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AN OPTIMAL ADAPTIVE FINITE ELEMENT METHOD *

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Abstract. Although existing adaptive finite element methods for solving second order elliptic equations often perform better in practical computations than non-adaptive ones, usually they are not even proven to converge. Only recently in the work of Dörfler ([11]) and that of Morin, Nochetto and Siebert ([15]), adaptive methods were constructed for which converge could be demonstrated. However, convergence alone does not imply that the method is more efficient than its non-adaptive counterpart. In [3], Binev, Dahmen and DeVore added a coarsening step to the routine from [15]. They could prove that the resulting method is quasi-optimal in the following sense: If the solution is such that for some $s > 0$, the error in energy norm of the best continuous piecewise linear approximations subordinate to any partition with n triangles is $\mathcal{O}(n^{-s})$, then given an $\varepsilon > 0$, the adaptive method produces an approximation with an error less than ε subordinate to a partition with $\mathcal{O}(\varepsilon^{-1/s})$ triangles, in only $\mathcal{O}(\varepsilon^{-1/s})$ operations.

In this paper, employing a different type of adaptive partitions, we develop an adaptive method with similar properties as in [3]. Unlike [3], our coarsening routine will be based on a transformation to a wavelet basis, and we expect it to have better quantitative properties. Furthermore, all our results are valid uniformly in the size of possible jumps of the diffusion coefficients. Since the algorithm uses solely approximations of the right-hand side, we can even allow right-hand sides in $H^{-1}(\Omega)$ that lie outside $L_2(\Omega)$, at least when they can be sufficiently well approximated by piecewise constants. In our final adaptive algorithm, all tolerances depend on an a posteriori estimate of the current error instead an a priori one; this can be expected to provide quantitative advantages.

AMS subject classifications. 65N30, 65N50, 65N15, 65Y20, 65T60, 41A25

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1. Introduction.

For solving elliptic boundary value problems for which the solution has singularities, the use of adaptive finite element methods potentially has the advantage of a significant reduction of the computational cost, compared to non-adaptive methods. Although the adaptive methods that can be found in the literature often exhibit such a reduction, they are, usually, not even proven to converge, let alone shown to outperform non-adaptive methods. Only quite recently, in the work of Dörfler ([11]), that was later extended by Morin, Nochetto and Siebert in ([15]), adaptive methods were constructed that were proven to converge. These methods are based on an adaptive refinement strategy that guarantees the so-called saturation property, namely that the difference between the solutions on two consecutive partitions is greater than some multiple of the error in the solution on the first partition. Exploiting Galerkin orthogonality, convergence then easily follows.

In [3], Binev, Dahmen and DeVore added a coarsening step to the method from [15]. Basically the idea of such a step, that has to be applied after each fixed number of refinement steps, is to undo refinements that ultimately hardly contribute to a better approximation. Thanks to this coarsening, under some conditions on the right-hand side, the resulting adaptive method was proven to be quasi-optimal in the following sense: if the solution is such that, for some $s > 0$, the error in energy norm of the best continuous piecewise linear approximations subordinate to any partition with n triangles is $\mathcal{O}(n^{-s})$, then given an $\varepsilon > 0$, the adaptive method produces an

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approximation with an error less than ε subordinate to a partition with $\mathcal{O}(\varepsilon^{-1/s})$ triangles, in only $\mathcal{O}(\varepsilon^{-1/s})$ operations.

In this paper, we consider a method, as developed in [15] and extended with a coarsening in [3], in a slightly different context: instead of working with conforming partitions produced with the so-called newest vertex bisection method, we consider generally non-conforming partitions produced by only 'red-refinement' steps, i.e., splittings of triangles into four congruent subtriangles. In this setting we extend the findings from [15] and [3] on some points.

- Following [15], we consider the model problem of Poisson's equation on a two-dimensional domain, generalized in the sense that a piecewise constant diffusion tensor is allowed. Assuming a so-called quasi-monotone distribution of possible jumps that this tensor may have, all results from this paper will be proven to hold uniformly in the size of such jumps.
- With an adaptive method, the right-hand side and the discrete Galerkin solution subordinate to the current partition together determine the next partition via an a posteriori error estimator. Aiming at proving optimal computational complexity, following [3] we consider inexact solutions of the discrete systems. In addition to that, we allow inexact right-hand sides to be used for both setting up the discrete systems and for the evaluation of the a posteriori error estimator. This generalization can be used to model the effect of the application of quadrature. Furthermore, it will allow us to prove quasi-optimality of the adaptive method even for right-hand sides in $H^{-1}(\Omega)$ outside $L_2(\Omega)$, at least when they can be sufficiently well approximated by piecewise constants.
- We introduce a new coarsening procedure, that unlike the procedure from [3] is based on a transformation to a wavelet basis for the space of continuous piecewise linears subordinate to the adaptively refined partition. We expect our procedure to have better quantitative properties, although admittedly a final answer can only be given after performing numerical tests. Both the coarsening from [3] and our coarsening rely on an adaptive tree approximation algorithm developed by Binev and DeVore in [5].
- The adaptive finite element method from [3] and our first routine **SOLVE1** require as input an a priori upper bound μ of the convergence rate of the algorithm without coarsening. When one supplies a μ that is too small, quasi-optimality as a consequence of the coarsening is not guaranteed. On the other hand, taking a μ that is unnecessarily close to 1 will result in a quantitatively less attractive algorithm, since due to the coarsening, the convergence rate will be limited by this μ. In this paper, we develop a second routine **SOLVE2** in which the tolerances allowed in the inexact Galerkin solutions and in the approximations of the right-hand side, and those required in the coarsening are all some fixed multiples of an a posteriori estimate of the current error. Apart from the fact that this releases the user from the task of supplying this most critical parameter, the new algorithm benefits from a better convergence rate than what might appear to be the case from an a priori worst case analysis.

