in \mathbb{N}$ *and* $\rho < 1$ *such that* $\|\mathrm{Id}-\alpha\mathbf{B}\| \le \rho$ *and* $2\rho^K < \theta$.

$i := 0$, $\mathbf{u}^{(0)} := 0$
`while` $\varepsilon_i > \varepsilon$ `do`
  $i := i+1$
  $\varepsilon_i := 2\rho^K\varepsilon_{i-1}/\theta$
  $\mathbf{v}^{(i,0)} := \mathbf{u}^{(i-1)}$
  `for` $j = 1,\dots,K$ `do`
    $\mathbf{v}^{(i,j)} := \mathbf{v}^{(i,j-1)} + \alpha(\mathbf{RHS}[\frac{\rho^j\varepsilon_{i-1}}{2\alpha K}] - \mathbf{APPLY}[\mathbf{v}^{(i,j-1)}, \frac{\rho^j\varepsilon_{i-1}}{2\alpha K}])$
  `enddo`
  $\mathbf{u}^{(i)} := \mathbf{COARSE}[(1-\theta)\varepsilon_i, \mathbf{v}^{(i,K)}]$
`enddo`
$\mathbf{u}_\varepsilon := \mathbf{u}^{(i)}$

Theorem 1 ([CDD02]). *Let* $\varepsilon_0 \ge \|\mathbf{u}\|$, *and* $\varepsilon > 0$, *then* $\mathbf{u}_\varepsilon :=$ **Rich**$[\varepsilon,\varepsilon_0]$ *satisfies* $\|\mathbf{u}-\mathbf{u}_\varepsilon\| \le \varepsilon$. *If for some* $s > 0$, $\mathbf{u} \in \mathscr{A}^s$, *then* $\#\operatorname{supp}\mathbf{u}_\varepsilon \lesssim \varepsilon^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}$. *If, additionally,* $\mathbf{B}$ *is* $\bar{s}$*-admissible,* $s \le \bar{s}$ *and* $\varepsilon < \varepsilon_0 \lesssim \|\mathbf{u}\|$, *then the number of operations used by the call* **Rich**$[\varepsilon,\varepsilon_0]$ *is bounded by an absolute multiple of* $\varepsilon^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}$. *In other words, if* $\bar{s} \ge s_{\max}$, *then the inexact Richardson iteration is (quasi-) optimal.*

Proof. For the first statement, it suffices to show that $\|\mathbf{u}-\mathbf{u}^{(i)}\| \le \varepsilon_i$. For $i = 0$, this is clearly valid. Now for some $i \ge 1$, let $\|\mathbf{u}-\mathbf{u}^{(i-1)}\| \le \varepsilon_{i-1}$. For $1 \le j \le K$, for some δ_j with $\|\delta_j\| \le \frac{\rho^j\varepsilon_{i-1}}{K}$, we have

$$\mathbf{u}-\mathbf{v}^{(i,j)} = (\mathrm{Id}-\alpha\mathbf{B})(\mathbf{u}-\mathbf{v}^{(i,j-1)}) + \delta_j,$$

and so

$$\mathbf{u}-\mathbf{v}^{(i,K)} = (\mathrm{Id}-\alpha\mathbf{B})^K(\mathbf{u}-\mathbf{u}^{(i-1)}) + \sum_{j=1}^{K}(\mathrm{Id}-\alpha\mathbf{B})^{K-j}\delta_j.$$

From $\|\mathrm{Id}-\alpha\mathbf{B}\| \le \rho$, we infer that

$$\|\mathbf{u}-\mathbf{v}^{(i,K)}\| \le \rho^K\varepsilon_{i-1} + \sum_{j=1}^{K}\rho^{K-j}\frac{\rho^j\varepsilon_{i-1}}{K} = 2\rho^K\varepsilon_{i-1} = \theta\varepsilon_i, \tag{19}$$

and conclude that

$$\|\mathbf{u}-\mathbf{u}^{(i)}\| \leq \theta\varepsilon_i + (1-\theta)\varepsilon_i = \varepsilon_i$$

as required.

Now for some $s>0$, let $\mathbf{u}\in\mathscr{A}^s$. From (19) and the definition of $\mathbf{u}^{(i)}$, Proposition 2 shows that

$$\#\operatorname{supp}\mathbf{u}^{(i)} \lesssim \varepsilon_i^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}, \quad \|\mathbf{u}^{(i)}\|_{\mathscr{A}^s} \lesssim \|\mathbf{u}\|_{\mathscr{A}^s},$$

which bounds, as we emphasize here, hold uniformly in i. Since $\varepsilon_i \gtrsim \varepsilon_{i-1}$, the first bound shows the second statement of the theorem.

Now let $\mathbf{B}$ is $\bar{s}$-admissible for some $\bar{s}\geq s$. Since K is fixed, Proposition 1 shows that $\|\mathbf{v}^{(i,j)}\|_{\mathscr{A}^s} \lesssim \|\mathbf{u}\|_{\mathscr{A}^s}$, uniformly in i and j. The properties from Definition 1, together with Corollary 1 show that $\#\operatorname{supp}\mathbf{v}^{(i,j)} \lesssim \varepsilon_i^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}$ and that the cost of computing it from the previous iterand is bounded by an absolute multiple of $\varepsilon_i^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}$. For the latter, we have used that by assumption on ε_0, $1 \lesssim \varepsilon_0^{-1/s}\|\mathbf{u}\|^{1/s} \leq \varepsilon_i^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}$. Since the cost of the call $\mathbf{COARSE}[(1-\theta)\varepsilon_i, \mathbf{v}^{(i,K)}]$ is bounded by an absolute multiple of $\#\operatorname{supp}\mathbf{v}^{(i,K)} + \max(\log(((1-\theta)\varepsilon_i)^{-1}\|\mathbf{v}^{(i,K)}\|, 1)$ $\lesssim \varepsilon_i^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}$, the proof is completed by using the linear decrease of ε_i as function of i. □

Remark 6. Although for any $s\in(0,\bar{s}]$, $\mathbf{APPLY}[\cdot,\varepsilon]:\mathscr{A}^s\to\mathscr{A}^s$ is bounded, even uniformly in ε, there is no guarantee that by a repeated application the $\|\cdot\|_{\mathscr{A}^s}$ (quasi-) norm of the iterands does not grow beyond any bound. This was the reason to add coarsening to this inexact Richardson iteration. Numerical experiments have shown that indeed generally **COARSE** is needed to ensure optimality of the inexact Richardson iteration.

3.3 The routines COARSE *and* APPLY

The obvious implementation of $\mathbf{COARSE}[\mathbf{w},\varepsilon]\to\mathbf{w}_\varepsilon$ would be to order the elements of $\mathbf{w}$ by non-increasing modulus, and then to define $\mathbf{w}_\varepsilon$ as the smallest possible head of $\mathbf{w}$ such that the discarded tail has norm not larger than ε. Unfortunately, with $M := \#\operatorname{supp}\mathbf{w}$, this ordering requires $\mathscr{O}(M\log M)$ operations, so that linear complexity cannot be realized. This is the reason that in [CDD01, CDD02] on many places the suboptimal complexity of the sorting was taken into account separately. Later, this problem was solved independently by Barinka and Metselaar in [Bar05, Met02], who proposed to apply an approximate "bucket" sorting:

BUCKETSORT$[\mathbf{w},\varepsilon]\to(\mathbf{w}_{[p]})_{1\leq p\leq P}$:
% Input: $\mathbf{w}\in\ell_0$, $\varepsilon>0$.
% Output: A distribution of the (largest) elements of $\mathbf{w}$ *over* P *"buckets".*

- *Let P be the smallest positive integer with* $2^{-P/2}\|\mathbf{w}\|_\infty\sqrt{\#\operatorname{supp}\mathbf{w}} \le \varepsilon$.
- *Store the indices of* $\mathbf{w}$ *in one of the P buckets, depending on the modulus of the corresponding coefficient to be in* $(\frac{1}{\sqrt{2}}\|\mathbf{w}\|_\infty, \|\mathbf{w}\|_\infty]$ *(first bucket),* $(\frac{1}{2}\|\mathbf{w}\|_\infty, \frac{1}{\sqrt{2}}\|\mathbf{w}\|_\infty]$, $\dots$, *or* $(2^{-P/2}\|\mathbf{w}\|_\infty, 2^{-(P-1)/2}\|\mathbf{w}\|_\infty]$, *and discard them otherwise.*
  Let $\mathbf{w}_{[p]}$ *denote the restriction of* $\mathbf{w}$ *to indices in bucket p.*

The number of buckets P is $\max(1, \lceil 2\log_2(\|\mathbf{w}\|_\infty\sqrt{\#\operatorname{supp}\mathbf{w}}/\varepsilon)\rceil)$. This number is chosen so that $\|\mathbf{w}-\sum_{p=1}^P \mathbf{w}_{[p]}\| \le \varepsilon$. This means that for the task of finding a (quasi) minimal Λ such that $\|\mathbf{w}-\mathbf{w}|_\Lambda\| \le \varepsilon$, these coefficients can be discarded anyway. This suggest the following coarsening routine:

COARSE$[\mathbf{w},\varepsilon] \to \mathbf{w}_\varepsilon$:
% Input: $\mathbf{w} \in \ell_0$, $\varepsilon > 0$.

$(\mathbf{w}_{[p]})_{1\le p\le P} :=$ **BUCKETSORT**$[\mathbf{w},\varepsilon]$
Build $\mathbf{w}_\varepsilon$ *by extracting indices from the buckets, starting with the first bucket and when it got empty continuing with the second one and so on, and within each bucket in arbitrary order, until* $\|\mathbf{w}-\mathbf{w}_\varepsilon\| \le \varepsilon$.

Note that for small ε, the number of buckets can be larger than $\#\operatorname{supp}\mathbf{w}$. Although then necessarily some buckets are empty, the computational cost of the call cannot be bounded on some absolute multiple of $\#\operatorname{supp}\mathbf{w}$ alone. This cost, however, can be bounded on some absolute multiple of $\#\operatorname{supp}\mathbf{w}$ plus the number of buckets. Further, since squared coefficients within one bucket differ at most a factor 2, $\#\operatorname{supp}\mathbf{w}_\varepsilon$ is at most twice as large as the length of the shortest approximation to $\mathbf{w}$ within tolerance ε. We conclude that the above implementation realizes all properties of **COARSE** that were mentioned in its description in the previous section (if necessary, consult [GHS07, Remark 2.3]).

To define a valid **APPLY** routine, we have to assume that $\mathbf{B}$ can be sufficiently well approximated by computable sparse matrices. We will assume to have available sequences $(e_j)_{j\in\mathbb{N}_0}, (c_j)_{j\in\mathbb{N}_0} \subset \mathbb{R}$, $(\mathbf{B}^{(j)})_{j\in\mathbb{N}_0} \subset \mathscr{L}(\ell_2,\ell_2)$, such that

- $\|\mathbf{B}-\mathbf{B}^{(j)}\| \le e_j$, $\lim_{j\to\infty} e_j = 0$,
- the number of non-zeros in each column of $\mathbf{B}^{(j)}$, as well as the number of operations needed to compute them, is bounded by c_j,
- $\mathbf{B}^{(0)} = 0$ (and thus $\|\mathbf{B}\| \le e_0$), $c_0 = 0$ and $\sup_{j\in\mathbb{N}_0} c_{j+1}/c_j < \infty$.

So the faster e_j decays as function of c_j, the closer is $\mathbf{B}$ to a computable sparse matrix. This motivates the following definition:

Definition 2. For $s^* > 0$, $\mathbf{B}$ will be called to be s^**-computable* when for any $s < s^*$, $\sup_j e_j c_j^s < \infty$.

By specifying an approximate matrix-vector multiplication routine **APPLY**, next we will show that an s^*-computable matrix $\mathbf{B}$ is $\bar{s}$-admissible for any $\bar{s} < s^*$.

In the **APPLY** routine proposed in [DSS08] and recalled below, for some P sufficiently large $\mathbf{w}$ is split into $\sum_{p=1}^P \mathbf{w}_{[p]}$ plus its tail $\mathbf{w}-\sum_{p=1}^P \mathbf{w}_{[p]}$, after which for

$1 \le p \le P$, $\mathbf{B}\mathbf{w}_{[p]}$ is approximated by $\mathbf{B}^{(j_p)}\mathbf{w}_{[p]}$, where (usually) j_p grows with decreasing p. On the tail $\mathbf{w} - \sum_{p=1}^{P} \mathbf{w}_{[p]}$, and possibly also on some $\mathbf{w}_{[p]}$ with p close to P, $\mathbf{B}$ is simply approximated by the zero operator. So the basic idea is to approximate columns of $\mathbf{B}$ that correspond to large entries in the input vector $\mathbf{w}$ more accurately that those that correspond to entries that are small. This means that **APPLY** is an adaptive routine, which depends non-linearly on the input $\mathbf{w}$.

A difference with the corresponding original routine proposed in [CDD01] is that instead of the splitting of $\mathbf{w}$ into buckets, each of them containing all entries of $\mathbf{w}$ with modulus in a certain range, there $\mathbf{w}$ was chopped into parts with prescribed lengths. Secondly, and more importantly, instead of taking as in [CDD01] an a priori distribution of the accuracies of the approximations of $\mathbf{B}$ over the parts, which distribution was chosen to yield an error below the prescribed tolerance in a worst case scenario, to enhance its quantitative performance, the current implementation is based on a minimization of the cost for yielding an error below the tolerance using a posteriori information.

APPLY$[\mathbf{w}, \varepsilon] \to \mathbf{z}_\varepsilon$:
% Input: $\mathbf{w} \in \ell_0$ *and* $\varepsilon > 0$.

1. $[(\mathbf{w}_{[p]})_p] := \mathbf{BUCKETSORT}[\mathbf{w}, \varepsilon/(2e_0)]$

2. *Compute the smallest* $\ell \in \mathbb{N}_0$ *with*

$$\delta := e_0 \|\mathbf{w} - \sum_{p=1}^{\ell} \mathbf{w}_{[p]}\| \le \varepsilon/2.$$

3. *Determine* $\mathbf{j} \in \mathbb{N}_0^\ell$ *such that* $\sum_{p=1}^{\ell} e_{\mathbf{j}_p} \|\mathbf{w}_{[p]}\| \le \varepsilon - \delta$ *and* $c_{\mathbf{j}_p} \lesssim c_{\tilde{\mathbf{j}}_p}$ $(p = 1, \dots, \ell)$, *where* $\tilde{\mathbf{j}} \in \mathbb{N}_0^\ell$ *is the solution of*

$$\sum_{p=1}^{\ell} c_{\tilde{\mathbf{j}}_p} \#\mathrm{supp}\, \mathbf{w}_{[p]} \to \min!, \quad \sum_{p=1}^{\ell} e_{\tilde{\mathbf{j}}_p} \|\mathbf{w}_{[p]}\| \le \varepsilon - \delta. \tag{20}$$

4. *Compute*

$$z_\varepsilon := \sum_{p=1}^{\ell} \mathbf{B}^{(\mathbf{j}_p)} \mathbf{w}_{[p]}.$$

In practice, the cost of solving the exact solution (i.e., $\mathbf{j} = \tilde{\mathbf{j}}$) of the small optimization problem in 3 is neglectable. By using that $\ell = \mathcal{O}(|\log \varepsilon|)$ (see the proof of Theorem 2) below), and by deriving some a priori bounds for $\|\tilde{\mathbf{j}}\|_\infty$, we expect it to be possible to prove that these cost are indeed always neglectable compared to the other cost of the algorithm. Instead of doing so, however, we show how to find analytically a near optimum in 2 common situations: If for some constants C and D, $c_j = C2^{j/s^*}$ and $e_j = D2^{-j}$, so that $\mathbf{B}$ is s^*-computable, then

$$\tilde{\mathbf{j}}_p = \log_2\left(\left(\frac{\|\mathbf{w}_{[p]}\|}{\#\mathrm{supp}\,\mathbf{w}_{[p]}}\right)^{\frac{s^*}{s^*+1}} \frac{\sum_{q=1}^{\ell}\|\mathbf{w}_{[q]}\|^{\frac{1}{s^*+1}}(\#\mathrm{supp}\,\mathbf{w}_{[q]})^{\frac{s^*}{1+s^*}}}{(\varepsilon-\delta)/D}\right)$$

is the solution of (20) when minimization is performed over $\mathbb{R}^\ell$. If for some constants C, D and $\omega > 0$, $c_j = Cj/\omega$ and $e_j = D2^{-j}$, so that $\mathbf{B}$ is even ∞-computable, then

$$\tilde{\mathbf{j}}_p = \log_2\left(\frac{\|\mathbf{w}_{[p]}\|\sum_{q=1}^{\ell}\#\mathrm{supp}\,\mathbf{w}_{[q]}}{\#\mathrm{supp}\,\mathbf{w}_{[p]}(\varepsilon-\delta)/D}\right)$$

is the solution of (20) when minimization is performed over $\mathbb{R}^\ell$. Assuming these $\tilde{\mathbf{j}}_p$ are non-negative, by rounding them up to the nearest value in $\mathbb{N}_0$ one obtains a valid $\mathbf{j}$.

Theorem 2. $\mathbf{z}_\varepsilon := \mathbf{APPLY}[\mathbf{w},\varepsilon]$ *satisfies* $\|\mathbf{B}\mathbf{w}-\mathbf{z}_\varepsilon\| \le \varepsilon$. *If* $\mathbf{B}$ *is* s^**-computable, then for any* $s < s^*$,

$$\#\mathrm{supp}\,\mathbf{z}_\varepsilon \lesssim \varepsilon^{-1/s}\|\mathbf{w}\|_{\mathscr{A}^s}^{1/s}, \tag{21}$$

where the number of operations required by the call is bounded by some absolute multiple of

$$\varepsilon^{-1/s}\|\mathbf{w}\|_{\mathscr{A}^s}^{1/s} + \#\mathrm{supp}\,\mathbf{w} + 1. \tag{22}$$

In other words, $\mathbf{B}$ *is* $\bar{s}$*-admissible for any* $\bar{s} < s^*$.

Proof. The estimates $\|\mathbf{B}\|\|\mathbf{w}-\sum_{p=1}^{\ell}\mathbf{w}_{[p]}\| \le \delta$ and $\sum_{p=1}^{\ell}\|\mathbf{B}-\mathbf{B}^{(\mathbf{j}_p)}\|\|\mathbf{w}_{[p]}\| \le \varepsilon-\delta$ show the first statement.

Let $s \in (0,s^*)$ and select $s < s_1 < s_2 < s^*$.

As we have seen, the cost of the call $\mathbf{BUCKETSORT}[\mathbf{w},\varepsilon/(2e_0)]$ is bounded by an absolute multiple of $\#\mathrm{supp}\,\mathbf{w}$ plus the number of buckets, the latter being not larger than $\max(1,\lceil 2\log_2(\|\mathbf{w}\|_\infty\sqrt{\#\mathrm{supp}\,\mathbf{w}}/(\varepsilon/(2e_0)))\rceil)$, so that the cost of the call is bounded by an absolute multiple of $\#\mathrm{supp}\,\mathbf{w}+1+\varepsilon^{-1/s}\|\mathbf{w}\|_{\mathscr{A}^s}^{1/s}$.