Finally, let us comment on the necessity of applying a coarsening routine. In the numerical experiments reported in [15], the partitions, although produced without coarsening, already seem to have a quasi-optimal cardinality. Of course, this does not exclude the possibility that there are other examples for which coarsening is necessary. On the other hand, it is also possible that coarsening is not a necessary ingredient of a quasi-optimal adaptive algorithm for solving these elliptic problems, but that a proof

of such a fact is eluding us. In any case, we do not consider our construction of a coarsening routine as being relevant for theoretical purposes only. Instead, we expect that coarsening will be very useful inside adaptive routines for solving non-stationary problems.

This paper is organized as follows: In §2 our model boundary value problem is described.

In §3, we introduce the class of admissible partitions, which is the subclass of all partitions that can be generated by red-refinements for which the generations of neighbouring triangles differ by at most by one. We show that any partition can be refined to an admissible one by increasing the number of triangles by at most a constant factor.

In §4, we introduce a wavelet basis for the space of continuous piecewise linears subordinate to any admissible partition. We show that both the basis transformation from wavelet to nodal basis and its inverse can be performed in optimal computational complexity.

Our coarsening routine is defined in §5. It is based on a transformation to wavelet basis, an application of the adaptive tree approximation routine from [5], and finally a construction of a reduced partition subordinate to which the remaining terms in the wavelet expansion are continuous piecewise linear functions.

In §6, an a posteriori error estimator is derived. A refinement strategy is developed that also for inexact, but sufficiently accurate, right-hand sides and discrete solutions is shown to be convergent.

In §7, the coarsening routine and the convergent adaptive refinement strategy are combined into an optimal adaptive finite element method.

Finally, in §8 we derive an optimal adaptive finite element method in which the tolerances for the errors in the right-side and in the discrete solution as well as for the coarsening routine are determined by an a posteriori estimate of the current error.

In order to avoid the repeated use of generic but unspecified constants, in this paper by $C \lesssim D$ we mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

2. Boundary value problem. Let Ω be a polygonal bounded domain in $\mathbb{R}^2$. We consider the following model boundary value problem in variational form: Given $f \in H^{-1}(\Omega)$, find $u \in H^1_0(\Omega)$ such that

$$a(u,w) := \int_\Omega \mathbf{A}\nabla u \cdot \nabla w = f(w), \quad (w \in H^1_0(\Omega)), \tag{2.1}$$

where $\mathbf{A} \in L_\infty(\Omega)$ is a symmetric 2×2 matrix with $\operatorname{ess\,inf}_{x\in\Omega}\lambda_{\min}(\mathbf{A}(x)) > 0$. Further assumptions on $\mathbf{A}$ are collected in the forthcoming Assumption 3.8. Defining $L : H^1_0(\Omega) \to H^{-1}(\Omega) = (H^1_0(\Omega))'$ by $(Lu)(w) = a(u,w)$, (2.1) can be rewritten as

$$Lu = f.$$

In some places we will assume that the right-hand side $f \in L_2(\Omega)$, in which case $f(w)$ should be interpreted as $\int_\Omega fw$.

Aiming at results that hold uniformly in the size of variations that the spectral radius $\rho := \rho(\mathbf{A})$ may have, we introduce a weighted $L_2(\Omega)$-scalar product

$$\langle u, w\rangle_0 = \int_\Omega \rho uw,$$

and define the weighted norms

$$\|w\|_0 = \langle w, w\rangle_0^{\frac{1}{2}}, \quad |w|_1 = a(w,w)^{\frac{1}{2}}, \quad \|g\|_{-1} = \sup_{0\neq w\in H_0^1(\Omega)} \frac{|g(w)|}{|w|_1}$$

on $L_2(\Omega)$, $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, respectively. Equipped with these norms, L is an isomorphism between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

For $\Sigma \subset \Omega$, we define $|w|_{1,\Sigma} := (\int_\Sigma \mathbf{A}|\nabla w|^2)^{\frac{1}{2}}$.

3. Partitions of Ω. We shall approximate the solution of (2.1) by continuous piecewise linear functions subordinate to a partition of Ω into triangles. In this subsection we precisely describe the type of partitions to be considered.