With $\tau := (\frac{1}{2}+s)^{-1}$, Lemma 2 shows that

$$\#\mathrm{supp}\,\mathbf{w}_{[p]} \le \#\{\lambda\in\nabla : |\mathbf{w}_\lambda| > 2^{-p/2}\|\mathbf{w}\|_\infty\} \lesssim 2^{p\tau/2}\|\mathbf{w}\|_\infty^{-\tau}\|\mathbf{w}\|_{\mathscr{A}^s}^{\tau},$$

so that

$$\|\mathbf{w}_{[p]}\| \lesssim 2^{-p/2}\|\mathbf{w}\|_\infty\sqrt{\#\mathrm{supp}\,\mathbf{w}_{[p]}} \lesssim 2^{-ps\tau/2}\|\mathbf{w}\|_\infty^{1-\tau/2}\|\mathbf{w}\|_{\mathscr{A}^s}^{\tau/2}.$$

The proof will be completed once we have shown that there exists *some* $\mathbf{j}\in\mathbb{N}_0^\ell$ with $\sum_{p=1}^{\ell}e_{\mathbf{j}_p}\|\mathbf{w}_{[p]}\| \le \varepsilon-\delta$ and $\sum_{p=1}^{\ell}c_{\mathbf{j}_p}\#\mathrm{supp}\,\mathbf{w}_{[p]} \lesssim \varepsilon^{-1/s}\|\mathbf{w}\|_{\mathscr{A}^s}^{1/s}$. For $\ell=0$ there is nothing to prove, so we assume that $\ell>0$.

First, we derive an upper bound for ℓ determined in step 2 of **APPLY**. By definition of ℓ, we have

$$\varepsilon/2 < e_0\|\mathbf{w}-\sum_{p=1}^{\ell-1}\mathbf{w}_{[p]}\| = e_0\sqrt{\sum_{p=\ell}^{\infty}\|\mathbf{w}_{[p]}\|^2} \lesssim e_0 2^{-\ell s\tau/2}\|\mathbf{w}\|_\infty^{1-\tau/2}\|\mathbf{w}\|_{\mathscr{A}^s}^{\tau/2},$$

or

$$2^{\ell\tau/2}\|\mathbf{w}\|_\infty^{-\tau}\|\mathbf{w}\|_{\mathscr{A}^s}^{\tau} \lesssim \varepsilon^{-1/s}\|\mathbf{w}\|_{\mathscr{A}^s}^{1/s}. \tag{23}$$

Note that here we used the notation $\mathbf{w}_{[p]}$ also to denote the restriction of $\mathbf{w}$ to indices in buckets beyond those that were generated by the call **BUCKETSORT**$[\mathbf{w}, \varepsilon/(2e_0)]$.

Next, let $J \geq \ell$ be defined as the smallest integer such that

$$\sum_{p=1}^{\ell} 2^{-(J-p)s_1\tau/2}\|\mathbf{w}_{[p]}\| \leq \varepsilon - \delta. \tag{24}$$

In case that $J > \ell$, from $s_1 > s$ we have

$$\begin{aligned}
\varepsilon/2 \leq \varepsilon - \delta &< \sum_{p=1}^{\ell} 2^{-(J-1-p)s_1\tau/2}\|\mathbf{w}_{[p]}\| \\
&< \sum_{p=1}^{\ell} 2^{-(J-1-p)s_1\tau/2} 2^{-ps\tau/2}\|\mathbf{w}\|_\infty^{1-\tau/2}\|\mathbf{w}\|_{\mathscr{A}^s}^{\tau/2} \\
\textit{synthesis} &\lesssim 2^{-(J-1-\ell)s_1\tau/2} 2^{-\ell s\tau/2}\|\mathbf{w}\|_\infty^{1-\tau/2}\|\mathbf{w}\|_{\mathscr{A}^s}^{\tau/2} \\
&\leq 2^{-(J-1)s\tau/2}\|\mathbf{w}\|_\infty^{1-\tau/2}\|\mathbf{w}\|_{\mathscr{A}^s}^{\tau/2},
\end{aligned}$$

or

$$2^{J\tau/2}\|\mathbf{w}\|_\infty^{-\tau}\|\mathbf{w}\|_{\mathscr{A}^s}^{\tau} \lesssim \varepsilon^{-1/s}\|\mathbf{w}\|_{\mathscr{A}^s}^{1/s}. \tag{25}$$

From (23) we see that the upper bound on J given by (25) is also valid when $J = \ell$.

Now we select $\mathbf{j}_p$ as to be the smallest integer such that $e_{\mathbf{j}_p} \leq 2^{-(J-p)s_1\tau/2}$. Then (24) shows that indeed $\sum_{p=1}^{\ell} e_{\mathbf{j}_p}\|\mathbf{w}_{[p]}\| \leq \varepsilon - \delta$. Because of $\sup_j e_j c_j^{s_2} < \infty$ and $\sup_j c_{j+1}/c_j < \infty$, we have $c_{\mathbf{j}_p} \lesssim c_{\mathbf{j}_p - 1} \lesssim e_{\mathbf{j}_p - 1}^{-1/s_2} < 2^{(J-p)(s_1/s_2)(\tau/2)}$. From (25), we conclude that

$$\begin{aligned}
\sum_{p=1}^{\ell} c_{\mathbf{j}_p}\#\operatorname{supp}\mathbf{w}_{[p]} &\lesssim \sum_{p=1}^{\ell} 2^{(J-p)(s_1/s_2)\tau/2} 2^{p\tau/2}\|\mathbf{w}\|_\infty^{-\tau}\|\mathbf{w}\|_{\mathscr{A}^s}^{\tau} \\
&\lesssim 2^{(J-\ell)(s_1/s_2)\tau/2} 2^{\ell\tau/2}\|\mathbf{w}\|_\infty^{-\tau}\|\mathbf{w}\|_{\mathscr{A}^s}^{\tau} \\
&\lesssim 2^{J\tau/2}\|\mathbf{w}\|_\infty^{-\tau}\|\mathbf{w}\|_{\mathscr{A}^s}^{\tau} \lesssim \varepsilon^{-1/s}\|\mathbf{w}\|_{\mathscr{A}^s}^{1/s},
\end{aligned}$$

which completes the proof. □

3.4 Non-coercive **B**

If **B** is non-coercive, then the Richardson iteration may not converge, and so the inexact Richardson iteration does not apply. A general applicable remedy is to apply the inexact Richardson iteration to the *normal equations*

$$\mathbf{B}^\top \mathbf{B}\mathbf{u} = \mathbf{B}^\top \mathbf{f}.$$

Clearly, $\mathbf{B}^\top\mathbf{B}$ is symmetric, positive definite, and boundedly invertible with $\|\mathbf{B}^\top\mathbf{B}\| = \|\mathbf{B}\|^2$ and $\|(\mathbf{B}^\top\mathbf{B})^{-1}\| = \|\mathbf{B}^{-1}\|^2$. In order to conclude that the inexact Richardson iteration applied to the normal equations is (quasi-) optimal, what is left to show is that for some $\bar{s} \geq s_{\max}$, $\mathbf{B}^\top\mathbf{B}$ is $\bar{s}$-admissible, and that we have a valid routine for approximating the right-hand side $\mathbf{B}^\top\mathbf{f}$ in the sense of Remark 5. Proposition 3 from [CDD02, Sect. 7] given below shows that both conditions are fulfilled when $\mathbf{B}$ and $\mathbf{B}^\top$ are $\bar{s}$-admissible for some $\bar{s} \geq s_{\max}$.

Concerning the latter, from Theorem 2, recall that $\mathbf{B}$ is $\bar{s}$-admissible for some $\bar{s} \geq s_{\max}$ when it is s^*-computable for some $s^* > s_{\max}$. The results demonstrating s^*-computability of $\mathbf{B}$, that will be given in Sect. 5, are symmetric in the sense that they also show s^*-computability of $\mathbf{B}^\top$ for the same value of s^*.

Proposition 3. (a). *If* $\mathbf{B}$ *and* $\mathbf{B}^\top$ *are* $\bar{s}$*-admissible, then so is* $\mathbf{B}^\top\mathbf{B}$*. With the* **APPLY** *routines for* $\mathbf{B}$ *and* $\mathbf{B}^\top$ *denoted as* $\mathbf{APPLY}_{\mathbf{B}}$ *and* $\mathbf{APPLY}_{\mathbf{B}^T}$*, respectively, and with* e_0 *being an upper bound for* $\|\mathbf{B}\|$*, a valid* **APPLY** *for* $\mathbf{B}^\top\mathbf{B}$ *is given by*

$$[\mathbf{w}, \varepsilon] \mapsto \mathbf{z}_\varepsilon := \mathbf{APPLY}_{\mathbf{B}^\top}[\mathbf{APPLY}_{\mathbf{B}}[\mathbf{w}, \varepsilon/(2e_0)], \varepsilon/2].$$

(b). *For* $\varepsilon > 0$, $\mathbf{g}_\varepsilon := \mathbf{APPLY}_{\mathbf{B}^\top}[\mathbf{RHS}[\varepsilon/(2e_0)], \varepsilon/2]$ *satisfies* $\|\mathbf{B}^\top\mathbf{f} - \mathbf{g}_\varepsilon\| \leq \varepsilon$. *If* $\mathbf{B}$ *and* $\mathbf{B}^\top$ *are* $\bar{s}$*-admissible, then whenever for some* $s \in (0, \bar{s}]$, $\mathbf{u} \in \mathscr{A}^s$*, it holds that* $\#\operatorname{supp}\mathbf{g}_\varepsilon \lesssim \varepsilon^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}$*, with the cost of producing it being bounded by some absolute multiple of* $\varepsilon^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s} + 1$.

Proof. (a).

$$\|\mathbf{B}^\top\mathbf{B}\mathbf{w} - \mathbf{z}_\varepsilon\| \leq \|\mathbf{B}^\top(\mathbf{B}\mathbf{w} - \mathbf{APPLY}_{\mathbf{B}}[\mathbf{w}, \varepsilon/(2e_0)])\| + \varepsilon/2 \leq \|\mathbf{B}\|\varepsilon/(2e_0) + \varepsilon/2 \leq \varepsilon.$$

Let $s \in (0, \bar{s}]$. Putting $\mathbf{t}_\varepsilon := \mathbf{APPLY}_{\mathbf{B}}[\mathbf{w}, \varepsilon/(2e_0)]$, from $\mathbf{B}$ being $\bar{s}$-admissible, we know that $\#\operatorname{supp}\mathbf{t}_\varepsilon \lesssim \varepsilon^{-1/s}\|\mathbf{w}\|_{\mathscr{A}^s}^{1/s}$, and that the cost of producing it is bounded by some absolute multiple of $\varepsilon^{-1/s}\|\mathbf{w}\|_{\mathscr{A}^s}^{1/s} + \#\operatorname{supp}\mathbf{w} + 1$. Moreover, Proposition 1 shows that $\|\mathbf{t}_\varepsilon\|_{\mathscr{A}^s} \lesssim \|\mathbf{w}\|_{\mathscr{A}^s}$, uniformly in ε (and in $\mathbf{w}$).

From $\mathbf{B}^\top$ being $\bar{s}$-admissible, we know that $\#\operatorname{supp}\mathbf{z}_\varepsilon \lesssim \varepsilon^{-1/s}\|\mathbf{t}_\varepsilon\|_{\mathscr{A}^s}^{1/s} \lesssim \varepsilon^{-1/s}\|\mathbf{w}\|_{\mathscr{A}^s}^{1/s}$, and that the cost of producing it from $\mathbf{t}_\varepsilon$ is bounded by a constant multiple of $\varepsilon^{-1/s}\|\mathbf{t}_\varepsilon\|_{\mathscr{A}^s}^{1/s} + \#\operatorname{supp}\mathbf{t}_\varepsilon + 1 \lesssim \varepsilon^{-1/s}\|\mathbf{w}\|_{\mathscr{A}^s}^{1/s} + 1$. We conclude that indeed $\mathbf{B}^\top\mathbf{B}$ is $\bar{s}$-admissible.

The proof of (b) is similar to that of (a). □

3.5 Alternatives for the Richardson iteration

As already appears from Lemma 3, the quantitative performance of the approximate Richardson scheme will depend on the spectral condition number of the matrix be-

ing inverted. In this respect, the approach, for non-coercive $\mathbf{B}$, of applying the inexact Richardson iteration to the normal equations, which gives rise to a squared condition number, might not always be the best possible choice.

For (symmetric) saddle point problems, as the Stokes equations from Sect. 2.2.3, as alternatives, in [CDD02] it was proposed to apply the inexact Richardson iteration to the reformulation introduced in [BP88] of the saddle point problem as a symmetric positive definite system, or to the Schur complement system (if necessary after first switching to the augmented Lagrangian formulation). In the latter case, each iteration requires the application of the Schur complement operator, and so in particular, the solution of an elliptic system. Necessarily, these systems can only be solved approximately, in which case the resulting scheme is known as the inexact Uzawa iteration. With the inner elliptic problems being solved with an adaptive wavelet method, (quasi-) optimality of the overall scheme in the sense of Theorem 1 was demonstrated in [DDU02].

Also in cases where the Richardson scheme applies directly to $\mathbf{Bu} = \mathbf{f}$, one may think of applying a more advanced iterative method. For symmetric and positive definite $\mathbf{B}$, in [CU05, DFR$^+$07b] it was shown that an approximate *Steepest Descent* method, with appropriate tolerances for the inexact matrix-vector and right-hand side evaluations, is (quasi-) optimal. Since the asymptotic convergence rate of the optimally damped Richardson iteration is equal to that of the Steepest Descent method, the main advantage of the latter scheme lies in the fact that it frees the user of the task of providing accurate estimates of the extremal eigenvalues of $\mathbf{B}$.

For $\mathbf{B}$ being only coercive, instead of the Steepest Descent method, the *Minimal Residual* method (see e.g. [Saa03]) might be applied. We envisage that (quasi-) optimality of an approximate Minimal Residual method can be proven along the same lines as for the Steepest Descent method. Since for $\mathbf{B}$ being only coercive it is even less obvious how to choose the damping parameter in the Richardson scheme, the advantage of the approximate Minimal Residual method is likely even bigger.

Even more advanced schemes than the Steepest Descent or Minimal Residual method are *Krylov subspace* methods, like the Conjugate Gradient method for symmetric positive definite systems. Clearly, in the infinite dimensional setting, these schemes can only be applied with inexact evaluations of the residuals. Numerical results are reported in [BK08]. With a suitable choice of the tolerances for these inexact evaluations, the approximate Conjugate Gradient method has been shown to converge ([vS04]). Yet, as far as we know, in the infinite dimensional setting it has not been proven that there exists a choice of the tolerances such that the resulting scheme is not only convergent but also (quasi-) optimal. Indeed, recall that the tolerances determine the support lengths of the iterands (except immediately after coarsening), and with that the cost of the algorithm. So in view of this observation, it is not necessarily true that a faster converging iteration gives rise, when applied approximately, to a quantitatively better performing adaptive wavelet scheme.

In the next section, we will study the Adaptive Wavelet-Galerkin Method proposed in [CDD01] and later modified in [GHS07]. As we will see, unlike the methods we discussed so far, this scheme can *not* be viewed as an inexact evaluation of some convergent iterative scheme applied to the bi-infinite matrix vector problem.

4 Adaptive wavelet schemes II: The Adaptive Wavelet-Galerkin Method ([CDD01, GHS07])

Throughout this section we will assume that $\mathbf{B}$ is symmetric and positive definite, i.e., $\mathbf{B} = \mathbf{B}^\top > 0$. On $\ell_2(\nabla)$, we define

$$|||\cdot||| := \langle \mathbf{B}\cdot, \cdot\rangle^{\frac{1}{2}}.$$

Remark 7. If $\mathbf{B}$ is not symmetric and positive definite, then the scheme presented here can be applied to the normal equations $\mathbf{B}^\top\mathbf{B}\mathbf{u} = \mathbf{B}^\top\mathbf{f}$, meaning that in the following everywhere $\mathbf{B}$ should be read as $\mathbf{B}^\top\mathbf{B}$ and $\mathbf{f}$ as $\mathbf{B}^\top\mathbf{f}$.

For any $\Lambda \subset \nabla$, with $\ell_2(\Lambda)$ we will mean the subspace of $\mathbf{v} \in \ell_2(\nabla)$ with supports in Λ. The trivial embedding of $\ell_2(\Lambda)$ into $\ell_2(\nabla)$ will be denoted by $\mathbf{I}_\Lambda$, and its adjoint with respect to $\langle\cdot,\cdot\rangle$, i.e., the operator that replaces coefficients outside Λ by zeros, will be denoted by $\mathbf{P}_\Lambda$. We set

$$\mathbf{B}_\Lambda := \mathbf{P}_\Lambda \mathbf{B} \mathbf{I}_\Lambda.$$

Using that $\mathbf{B}$ is symmetric and positive definite, one verifies that for any $\Lambda \subseteq \nabla$,

$$\begin{aligned} \|\mathbf{B}_\Lambda^{-1}\|^{-\frac{1}{2}}\|\cdot\| \le \ |||\cdot||| \ &\le \|\mathbf{B}_\Lambda\|^{\frac{1}{2}}\|\cdot\| \quad \text{on } \ell_2(\Lambda), \\ \|\mathbf{B}_\Lambda^{-1}\|^{-\frac{1}{2}}|||\cdot||| \le \|\mathbf{B}_\Lambda\cdot\| &\le \|\mathbf{B}_\Lambda\|^{\frac{1}{2}}|||\cdot||| \quad \text{on } \ell_2(\Lambda), \end{aligned}$$

as well as $\|\mathbf{B}_\Lambda\| \le \|\mathbf{B}\|$ and $\|\mathbf{B}_\Lambda^{-1}\| \le \|\mathbf{B}^{-1}\|$, which properties will be used often in the following.

4.1 *The Adaptive Wavelet-Galerkin Method (AWGM) in a idealized setting*

The solution $\mathbf{u}_\Lambda \in \ell_2(\Lambda)$ of $\mathbf{B}_\Lambda\mathbf{u}_\Lambda = \mathbf{P}_\Lambda\mathbf{f}$ is known as the *Galerkin approximation* to $\mathbf{u}$ from $\ell_2(\Lambda)$. With respect to $|||\cdot|||$, it is the best approximation to $\mathbf{u}$ from this subspace.

The idea of the AWGM is to loop over the following 2 steps: Given $\Lambda \subset \nabla$, compute the Galerkin approximation $\mathbf{u}_\Lambda$. Enlarge Λ to a set $\tilde{\Lambda} \subset \nabla$ such that for some constant $\rho < 1$, $|||\mathbf{u} - \mathbf{u}_{\tilde{\Lambda}}||| \le \rho|||\mathbf{u} - \mathbf{u}_\Lambda|||$. This loop is similar to the one that underlies the Adaptive Finite Element Method (AFEM), where the enlargement of Λ corresponds to mesh-refinement. The AFEM is discussed by R. Nochetto in another chapter of this book.

In the AFEM, a refinement that guarantees error reduction is obtained by computing an a posteriori error estimator, being the square root of the sum of local error indicators associated to the elements, and by refining those elements that carry the largest error indicators and whose joint sum can be bounded from below on a con-

stant multiple of the total squared a posteriori error estimator (this is known as the so-called *bulk criterion*). The AWGM works according to the same principle, with the role of the a posteriori error estimator being played by the residual $\mathbf{f}-\mathbf{B}\mathbf{u}_\Lambda$, where for the moment we ignore the fact that this residual cannot be computed exactly.