We will use P to denote a *partition* of Ω, defined as a collection of closed triangles $\triangle$ such that $\overline{\Omega} = \cup_{\triangle\in P}\triangle$ and $\mathrm{meas}(\triangle \cap \tilde{\triangle}) = 0$ for any two different $\triangle, \tilde{\triangle} \in P$. When $\triangle \cap \tilde{\triangle} \neq \emptyset$, such triangles will be called *neighbours*. A partition $\tilde{P}$ is called a *refinement* of P, when $\tilde{P}$ can be constructed by, for zero or more $\triangle \in P$, replacing $\triangle$ by the four subtriangles created by connecting the midpoints of edges of $\triangle$, or by a recursive application of this elementary 'red' refinement step. The above $\triangle$ will be referred to as being the *parent* of its four subtriangles, called *children* of $\triangle$. As expected, children of children of $\triangle$ are called *grandchildren* of $\triangle$.

Throughout this paper we consider only partitions P that are refinements of some *fixed initial partition* P_0 of Ω. Many of our statements will involve constants that actually depend on P_0. However since P_0 is assumed to be fixed, for ease of presentation we ignore these dependencies. Clearly, any $\triangle \in P$ is *similar* to a triangle from P_0. For $\triangle \in P$, $\mathrm{gen}(\triangle)$ will denote the number of elementary refinement steps needed to create $\triangle$ starting from some $\tilde{\triangle} \in P_0$, where $\mathrm{gen}(\tilde{\triangle}) := 0$.

We call v a *vertex of* P, when there exists a $\triangle \in P$ such that v is a vertex of $\triangle$. A vertex v of P is called *non-hanging* when it is a vertex of all $\triangle \in P$ that contain v, otherwise it is called a *hanging* vertex of P. With $\bar{V}_P$ or V_P we will denote the set of all *non-hanging* vertices of P or all *non-hanging, interior* vertices of P respectively. We assume that P_0 is *conforming*, i.e., all its vertices are non-hanging.

A vertex v of P is called *regular* when for all $\triangle \in P$ that contain v, $\mathrm{gen}(\triangle)$ has the same value. Note that a regular vertex is non-hanging.

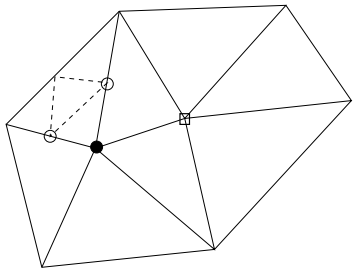

FIG. 3.1. *Regular (□), non-hanging but non-regular (•), and hanging vertices (○).*

For a vertex v of P, the number of $\triangle \in P$ that contain v is called the *valence* of v in P. The valence of any v of P is less or equal to the maximum of 6 and the maximum valence of all vertices of P_0. If for a $\triangle \in P$, $\mathrm{gen}(\triangle) = \max_{\tilde{\triangle}\in P} \mathrm{gen}(\tilde{\triangle})$, then its edges cannot contain hanging nodes. As a consequence, for such $\triangle$, the number of its neighbours in P is given by the sum of the valences of its vertices minus

the sum of 6 and the number of edges of $\triangle$ on $\partial\Omega$; this shows, in particular, that the number of neighbours is uniformly bounded.

PROPOSITION 3.1. *For any partition P of the type we consider, there exists a unique sequence of partitions*

$$P_0, P_1, \ldots, P_n$$

with $\max_{\triangle\in P_i} \mathrm{gen}(\triangle) = i$, $P_n = P$, *and where P_{i+1} is created from P_i by refining some $\triangle \in P_i$ with* $\mathrm{gen}(\triangle) = i$. *For convenience, we set $P_{-1} = \emptyset$ and so $V_{P_{-1}} = \emptyset$. The following properties are valid:*

(i) $V_{P_{i-1}} \subset V_{P_i}$ *and so* $V_P = \cup_{i=0}^n V_{P_i}\backslash V_{P_{i-1}}$ *with empty mutual intersections;*

(ii) *a $v \in V_{P_i}\backslash V_{P_{i-1}}$ is not a vertex of P_{i-1}, and so it is a regular vertex of P_i.*

Proof. The existence and uniqueness of the sequence $(P_i)_i$ and also (i) are obvious.

Suppose that for some $1 \le i \le n$, $v \in V_{P_i}\backslash V_{P_{i-1}}$ is a vertex of P_{i-1}; then it is a hanging vertex of P_{i-1}. So there is a $\triangle \in P_{i-1}$ with $\mathrm{gen}(\triangle) < i-1$ that contains v. However, by definition of the sequence $(P_i)_i$, such a $\triangle$ is not refined when going to P_i, meaning that v is also a hanging vertex of P_i, which gives a contradiction. □

NOTATION 3.2. *Throughout this paper, for any partition P (or $\hat{P}$, $\tilde{P}$, etc.), by $(P_i)_i$ (or $(\hat{P}_i)_i$, $(\tilde{P}_i)_i$, etc.) we will always mean the corresponding sequence, as in Proposition 3.1. When we write $P = P_n$, we mean that P is a partition with* $\max_{\triangle\in P} \mathrm{gen}(\triangle) = n$.

In view of the forthcoming discussion on adaptively refined partitions, we emphasize here that given any partition P, the definition of the corresponding sequence $(P_i)_i$ is independent of the way P has been constructed.

DEFINITION 3.3. *A partition P^a is called* admissible *when for all neighbours $\triangle, \tilde{\triangle} \in P^a$, $|\mathrm{gen}(\triangle) - \mathrm{gen}(\tilde{\triangle})| \le 1$.* As will turn out later, the reason to consider this restricted class of admissible partitions is given by the following proposition.

PROPOSITION 3.4. *Let P^a be admissible. For any $\triangle \in P^a$ with $i := \mathrm{gen}(\triangle) > 0$ the vertices of the parent $\tilde{\triangle} \in P^a_{i-1}$ of $\triangle$ are regular vertices of P^a_{i-1}.*

Proof. For $i = 1$ the statement is true. Now let $i > 1$. Suppose that some vertex v of $\tilde{\triangle}$ is not a regular vertex of P^a_{i-1}. Then, there exists a $\hat{\triangle} \in P^a_{i-1}$ with $v \in \hat{\triangle}$ and $\mathrm{gen}(\hat{\triangle}) < i-1$. Since, by definition of the sequence $(P^a_i)_i$, this $\hat{\triangle}$ will never be refined, we get a contradiction with the fact that P^a is admissible. □

PROPOSITION 3.5. *If $P^a = P^a_n$ is admissible, then for any $0 \le i \le n$, P^a_i is also admissible.*

Proof. Suppose P^a_i is not admissible; then it contains neighbours $\hat{\triangle}, \triangle$ with $\mathrm{gen}(\hat{\triangle}) < \mathrm{gen}(\triangle) - 1$. Since $\mathrm{gen}(\hat{\triangle}) < i$, it will never be refined and so P^a cannot be admissible. □

Although not all partitions are admissible, any partition has an admissible refinement with a number of triangles which is at most a constant factor larger. Indeed, given a partition $P = P_n$, consider the following algorithm to compute the partition $P^a = P^a_n$:

ALGORITHM 3.6.

$P^a_0 := P_0$

`for` $i = 0, \ldots, n$ `do`

define P^a_{i+1} as the union of P_{i+1} and, when $i \le n-2$, the collection children of those $\triangle \in P^a_i$ that have a neighbour in P_i with grandchildren in P_{i+2}

`od`

PROPOSITION 3.7. *The partition P^a produced by Algorithm 3.6 is an admissible refinement of P with $\#P^a \lesssim \#P$.*

Proof. The criterion to add children of a $\triangle \in P_i^a$ to P_{i+1} with the construction of P_{i+1}^a can only be fulfilled when $\triangle$ is neighbour of a $\tilde{\triangle} \in P_i$ which was refined when going to P_{i+1}, and thus with $\mathrm{gen}(\tilde{\triangle}) = i$. As we have seen, since $\max_{\triangle \in P_i^a} \mathrm{gen}(\triangle) = i$, the number of neighbours in P_i^a of such a $\tilde{\triangle}$ is uniformly bounded. So defining λ_i or λ_i^a as the number of triangles that were refined when going from P_i to P_{i+1} or from P_i^a to P_{i+1}^a respectively, we have $\lambda_i^a \lesssim \lambda_i$.

Note that $(P_i^a)_{0 \le i \le n}$ corresponds to P^a in the sense of Proposition 3.1. Since each time a triangle in a partition is refined the number of triangles is increased by 3, we conclude that

$$\#P^a = \#P_0 + 3\sum_{i=0}^{n-1} \lambda_i^a \lesssim \#P_0 + 3\sum_{i=0}^{n-1} \lambda_i = \#P.$$

What is left to show is that P^a is admissible. Obviously the partitions P_0^a and P_1^a are admissible. Suppose that there exists an $1 \le i \le n-1$ such that P_i^a is admissible, whereas P_{i+1}^a is not. Then, there exist neighbours $\triangle, \tilde{\triangle} \in P_{i+1}^a$ with $\mathrm{gen}(\triangle) - \mathrm{gen}(\tilde{\triangle}) > 1$. Since P_i^a is admissible, necessarily $\mathrm{gen}(\triangle) = i+1$ and $\mathrm{gen}(\tilde{\triangle}) = i-1$.

If $\triangle \in P_{i+1}$, then $\tilde{\triangle} \in P_{i-1}^a$ has a neighbour in P_{i-1} with grandchildren in P_{i+1}. So, by construction, $\tilde{\triangle}$ would have been refined when going to P_i^a which gives a contradiction with the assumption that $\tilde{\triangle} \in P_{i-1}^a$.

If $\triangle \in P_{i+1}^a \backslash P_{i+1}$, then by construction, its parent $\triangle_f \in P_i^a$ has a neighbour $\hat{\triangle} \in P_i$ with grandchildren in P_{i+2}, whereas obviously also $\triangle_f \in P_i^a$ and $\tilde{\triangle}$ are neighbours. Let $\triangle_{ff} \in P_{i-1}^a$ and $\hat{\triangle}_f \in P_{i-1}$ denote the parents of $\triangle_f$ and $\hat{\triangle}$ respectively. We are going to show that $\hat{\triangle}_f$ and $\tilde{\triangle}$ are neighbours in P_{i-1}^a, meaning, because $\hat{\triangle}_f$ has grandchildren in P_{i+1}, that $\tilde{\triangle}$ must have been refined when going to P_i^a, resulting in a contradiction with the assumption that $\tilde{\triangle} \in P_{i+1}^a$. We have to distinguish between two cases: If $\triangle_f$ is the central subtriangle of $\triangle_{ff}$, then both $\triangle_{ff}$ and $\hat{\triangle}_f$, and $\triangle_{ff}$ and $\tilde{\triangle}$ share an edge, and so $\hat{\triangle}_f$ and $\tilde{\triangle}$ are neighbours (cf. left picture in Figure 3.2). If $\triangle_f$ is a corner subtriangle of $\triangle_{ff}$, i.e, $\triangle_f$ and $\triangle_{ff}$ share a vertex v, then v is also a vertex of both $\hat{\triangle}_f$ and $\tilde{\triangle}$, again showing that they are neighbours (cf. right picture in Figure 3.2). □