The next lemma, being [CDD01, Lemma 4.1], shows convergence of the AWGM. Although in this lemma $\mathbf{w}$ can be a general function in $\ell_2(\Lambda)$, we have in mind it to be (an approximation to) the Galerkin approximation $\mathbf{u}_\Lambda$.

Lemma 5. *Let $\mu \in (0,1]$, $\mathbf{w} \in \ell_2(\Lambda)$ and $\Lambda \subset \tilde{\Lambda} \subset \nabla$ such that*

$$\|\mathbf{P}_{\tilde{\Lambda}}(\mathbf{f}-\mathbf{B}\mathbf{w})\| \geq \mu \|\mathbf{f}-\mathbf{B}\mathbf{w}\|. \tag{26}$$

Then, for $\mathbf{u}_{\tilde{\Lambda}} \in \ell_2(\tilde{\Lambda})$ being the solution of $\mathbf{B}_{\tilde{\Lambda}}\mathbf{u}_{\tilde{\Lambda}} = \mathbf{P}_{\tilde{\Lambda}}\mathbf{f}$, we have

$$|||\mathbf{u}-\mathbf{u}_{\tilde{\Lambda}}||| \leq \left[1-\mu^2\kappa(\mathbf{B})^{-1}\right]^{\frac{1}{2}} |||\mathbf{u}-\mathbf{w}|||.$$

Proof. We have

$$\begin{aligned} |||\mathbf{u}_{\tilde{\Lambda}}-\mathbf{w}||| &\geq \|\mathbf{B}\|^{-\frac{1}{2}}\|\mathbf{B}(\mathbf{u}_{\tilde{\Lambda}}-\mathbf{w})\| \geq \|\mathbf{B}\|^{-\frac{1}{2}}\|\mathbf{P}_{\tilde{\Lambda}}(\mathbf{f}-\mathbf{B}\mathbf{w})\| \\ &\geq \|\mathbf{B}\|^{-\frac{1}{2}}\mu\|\mathbf{f}-\mathbf{B}\mathbf{w}\| \geq \mu\kappa(\mathbf{B})^{-\frac{1}{2}}|||\mathbf{u}-\mathbf{w}|||. \end{aligned}$$

The proof of is completed by using the Galerkin orthogonality

$$|||\mathbf{u}-\mathbf{w}|||^2 = |||\mathbf{u}-\mathbf{u}_{\tilde{\Lambda}}|||^2 + |||\mathbf{u}_{\tilde{\Lambda}}-\mathbf{w}|||^2. \qquad \square$$

In Lemma 5, $\tilde{\Lambda}$ is some enlargement of Λ such that the bulk criterion (26) is satisfied. The natural approach is to construct $\tilde{\Lambda}$ by gathering the indices of the *largest* elements in modulus of the residual. In [CDD01], the corresponding practical algorithm –i.e., with the inexact solution of the arising Galerkin systems and the inexact evaluation of the residuals using the **APPLY** and **RHS** routines– was shown to be (quasi-) optimal by the addition of a recurrent application of **COARSE**, similar to the inexact Richardson iteration from Sect. 3.

In the next lemma, being [GHS07, Lemma 2.1], it is shown that when μ is taken to be sufficiently small, then the cardinality of the expansion $\tilde{\Lambda}\backslash\Lambda$ can be controlled. This lemma will be the key to show that the algorithm from [CDD01] *without* a recurrent coarsening of the iterands is already (quasi-) optimal (coarsening will still be used to find the large entries from the approximate residuals). Later, basically the same technique was used to show that the standard adaptive finite element method, so without coarsening, is (quasi-) optimal, see [Ste07].

Lemma 6. *If, in the situation of Lemma 5, $\mu < \kappa(\mathbf{B})^{-\frac{1}{2}}$ and $\tilde{\Lambda} \supset \Lambda$ is the* smallest *set satisfying* (26)*, then*

$$\#(\tilde{\Lambda}\backslash\Lambda) \leq \min\{N : |||\mathbf{u}-\mathbf{u}_N||| \leq [1-\mu^2\kappa(\mathbf{B})]^{\frac{1}{2}}|||\mathbf{u}-\mathbf{w}|||\}. \tag{27}$$

Proof. For an N as in the right-hand side of (27), let $\breve{\Lambda} := \Lambda \bigcup \operatorname{supp} \mathbf{u}_N$. Then, for the solution of $\mathbf{B}_{\breve{\Lambda}} \mathbf{u}_{\breve{\Lambda}} = \mathbf{P}_{\breve{\Lambda}} \mathbf{f}$, we have $|||\mathbf{u} - \mathbf{u}_{\breve{\Lambda}}||| \leq |||\mathbf{u} - \mathbf{u}_N|||$, and so by Galerkin orthogonality

$$|||\mathbf{u}_{\breve{\Lambda}} - \mathbf{w}||| \geq \mu \kappa(\mathbf{B})^{\frac{1}{2}} |||\mathbf{u} - \mathbf{w}|||,$$

giving

$$\begin{aligned} \|\mathbf{P}_{\breve{\Lambda}}(\mathbf{f} - \mathbf{B}\mathbf{w})\| &= \|\mathbf{B}_{\breve{\Lambda}}(\mathbf{u}_{\breve{\Lambda}} - \mathbf{w})\| \geq \|\mathbf{B}^{-1}\|^{-\frac{1}{2}} |||\mathbf{u}_{\breve{\Lambda}} - \mathbf{w}||| \\ &\geq \|\mathbf{B}^{-1}\|^{-\frac{1}{2}} \mu \kappa(\mathbf{B})^{\frac{1}{2}} |||\mathbf{u} - \mathbf{w}||| \geq \mu \|\mathbf{f} - \mathbf{B}\mathbf{w}\|. \end{aligned}$$

By our assumption on $\tilde{\Lambda}$, we conclude that $\#(\tilde{\Lambda} \backslash \Lambda) \leq \#(\breve{\Lambda} \backslash \Lambda) \leq N$. □

Lemmas 5 and 6 suggest the following routine:

exact-AWGM:
% Parameter: $\mu \in (0, \kappa(\mathbf{B})^{-\frac{1}{2}})$.

$\Lambda_0 := \emptyset$, $\mathbf{u}_{\Lambda_0} := 0$,
`for` $i = 1, 2, \ldots$ `do`
  find the smallest $\Lambda_{i+1} \supset \Lambda_i$ *with* $\|\mathbf{P}_{\Lambda_{i+1}}(\mathbf{f} - \mathbf{B}\mathbf{u}_{\Lambda_i})\| \geq \mu \|\mathbf{f} - \mathbf{B}\mathbf{u}_{\Lambda_i}\|$
  solve $\mathbf{B}_{\Lambda_{i+1}} \mathbf{u}_{\Lambda_{i+1}} = \mathbf{P}_{\Lambda_{i+1}} \mathbf{f}$
`enddo`

Proposition 4. *For* $(\mathbf{u}_{\Lambda_i})_i$ *produced by* **exact-AWGM**, *we have*

$$|||\mathbf{u} - \mathbf{u}_{\Lambda_i}||| \leq \left[1 - \mu^2 \kappa(\mathbf{B})^{-1}\right]^{i/2} |||\mathbf{u}|||,$$

and if for some $s > 0$, $\mathbf{u} \in \mathscr{A}^s$, *then*

$$\#\operatorname{supp} \mathbf{u}_{\Lambda_i} \lesssim \|\mathbf{u} - \mathbf{u}_{\Lambda_{i-1}}\|^{-1/s} \|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}.$$

Proof. For $0 \leq k \leq i$, Lemma 5 shows that $|||\mathbf{u} - \mathbf{u}_{\Lambda_i}||| \leq \rho^{i-k} |||\mathbf{u} - \mathbf{u}_{\Lambda_k}|||$ where $\rho := \left[1 - \mu^2 \kappa(\mathbf{B})^{-1}\right]^{\frac{1}{2}}$, which in particular shows the first statement.

Assuming that $\mathbf{u} \in \mathscr{A}^s$ for some $s > 0$, with $\sigma := [1 - \mu^2 \kappa(\mathbf{B})]^{\frac{1}{2}}$, Lemma 6 shows that

$$\begin{aligned} \#(\boldsymbol{\Lambda}_k \backslash \boldsymbol{\Lambda}_{k-1}) &\leq \min\{N : |||\mathbf{u} - \mathbf{u}_N||| \leq \sigma |||\mathbf{u} - \mathbf{u}_{\Lambda_{k-1}}|||\} \\ &\leq \min\{N : \|\mathbf{u} - \mathbf{u}_N\| \leq \|\mathbf{B}\|^{-\frac{1}{2}} \sigma |||\mathbf{u} - \mathbf{u}_{\Lambda_{k-1}}|||\} \\ &\leq [\|\mathbf{B}\|^{-\frac{1}{2}} \sigma |||\mathbf{u} - \mathbf{u}_{\Lambda_{k-1}}|||]^{-1/s} \|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}, \end{aligned}$$

by $|||\cdot||| \leq \|\mathbf{B}\|^{\frac{1}{2}} \|\cdot\|$ and the definition of $\|\cdot\|_{\mathscr{A}^s}$.

By combining both estimates, for $i \in \mathbb{N}$ we have

$$\begin{aligned}\#\mathrm{supp}\,\mathbf{u}_{\Lambda_i} \le \#\boldsymbol{\Lambda}_i &= \sum_{k=1}^{i} \#(\boldsymbol{\Lambda}_k \backslash \boldsymbol{\Lambda}_{k-1}) \le \|\mathbf{B}\|^{1/2s}\sigma^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s} \sum_{k=1}^{i} |||\mathbf{u}-\mathbf{u}_{\Lambda_{k-1}}|||^{-1/s} \\ &\le \|\mathbf{B}\|^{1/2s}\sigma^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}|||\mathbf{u}-\mathbf{u}_{\Lambda_{i-1}}|||^{-1/s} \sum_{k=1}^{i} (\rho^{i-k})^{1/s} \\ &\le \kappa(\mathbf{B})^{1/2s}\frac{\sigma^{-1/s}}{1-\rho^{1/s}}\|\mathbf{u}-\mathbf{u}_{\Lambda_{i-1}}\|^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}, \end{aligned} \tag{28}$$

by $\|\cdot\| \le \|\mathbf{B}^{-1}\|^{\frac{1}{2}}|||\cdot|||$. □

In view of the definition of $\mathscr{A}^s$, note that the bound on $\#\mathrm{supp}\,\mathbf{u}_{\Lambda_i}$ derived in Proposition 4 is, up to some constant multiple, the generally best possible one. That is, not taking into account the computational cost, the routine **exact-AWGM** is *(quasi-) optimal*.

4.2 Practical scheme

In this subsection, we turn **exact-AWGM** into a practical scheme by

- computing residuals only approximately,
- allowing that for the enlargement Λ_{i+1} of Λ_i, which satisfies the "bulk criterion", $\#(\Lambda_{i+1}\backslash\Lambda_i)$ is only minimal up to some constant multiple,
- solving the arising Galerkin problems only approximately.

The following proposition extends Lemmas 5 and 6 to this setting.

Proposition 5. *Let $\delta \in (0,\alpha)$, $\gamma > 0$ be constants such that $\mu := \frac{\alpha+\delta}{1-\delta} < \kappa(\mathbf{B})^{-\frac{1}{2}}$ and $\gamma < \frac{(1-\delta)(\alpha-\delta)}{1+\delta}\kappa(\mathbf{B})^{-1}$. Given $\Lambda \subset \nabla$ and $\mathbf{w} \in \ell_2(\Lambda)$, let $\mathbf{r} \in \ell_2(\nabla)$ be such that*

$$\|\mathbf{f}-\mathbf{B}\mathbf{w}-\mathbf{r}\| \le \delta\|\mathbf{r}\|. \tag{29}$$

Let $\Lambda \subset \tilde{\Lambda} \subset \nabla$ be such that

$$\|\mathbf{P}_{\tilde{\Lambda}}\mathbf{r}\| \ge \alpha\|\mathbf{r}\| \tag{30}$$

and such that, up to some absolute multiple, $\#(\tilde{\Lambda}\backslash\Lambda)$ is minimal among all such $\tilde{\Lambda}$. Let $\tilde{\mathbf{w}} \in \ell_2(\tilde{\Lambda})$ be an approximation to $\mathbf{u}_{\tilde{\Lambda}}$ such that

$$\|\mathbf{P}_{\tilde{\Lambda}}\mathbf{f}-\mathbf{B}_{\tilde{\Lambda}}\tilde{\mathbf{w}}\| \le \gamma\|\mathbf{r}\|. \tag{31}$$

Then it holds that[1]

$$|||\mathbf{u}-\tilde{\mathbf{w}}||| \le \rho|||\mathbf{u}-\mathbf{w}|||, \tag{32}$$

where $\rho := \left[1-\left(\frac{\alpha-\delta}{1+\delta}\right)^2\kappa(\mathbf{B})^{-1}+\frac{\gamma^2}{(1-\delta)^2}\kappa(\mathbf{B})\right]^{\frac{1}{2}} < 1$, and

[1] Under the milder condition $\gamma < \frac{1}{3}(\alpha-\delta)\kappa(\mathbf{B})^{-\frac{1}{2}}$, a more complicated proof ([Gan06, Proposition 3.2.2] or [GHS07, Theorem 2.7]) shows (32) for another $\rho < 1$.

$$\#(\tilde{\Lambda}\backslash\Lambda) \lesssim \min\{N : |||\mathbf{u}-\mathbf{u}_N||| \le [1-\mu^2\kappa(\mathbf{B})]^{\frac{1}{2}}|||\mathbf{u}-\mathbf{w}|||\}.$$

Proof. From $\|\mathbf{f}-\mathbf{B}\mathbf{w}\| \le (1+\delta)\|\mathbf{r}\|$ and $\|\mathbf{P}_{\tilde{\Lambda}}\mathbf{r}\| \le \|\mathbf{P}_{\tilde{\Lambda}}(\mathbf{f}-\mathbf{B}\mathbf{w})\| + \delta\|\mathbf{r}\|$, we have $\|\mathbf{P}_{\tilde{\Lambda}}(\mathbf{f}-\mathbf{B}\mathbf{w})\| \ge (\alpha-\delta)\|\mathbf{r}\| \ge \frac{\alpha-\delta}{1+\delta}\|\mathbf{f}-\mathbf{B}\mathbf{w}\|$, so that Lemma 5 shows that

$$|||\mathbf{u}-\mathbf{u}_{\tilde{\Lambda}}||| \le [1-(\tfrac{\alpha-\delta}{1+\delta})^2\kappa(\mathbf{B})^{-1}]^{\frac{1}{2}}|||\mathbf{u}-\mathbf{w}|||. \tag{33}$$

We have

$$\begin{aligned}|||\mathbf{u}_{\tilde{\Lambda}}-\tilde{\mathbf{w}}||| &\le \|\mathbf{B}^{-1}\|^{\frac{1}{2}}\|\mathbf{P}_{\tilde{\Lambda}}\mathbf{f}-\mathbf{B}_{\tilde{\Lambda}}\tilde{\mathbf{w}}\| \le \|\mathbf{B}^{-1}\|^{\frac{1}{2}}\gamma\|\mathbf{r}\| \\ &\le \|\mathbf{B}^{-1}\|^{\frac{1}{2}}\tfrac{\gamma}{1-\delta}\|\mathbf{f}-\mathbf{B}\mathbf{w}\| \le \tfrac{\gamma}{1-\delta}\kappa(\mathbf{B})^{\frac{1}{2}}|||\mathbf{u}-\mathbf{w}|||.\end{aligned}$$

The last two displayed formulas together with $|||\mathbf{u}-\tilde{\mathbf{w}}|||^2 = |||\mathbf{u}-\mathbf{u}_{\tilde{\Lambda}}|||^2 + |||\mathbf{u}_{\tilde{\Lambda}}-\tilde{\mathbf{w}}|||^2$ show (32). The condition on γ shows that $\rho < 1$.

Let $\Lambda \subset \hat{\Lambda} \subset \nabla$ be the smallest set with

$$\|\mathbf{P}_{\hat{\Lambda}}(\mathbf{f}-\mathbf{B}\mathbf{w})\| \ge \mu\|\mathbf{f}-\mathbf{B}\mathbf{w}\|.$$

Then

$$\mu\|\mathbf{r}\| \le \mu\|\mathbf{f}-\mathbf{B}\mathbf{w}\| + \mu\delta\|\mathbf{r}\| \le \|\mathbf{P}_{\hat{\Lambda}}(\mathbf{f}-\mathbf{B}\mathbf{w})\| + \mu\delta\|\mathbf{r}\| \le \|\mathbf{P}_{\hat{\Lambda}}\mathbf{r}\| + (1+\mu)\delta\|\mathbf{r}\|$$

or $\|\mathbf{P}_{\hat{\Lambda}}\mathbf{r}\| \ge (\mu-(1+\mu)\delta)\|\mathbf{r}\| = \alpha\|\mathbf{r}\|$. We conclude that

$$\#(\tilde{\Lambda}\backslash\Lambda) \lesssim \#(\hat{\Lambda}\backslash\Lambda) \le \min\{N : |||\mathbf{u}-\mathbf{u}_N||| \le [1-\mu^2\kappa(\mathbf{B})]^{\frac{1}{2}}|||\mathbf{u}-\mathbf{w}|||\},$$

where the last inequality follows from Lemma 6 using that $\mu < \kappa(\mathbf{B})^{-\frac{1}{2}}$. □

The selection of a $\tilde{\Lambda}$ as in (30) will be performed by a call of the following routine.

EXPAND$[\Lambda,\mathbf{r},\alpha] \to \tilde{\Lambda}$:
% Input: $\Lambda \subset \nabla$, $\#\Lambda < \infty$, $\mathbf{r} \in \ell_0$, $\alpha \in [0,1]$.

$\bar{\mathbf{r}} :=$ **COARSE**$[\mathbf{r}|_{\nabla\backslash\Lambda}, \sqrt{1-\alpha^2}\|\mathbf{r}\|]$
$\tilde{\Lambda} := \Lambda \cup \operatorname{supp}\bar{\mathbf{r}}$

Proposition 6. $\tilde{\Lambda} :=$ **EXPAND**$[\Lambda,\mathbf{r},\alpha]$ *satisfies* $\tilde{\Lambda} \supset \Lambda$, $\|\mathbf{P}_{\tilde{\Lambda}}\mathbf{r}\| \ge \alpha\|\mathbf{r}\|$, *and*

$$\#(\tilde{\Lambda}\backslash\Lambda) \lesssim \min\{\#(\breve{\Lambda}\backslash\Lambda) : \|\mathbf{P}_{\breve{\Lambda}}\mathbf{r}\| \ge \alpha\|\mathbf{r}\|, \Lambda \subset \breve{\Lambda} \subset \nabla\}.$$

The number of operations used by the call **EXPAND**$[\Lambda,\mathbf{r},\alpha]$ *is bounded by some absolute multiple of* $\#\Lambda + \#\operatorname{supp}\mathbf{r} + 1$.