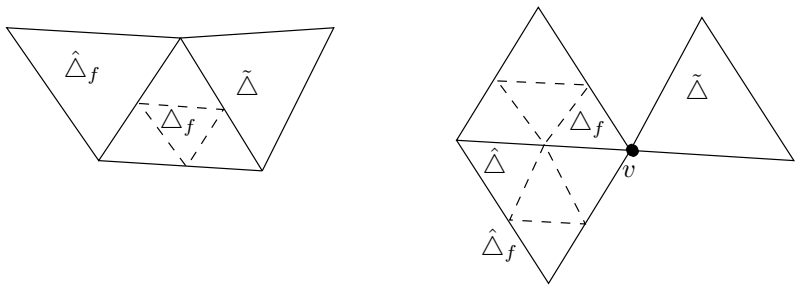

FIG. 3.2. *Illustration with the proof of Proposition 3.7.*

With $P_0^* = P_0$, by induction on i we construct the partition P_i^* from P_{i-1}^* by applying a red refinement step to *all* $\triangle \in P_{i-1}^*$, i.e., P_i^* is the result of applying recursively i *uniform refinement steps* to P_0. Note that these definitions are in accordance

with Notation 3.2. We define

$$V_* = \cup_{i\geq 0} V_{P_i^*} \backslash V_{P_{i-1}^*};$$

this set contains V_P for any partition $P = P_n$. Obviously, for $0 \leq i \leq n$, $V_{P_i} \subset V_{P_i^*}$, and Proposition 3.1(ii) shows that $V_{P_i} \backslash V_{P_{i-1}} \subset V_{P_i^*} \backslash V_{P_{i-1}^*}$.

Finally in this subsection, having defined the initial partition P_0, we are able to formulate all assumptions on the coefficient matrix $\mathbf{A}$ that we will need.

ASSUMPTION 3.8. *In addition to assuming that* $\mathbf{A} \in L_\infty(\Omega)$ *is a symmetric* 2×2 *matrix with* $\operatorname{ess\,inf}_{x\in\Omega} \lambda_{\min}(\mathbf{A}(x)) > 0$, *we assume that* $\mathbf{A} \eqsim \rho(\mathbf{A})\mathbf{id}$, *uniformly over the domain (*isotropic diffusion*), and that* $\mathbf{A}$ *is* piecewise constant with respect to P_0. *Further, following [12], we assume that* $\rho = \rho(\mathbf{A})$ *is* quasi-monotone *with respect to* P_0. *That is, defining for* $v \in \bar{V}_{P_0}$, $P_0(v) = \{\triangle \in P_0 : v \in \triangle\}$ *and* $\triangle(v) = \operatorname{argmax}\{\rho|_\triangle : \triangle \in P_0(v)\}$, *we assume that for some absolute constant* $c > 0$, *for all* $v \in \bar{V}_{P_0}$ *and* $\triangle \in P_0(v)$ *there exist* $\triangle = \triangle_1, \ldots, \triangle_m = \triangle(v)$ *such that* $\triangle_i$ *shares an edge with* $\triangle_{i+1}$ *and* $\rho|_{\triangle_i} \leq c\rho|_{\triangle_{i+1}}$. *Moreover, if* $v \in \bar{V}_{P_0} \backslash V_{P^0}$, *then in addition we assume that there exists a* $\triangle \in P_0(v)$ *having an edge on* $\partial\Omega$ *such that* $\rho|_{\triangle(v)} \leq c\rho|_\triangle$. *Under these assumptions, all of our results that depend on* $\mathbf{A}$ *should be interpreted as ones that hold* uniformly in $(\rho|_\triangle)_{\triangle\in P_0}$.

4. Finite element spaces and bases, and Galerkin approximations. For a given partition P, let $\mathcal{S}_P \subset H_0^1(\Omega)$ denote the space of continuous, piecewise linear functions subordinate to P which vanish at $\partial\Omega$. The solution $u_P \in \mathcal{S}_P$ of

$$a(u_P, w_P) = f(w_P), \quad (w_P \in \mathcal{S}_P), \tag{4.1}$$

is called the *Galerkin approximation* of the solution u of (2.1). Defining $L_P : \mathcal{S}_P \to (\mathcal{S}_P)' \supset H^{-1}(\Omega)$ by $(L_P u_P)(w_P) = a(u_P, w_P)$, the solution of (4.1) is $L_P^{-1} f$.

On some places we will replace the right-hand side f by some approximation from $\mathcal{S}_P^0$, being defined as the space of functions that are piecewise constant with respect to P.

If $\tilde{P}$ is a refinement of P, then $\mathcal{S}_P \subset \mathcal{S}_{\tilde{P}}$ and $\mathcal{S}_P^0 \subset \mathcal{S}_{\tilde{P}}^0$. Each $u \in \mathcal{S}_P$ is uniquely determined by its values on V_P, and so, in particular, $\#V_P = \dim \mathcal{S}_P$. Defining, for $v \in V_P$, $\phi_P^v \in \mathcal{S}_P$ by

$$\phi_P^v(\tilde{v}) = \begin{cases} 1 & v = \tilde{v}, \\ 0 & v \neq \tilde{v} \in V_P, \end{cases}$$

the set

$$\{\phi_P^v : v \in V_P\}$$

is a basis for $\mathcal{S}_P$, called *nodal basis*.

One easily verifies the following result.

LEMMA 4.1. *Let* v *be a regular vertex of a partition* P. *Then, with* $i := \operatorname{gen}(\triangle)$ *for any (and thus all)* $\triangle \in P$ *that contain* v, *we have that* $\phi_{P_i^*}^v \in \mathcal{S}_P$.

PROPOSITION 4.2. *For any partition* $P = P_n$, $\cup_{i=0}^n \{\phi_{P_i^*}^v : v \in V_{P_i} \backslash V_{P_{i-1}}\}$ *is a basis for* $\mathcal{S}_P$, *called* hierarchical basis.

Proof. Proposition 3.1(ii) and Lemma 4.1 show that for $v \in V_{P_i} \backslash V_{P_{i-1}}$, $\phi_{P_i^*}^v \in \mathcal{S}_{P_i} \subset \mathcal{S}_P$. Since $\phi_{P_i^*}^v$ vanishes on $V_{P_{i-1}}$, by induction on i we conclude that for given scalars $(d_v)_{v\in V_P}$ the interpolation problem of finding scalars $(c_v)_{v\in V_P}$ with

$\sum_{i=0}^{n}\sum_{v\in V_{P_i}\backslash V_{P_{i-1}}} c_v\phi^v_{P_i^*}(\tilde v) = d_{\tilde v}$ $(\tilde v\in V_P)$ has a unique solution. Since $\dim\mathcal{S}_P = \#V_P = \sum_{i=0}^{n}\#(V_{P_i}\backslash V_{P_{i-1}})$ by Proposition 3.1(i), the proof is completed. □

Besides the nodal and hierarchical bases, for admissible partitions P^a we introduce another basis for $\mathcal{S}_{P^a}$, which, as we shall see, is appropriately called a *wavelet basis*.

Let $v\in V_*$; then there exists a unique $i\in\mathbb{N}$ such that $v\in V_{P_i^*}\backslash V_{P_{i-1}^*}$. When $i>0$, v is the midpoint of the common edge of two triangles $\triangle_1,\triangle_2\in P^*_{i-1}$. Let us denote by $v_1(v),\dots,v_4(v)$ the vertices of these $\triangle_1,\triangle_2$, with $v_2(v),v_3(v)$ being the vertices on the edge containing v, see Figure 4.1. For some scalars $\mu_{v,j}$ which will be specified later on, with $\mu_{v,j}:=0$ when $v_j(v)\in\partial\Omega$, we define

$$\psi^v := \phi^v_{P_i^*} - \sum_{j=1}^{4}\mu_{v,j}\phi^{v_j(v)}_{P^*_{i-1}}, \tag{4.2}$$

and, for convenience, for $v\in V_{P_0^*}$ we set $\psi^v=\phi^v_{P_0^*}$.

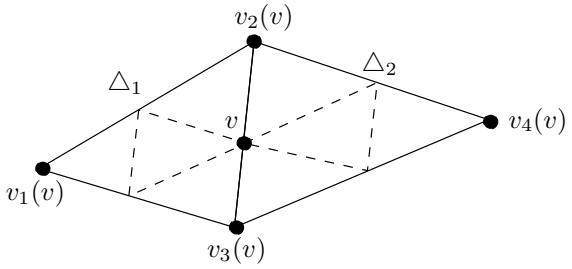

FIG. 4.1. *Definition of* $v_j(v)$.

PROPOSITION 4.3. *If* P^a *is* admissible*, then*

$$\{\psi^v : v\in V_{P^a}\} \tag{4.3}$$

is a basis for $\mathcal{S}_{P^a}$*.*

Proof. Obviously $\{\psi^v : v\in V_{P_0}\}$ is a basis for $\mathcal{S}_{P_0}$. Assuming that $\{\psi^v : v\in V_{P^a_{n-1}}\}$ is a basis for $\mathcal{S}_{P^a_{n-1}}$, the same argument as the one that was applied in the proof of Proposition 4.2 shows that $\{\psi^v : v\in V_{P^a_{n-1}}\}\cup\{\phi^v_{P_n^*} : v\in V_{P^a_n}\backslash V_{P^a_{n-1}}\}$ is basis for $\mathcal{S}_{P^a_n}$. Proposition 3.1(ii) shows that each $v\in V_{P^a_n}\backslash V_{P^a_{n-1}}$ is the midpoint of $\triangle_1,\triangle_2\in P^a_{n-1}$ with $\mathrm{gen}(\triangle_1)=\mathrm{gen}(\triangle_2)=n-1$, both of which are refined in the transition to P^a_n. Proposition 3.4 shows that each of $v_1(v),\dots,v_4(v)$, which are the vertices of $\triangle_1$ and $\triangle_2$, is a regular vertex of P^a_{n-1}, obviously with $\mathrm{gen}(\hat\triangle)=n-1$ for any, and thus all $\hat\triangle\in P^a_{n-1}$ which contain this vertex. From Lemma 4.1 we now conclude that $\sum_{j=1}^{4}\mu_{v,j}\phi^{v_j(v)}_{P^*_{i-1}}\in\mathcal{S}_{P^a_{n-1}}$, from which it follows that also $\{\psi^v : v\in V_{P^a_n}\}$ is a basis for $\mathcal{S}_{P^a_n}$. □