Proof. We have $\|\mathbf{r}-\mathbf{P}_{\tilde{\Lambda}}\mathbf{r}\| = \|\mathbf{r}|_{\nabla\backslash\Lambda} - \bar{\mathbf{r}}\| \le \sqrt{1-\alpha^2}\|\mathbf{r}\|$, which is equivalent to $\|\mathbf{P}_{\tilde{\Lambda}}\mathbf{r}\| \ge \alpha\|\mathbf{r}\|$. The properties of **COARSE** imply the statement about the work as well as that

$$\begin{aligned}\#(\tilde{\Lambda}\backslash\Lambda) = \#\mathrm{supp}\,\bar{\mathbf{r}} &\lesssim \min\{\#\bar{\Lambda} : \bar{\Lambda} \subset \nabla\backslash\Lambda,\ \|\mathbf{r}|_{\nabla\backslash\Lambda} - \mathbf{P}_{\bar{\Lambda}}(\mathbf{r}|_{\nabla\backslash\Lambda})\| \le \sqrt{1-\alpha^2}\|\mathbf{r}\|\}\\ &= \min\{\#\bar{\Lambda} : \bar{\Lambda} \subset \nabla\backslash\Lambda,\ \|\mathbf{P}_{\Lambda\cup\bar{\Lambda}}\bar{\mathbf{r}}\| \ge \alpha\|\mathbf{r}\|\},\end{aligned}$$

which completes the proof. □

The arising Galerkin systems will be solved approximately by the application of an iterative scheme. Since the (approximate) solution of the previous Galerkin system will be used as the starting vector, a uniformly bounded number of iterations will suffice. Each iteration requires the application of $\mathbf{B}_\Lambda$. Although this matrix is close to being sparse, generally its number of non-zero entries is not of the order of $\#\Lambda$. Therefore, the iterative scheme will be executed only approximately. Below we consider the simplest option of applying an inexact Richardson iteration.

GALERKIN$[\Lambda, \bar{\mathbf{w}}_\Lambda, \delta, \varepsilon] \to \mathbf{w}_\Lambda$:
% Input: $\delta, \varepsilon > 0$, $\Lambda \subset \nabla$, $\#\nabla < \infty$, $\bar{\mathbf{w}}_\Lambda \in \ell_2(\Lambda)$ *with* $\|\mathbf{P}_\Lambda \mathbf{f} - \mathbf{B}_\Lambda \bar{\mathbf{w}}_\Lambda\| \le \delta$.
% Parameters: $\rho, \alpha, e_0 \in \mathbb{R}$, $K \in \mathbb{N}$ *such that* $\|\mathrm{Id} - \alpha\mathbf{B}\| \le \rho < 1$, $2\rho^K \le \varepsilon/\delta$,
% and $\|\mathbf{B}\| \le e_0$.

$\mathbf{v}^{(0)} := \bar{\mathbf{w}}_\Lambda$
`for` $i = 1, \dots, K$, `do`
$\quad \mathbf{v}^{(i)} := \mathbf{v}^{(i-1)} + \alpha\mathbf{P}_\Lambda(\mathbf{RHS}[\frac{\rho^i\delta}{2\alpha K e_0}] - \mathbf{APPLY}[\mathbf{v}^{(i-1)}, \frac{\rho^i\delta}{2\alpha K e_0}])$
`enddo`
$\mathbf{w}_\Lambda := \mathbf{v}^{(K)}$

The following proposition is essentially [CDD01, Prop. 6.7].

Proposition 7. $\mathbf{w}_\Lambda :=$ **GALERKIN**$[\Lambda, \bar{\mathbf{w}}_\Lambda, \delta, \varepsilon]$ *satisfies* $\|\mathbf{P}_\Lambda \mathbf{f} - \mathbf{B}_\Lambda \mathbf{w}_\Lambda\| \le \varepsilon$. *Let* $\mathbf{B}$ *be* $\bar{s}$*-admissible, and for some* $s \in (0, \bar{s}]$, $\mathbf{u} \in \mathscr{A}^s$. *Then the cost of the call can be bounded on some absolute multiple of*

$$\eta(\delta/\varepsilon)(\delta^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s} + \delta^{-1/s}\|\bar{\mathbf{w}}_\Lambda\|_{\mathscr{A}^s}^{1/s} + \#\Lambda + 1),$$

where $\eta : (0,\infty) \to [1,\infty)$ *is some non-decreasing function.*

Proof. For some $\delta_1, \dots, \delta_K \in \ell_2(\Lambda)$ with $\|\delta_i\| \le \frac{\rho^i\delta}{Ke_0}$,

$$\begin{aligned}\|\mathbf{P}_\Lambda \mathbf{f} - \mathbf{B}_\Lambda \mathbf{v}^{(K)}\| &= \|(\mathrm{Id} - \alpha\mathbf{B}_\Lambda)^K(\mathbf{P}_\Lambda \mathbf{f} - \mathbf{B}_\Lambda \mathbf{v}^{(0)}) + \mathbf{B}_\Lambda \sum_{i=1}^K (\mathrm{Id} - \alpha\mathbf{B}_\Lambda)^{K-i}\delta_i\|\\ &\le \rho^K\delta + e_0\sum_{i=1}^K \rho^{K-i}\frac{\rho^i\delta}{Ke_0} \le \varepsilon\end{aligned}$$

(cf. proof of Theorem 1). The statement about the cost follows from the $\bar{s}$-admissibility of $\mathbf{B}$ and the assumptions on **RHS**, in particular (16), Corollary 1 and Proposition 1, as well as from the fact that $K < \infty$ depending on δ/ε. □

Remark 8. The above implementation of **GALERKIN** can be improved. Instead of computing $\mathbf{P}_\Lambda \mathbf{RHS}[\eta]$ for a decreasing sequence of η's, it is better to compute once an approximation $\bar{\mathbf{f}}_\Lambda \in \ell_2(\Lambda)$ with $\|\mathbf{P}_\Lambda \mathbf{f} - \bar{\mathbf{f}}_\Lambda\| \le \eta$ for the final accuracy η (actually, then an even less accurate approximation suffices). Further, instead of approximating the application of $\mathbf{B}_\Lambda$ by using the **APPLY** routine and by afterwards restricting the result to Λ, obviously it is better not to compute any entry with index outside Λ. Also with these improvements, the routine remains quantitatively demanding because of the relatively expensive adaptive approximate matrix vector applications.

A more efficient Galerkin routine can be constructed using a *defect correction* principle. Let $\tilde{\mathbf{B}}_\Lambda$ be a *fixed* sparse matrix with $\|\mathrm{Id} - \mathbf{B}_\Lambda \tilde{\mathbf{B}}_\Lambda^{-1}\| \le \varepsilon/\delta$. Existence of such a matrix follows by assuming s^*-computability of $\mathbf{B}$. Then

$$\mathbf{w}_\Lambda := \bar{\mathbf{w}}_\Lambda + \tilde{\mathbf{B}}_\Lambda^{-1}(\mathbf{P}_\Lambda \mathbf{f} - \mathbf{B}_\Lambda \bar{\mathbf{w}}_\Lambda)$$

satisfies

$$\|\mathbf{P}_\Lambda \mathbf{f} - \mathbf{B}_\Lambda \mathbf{w}_\Lambda\| = \|(\mathrm{Id} - \mathbf{B}_\Lambda \tilde{\mathbf{B}}_\Lambda^{-1})(\mathbf{P}_\Lambda \mathbf{f} - \mathbf{B}_\Lambda \bar{\mathbf{w}}_\Lambda)\| \le \tfrac{\varepsilon}{\delta}\delta = \varepsilon.$$

By taking $\tilde{\mathbf{B}}_\Lambda$ to be somewhat more accurate, say with $\|\mathrm{Id} - \mathbf{B}_\Lambda \tilde{\mathbf{B}}_\Lambda^{-1}\| \le \varepsilon/(2\delta)$, room is left to compute the initial *defect* $\mathbf{P}_\Lambda \mathbf{f} - \mathbf{B}_\Lambda \bar{\mathbf{w}}_\Lambda$ approximately, and to approximate the application of $\tilde{\mathbf{B}}_\Lambda^{-1}$. The first task requires single calls of **RHS** and **APPLY**, and for the second task a few iterations of a fast iterative method can be applied, e.g., the conjugate residual method. Details can be found in [GHS07].

We are ready to formulate the practical AWGM. It works according to the principles outlined in Proposition 5. The tasks (30) and (31) are realized by calls of the routines **EXPAND** and **GALERKIN**, respectively. The first task (29) amounts to finding an approximation of the residual of the current iterand with a *relative* error not larger than δ. This will be implemented by initially approximating this residual with an *absolute* tolerance equal to some multiple θ of the norm of the previous residual. If this tolerance turns out to be too large, in the sense that it is not less than δ times the norm of the so computed residual, in an inner loop it is halved until the criterion is met.

In view of obtaining a quantitatively efficient implementation, one would like to choose this θ not too small, but sufficiently small such that "usually" one residual computation suffices. It can, however, never be excluded that by sheer chance at an early stage the (favourable) situation is encountered that the current iterand $\mathbf{u}^{(i)}$ is equal or exceptionally close to the solution $\mathbf{u}$. Then the algorithm will continue halving the tolerance until the norm of the computed residual plus tolerance is not larger than the target ε, showing that the true residual is not larger than ε. Since in this case there is no point in expanding the index set or computing the Galerkin solution more accurately, this behaviour of the algorithm is desired.

AWGM$[\varepsilon, \varepsilon_{-1}] \to \mathbf{u}_\varepsilon$:
% Input: $\varepsilon, \varepsilon_{-1} > 0$.
% Parameters: $\alpha, \delta, \gamma, \theta$ *such that* $\delta \in (0, \alpha)$, $\frac{\alpha+\delta}{1-\delta} < \kappa(\mathbf{B})^{-\frac{1}{2}}$, $\theta > 0$ *and*
% $\gamma \in \big(0, \frac{(1-\delta)(\alpha-\delta)}{1+\delta}\kappa(\mathbf{B})^{-1}\big)$.

```
i := 0, u^(i) := 0, Λ_i := ∅
do ζ := θε_{i-1}
   do ζ := ζ/2, r^(i) := RHS[ζ/2] − APPLY[u^(i), ζ/2]
      if ε_i := ‖r^(i)‖ + ζ ≤ ε then u_ε := u^(i) stop endif
   until ζ ≤ δ‖r^(i)‖
   Λ_{i+1} := EXPAND[Λ_i, r^(i), α]
   u^(i+1) := GALERKIN[Λ_{i+1}, u^(i), ε_i, γ‖r^(i)‖]
   i := i+1
enddo
```

Theorem 3 ([GHS07]). *Let $\varepsilon_{-1}, \varepsilon > 0$, then for $\mathbf{u}_\varepsilon := \mathbf{AWGM}[\varepsilon, \varepsilon_{-1}]$ we have that $\|\mathbf{f} - \mathbf{B}\mathbf{u}_\varepsilon\| \le \varepsilon$. If for some $s > 0$, $\mathbf{u} \in \mathscr{A}^s$, then $\#\operatorname{supp}\mathbf{u}_\varepsilon \lesssim \varepsilon^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}$. If, additionally, $\mathbf{B}$ is $\bar{s}$-admissible, $s \le \bar{s}$ and $\varepsilon \lesssim \varepsilon_{-1} \eqsim \|\mathbf{f}\|$, then the number of operations used by the call $\mathbf{AWGM}[\varepsilon, \varepsilon_0]$ is bounded by an absolute multiple of $\varepsilon^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}$. In other words, if $\bar{s} \ge s_{\max}$, then this AWGM is (quasi-) optimal.*

Proof. By definition of ε_i, we have

$$\|\mathbf{f} - \mathbf{B}\mathbf{u}^{(i)}\| \le \varepsilon_i, \tag{34}$$

so that ε_i is a valid parameter for the later call $\mathbf{GALERKIN}[\Lambda_{i+1}, \mathbf{u}^{(i)}, \varepsilon_i, \gamma\|\mathbf{r}^{(i)}\|]$.

Since ζ is halved in each iteration, if the inner loop does not terminate because of $\zeta \le \delta\|\mathbf{r}^{(i)}\|$, then at some point it will terminate because of $\varepsilon_i \le \varepsilon$.

If the inner loop terminates because of $\zeta \le \delta\|\mathbf{r}^{(i)}\|$, then, because of $\delta < 1$,

$$\varepsilon_i \eqsim \|\mathbf{r}^{(i)}\| \eqsim \|\mathbf{f} - \mathbf{B}\mathbf{u}^{(i)}\| \tag{35}$$

and $\|\mathbf{f} - \mathbf{B}\mathbf{u}^{(i)} - \mathbf{r}^{(i)}\| \le \zeta \le \delta\|\mathbf{r}^{(i)}\|$. Since after the subsequent calls of **EXPAND** and **GALERKIN**, $\|\mathbf{P}_{\Lambda_{i+1}}\mathbf{r}^{(i)}\| \ge \alpha\|\mathbf{r}^{(i)}\|$ and $\|\mathbf{P}_{\Lambda_{i+1}}\mathbf{f}^{(i)} - \mathbf{B}_{\Lambda_{i+1}}\mathbf{u}^{(i+1)}\| \le \gamma\|\mathbf{r}^{(i)}\|$, an application of Proposition 5 shows that, with $\rho < 1$ from that proposition,

$$|||\mathbf{u} - \mathbf{u}^{(i+1)}||| \le \rho|||\mathbf{u} - \mathbf{u}^{(i)}||| \tag{36}$$

and

$$\#(\Lambda_{i+1}\backslash\Lambda_i) \lesssim \min\{N : |||\mathbf{u} - \mathbf{u}_N||| \le [1 - \mu^2\kappa(\mathbf{B})]^{\frac{1}{2}}|||\mathbf{u} - \mathbf{u}^{(i)}|||\}. \tag{37}$$

Because $\varepsilon_i = \|\mathbf{r}^{(i)}\| + \zeta \le \|\mathbf{f} - \mathbf{B}\mathbf{u}^{(i)}\| + 2\zeta \le \|\mathbf{f} - \mathbf{B}\mathbf{u}^{(i)}\| + 2\theta\varepsilon_{i-1}$, from (35) and (36), we conclude that eventually the inner loop, and thus the algorithm, will terminate because of $\varepsilon_i \le \varepsilon$. By (34), this proves the first statement of the theorem.

Fully analogous to the proof of Proposition 4, from (36) and (37) we conclude that if for some $s > 0$, $\mathbf{u} \in \mathscr{A}^s$, then

$$\#\operatorname{supp}\mathbf{u}^{(i+1)} \le \#\Lambda_{i+1} \lesssim \|\mathbf{u} - \mathbf{u}^{(i)}\|^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}. \tag{38}$$

With K denoting the value of i at termination, i.e., $\mathbf{u}_\varepsilon = \mathbf{u}^{(K)}$, if $K = 0$ then $\#\operatorname{supp}\mathbf{u}_\varepsilon = 0$, and the second statement of the theorem is obviously true. If $K > 0$,

then this second statement follows from $\varepsilon < \varepsilon_{K-1} \eqsim \|\mathbf{u}-\mathbf{u}^{(K-1)}\|$ and (38). Together with Lemma 1, the same arguments also show that

$$\|\mathbf{u}^{(i)}\|_{\mathscr{A}^s} \lesssim \|\mathbf{u}\|_{\mathscr{A}^s}. \tag{39}$$

Now let $\mathbf{B}$ be $\bar{s}$-admissible for some $\bar{s} \geq s$, and let $\varepsilon \lesssim \varepsilon_{-1} \eqsim \|\mathbf{f}\|$. By definition of $\bar{s}$-admissibility and Corollary 1, with C_i denoting the cost of the evaluation of $\mathbf{r}^{(i)} := \mathbf{RHS}[\zeta/2] - \mathbf{APPLY}[\mathbf{u}^{(i)}, \zeta/2]$, we have

$$\begin{aligned}\#\mathrm{supp}\,\mathbf{r}^{(i)} \lesssim C_i &\lesssim (\zeta/2)^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s} + 1 + (\zeta/2)^{-1/s}\|\mathbf{u}^{(i)}\|_{\mathscr{A}^s}^{1/s} + \#\mathrm{supp}\,\mathbf{u}^{(i)} + 1 \\ &\lesssim \zeta^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s} + \varepsilon_{i-1}^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s},\end{aligned} \tag{40}$$

by (39) and, for $i > 1$, by (38), (35) and $\varepsilon_{i-1} \lesssim \varepsilon_0 \eqsim \|\mathbf{f}\| \lesssim \|\mathbf{u}\|_{\mathscr{A}^s}$ (and thus $1 \lesssim \varepsilon_{i-1}^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}$), and, for $i = 0$, by $\#\mathrm{supp}\,\mathbf{u}^{(0)} = 0$ and $\varepsilon_{-1} \lesssim \|\mathbf{f}\| \lesssim \|\mathbf{u}\|_{\mathscr{A}^s}$.

To proceed, we claim that for $0 \leq i < K$, at termination of the inner loop, $\zeta \gtrsim \varepsilon_i$. Indeed, if the inner loop terminates at the first evaluation of the `until`-clause, then $\zeta = \theta\varepsilon_{i-1} \gtrsim \varepsilon_i$, the latter for $i = 0$ being valid by assumption. Otherwise, at the previous evaluation of the `until`-clause, we had $\|\mathbf{f} - \mathbf{B}\mathbf{u}^{(i)}\| \leq \|\mathbf{r}^{(i)}\| + \zeta < (\delta^{-1}+1)\zeta$. Since this ζ is twice the final one, (35) shows that the latter satisfies $\zeta \gtrsim \varepsilon_i$.

From the above claim, (40) and the successive halvings of ζ starting from $\zeta = \theta\varepsilon_{i-1}$, we conclude that for $0 \leq i < K$, at termination of the inner loop

$$\#\mathrm{supp}\,\mathbf{r}^{(i)} \lesssim \bar{C}_i \lesssim \varepsilon_i^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s},$$

where $\bar{C}_i$ denotes the *total* cost of the inner loop that produced this $\mathbf{r}^{(i)}$. Propositions 6 and 7 show that the cost of subsequent calls of **EXPAND** and **GALERKIN** is bounded by an absolute multiple of $\#\Lambda_i + \varepsilon_i^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s} \lesssim \varepsilon_i^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}$ and, since $\varepsilon_i \lesssim \gamma\|\mathbf{r}^{(i)}\|$, of $\#\Lambda_{i+1} \lesssim \varepsilon_i^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}$, respectively.

From $\varepsilon_i \lesssim \rho^{i-j}\varepsilon_j$ $(i \leq j)$, being a consequence of (36) and (35), and, when $K > 0$, $\varepsilon_{K-1} > \varepsilon$, we may conclude that the total cost of the call $\mathbf{AWGM}[\varepsilon, \varepsilon_0]$ is bounded by an absolute multiple of $\varepsilon^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}$, once we have shown that the cost of the final run of the inner loop can be bounded an absolute multiple of this expression. For this goal, it suffices to show that at termination of this last inner loop, $\zeta \gtrsim \varepsilon$.