For $P^a=P^a_n$ and $w_{P^a}\in\mathcal{S}_{P^a}$ given by its values $(w_{P^a}(v))_{v\in\bar V_{P^a}}$, i.e., the coefficients of its representation with respect to the nodal basis, where $w_{P^a}(v):=0$ when $v\in\bar V_{P^a}\backslash V_{P^a}$, an application of the following routine yields the coefficients $(c_v)_{v\in V_{P^a}}$ of its representation with respect to the wavelet basis (4.3):

ALGORITHM 4.4.

$c_v := w_{P^a}(v)$ *(*$v\in\bar V_{P^a}$*)*

`for` $i=n,\dots,1$ `do`

$\quad c_v := c_v - \frac12(c_{v_2(v)}+c_{v_3(v)})$ *(*$v\in V_{P_i^a}\backslash V_{P^a_{i-1}}$*)*

$c_{v_j(v)} := c_{v_j(v)} + c_v \mu_{v,j}$ *(*$v \in V_{P_i^a} \backslash V_{P_{i-1}^a}$, $1 \le j \le 4$*)*

`od`

Conversely, if $w_{P^a} \in \mathcal{S}_{P^a}$ is given by its coefficients $(c_v)_{v \in \bar{V}_{P^a}}$ with respect to the wavelet basis (4.3), where $c_v := 0$ when $v \in \bar{V}_{P^a} \backslash V_{P^a}$, then the values $(w_{P^a}(v))_{v \in V_{P^a}}$ are obtained from the following routine:

ALGORITHM 4.5.

`for` $i = 1, \dots, n$ `do`

$c_{v_j(v)} := c_{v_j(v)} - c_v \mu_{v,j}$ *(*$v \in V_{P_i^a} \backslash V_{P_{i-1}^a}$, $1 \le j \le 4$*)*

$c_v := c_v + \frac{1}{2}(c_{v_2(v)} + c_{v_3(v)})$ *(*$v \in V_{P_i^a} \backslash V_{P_{i-1}^a}$*)*

`od`

$w_{P^a}(v) := c_v$ *(*$v \in V_{P^a}$*)*

One directly infers the following result.

PROPOSITION 4.6. *Both of the above algorithms for switching between the representation of a* $w \in \mathcal{S}_{P^a}$ *in terms of the nodal basis to its representation in terms of* (4.3)*, and vice versa, take* $\mathcal{O}(\dim \mathcal{S}_{P^a})$ *operations.*

Now we come to the specification of the coefficients $\mu_{v,j}$ from (4.2). We take

$$\mu_{v,j} = \frac{3(\rho|_{\triangle_1} \mathrm{meas}(\triangle_1) + \rho|_{\triangle_2} \mathrm{meas}(\triangle_2))}{8 \sum_{\{\triangle \in P_{i-1}^*, v_j(v) \in \triangle\}} \rho|_{\triangle} \mathrm{meas}(\triangle)} \text{ when } j \in \{2,3\} \text{ and } v_j(v) \notin \partial\Omega, \quad (4.4)$$

and $\mu_{v,j} = 0$ otherwise. An alternative choice of the coefficients will be discussed later, in Remark 4.9. When both $v_2(v), v_3(v) \notin \partial\Omega$, a simple calculation reveals that, for $i \ge 1$ and $v \in V_{P_i^*} \backslash V_{P_{i-1}^*}$, $\int_\Omega \rho \psi^v = 0$, so that it is appropriate to call ψ^v a *wavelet*.

For coefficient matrices $\mathbf{A}$ that satisfy Assumption 3.8, a combination of results from [19, 18, 12] shows the following result.

THEOREM 4.7. *With* $\bar{\psi}^v := \psi^v / |\psi^v|_1$*,*

$$\{\bar{\psi}^v : v \in V_*\} \quad (4.5)$$

is a Riesz basis for $H_0^1(\Omega)$ *equipped with* $|\cdot|_1$*. For the sake of completeness we emphasize that this result is valid uniformly in* $(\rho|_{\triangle})_{\triangle \in P_0}$*.*

Defining $\|\!|w|\!\|_1 = (\sum_{v \in V_*} c_v^2)^{\frac{1}{2}}$, where $w = \sum_{v \in V_*} c_v \bar{\psi}^v$ is the unique expansion of $w \in H_0^1(\Omega)$, with $\lambda_{\bar{\Psi}}, \Lambda_{\bar{\Psi}} > 0$ we will denote the largest or smallest constant for which

$$\lambda_{\bar{\Psi}} \|\!| \cdot |\!\|_1^2 \le |\cdot|_1^2 \le \Lambda_{\bar{\Psi}} \|\!| \cdot |\!\|_1^2, \quad (4.6)$$

and $\kappa_{\bar{\Psi}} := \frac{\Lambda_{\bar{\Psi}}}{\lambda_{\bar{\Psi}}}$. We note that this equivalence between the $|\cdot|_1$-norm of a function in $H_0^1(\Omega)$ and the ℓ_2-norm of its, generally infinite, coefficient vector holds, in particular, for functions from $\mathcal{S}_{P^a}$ for admissible P^a, which by Proposition 4.3 have a finite wavelet expansion.