If this inner loop terminates by the first evaluation of the `if`-clause, then $\zeta = \theta\varepsilon_{K-1} \gtrsim \varepsilon$, for $K = 0$ by assumption. Otherwise, the previous value of ζ, being twice the final one, satisfies both $\|\mathbf{r}^{(i)}\| + \zeta > \varepsilon$ and $\zeta \geq \delta\|\mathbf{r}^{(i)}\|$, and so $(1+\delta^{-1})\zeta > \varepsilon$, with which the proof is completed. □

4.3 Discussion

As we have seen, both the adaptive inexact Richardson scheme **Rich** from Sect. 3 and the Adaptive Wavelet Galerkin Method **AWGM** discussed in the present section are (quasi-) optimal. Practical experiments, see [GHS07] and [DHS07, Sect. 4], show that the **AWGM** is quantitatively more efficient. One reason could be the need for coarsening in **Rich**. Indeed, without coarsening generally this algorithm turns out not to be (quasi-) optimal. This means that in between two coarsening steps, the error as function of the support size does not decay with the optimal rate. As a consequence, in each coarsening step many previously computed coefficients are thrown away. Another possible explanation is that in both algorithms, the expansion of the current wavelet index set via an approximate residual computation is the most costly part. In view of this, given such an index set, it seems most efficient to compute a (near) best approximation from the span of the corresponding wavelets, being the Galerkin approach.

Apart from the aforementioned references, practical experiments with (variants of) the **AWGM** can be found in [BBC$^+$01, Bar01, BK06, BK08]. Numerical results with (variants of) the adaptive inexact Richardson scheme applied to the Schur complement of the Stokes equations (Uzawa scheme) can be found in [DUV02].

5 The approximation of operators in wavelet coordinates by computable sparse matrices

From the main theorems 1 and 3, recall that the inexact Richardson iteration **Rich** and the Adaptive Wavelet Galerkin Method **AWGM** applied to $\mathbf{Bu} = \mathbf{f}$ are (quasi-) optimal under the condition that $\mathbf{B}$ is $\bar{s}$-admissible (cf. Definition 1) for some $\bar{s} \geq s_{\max}$. Consequently, if either of the adaptive wavelet schemes is applied to the normal equations, both $\mathbf{B}$ and $\mathbf{B}^\top$ have to be $\bar{s}$-admissible for some $\bar{s} \geq s_{\max}$. With B being a boundedly invertible operator between $\mathscr{X}$ and $\mathscr{Y}'$, recall that $s_{\max}$ is the generally best possible approximation rate from span $\Psi^{\mathscr{X}}$ of a function in $\mathscr{X}$. Furthermore, from Theorem 2, recall that if $\mathbf{B}$ is s^*-computable (cf. Definition 2), then it is $\bar{s}$-admissible for any $\bar{s} < s^*$. In view of these results, our task is therefore to show s^*-computability of $\mathbf{B}$ and possibly $\mathbf{B}^\top$ for some $s^* > s_{\max}$.

The question whether $\mathbf{B}$ (and $\mathbf{B}^\top$) is s^*-computable for some $s^* > s_{\max}$ depends on the operator B and the wavelets at hand. So far, apart from the boundedly invertibility of B, we only assumed that $\Psi^{\mathscr{X}}$ and $\Psi^{\mathscr{Y}}$ are Riesz bases for $\mathscr{X}$ and $\mathscr{Y}$, respectively. In this section, we study the issue of s^*-computability for B resulting from a *scalar PDE* or a *system of PDE's* on a domain $\Omega \subset \mathbb{R}^n$, and for $\Psi^{\mathscr{X}}$ and $\Psi^{\mathscr{Y}}$ being collections of commonly applied, locally supported, piecewise smooth wavelets. In Section 7, we comment on the case of $\Psi^{\mathscr{X}}$ and $\Psi^{\mathscr{Y}}$ being collections of tensor product wavelets.

Also for classes for *singular integral operators* and suitable wavelets s^*-computability with $s^* > s_{\max}$ is valid. We refer to [Ste04, GS06b, DHS07] and on the chapter "Rapid Solution of Boundary Integral Equations" by H. Harbrecht and R. Schneider in this book.

5.1 Near-sparsity of partial differential operators in wavelet coordinates

This subsection is devoted to the question how well the representation $\mathbf{B}$ of a partial differential operator with respect to wavelet bases can be approximated by sparse matrices. We will not be concerned with the question how to compute, or more generally, how to approximate the entries of these sparse matrices, and at which cost. These issues will be postponed to the next subsections. Our current task motivates the following definition.

Definition 3. For $s^* > 0$, $\mathbf{B} \in \mathscr{L}(\ell_2, \ell_2)$ will be called to be s^*-*compressible* when we have available sequences $(e_j)_{j\in\mathbb{N}_0}, (c_j)_{j\in\mathbb{N}_0} \subset \mathbb{R}$, $(\mathbf{B}^{(j)})_{j\in\mathbb{N}_0} \subset \mathscr{L}(\ell_2, \ell_2)$, such that

- $\|\mathbf{B} - \mathbf{B}^{(j)}\| \le e_j$, $\lim_{j\to\infty} e_j = 0$,
- the number of non-zero entries in each column of $\mathbf{B}^{(j)}$ is bounded by c_j,
- $\mathbf{B}^{(0)} = 0$ (and thus $\|\mathbf{B}\| \le e_0$), $c_0 = 0$ and $\sup_{j\in\mathbb{N}_0} c_{j+1}/c_j < \infty$.

and such that for any $s < s^*$, $\sup_j e_j c_j^s < \infty$.

So compared to the definition of s^*-computability (Definition 2), the only difference is that we do not require that number of operations needed to *compute* the non-zero entries in each column of $\mathbf{B}^{(j)}$ is bounded by c_j.

For some $\alpha_l \in \mathbb{N}_0^n$ ($l \in \{1,2\}$), we consider the representation as a bi-infinite matrix of a bounded linear operator $E : H_0^{|\alpha_1|}(\Omega) \to (H_0^{|\alpha_2|}(\Omega))'$ defined by

$$(Eu_1)(u_2) = \int_\Omega g\partial^{\alpha_1} u_1 \partial^{\alpha_2} u_2 \quad (u_l \in H_0^{|\alpha_l|}(\Omega)),$$

with respect to wavelet collections

$$\Psi^{(l)} = \{\psi_\lambda^{(l)} : \lambda \in \nabla\} \subset H_0^{|\alpha_l|}(\Omega).$$

We will assume that the coefficient g is *sufficiently smooth*.

In this paper, we do not discuss the *construction* of wavelet bases on domains, but refer to the numerous papers written on that topic. Some references are included at the end of this subsection. Following standard conventions, $|\lambda| \in \mathbb{N}_0$ will denote the *level* of the wavelet $\psi_\lambda^{(l)}$. Here thinking of the wavelets being *normalized in* $L_2(\Omega)$ and constructed using *dyadic* dilations, for $s \ge 0$ up to some upper bound

determined by the smoothness of the wavelets, it holds that $\|\psi_\lambda^{(l)}\|_{H^s(\Omega)} \eqsim 2^{s|\lambda|}$. In view of this, we investigate the approximation of

$$\mathbf{E} := \left[2^{-|\mu||\alpha_1|-|\lambda||\alpha_2|}(A\psi_\mu^{(1)})(\psi_\lambda^{(2)})\right]_{\lambda,\mu\in\nabla}$$

by sparse matrices.

The representation of a scalar PDE will result into a sum of such matrices (where, because of the eventual normalization of the wavelets in higher order Sobolev norms, matrices corresponding to lower order terms will be multiplied from left and right by $\mathrm{diag}[2^{-|\lambda|s}]_{\lambda\in\nabla}$ or $\mathrm{diag}[2^{-|\lambda|t}]_{\lambda\in\nabla}$ for some $s,t\geq 0$ with $s+t>0$) and the representation of a system of PDE's will consist of blocks, each of them being a sum of such matrices.

We will assume that the wavelets are *local*, *locally finite* and *piecewise smooth*, where for an easy treatment of the quadrature issue, we assume that the wavelets from both collections are piecewise smooth with respect to the *same* partitions, moreover which are *nested* as function of the level (for the general case, see [SW08]): We assume that for all $k\in\mathbb{N}_0$, there exists a collection $\{\Omega_k^{(\nu)} : \nu\in\mathscr{O}_k\}$ of disjoint, uniformly shape regular, open subdomains, with $\overline{\Omega} = \cup_{\nu\in\mathscr{O}_k}\overline{\Omega}_k^{(\nu)}$, $\mathrm{diam}(\Omega_k^{(\nu)}) \eqsim 2^{-k}$ and $\Omega_k^{(\nu)}$ being the union of some $\Omega_{k+1}^{(\tilde{\nu})}$. These subdomains will be such that $\mathrm{supp}\,\psi_\lambda^{(l)}$ ($l\in\{1,2\}$), which is assumed to be connected, is the union of a uniformly bounded number of $\Omega_{|\lambda|}^{(\nu)}$ (*locality*), and such that each $\Omega_k^{(\nu)}$ has non-empty intersection with the supports of a uniformly bounded number of $\psi_\lambda^{(l)}$ with $|\lambda|=k$ (*locally finiteness*). Typical examples of the $\Omega_k^{(\nu)}$ are n-cubes or n-simplices, or smooth images of such. We assume that $\psi_\lambda^{(l)}|_{\Omega_{|\lambda|}^{(\nu)}}$ is smooth with, for any $\gamma\in\mathbb{N}_0^n$,

$$\sup_{x\in\Omega_{|\lambda|}^{(\nu)}} |\partial^\gamma\psi_\lambda^{(l)}(x)| \lesssim 2^{|\lambda|(\frac{n}{2}+|\gamma|)} \tag{41}$$

(*piecewise smoothness*), the latter being a consequence of the smoothness of $\psi_\lambda^{(l)}|_{\Omega_{|\lambda|}^{(\nu)}}$, the normalization of the wavelets in $L_2(\Omega)$ and their construction using dyadic dilations. Note that the singular support of $\psi_\lambda^{(l)}$ is part of the skeleton $\cup_{\nu\in\mathscr{O}_k}\partial\Omega_{|\lambda|}^{(\nu)}$.

We will also need that the wavelets satisfy some *global smoothness* conditions: For some

$$\mathbb{N}_0\cup\{-1\}\ni r_l\geq|\alpha_l|-1,$$

we assume that

$$\|\psi_\lambda^{(l)}\|_{W_\infty^t(\Omega)} \lesssim 2^{|\lambda|(\frac{n}{2}+t)} \quad (t\in[0,r_l+1]). \tag{42}$$

For $r_l>-1$, this estimate follows from (41) when $\psi_\lambda^{(l)}\in C^{r_l}(\Omega)$.

We assume that the wavelets have *cancellation properties of order* $\tilde{d}_l\in\mathbb{N}_0$, meaning that

$$\Big|\int_\Omega u\psi_\lambda^{(l)}\Big| \lesssim 2^{-|\lambda|t}\|u\|_{W^t_\infty(\operatorname{supp}\psi_\lambda^{(l)})}\|\psi_\lambda^{(l)}\|_{L_1(\Omega)} \quad (t\in[0,\tilde{d}_l], u\in W^t_\infty(\Omega)). \tag{43}$$

Actually, with some constructions, here $\operatorname{supp}\psi_\lambda^{(l)}$ should read as a neighbourhood of $\operatorname{supp}\psi_\lambda^{(l)}$ with diameter $2^{-|\lambda|}$. For convenience we ignore this fact, but our results extend trivially to this situation.

Finally, for any $\gamma\le\alpha_l$, $\gamma\ne\alpha_l$, we assume the homogeneous Dirichlet boundary conditions

$$\partial^\gamma\psi_\lambda^{(l)} = 0 \quad \text{at } \partial\Omega, \tag{44}$$

actually being a consequence of our earlier assumption that $\Psi^{(l)}\subset H_0^{|\alpha_l|}(\Omega)$.

We split

$$\mathbf{E} = \mathbf{E}^{(r)} + \mathbf{E}^{(s)},$$

where $\mathbf{E}^{(r)}_{\lambda,\mu} = \mathbf{E}_{\lambda,\mu}$ when either $|\lambda|>|\mu|$ and $\operatorname{supp}\psi_\lambda^{(2)}\subset\bar{\Omega}^{(\nu)}_{|\mu|}$ for some $\nu\in\mathscr{O}_{|\mu|}$ or $|\lambda|<|\mu|$ and $\operatorname{supp}\psi_\mu^{(1)}\subset\bar{\Omega}^{(\nu)}_{|\lambda|}$ for some $\nu\in\mathscr{O}_{|\lambda|}$, and $\mathbf{E}^{(r)}_{\lambda,\mu}$ is zero otherwise. So $\mathbf{E}^{(r)}$ contains the *regular entries* of $\mathbf{E}$, i.e., the non-zero entries for which the the interior of the support of the wavelet on the higher level does not intersect the singular support of the wavelet on the lower level. The remaining *singular entries* are gathered in $\mathbf{E}^{(s)}$. As we will see, the size of the singular entries decays less fast as function of the difference in the levels of the indices than with the regular entries, but this will be compensated by a smaller increase of their number.

We write $\mathbf{E}^{(r)} = (\mathbf{E}^{(r)}_{\ell,k})_{\ell,k\in\mathbb{N}_0}$, where $\mathbf{E}^{(r)}_{\ell,k} = (\mathbf{E}^{(r)}_{\lambda,\mu})_{|\lambda|=\ell,|\mu|=k}$ and similarly $\mathbf{E}^{(s)} = (\mathbf{E}^{(s)}_{\ell,k})_{\ell,k\in\mathbb{N}_0}$.

Proposition 8. *The number of non-zero entries in each row of* $\mathbf{E}^{(r)}_{\ell,k}$ $(\mathbf{E}^{(s)}_{\ell,k})$ *or column of* $\mathbf{E}^{(r)}_{k,\ell}$ $(\mathbf{E}^{(s)}_{k,\ell})$ *is bounded by an absolute multiple of* $2^{\max(k-\ell,0)n}$ $(2^{\max(k-\ell,0)(n-1)})$.

With

$$\rho_r := \tilde{d}_2 + |\alpha_2|, \quad \rho_s := \tfrac{1}{2} + \min(\tilde{d}_2 + |\alpha_2|, r_1 + 1 - |\alpha_1|),$$

for $|\lambda|>|\mu|$, *we have*

$$|\mathbf{E}^{(s)}_{\lambda,\mu}| \lesssim \|g\|_{W^{\rho_s-1/2}_\infty(\Omega)} 2^{-(\left||\lambda|-|\mu|\right|)(\frac{n-1}{2}+\rho_s)}, \quad |\mathbf{E}^{(r)}_{\lambda,\mu}| \lesssim \|g\|_{W^{\rho_r}_\infty(\Omega)} 2^{-(\left||\lambda|-|\mu|\right|)(\frac{n}{2}+\rho_r)}$$

The same statement is valid for $|\lambda|<|\mu|$ *when* $(\alpha_1,\alpha_2,r_1,\tilde{d}_2)$ *is replaced by* $(\alpha_2,\alpha_1,r_2,\tilde{d}_1)$ *in the definitions of* ρ_r *and* ρ_s.

Proof. The first statement follows by the localness and locally finiteness of both wavelet collections, and concerning $\mathbf{E}^{(s)}$, by their piecewise smoothness.

When $r_1+1\le|\alpha_1|+|\alpha_2|$, select a $\gamma\le\alpha_2$ with $|\alpha_1+\gamma| = r_1+1$ and so $|\alpha_2-\gamma| = |\alpha_1|+|\alpha_2|-(r_1+1)$. Using (44) for the case that $\operatorname{supp}\psi_\lambda^{(2)}\cap\operatorname{supp}\psi_\mu^{(1)}\cap\partial\Omega\ne\emptyset$, integration by parts, $\operatorname{vol}(\operatorname{supp}\psi_\lambda^{(2)})\lesssim 2^{-|\lambda|n}$ and (42) show that

$$
\begin{aligned}
|\mathbf{E}_{\lambda,\mu}| &= 2^{-|\mu||\alpha_1|-|\lambda||\alpha_2|}\Big|\int_{\operatorname{supp}\psi_\lambda^{(2)}}(-1)^{|\gamma|}\partial^\gamma(g\partial^{\alpha_1}\psi_\mu^{(1)})\partial^{\alpha_2-\gamma}\psi_\lambda^{(2)}\Big| \\
&\lesssim 2^{-|\mu||\alpha_1|-|\lambda||\alpha_2|}\|g\|_{W_\infty^{r_1+1-|\alpha_1|}(\Omega)}2^{-|\lambda|n}2^{|\mu|(\frac{n}{2}+r_1+1)}2^{|\lambda|(\frac{n}{2}+|\alpha_1|+|\alpha_2|-(r_1+1))} \\
&= \|g\|_{W_\infty^{r_1+1-|\alpha_1|}(\Omega)}2^{-(|\lambda|-|\mu|)(\frac{n}{2}+r_1+1-|\alpha_1|)}.
\end{aligned}
$$

For $r_1+1>|\alpha_1|+|\alpha_2|$ by additionally using that the $\psi_\lambda^{(2)}$ have $\tilde{d}_2$ vanishing moments ((43)) and taking into account that $\psi_\mu^{(1)}\in W_\infty^{r_1+1}(\Omega)$ ((42)), we have

$$
\begin{aligned}
|\mathbf{E}_{\lambda,\mu}| &= 2^{-|\mu||\alpha_1|-|\lambda||\alpha_2|}\Big|\int_{\operatorname{supp}\psi_\lambda^{(2)}}(-1)^{|\alpha_2|}\partial^{\alpha_2}(g\partial^{\alpha_1}\psi_\mu^{(1)})\psi_\lambda^{(2)}\Big| \\
&\lesssim 2^{-|\mu||\alpha_1|-|\lambda||\alpha_2|}2^{-|\lambda|\min(\tilde{d}_2,r_1+1-|\alpha_1|-|\alpha_2|)} \\
&\qquad\qquad \times \|\partial^{\alpha_2}(g\partial^{\alpha_1}\psi_\mu^{(1)})\|_{W_\infty^{\min(\tilde{d}_2,r_1+1-|\alpha_1|-|\alpha_2|)}(\operatorname{supp}\psi_\lambda^{(2)})}\|\psi_\lambda^{(2)}\|_{L_1(\Omega)} \\
&\lesssim 2^{-|\mu||\alpha_1|-|\lambda||\alpha_2|}2^{-|\lambda|\min(\tilde{d}_2,r_1+1-|\alpha_1|-|\alpha_2|)} \\
&\qquad\qquad \times \|g\|_{W_\infty^{\min(\tilde{d}_2+|\alpha_2|,r_1+1-|\alpha_1|)}(\Omega)}2^{|\mu|(\frac{n}{2}+\min(\tilde{d}_2+|\alpha_1|+|\alpha_2|,r_1+1)}2^{-|\lambda|\frac{n}{2}} \\
&\eqsim \|g\|_{W_\infty^{\min(\tilde{d}_2+|\alpha_2|,r_1+1-|\alpha_1|)}(\Omega)}2^{-(|\lambda|-|\mu|)(\frac{n}{2}+\min(\tilde{d}_2+|\alpha_2|,r_1+1-|\alpha_1|))},
\end{aligned}
$$

which completes the proof of the first estimate.

Finally, when $\operatorname{supp}\psi_\lambda^{(2)}\subset\bar{\Omega}_{|\mu|}^{(\nu)}$ for some $\nu\in\mathscr{O}_{|\mu|}$, by estimating $\mathbf{E}_{\lambda,\mu}^{(r)}$ as above, but now applying (41) for sufficiently large γ instead of (42), we obtain the second estimate. □

In the next proposition, we construct sparse approximation for matrices like $\mathbf{E}^{(r)}$ or $\mathbf{E}^{(s)}$.

Proposition 9. *Let $\mathbf{C}=(\mathbf{C}_{\ell,k})_{\ell,k\in\mathbb{N}_0}$ with $\mathbf{C}_{\ell,k}=(\mathbf{C}_{\lambda,\mu})_{|\lambda|=\ell,|\mu|=k}$ be such that for some $q\in\mathbb{N}_0$ and $\rho>0$, the number of non-zero entries in each row of $\mathbf{C}_{\ell,k}$ or column of $\mathbf{C}_{k,\ell}$ is bounded by an absolute multiple of $2^{\max(k-\ell,0)q}$ and*

$$|\mathbf{C}_{\lambda,\mu}|\lesssim 2^{-\big||\lambda|-|\mu|\big|(\frac{q}{2}+\rho)}.$$

Then with $\mathbf{C}^{(j)}$ constructed from $\mathbf{C}$ by dropping $\mathbf{C}_{\lambda,\mu}$ when

$$\big||\lambda|-|\mu|\big|>j/\rho,$$

we have

$$\|\mathbf{C}-\mathbf{C}^{(j)}\|\lesssim 2^{-j},$$

where the number of non-zero entries per row and column of $\mathbf{C}^{(j)}$ is bounded by some absolute multiple of

$$\max(2^{qj/\rho},j/\rho).$$

Proof. By two applications of the Schur lemma, we have

$$\|\mathbf{C}_{\ell,k}\|^2 \le \max_{|\lambda|=\ell} \sum_{|\mu|=k} |\mathbf{C}_{\lambda,\mu}| \cdot \max_{|\mu|=k} \sum_{|\lambda|=\ell} |\mathbf{C}_{\lambda,\mu}| \lesssim 4^{-(\ell-m)\rho},$$

$$\|\mathbf{C}-\mathbf{C}^{(j)}\|^2 \le \max_{\ell} \sum_{\{k:|\ell-k|>j/\rho\}} \|\mathbf{C}_{\ell,k}\| \cdot \max_{k} \sum_{\{\ell:|\ell-k|>j/\rho\}} \|\mathbf{C}_{\ell,k}\| \lesssim 4^{-j}. \qquad \square$$

So the result of the last proposition shows that $\mathbf{C}$ and $\mathbf{C}^\top$ are s^*-*compressible* with

$$s^* = \rho/q$$

(or $s^* = \infty$ when $q = 0$). We exemplify our findings concerning s^*-compressibility in the model case of an (elliptic) scalar PDE of order $2m$:

Example 3. For some bounded domain $\Omega \subset \mathbb{R}^n$, with $n \ge 2$, and $m \in \mathbb{N}$, let $B : H_0^m(\Omega) \to H_0^m(\Omega)'$ be defined as

$$(Bu)(v) = \int_\Omega \sum_{|\alpha|,|\beta|\le m} a_{\alpha,\beta} \partial^\alpha u \partial^\beta v,$$

with coefficients such that B is boundedly invertible and that are sufficiently smooth.

Let $\Psi^{\mathscr{X}} = \Psi^{\mathscr{Y}} = \{\psi_\lambda : \lambda \in \nabla\} \subset H_0^m(\Omega)$ be a dyadic wavelet collection, normalized in $L_2(\Omega)$, such that for some $\mathbb{N} \ni d > m$, $\tilde{d} \in \mathbb{N}_0$, $\mathbb{N}_0 \cup \{-1\} \ni r \ge m-1$,

a). $\inf_{v_i \in \mathrm{span}\{\psi_\lambda : |\lambda| \le i\}} \|u - v_i\|_{H^m(\Omega)} \lesssim 2^{-(d-m)i} \|u\|_{H^d(\Omega)} \quad (u \in H^d(\Omega) \cap H_0^m(\Omega))$,
b). the wavelets are local, locally finite and piecewise smooth (and thus satisfy (41)),
c). the wavelets are in $C^r(\Omega)$ (and thus satisfy (42) with $r_l = r$),
d). the wavelets have cancellation properties of order $\tilde{d}$ ((43)),
e). $\{2^{-|\lambda|m}\psi_\lambda : \lambda \in \nabla\}$ is a Riesz basis for $H_0^m(\Omega)$.

The representation of B with respect to the wavelet basis from e) reads as

$$\mathbf{B} := \sum_{|\alpha|,|\beta|\le m} \big[2^{-(|\lambda|+|\mu|)m} \int_\Omega a_{\alpha,\beta} \partial^\alpha \psi_\mu \partial^\beta \psi_\lambda\big]_{\lambda,\mu\in\nabla}.$$

Due to the scaling factor $2^{-(|\lambda|+|\mu|)m}$, one may verify that it suffices to analyze the s^*-compressibilty of the highest order terms. By applying Propositions 8 and 9 to those terms, we infer that $\mathbf{B}$ and $\mathbf{B}^\top$ are s^*-compressible with

$$s^* = \min\Big(\frac{\tilde{d}+m}{n}, \frac{\frac{1}{2}+\min(\tilde{d}+m, r+1-m)}{n-1}\Big).$$

As a consequence of the dyadic construction, we have that $\#\{\lambda \in \nabla : |\lambda| \le i\} \approx 2^{ni}$, which together with a) shows that

$$s_{\max} = \frac{d-m}{n}.$$

We conclude that $s^* > s_{\max}$ when $\tilde{d} > d - 2m$ and $\frac{r+\frac{3}{2}-m}{n-1} > \frac{d-m}{n}$ (the third condition $\tilde{d} + m \geq r + 1 - m$ follows already from the first one using that always $r \leq d-2$).

On $(0,1)^n$, or on a smooth image of that, *biorthogonal spline* wavelets can be constructed that satisfy a)-e) for arbitrary $\tilde{d} \geq d$ with $d + \tilde{d}$ even and $r = d-2$ ([DS98]). Because of $r = d-2$, the conditions for $s^* > s_{\max}$ read as $\tilde{d} > d - 2m$ and $\frac{d-m}{n} > \frac{1}{2}$.

For general domains, these wavelets can be applied in combination with non-overlapping *domain decomposition* techniques. The existing techniques fall into 2 categories: With the technique based on *extension operators* proposed in [DS99b], all above conditions can be satisfied. The condition number of the resulting basis, however, turns out to increase rapidly with d. The other technique amounts to a *continuous* gluing of multiresolution analyses over the interfaces between patches, see [DS99a, CTU99]. As a result, wavelets with supports that extend to more than one patches are only continuous, and thus for $d > 2$ not in C^{d-2}, resulting in a reduced value of s^*. For problems of order $2m = 2$, this limitation can be overcome with a construction of wavelets that have *patchwise vanishing moments*, see [HS06].

5.2 The approximate computation of the significant entries

For a non-constant coefficient g, generally the entries of $\mathbf{E}^{(r)}$ and $\mathbf{E}^{(s)}$ have to be approximated by suitable quadrature. In this subsection, we show that such approximations can be made that keep the error on the same level, while taking in each row and column *on average* $\mathcal{O}(1)$ operations per entry. This means these matrices are s^*-computable for the same value of s^* as they were shown to be s^*-compressible. The key observation is that this restriction on the work does allow to spend quite some operations, up to the number of entries in the row or column, to the approximation of the few largest entries with indices that have equal level, as long as the work per entry decays sufficiently fast as function of the difference in the levels of the indices. For simplicity, we exclude the special, although easy case that $q = 0$ in Proposition 9. Since with $\mathbf{E}^{(s)}$ the role of q is played by $n-1$, we thus assume that $n > 1$.

Proposition 10. *Let* $\mathbf{C}$ *and* $\mathbf{C}^{(j)}$ *be as in Proposition 9 assuming that* $q > 0$. *Suppose that for some constants* $\xi, \omega > 0$, $\xi \neq \omega$, *for any* $\lambda, \mu \in \nabla$ *one can compute an approximation* $\tilde{\mathbf{C}}_{\lambda,\mu}$ *to* $\mathbf{C}_{\lambda,\mu}$ *in* $\mathcal{O}(N)$ *operations with*

$$|\mathbf{C}_{\lambda,\mu} - \tilde{\mathbf{C}}_{\lambda,\mu}| \lesssim N^{-\omega} 2^{-\left||\lambda|-|\mu|\right|(\frac{q}{2}+\xi q)}. \tag{45}$$

Now for some $\sigma \in (1, \xi/\omega)$ *when* $\xi > \omega$, *and* $\sigma \in (\xi/\omega, 1)$ *when* $\xi < \omega$, *and* $\theta \leq \min(1, \sigma)$, *build* $\tilde{\mathbf{C}}^{(j)}$ *by approximating each non-zero entry of* $\mathbf{C}^{(j)}$ *as above by taking*

$$N = N_{j,\lambda,\mu} \eqsim \max\left(1, 2^{qj\theta/\rho - \left||\lambda|-|\mu|\right|\sigma q}\right)$$

operations. Then the work for computing each row or column of $\tilde{\mathbf{C}}^{(j)}$ *is bounded by some absolute multiple of* $2^{qj/\rho}$*, and*

$$\|\mathbf{C}^{(j)} - \tilde{\mathbf{C}}^{(j)}\| \lesssim \begin{cases} 2^{-qj\omega\theta/\rho} & \text{when } \xi > \omega, \\ 2^{-qj(\xi+(\theta-\sigma)\omega)/\rho} & \text{when } \xi < \omega. \end{cases} \tag{46}$$

In particular, taking $\theta = \min(1,\sigma)$*, we have* $\|\mathbf{C}^{(j)} - \tilde{\mathbf{C}}^{(j)}\| \lesssim 2^{-qj\min(\omega,\xi)/\rho}$.

Proof. The work per row or column is bounded by an absolute multiple of

$$\begin{aligned}\sum_{i=0}^{j/\rho} 2^{iq} \max\left(1, 2^{qj\theta/\rho - i\sigma q}\right) &\eqsim 2^{qj/\rho} + 2^{qj\theta/\rho} \sum_{i=0}^{j/\rho} 2^{iq(1-\sigma)} \\ &\eqsim 2^{qj/\rho} + 2^{qj\theta/\rho} \max(1, 2^{qj(1-\sigma)/\rho}) \eqsim 2^{qj/\rho},\end{aligned}$$

because of $\theta \le \min(1,\sigma)$.

Taking into account the selection of $N_{j,\lambda,\mu}$, two applications of the Schur lemma show that

$$\begin{aligned}\|\mathbf{C}^{(j)}_{\ell,m} - (\tilde{\mathbf{C}}^{(j)}_{\lambda,\mu})_{|\lambda|=\ell,|\mu|=m}\|^2 &\lesssim 2^{|\ell-m|q}(2^{qj\theta/\rho - |\ell-m|\sigma q})^{-2\omega} 2^{-|\ell-m|(q+2\xi q)} \\ &= 2^{-2qj\theta\omega/\rho} 2^{-|\ell-m|2q(\xi-\sigma\omega)}, \\ \|\mathbf{C}^{(j)} - \tilde{\mathbf{C}}^{(j)}\| &\lesssim \sum_{0\le i\le j/\rho} 2^{-qj\theta\omega/\rho} 2^{-iq(\xi-\sigma\omega)},\end{aligned}$$

which shows (46). □

Comparing Propositions (9) and (10), we see that in order to prove our earlier claim that $\mathbf{C} = \mathbf{E}^{(r)}$ or $\mathbf{C} = \mathbf{E}^{(s)}$ are s^*-computable for the same value of s^* as they were shown to be s^*-computable, it suffices to have available a family of quadrature formulas satisfying (45) with

$$\min(\omega,\xi) \ge \rho \quad \text{and} \quad \max(\omega,\xi) > \rho.$$

Below, under some mild additional assumption ((48)), we verify this by showing that for any $a,b > 0$, we can construct a family of approximations $(\tilde{\mathbf{E}}_{\lambda,\mu,N})_{N\in\mathbb{N}}$, where $\tilde{\mathbf{E}}_{\lambda,\mu,N}$ requires $\mathscr{O}(N)$ evaluations of $g\partial^{\alpha_1}\psi^{(1)}_\mu \partial^{\alpha_2}\psi^{(2)}_\lambda$, such that for some $t \in \mathbb{N}$,

$$|\mathbf{E}_{\lambda,\mu} - \tilde{\mathbf{E}}_{\lambda,\mu,N}| \lesssim N^{-a} 2^{-\left|\lambda|-|\mu|\right|(\frac{n}{2}+b)} \|g\|_{W^t_\infty(\Omega)}. \tag{47}$$

This means that (45) is valid with $\omega = a$ and $\xi = b/n$ or $\xi = (b+\frac{1}{2})/(n-1)$ for $q = n$ or $q = n-1$, respectively.

Without loss of generality let us assume that

$$|\lambda| \ge |\mu|.$$

Suppose that for any $k \in \mathbb{N}_0$ and $\nu \in \mathscr{O}_k^{(\nu)}$, there exists a sufficiently smooth transformation of coordinates κ, with derivatives bounded uniformly in k and ν, such that for some $e \in \mathbb{N}$, and all $|\lambda| = k$,

$$\psi_\lambda^{(2)} \circ \kappa|_{\kappa^{-1}(\Omega_k^{(\nu)})} \in P_{e-1}. \tag{48}$$

In the following, for notational convenience, without loss of generality we take $\kappa = \mathrm{id}$.

To approximate an integral $\int_{\Omega_k^{(\nu)}} f$, for any $p \in \mathbb{N}$ we consider internal, uniformly stable, composite quadrature rules $Q_{\Omega_k^{(\nu)},N}(f)$ of *fixed order* (i.e, the degree of polynomial exactness plus one) p, and *variable rank* N. The rank N of a composite quadrature formula denotes the number of subdomains on which the elementary quadrature formula is applied. Since the order p of $Q_{\Omega_k^{(\nu)},N}$ is fixed, the number of abscissae in the composite rule $Q_{\Omega_k^{(\nu)},N}$ is $\mathscr{O}(N)$. For such rules, the following error estimate is valid

$$|\int_{\Omega_k^{(\nu)}} f - Q_{\Omega_k^{(\nu)},N}(f)| \lesssim \mathrm{vol}(\Omega_k^{(\nu)}) N^{-p/n} \mathrm{diam}(\Omega_k^{(\nu)})^p \|f\|_{W_\infty^p(\Omega_k^{(\nu)})} \tag{49}$$

(e.g., see [GS06a, §2]).

To find an upper bound for the quadrature error when these rules are applied with integrand $2^{-|\mu||\alpha_1|-|\lambda||\alpha_2|} g \partial^{\alpha_1} \psi_\mu^{(1)} \partial^{\alpha_2} \psi_\mu^{(2)}$, we have to bound the expression $(\partial^\rho g)(\partial^\sigma \partial^{\alpha_1} \psi_\mu^{(1)})(\partial^\tau \partial^{\alpha_2} \psi_\lambda^{(2)})$ for all multi-indices with $|\rho + \sigma + \tau| \leq p$. Since g is assumed to be sufficiently smooth, $|\lambda| \geq |\mu|$ and $\partial^\tau \partial^{\alpha_2} \psi_\lambda^{(2)}$ vanishes when $|\tau + \alpha_2| \geq e$, by invoking (41) we see that the worst case occurs when $\rho = 0$ and $|\tau + \alpha_2| = z := \min(e-1, p + |\alpha_2|)$, and thus when $|\sigma| = p - z + |\alpha_2|$, yielding

$$\begin{aligned} 2^{-|\mu||\alpha_1|-|\lambda||\alpha_2|} \|g \partial^{\alpha_1} \psi_\mu^{(1)} \partial^{\alpha_2} \psi_\lambda^{(2)}\|_{W_\infty^p(\Omega_k^{(\nu)})} &\lesssim \\ 2^{(|\mu|+|\lambda|)\frac{n}{2}} 2^{|\mu|(p-z+|\alpha_2|)} 2^{|\lambda|(z-|\alpha_2|)} &\|g\|_{W_\infty^p(\Omega_k^{(\nu)})}. \end{aligned}$$

By substituting this result into (49), using that $\mathrm{diam}(\Omega_k^{(\nu)}) \eqsim 2^{-|\lambda|}$ and $\mathrm{vol}(\Omega_k^{(\nu)}) \eqsim 2^{-|\lambda|n}$, by taking p satisfying

$$p \geq \max(na, b - |\alpha_2| + e - 1)$$

and by summing over the uniformly bounded number of $\overline{\Omega}_k^{(\nu)}$ that make up supp $\psi_\lambda^{(2)}$ we end up with (47).

This completes the proof of our claim made at the beginning of this subsection that that $\mathbf{C} = \mathbf{E}^{(r)}$ or $\mathbf{C} = \mathbf{E}^{(s)}$ are s^*-computable for the same value of s^* as they were shown to be s^*-computable.

Remark 9. The estimate for the quadrature error obtained by summing the error estimates for the quadrature errors over those ν with $\Omega_\ell^{(\nu)} \subset \operatorname{supp}\psi_\lambda^{(2)}$ can be orders of magnitude too pessimistic. The point is that it has not been used that $\psi_\lambda^{(2)}$ is a wavelet and thus is oscillating, which causes cancellation of errors, in particular when $\psi_\mu^{(1)}$ is smooth on the interior of $\operatorname{supp}\psi_\lambda^{(2)}$, i.e, when it concerns a *regular* entry. For that case, much sharper estimates can be found in [SW08], see also [BBD$^+$02].

5.3 Trees

Although, as we demonstrated, it can be done whilst retaining optimal computational complexity, the approximate computation using quadrature of the required entries of the stiffness matrix that may involve wavelets on largely different levels is a rather delicate process. Such computations can be avoided by restricting to wavelet approximations where the underlying index sets form a tree. In this subsection, we briefly indicate the main ingredients of this approach.

We restrict ourselves to the case that $\Psi = \Psi^{\mathscr{X}} = \Psi^{\mathscr{Y}} = \{\psi_\lambda : \lambda \in \nabla\}$ is a Riesz basis for $\mathscr{X} = \mathscr{Y}$. Apart from wavelets, here we will need scaling functions. A set $\Phi_k \subset \mathscr{X}$ is called a collection of scaling functions on level k when $\operatorname{span}\{\psi_\lambda : |\lambda| \leq k\} = \operatorname{span}\Phi_k$. We assume that the Φ_k are (uniformly) local and locally finite (cf. definitions in Subsec. 5.1), and that each wavelet ψ_λ is a linear combination of a uniformly bounded number of scaling functions on level $|\lambda|$ (and that $\operatorname{supp}\psi_\lambda$ is connected).

We equip the index set ∇ with a tree structure by assigning to each $\lambda \in \nabla$ with $|\lambda| > 0$ a *parent* μ with $|\mu| = |\lambda| - 1$ and $\operatorname{supp}\psi_\lambda \cap \operatorname{supp}\psi_\mu \neq \emptyset$. By our assumptions, the number of children of any parent is uniformly bounded. We call $\Lambda \subset \nabla$ a *tree*, when all $\lambda \in \nabla$ with $|\lambda| = 0$ are in Λ (the "roots"), and when whenever $\lambda \in \nabla$ with $|\lambda| > 0$ is in Λ then so is its parent.

Analogously to (2), we define approximation classes $\mathscr{A}^s$, and corresponding (quasi-) norms $\|\cdot\|_{\mathscr{A}^s}$, where we now consider only best N-term approximations $\mathbf{u}_N$ to $\mathbf{u} \in \ell_2$ whose supports, apart from having a length not larger than N, form a tree. For $\mathscr{X}$ being a Sobolev space, it has been shown that the resulting classes are only slightly smaller than those one obtains with unconstrained best N-term approximation, see [CDDD01] for details.

The reason to consider tree approximation is that any $\mathbf{w} \in \ell_0$ whose support forms a tree, can be expressed as a linear combination of K scaling functions, where $K \lesssim \#\operatorname{supp}\mathbf{w}$ and where the supports of any two scaling functions in this expansion can only intersect when their difference in levels is not larger than 1. Moreover, this scaling function representation can be found in $\mathscr{O}(\#\operatorname{supp}\mathbf{w})$ operations, see [DSX00b].

As an application, now let $\mathbf{B} \in \mathscr{L}(\ell_2, \ell_2)$ be s^*-compressible, let the support of $\mathbf{w} \in \ell_0$ form a tree, and let $\varepsilon > 0$ be given. Then as shown in [DHS07], using

the near best N-term tree approximation algorithm from [BD04], trees $\Lambda_j \subset \cdots \subset \Lambda_2 \subset \Lambda_1 \subset \operatorname{supp}\mathbf{w}$ can be found such that, with $\mathbf{w}_{[p]} := \mathbf{w}|_{\Lambda_p \setminus \Lambda_{p+1}}$ ($\Lambda_{j+1} := \emptyset$) and suitable $j_p \in \mathbb{N}_0$, $\mathbf{z}_\varepsilon := \sum_{p=1}^{j} \mathbf{B}^{(j_p)}\mathbf{w}_{[p]}$ satisfies $\|\mathbf{B}\mathbf{w} - \mathbf{z}_\varepsilon\| \le \varepsilon$ and, for any $s < s^*$, $\#\operatorname{supp}\mathbf{z}_\varepsilon \lesssim \varepsilon^{-1/s}\|\mathbf{z}_\varepsilon\|_{\mathscr{A}^s}^{-1/s}$, where the cost of determining $\operatorname{supp}\mathbf{z}_\varepsilon$ is bounded by some absolute multiple of $\varepsilon^{-1/s}\|\mathbf{z}_\varepsilon\|_{\mathscr{A}^s}^{-1/s} + \#\operatorname{supp}\mathbf{w} + 1$. What is more, taking the construction of the sparse matrices $\mathbf{B}^{(j)}$ into account, for both partial differential and singular integral operators, $\operatorname{supp}\mathbf{z}_\varepsilon$ forms a tree.

Instead of approximating the required entries of the involved matrices $\mathbf{B}^{(j_p)}$, this opens another possibility to approximate $\mathbf{B}\mathbf{w}$. Since $\|\mathbf{B}\mathbf{w} - \mathbf{z}_\varepsilon\| \le \varepsilon$ is shown by estimating $\|\mathbf{B}\mathbf{w} - \mathbf{z}_\varepsilon\| \le \sum_{p=1}^{j} \|\mathbf{B} - \mathbf{B}^{(j_p)}\|\|\mathbf{w}_{[p]}\|$, and by bounding $\|\mathbf{B} - \mathbf{B}^{(j_p)}\|$ by summing over upper bounds for the entries of $\mathbf{B}$ that were dropped in the definition of $\mathbf{B}^{(j_p)}$, one infers that also $\|\mathbf{B}\mathbf{w} - (\mathbf{B}\mathbf{w})|_{\operatorname{supp}\mathbf{z}_\varepsilon}\| \le \varepsilon$ as well as that $\|\mathbf{B}\mathbf{w} - (\mathbf{B}\mathbf{w})|_{\bar\Lambda}\| \le \varepsilon$, where $\bar\Lambda := \operatorname{supp}\mathbf{w} \cup \operatorname{supp}\mathbf{z}_\varepsilon$ is a tree.

Now with $\bar\Phi$ denoting the collection of the single scale functions with $\operatorname{span}\bar\Phi = \{\psi_\lambda : \lambda \in \bar\Lambda\}$ and $T_{\bar\Lambda}$ the corresponding basis transformation from multiscale to single scale representation, we have $\mathbf{B}|_{\bar\Lambda\times\bar\Lambda} = T_{\bar\Lambda}^\top \mathbf{B}(\bar\Phi,\bar\Phi)T_{\bar\Lambda}$, thus with $\mathbf{B}(\bar\Phi,\bar\Phi)$ being the single-scale representation of $\mathbf{B}|_{\bar\Lambda\times\bar\Lambda}$. Since $(\mathbf{B}\mathbf{w})|_{\bar\Lambda} = T_{\bar\Lambda}^\top \mathbf{B}(\bar\Phi,\bar\Phi)T_{\bar\Lambda}\mathbf{w}$, in order to construct a valid **APPLY**, what is left is to approximate the multiplication with $\mathbf{B}(\bar\Phi,\bar\Phi)$ in $\mathscr{O}(\varepsilon^{-1/s}\|\mathbf{z}_\varepsilon\|_{\mathscr{A}^s}^{-1/s} + \#\operatorname{supp}\mathbf{w} + 1)$ operations, while keeping the error on the level of a multiple of ε. For partial differential operators, the advantage is that non-zeros entries of $\mathbf{B}(\bar\Phi,\bar\Phi)$ only involve pairs of scaling functions on equal or consecutive levels. For singular integral operators, to approximate the multiplication with $\mathbf{B}(\bar\Phi,\bar\Phi)$ one may think of the application of *panel clustering* ([HN89]) or *multipole expansions* ([GR87]).

Finally, whereas for the optimal adaptive solution of *linear* operator equations, the restriction to tree approximations is not really necessary, for such a solution of *nonlinear* operator equations it seems indispensable (see [CDD03a]). Indeed, note that for a nonlinear operator of the form $f(v)(x) = g(v(x))$, the evaluation of $f(\mathbf{w}^\top\Psi)(x)$ already requires a number of operations of the order of the number of wavelets in the expansion that are non-zero in x. If $\operatorname{supp}\mathbf{w}$ is a tree, however, then after transformation to the locally finite single scale representation, any of such a point evaluations can be done in $\mathscr{O}(1)$ operations.

6 Adaptive frame methods

6.1 Introduction

A drawback of wavelet methods for solving operator equations is the rather complicated construction of wavelet bases on non-product domains. As was already mentioned at the end of Sect. 5.1, the usual construction is via a non-overlapping decomposition of the n-dimensional domain or manifold into subdomains, each of

them being a smooth parametric image of the n-dimensional unit cube. Loosely speaking, wavelets or scaling functions constructed on this n-cube are lifted to the subdomains, after which those functions that do not vanish at an interface between subdomains are either continuously connected to functions from neighbouring subdomains or are smoothly extended into these subdomains. Apart from the fact that these constructions are not that easy to implement, another disadvantage is that the condition numbers of the resulting bases are quite somewhat larger than that of the corresponding bases on the n-cube.

As an alternative, for $\mathscr{X}$ being a Sobolev space, in [Ste03] it was suggested to use an *overlapping* domain decomposition, and to define $\Psi^{\mathscr{X}}$ simply as the union of the wavelet bases on the subdomains. By a proper choice of the bases on these subdomains, the span of $\Psi^{\mathscr{X}}$ will be dense in $\mathscr{X}$, but due to the overlap regions, it cannot be a basis for $\mathscr{X}$. Instead it will be a *frame* for $\mathscr{X}$. In [DFR07a], such a frame was called an *aggregated wavelet frame*.

6.2 Frames

Let $\mathscr{X}$ be a separable Hilbert space. A collection $\Psi = \{\psi_\lambda : \lambda \in \nabla\} \subset \mathscr{X}$ is called a *frame* for $\mathscr{X}$ when the *analysis operator*

$$\mathscr{F} : \mathscr{X}' \to \ell_2 : g \mapsto [g(\psi_\lambda)]_{\lambda\in\nabla},$$

is a boundedly invertible mapping between $\mathscr{X}'$ and *its range* $\operatorname{ran}\mathscr{F}$. From Sect. 2, recall that its adjoint, known as the *synthesis operator*, reads as

$$\mathscr{F}' : \ell_2 \to \mathscr{X} : \mathbf{c} \mapsto \mathbf{c}^\top \Psi.$$

We set the *frame constants*

$$\Lambda_\Psi := \|\mathscr{F}\|_{\mathscr{X}'\to\ell_2}, \quad \lambda_\Psi := \inf_{0\neq g\in\mathscr{X}'} \frac{\|\mathscr{F}g\|_{\ell_2}}{\|g\|_{\mathscr{X}}}.$$

The composition $\mathscr{F}'\mathscr{F} : \mathscr{X}' \to \mathscr{X}$ is boundedly invertible with $\|\mathscr{F}'\mathscr{F}\|_{\mathscr{X}'\to\mathscr{X}} = \Lambda_\Psi^2$ and $\|(\mathscr{F}'\mathscr{F})^{-1}\|_{\mathscr{X}\to\mathscr{X}'} = \lambda_\Psi^{-2}$.

The collection $\tilde{\Psi} := (\mathscr{F}'\mathscr{F})^{-1}\Psi$ is a frame for $\mathscr{X}'$, known as the canonical dual frame, with analysis operator $\tilde{\mathscr{F}} := \mathscr{F}(\mathscr{F}'\mathscr{F})^{-1}$ and frame constants λ_Ψ^{-1} and Λ_Ψ^{-1}. From $\mathscr{F}'\tilde{\mathscr{F}} = I$, one infers that any $v \in \mathscr{X}$ has a representation $v = \mathbf{v}^\top\Psi$ with $\Lambda_\Psi^{-1} \le \|\mathbf{v}\|_{\ell_2}/\|v\|_{\mathscr{X}} \le \lambda_\Psi^{-1}$, actually a property that is equivalent to Ψ being a frame with frame constants Λ_Ψ and λ_Ψ. Note that generally a representation of $v \in \mathscr{X}$ in frame coordinates is not unique (unless Ψ is a Riesz basis).

We have $\ell_2 = \operatorname{ran}\mathscr{F} \oplus^\perp \ker\mathscr{F}'$ and $\mathbf{Q} := \tilde{\mathscr{F}}\mathscr{F}'$ is the orthogonal projector onto $\operatorname{ran}\mathscr{F}$. The frame Ψ is a Riesz basis for $\mathscr{X}$ if and only if $\ker\mathscr{F}' = 0$ or equivalently $\operatorname{ran}\mathscr{F} = \ell_2$.

Many examples of frames can be given. Besides aggregated wavelet frames, here we only mention curvelets ([CD04]) and shearlets ([LLKW05]).

For a given $f \in \mathscr{X}'$ and a boundedly invertible $B \in \mathscr{L}(\mathscr{X}, \mathscr{X}')$, let us consider the problem of finding $u \in \mathscr{X}$ such that

$$Bu = f. \tag{50}$$

Writing $u = \mathscr{F}'\mathbf{u}$ for *some* $\mathbf{u} \in \ell_2$, this $\mathbf{u}$ solves

$$\mathbf{B}\mathbf{u} = \mathbf{f}, \tag{51}$$

where

$$\mathbf{B} := \mathscr{F}B\mathscr{F}', \quad \mathbf{f} := \mathscr{F}f.$$

Obviously, we have $\|\mathbf{B}\| \leq \Lambda_\Psi^2 \|B\|_{\mathscr{X}\to\mathscr{X}'}$. With respect to the decomposition $\ell_2 = \operatorname{ran}\mathscr{F} \oplus^\perp \ker\mathscr{F}'$, $\mathbf{B}$ is of the form $\left[\begin{smallmatrix}\mathbf{B}_0 & 0\\ 0 & 0\end{smallmatrix}\right]$. From $\tilde{\mathscr{F}}B^{-1}\tilde{\mathscr{F}}'\mathbf{B} = \mathbf{B}\tilde{\mathscr{F}}B^{-1}\tilde{\mathscr{F}}' = \mathbf{Q}$, we conclude that $\mathbf{B}_0 = \mathbf{B}|_{\operatorname{ran}\mathscr{F}} : \operatorname{ran}\mathscr{F} \to \operatorname{ran}\mathscr{F}$ is boundedly invertible with $\|\mathbf{B}_0^{-1}\| \leq \lambda_\Psi^{-2}\|B^{-1}\|_{\mathscr{X}'\to\mathscr{X}}$. Finally, we note that for $\mathbf{v}, \mathbf{w} \in \operatorname{ran}\mathscr{F}$,

$$\langle \mathbf{B}_0\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{B}\mathbf{v}, \mathbf{w}\rangle = \langle \mathscr{F}B\mathscr{F}'\mathbf{v}, \mathbf{w}\rangle = (Bv)(w), \tag{52}$$

where $v = \mathscr{F}'\mathbf{v}$ and $w = \mathscr{F}'\mathbf{w}$, or equivalently because $\mathbf{v}, \mathbf{w} \in \operatorname{ran}\mathscr{F}$, $\mathbf{v} = \tilde{\mathscr{F}}v$ and $\mathbf{w} = \tilde{\mathscr{F}}w$.

6.3 *The adaptive solution of an operator equation in frame coordinates*

In case the operator B in (50) is symmetric and positive definite, one may think of applying the adaptive wavelet Galerkin approach discussed in Sect. 4 onto $\mathbf{B}\mathbf{u} = \mathbf{f}$ from (51). Since, however, for a "true" frame, $\mathbf{B}$ has a non-trivial kernel, for $\Lambda \subsetneq \nabla$ the generalized condition number of $\mathbf{B}|_{\Lambda\times\Lambda}$, i.e., the quotient of its largest and its smallest non-negative eigenvalue, can be arbitrarily large. This makes this approach unfeasible.

Therefore, we return to the damped Richardson iteration discussed in Sect. 3.1. Denoting its ith iterand as $\mathbf{u}^{(i)}$, and with $\mathbf{u}$ *some* solution of $\mathbf{B}\mathbf{u} = \mathbf{f}$, we have

$$\mathbf{u} - \mathbf{u}^{(i)} = (\mathbf{I} - \alpha\mathbf{B})(\mathbf{u} - \mathbf{u}^{(i-1)}),$$

which, due to the non-trivial kernel of $\mathbf{B}$, shows no convergence. By applying $\mathbf{Q}$, however, we obtain

$$\mathbf{Q}(\mathbf{u} - \mathbf{u}^{(i)}) = (\mathbf{I} - \alpha\mathbf{B}_0)\mathbf{Q}(\mathbf{u} - \mathbf{u}^{(i-1)}).$$

If B is symmetric and positive definite or only coercive, then in view of (52), analogously to the analysis from Sect. 4, we infer that by a proper choice of α, $\|\mathbf{I}-\alpha\mathbf{B}_0\|<1$. Since $u-\mathscr{F}'\mathbf{u}^{(i)}=\mathscr{F}'\mathbf{Q}(\mathbf{u}-\mathbf{u}^{(i)})$, we conclude linear convergence of $\mathscr{F}'\mathbf{u}^{(i)}$ to u in $\mathscr{X}$. For non-coercive B, the iteration can be applied to the normal equations.

When applying the damped Richardson iteration with an inexact evaluation of the matrix-vector multiplication and that of the right-hand side $\mathbf{f}$, then, with a proper choice of decaying tolerances, for the resulting iteration a linear decrease of the projected error $\mathbf{Q}(\mathbf{u}-\mathbf{u}^{(i)})$ can still be shown. These inexact evaluations, however, generally produce error components that are in $\ker\mathscr{F}'$. Since $\ker\mathscr{F}'=\ker\mathbf{B}$, these error components will not be changed by subsequent Richardson steps. Although these error components do not affect the projected error, generally they do affect the $\mathscr{A}^s$-norms of the iterands, and with that, the cost of the applications of the **APPLY** routine.

In spite of this, in [Ste03] it was proved that the algorithm **Rich**, as given in Sect. 3.2 but with a modified choice of the tolerances (see [Ste03] for details), is again (quasi-) optimal in the sense of Theorem 1: *Given an* $\varepsilon>0$, *it produces an* $\mathbf{u}_\varepsilon$ *with* $\|\mathbf{Q}(\mathbf{u}-\mathbf{u}_\varepsilon)\|\le\varepsilon$. *If for some* $s>0$, $\mathbf{Bu}=\mathbf{f}$ *has* some *solution* $\mathbf{u}\in\mathscr{A}^s$, *then* $\#\operatorname{supp}\mathbf{u}_\varepsilon\lesssim\varepsilon^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}$. *If, additionally, for some* $\bar{s}>s$, $\mathbf{B}$ *is* $\bar{s}$*-admissible and* $\mathbf{Q}:\mathscr{A}^{\bar{s}}\to\mathscr{A}^{\bar{s}}$ *is bounded, then the number of operations used by the call is bounded by an absolute multiple of* $\varepsilon^{-1/s}\|\mathbf{u}\|_{\mathscr{A}^s}^{1/s}$. *In order words, if* $\bar{s}>s_{\max}$, *with* $s_{\max}$ *defined similarly as in the basis case* (*see Sect. 1.1*), *then this inexact Richardson iteration is* (quasi-) optimal.

The additional condition that $\mathbf{Q}:\mathscr{A}^{\bar{s}}\to\mathscr{A}^{\bar{s}}$ is bounded is satisfied when $\mathbf{Q}$ is $\bar{s}$-admissable (cf. Definition 1 and Proposition 1), which in turn is satisfied when, for some $s^*>\bar{s}$, $\mathbf{Q}$ is s^*-compressible (cf. Definitions 2, 3 and Theorem 2, and realize that the question about cost of computing entries of $\mathbf{Q}$ is not relevant, since $\mathbf{Q}$ does not enter the algorithm, but its boundedness in $\mathscr{A}^s$ is only needed for the proof of optimality).

Unfortunately, although we expect it to hold more generally, in the aggregated wavelet frame case so far the s^*-compressibility of $\mathbf{Q}$ was proved (in [Ste03, §4.3]) only in the case that the wavelets on each subdomain are L_2-orthogonal and that, before aggregation, they were multiplied by a smooth function that is positive on the subdomain and that vanishes outside the subdomain. Numerical results reported in [DFR$^+$07b] indicate (quasi-) optimality in other cases. In [DFR07a], the boundedness of $\mathbf{Q}:\mathscr{A}^{\bar{s}}\to\mathscr{A}^{\bar{s}}$ was shown for time-frequency localized Gabor frames.

Sufficient for $\bar{s}$-admissability of $\mathbf{B}$ is that it is s^*-computable for some $s^*>\bar{s}$. For aggregated wavelet frames, a proof of s^*-compressiblity of $\mathbf{B}$ can follows the same lines as in Sect. 5.1 for the basis case. In the aggregated wavelet frame case, the approximate computation using quadrature of the significant entries of $\mathbf{B}$ is a harder task. Indeed, wavelets from different subdomains whose supports overlap will be piecewise smooth with respect to different underlying partitions. Nevertheless, in [SW08], for partial differential operators with smooth coefficients, s^*-computability for an $s^*>s_{\max}$ was demonstrated.

Thinking of a symmetric and positive definite B, the selection of a suitable damping parameter α for the Richardson iteration requires estimating the smallest non-negative eigenvalue of $\mathbf{B}$. Other than in the Riesz basis case where $\mathbf{B}$ has no zero eigenvalues, in the true frame case it is difficult to estimate this eigenvalue numerically. In [DFR$^+$07b], it was shown that an approximate steepest descent iteration, which does not require information about the spectrum of $\mathbf{B}$, is (quasi-) optimal under the same conditions as the approximate Richardson iteration.

6.4 An adaptive Schwarz method for aggregated wavelet frames

Let $B \in \mathscr{L}(\mathscr{X}, \mathscr{X}')$ be symmetric and positive definite, where $\mathscr{X}$ is a Sobolev space with positive smoothness index on a domain Ω. Let Ψ be an aggregated wavelet frame being the union of wavelet bases $\Psi_1, \dots, \Psi_m$ on overlapping subdomains $\Omega_1, \dots, \Omega_m$, respectively. Each of these bases is a Riesz basis of the corresponding Sobolev space on the subdomain, with homogeneous Dirichlet boundary conditions on the internal boundary.

The partition of the domain into overlapping subdomains, or that of the frame into the different Riesz systems, suggest the application of a Schwarz method to solve $\mathbf{B}\mathbf{u} = \mathbf{f}$, being the representation of the operator equation $Bu = f$ in frame coordinates. An multiplicative adaptive Schwarz method was studied in [SW09].

Let $\mathbf{B} = (\mathbf{B}_{k\ell})_{1 \le k,\ell \le m}$ and $\mathbf{v} = (\mathbf{v}_k)_{1 \le k \le m}$ denote the corresponding partitions of the system matrix $\mathbf{B}$ and any vector of frame coordinates, respectively. Then the (exact) multiplicative Schwarz algorithm reads as follows:

```
for i = 1,2,... do
    for k = 1 to m do
        solve B_kk u_k^(i) = f_k − Σ_{ℓ=1}^{k−1} B_kℓ u_ℓ^(i) − Σ_{ℓ=k+1}^{m} B_kℓ u_ℓ^(i−1)
    enddo
enddo
```

Using the general theory of Schwarz methods (e.g. see [Xu92]), one shows that $\mathscr{F}'\mathbf{u}^{(i)} = \mathbf{u}^{(i)\top}\Psi$ converges linearly to u in $\mathscr{X}$.

The idea behind an inexact, adaptive variant is to find an approximation to $\mathbf{u}_k^{(i)}$ by the application of an adaptive *wavelet* method on subdomain Ω_k (either of inexact Richardson type or an adaptive wavelet Galerkin method). By a suitable choice of decaying tolerances, the resulting method will still be linearly convergent.

For each k, the sequence $(\mathbf{u}_k^{(i)})_i$ of approximate solutions of the subdomain problems on Ω_k converges to some $\mathbf{u}_k$, that depends on the choice of the initial vectors $(\mathbf{u}_\ell^{(0)})_{1 \le \ell \le m}$. With $\mathbf{u}$ being *some* representation of u, i.e., $\mathbf{u}^\top \Psi = u$, it is not clear that the splitting $\mathbf{u} = \mathbf{u}_k + (\mathbf{u} - \mathbf{u}_k)$ is smoothness preserving, in the sense that if $\mathbf{u} \in \mathscr{A}^s$, then $\mathbf{u}_k \in \mathscr{A}^s$ with $\|\mathbf{u}_k\|_{\mathscr{A}^s} \lesssim \|\mathbf{u}\|_{\mathscr{A}^s}$. From our considerations about the cost of the **APPLY** routine that is part of the adaptive wavelet method, it is however clear that such a smoothness preservation would be needed to conclude (quasi-) optimality of

the resulting method. Actually, numerical experiments indicated that generally this splitting is not smoothness preserving.

In order to solve this problem, again consider the system $\mathbf{B}_{kk}\mathbf{u}_k^{(i)} = \mathbf{f}_k - \sum_{\ell=1}^{k-1}\mathbf{B}_{k\ell}\mathbf{u}_\ell^{(i)} - \sum_{\ell=k+1}^{m}\mathbf{B}_{k\ell}\mathbf{u}_\ell^{(i-1)}$. Note that if, before solving, coefficients from $(\mathbf{u}_\ell^{(i)})_{1\le\ell\le k-1}$ and $(\mathbf{u}_\ell^{(i-1)})_{k+1\le\ell\le m}$ that correspond to wavelets that are fully supported in Ω_k are modified, in particular, are *deleted*, then this will not change the approximation $\mathbf{u}^{(i)\top}\Psi = \sum_{\ell=1}^{k}\mathbf{u}_\ell^{(i)\top}\Psi_\ell + \sum_{\ell=k+1}^{m}\mathbf{u}_\ell^{(i-1)\top}\Psi_\ell$ after this solving, although the vectors $(\mathbf{u}_\ell^{(i)})_{1\le\ell\le k}$ and $(\mathbf{u}_\ell^{(i-1)})_{k+1\le\ell\le m}$ generally do change. For this process, but then with an inexact adaptive solving, it was shown that if the sizes of the overlap regions are sufficiently large compared to the maximal diameter of the support of any wavelet, then the aforementioned splitting *is* smoothness preserving. Using this result, the overall method was shown to be (quasi-) optimal assuming that $\mathbf{B}$ is $\bar{s}$-admissable for some $\bar{s} \ge s_{\max}$ (cf. the discussion in Sect. 6.3). The boundedness of $\mathbf{Q} : \mathscr{A}^{\bar{s}} \to \mathscr{A}^{\bar{s}}$ is *not* required.

Note that the method with the deletion of the coefficients that correspond to wavelets associated to other subdomains, but that are fully supported in the current subdomain is actually closer to the original Schwarz method from [Sch90] than the method we described first. Indeed, what is left after this deletion process is essentially only boundary data for the problem on the current subdomain. The method with deletion is also cheaper to implement since it requires the computation of less entries in the system matrix corresponding to pairs of wavelets associated to different subdomains. Recall that the quadrature problem to approximate those entries is more demanding.

Numerical results reported in [SW09] show that quantitatively this multiplicative adaptive Schwarz method is much more efficient that the adaptive steepest descent method described in Sect. 6.3.

7 Adaptive methods based on tensor product wavelet bases

7.1 Tensor product wavelets

Let Ω be a product domain, i.e., $\Omega = \Omega_1 \times \cdots \times \Omega_n$, then for $t \ge 0$,

$$H^t(\Omega) = H^t(\Omega_1)\otimes L_2(\Omega_2)\otimes\cdots\otimes L_2(\Omega_n)\cap\cdots\cap L_2(\Omega_1)\otimes\cdots\otimes L_2(\Omega_{n-1})\otimes H^t(\Omega_n)$$

For $t \notin \mathbb{N}_0 + \frac{1}{2}$, the same holds true with $H^t(\Omega)$ reading as $H_0^t(\Omega)$ and $H^t(\Omega_i)$ as $H_0^t(\Omega)$. Similar statements involving boundary conditions of lower order, or with boundary conditions on a part of the boundary (of product type) are also valid ([DS09a]).

Now for $1 \le i \le n$, let $\Psi^{(i)} = \{\psi_\lambda^{(i)} : \lambda \in \nabla_i\} \subset H^t(\Omega_i)$ be a Riesz basis for $L_2(\Omega_i)$ that, when normalized in $H^t(\Omega_i)$, is a Riesz basis for $H^t(\Omega_i)$. Wavelet bases are known to have this property for a range of t. Then using above characterization

of $H^t(\Omega)$, it can be shown (cf. [GO95]) that the tensor product wavelet basis

$$\boldsymbol{\Psi} := \Psi^{(1)} \otimes \cdots \otimes \Psi^{(n)} = \{\boldsymbol{\psi}_{\boldsymbol{\lambda}} := \psi^{(1)}_{\lambda_1} \otimes \cdots \psi^{(n)}_{\lambda_n} : \boldsymbol{\lambda} \in \boldsymbol{\nabla} := \nabla^{(1)} \times \cdots \times \nabla^{(n)}\}$$

is a Riesz basis for $H^t(\Omega)$.

Note that the widths of the support of a tensor product wavelet measured in the coordinate directions can differ to an arbitrarily large extend. Furthermore, other than with a (standard) wavelet basis, there exists no multiresolution analysis on Ω such that (biorthogonal) complement spaces are spanned by a subset of $\boldsymbol{\Psi}$.

In spite of these differences, tensor product wavelet bases can be applied in adaptive wavelet algorithms. In order to show that these algorithms give (quasi-) optimal results, what is needed to verify is that the representation of the operator under consideration in tensor product wavelet coordinates can be sufficiently well approximated by computable sparse matrices in relation to the best possible convergence rate that can be expected. That is, what is needed to check is whether $s^* > s_{\max}$, with $s_{\max}$ being defined in Sect. 1.1 and s^* from Definition 2 in Sect. 3.3.

7.2 Non-adaptive approximation

Let Ω_i be a domain of dimension n_i and $\Psi^{(i)}$ be a wavelet basis of order $d_i > t$, cf. Example 1. Then it is well-known that a sufficiently smooth function on Ω can be approximated in $H^t(\Omega)$ from the sequence of spaces $(\mathrm{span}\{\boldsymbol{\psi}_{\boldsymbol{\lambda}} : \sum_{i=1}^n |\lambda_i| \leq \ell\})_\ell$ with rate $s_{\max} = \max_i \frac{d_i - t}{n_i}$, up to some log-factors (the error bound reads as $N^{-\max_i \frac{d_i-t}{n_i}} (\log N)^q$ for some $q > 0$ with N being the number of unknowns). This type of approximation is known as *sparse-grid* or *hyperbolic cross* approximation (see [Zen91, DKT98, BG04]). For $t > 0$, the aforementioned log-factors can even be removed by considering slightly modified approximation spaces, known as *optimized sparse-grid* spaces ([GK00]). In particular, from now on thinking of $n_1 = \ldots = n_n = 1$ and $d_1 = \ldots = d_n =: d > t$, a sufficiently smooth function on an n-rectangle is approximated in H^t for $t > 0$ by optimized sparse grids with rate $s_{\max} = d - t$. That is, the so-called "curse of dimensionality" – the fact that with standard wavelet (or finite element) approximation the rate is inversely proportional with the space dimension – is completely removed.

7.3 Best N-term approximation and regularity

Sparse grid approximation is *non-adaptive*, and the aforementioned high convergence rate requires a smoothness of the function being approximated that the solution of an operator equation may not possess. Indeed, in [DS09a] it was shown that for the Poisson problem on the n-rectangle with homogeneous Dirichlet boundary

conditions and a smooth right-hand side, the optimized sparse grid convergence rate in H^1 is $\frac{1}{2}+\frac{1}{n}$, instead of $s_{\max}=d-1$ that would be obtained when the solution was sufficiently smooth. Only if the right-hand side vanishes to a sufficiently high order at the non-smooth parts of the boundary, the best possible rate is obtained.

The requirements to approximate a function on the n-rectangle with a certain rate $s \le s_{\max}=d-t$ with *best N-term approximation* from the tensor product basis, i.e., the requirements for the function to be in $\mathscr{A}^s$, are (much) milder than the requirements to obtain this rate with (optimized) sparse grid approximation. For $s<s_{\max}$, a characterization of $\mathscr{A}^s$ in terms of intersections of tensor products of Besov spaces was given in [Nit06]. Following earlier work in [Nit05], for $t \in \mathbb{N}$ in [DS09a] it was shown that if a function u on the n-rectangle has partial derivatives up to order nd in certain weighted L_2 spaces, with weights that vanish at the boundary, then $u \in \mathscr{A}^{d-t}$. What is more, additionally it was shown that the solution of an elliptic boundary value problem of order $2t$ on the n-rectangle with smooth coefficients, homogeneous Dirichlet boundary conditions and a smooth right-hand side satisfies these regularity conditions.

Here we emphasize that for sufficiently large n and d, a rate $d-t$ cannot be realized with best N-term standard wavelet approximation. Indeed, with wavelets of order $\hat{d}$, in n space dimensions the best possible rate is $\frac{\hat{d}-t}{n}$. A (near) characterization of $\mathscr{A}^{\frac{\hat{d}-t}{n}}$ can be given in terms of certain Besov spaces. It is known, however, that for $n \ge 3$, the solution of an elliptic boundary value problem has limited smoothness in this scale of Besov spaces. In other words, one cannot simply choose the rate at one's convenience by increasing the order $\hat{d}$. In any case in three dimensions, with finite elements of order $\hat{d}$ one can realize the best possible rate $\frac{\hat{d}-t}{3}$ by including *anisotropic* refinements towards the boundary ([Ape99]). The tensor product wavelet approach has the unique additional feature that the rate $s_{\max}$ does not deteriorate with an increasing space dimension.

7.4 s^*-computability

In order to conclude that the adaptive tensor product wavelet method converges at the same rate as the sequence of best N-term approximations with respect to the tensor product basis in linear complexity, it is needed that $s^* > s_{\max}=d-t$. For boundary value problems with homogeneous Dirichlet boundary conditions and smooth coefficients and piecewise smooth, sufficiently globally smooth univariate wavelets with sufficiently many vanishing moments, this has been verified in [SS08]. Thinking of the arbitrarily stretched supports of the tensor product wavelets, one might consider it as counterintuitive that an operator is better compressible in a tensor product wavelet basis than it is in a standard wavelet basis. The key is that the sizes of the entries decay exponentially as function of the *sum* of the absolute differences in levels of the tensor product wavelets involved. Compressiblity of integrodifferen-

tial operators has been investigated in [Rei08].

7.5 Truly sparse stiffness matrices

Recently, in [DS09b] a univariate wavelet basis of cubic Hermite splines was constructed that has the property that any second order boundary value problem with constant coefficients and homogeneous Dirichlet boundary conditions on the n-cube with respect to the n-fold tensor product basis is *truly sparse*. As a consequence, the application of an adaptive wavelet method simplifies enormously. Indeed, the application of the stiffness matrix to any finitely supported vector can be performed exactly in linear complexity. Also with non-constant, smooth coefficients, the application of this basis in the adaptive wavelet Galerkin method is advantageous. For the approximate residual computation, being the most time consuming part of the algorithm, entries outside the nonzero pattern of a constant coefficient operator, except those that correspond to wavelets on a few coarsest levels, are an order of magnitude smaller than those inside this pattern, and so can be discarded.

7.6 Problems in space high dimension

We have seen that the sequence of approximations produced by an adaptive tensor product wavelet method converges with the same *rate* as the sequence of best N-term approximations with respect to the tensor product basis. This does not exclude the possibility that the quotient of the error produced by the adaptive method and that of the best N-term approximation of the same length grows with increasing n. Actually, generally in any case any available upper bound for this quotient will grow exponentially as function of n. A reason is that various estimates to bound the error for the adaptive method depend critically on the condition number of the n-fold tensor product basis. If and only if the univariate wavelets are chosen to be L_2-orthogonal, this condition number is bounded uniformly in n, whereas it grows exponentially in n otherwise.

In [DSS08], the n-fold tensor product of the univariate piecewise polynomial L_2-orthogonal wavelet basis from [DGH96] is applied to solve constant coefficient elliptic boundary value problems on the n-rectangle. For this case, it was shown that even the *factor* that the adaptive method might lose compared to the best N-term approximations is bounded by an absolute constant. Experiments for the Poisson problem on the n-cube with right-hand side 1 show, however, that the best N-term approximations themselves still suffer from another, although much milder curse of dimensionality. Although for any dimension n, the rate of approximation in H^1 is $d-1$, the number of unknowns needed to achieve a relative error below some given tolerance grows exponentially with n. Apparently, the constant C in the error

bound CN^{d-1} grows exponentially with n. In view of the result from [NW08] saying that the approximation of a general infinitely differentiable multivariate function is *intractable*, this exponential growth of the constant is not surprising.

Likely, to approximate a function in high space dimensions, with the current hardware think of dimensions higher than say 8-10, one should exploit more information about the function than only that it is the solution of a boundary value problem with some *general* smooth right-hand side. As demonstrated in [Gra04, BM02, HK07], a class of functions that can be accurately approximated in high space dimensions are the solutions of boundary value problems with right-hand sides that can be well approximated by a small number of separable functions.

7.7 Non-product domains

The application of tensor product wavelet bases is not restricted to Sobolev spaces $H^t(\Omega)$ with $t \geq 0$ where Ω is product domain. Indeed, recall that the commonly applied approaches to construct wavelet bases on a non-product domain start with writing this domain as a non-overlapping union of subdomains, each of them being a smooth parametric image of the n-cube. With the approach based on extension operators, wavelet bases on the n-cube are lifted to the subdomains, after which those that do not vanish at an interface between subdomains are smoothly extended into neighbouring subdomains. This approach can be applied verbatim to tensor product wavelet bases on the n-cube.

Using a *non-overlapping* domain decomposition, one may also think of constructing an aggregated frame based on tensor product wavelet bases on the subdomains. In the general case, however, where the underlying partitions in the overlap regions are not aligned, the compressibility of the resulting system matrix will be too low.

7.8 Other, non-elliptic problems

We considered well-posed linear operator equations of the form $B : \mathscr{X} \to \mathscr{X}'$, where $\mathscr{X} = H^t(\Omega)$ or $H^t_0(\Omega)$ and Ω is a product domain. In this case, $H^t(\Omega)$ is an intersection of tensor product of Sobolev spaces. Well-posed operator equations $B : \mathscr{X} \to \mathscr{Y}'$, where $\mathscr{X}$ and $\mathscr{Y}$ are of this type arise more generally. We mention here the "unfolding" of elliptic n-scale homogenization problems (cf. [AB96, HS05]) as well as the higher dimensional partial differential equations for the mean field, two-point correlation and possibly higher order moments of the random solution of an elliptic PDE with stochastic input data (cf. e.g. [ST03, HSS08, vPS06]).

Another example is given by the space-time variational formulation of the parabolic initial boundary value problem presented in Sect. 2.2.4. In this case $\mathscr{X} = L_2(0,T) \otimes H_0^1(\Omega) \cap H^1(0,T) \otimes H^{-1}(\Omega)$ and $\mathscr{Y} = (L_2(0,T) \otimes H_0^1(\Omega)) \times L_2(\Omega)$.

A classical approach to the numerical solution of the parabolic initial boundary value problem is the *Method of Lines*, which reduces the problem by spatial semidiscretization to a system of coupled ordinary differential equations to be solved numerically in $(0,T)$. Conversely, in Rothe's Method the problem is reduced by time semidiscretization to a sequence of coupled spatial, elliptic problems to be solved. Both these approaches, and the more recently proposed discontinuous Galerkin method are essentially time marching methods. The ultimate aim of adaptive methods is to achieve an approximate solution with an error below a prescribed tolerance at the expense of, up to an absolute multiple, minimal amount of computer time and storage. Due to the character of time stepping this seems hard to realize and, unlike for elliptic problems, so far no optimality results seem to be known.

In [SS09], the aforementioned spaces $\mathscr{X}$ and $\mathscr{Y}$ were equipped with tensor product wavelet bases. The resulting system matrix was proven to be sufficiently compressible and so the adaptive wavelet method applied to the simultaneously space-time variational formulation converges with the rate as that of the best N-term approximations. While keeping discrete solutions on all time levels is prohibitive for time marching methods, thanks to the use of tensor product bases, with the method in [SS09] there is no penalty in complexity because of the additional time dimension.

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