Remark 4.8. Since the proof of Theorem 4.7 can only be deduced by combining results from different papers, we briefly comment on its derivation. For any $\triangle \in \cup_{i \ge 0} P_i^*$, let $I_\triangle : C(\triangle) \to P_1(\triangle)$ be the nodal value interpolant, consider the bilinear form $\langle\!\langle u, w \rangle\!\rangle_\triangle := \frac{1}{3}\mathrm{meas}(\triangle) \cdot \sum_{v \text{ vertex of } \triangle} u(v) w(v)$, and let

$$\langle u, w \rangle_\triangle := \langle\!\langle I_\triangle u, I_\triangle w \rangle\!\rangle_\triangle + \sum_{k=1}^4 [\langle\!\langle u, w \rangle\!\rangle_{\triangle_k} - \langle\!\langle I_\triangle u, I_\triangle w \rangle\!\rangle_{\triangle_k}],$$

where $\triangle_1, \dots, \triangle_4$ are the children of $\triangle$. Note that for $u, w \in P_1(\triangle)$, $\langle u, w \rangle_\triangle = \langle\!\langle u, w \rangle\!\rangle_\triangle$ which is a scalar product on $P_1(\triangle)$. Let $Y_\triangle : C(\triangle) \cap \prod_{k=1}^4 P_1(\triangle_k) \to P_1(\triangle)$

be the orthogonal projector with respect to $\langle\cdot,\cdot\rangle_{\triangle}$. Finally, for $i\geq 1$, on $\mathcal{S}_{P_i^*}\times\mathcal{S}_{P_i^*}$ let

$$\langle u,w\rangle_{\mathcal{S}_{P_i^*}}:=\sum_{\triangle\in P_{i-1}^*}\rho|_{\triangle}\langle u,w\rangle_{\triangle}.$$

Using that $\{\phi^v_{P^*_{i-1}}:v\in V_{P^*_{i-1}}\}$ is an orthogonal set with respect to $\langle\cdot,\cdot\rangle_{\mathcal{S}_{P_i^*}}$, in [18, §6] it was shown that for $i\geq 1$ the sets

$$\{\psi^v/\|\psi^v\|_0:v\in V_{P_i^*}\backslash V_{P^*_{i-1}}\},$$

defined by (4.2), (4.4), are uniform Riesz bases for $\mathcal{S}_{P_i^*}\cap\mathcal{S}_{P^*_{i-1}}^{\perp_{\langle\,,\,\rangle_{\mathcal{S}_{P_i^*}}}}$ equipped with $\|\cdot\|_0$, where 'uniform' refers to both the parameter i and to $(\rho|_{\triangle})_{\triangle\in P_0}$.

With

$$t:=\sup_{0\neq w\in C(\triangle)\cap\prod_{k=1}^4 P_1(\triangle_k)}\frac{\langle w,w\rangle_{\triangle}-\langle(I-Y_{\triangle})w,(I-Y_{\triangle})w\rangle_{\triangle}}{\sum_{k=1}^4\langle\!\langle w,w\rangle\!\rangle_{\triangle_k}},$$

whose value is independent of the triangle $\triangle$, it follows from [19, Th. 3.1 and (4.7)] that in case of *constant* $\mathbf{A}=\mathbf{id}$, for $\frac{3}{2}>s>\log_2\sqrt{t}$ the infinite collection

$$\cup_{i\geq 0}\{2^{(s-1)i}\psi^v:v\in V_{P_i^*}\backslash V_{P^*_{i-1}}\}$$

is a Riesz basis for $H^s_0(\Omega)$. Some calculations show that $t=\frac{181}{64}$, and so $\sqrt{t}\approx .7499$, meaning in particular that (4.5) is a Riesz basis for $H^1_0(\Omega)$.

With $Q_i:L_2(\Omega)\to\mathcal{S}_{P_i^*}$ being the $\langle\,,\,\rangle_0$-orthogonal projector onto $\mathcal{S}_{P_i^*}$, and $Q_{-1}:=0$, for coefficient matrices $\mathbf{A}$ that satisfy Assumption 3.8, it was shown in [12] that

$$\|w\|_1^2\eqsim\sum_{i=0}^{\infty}4^i\|(Q_i-Q_{i-1})w\|_0^2\quad(w\in H^1_0(\Omega)),$$

uniformly in $(\rho|_{\triangle})_{\triangle\in P_0}$. As was shown in [19], from this result and the fact that $t<1$ it even follows that (4.5) is a Riesz basis for $H^1_0(\Omega)$ equipped with $|\cdot|_1$, uniformly in $(\rho|_{\triangle})_{\triangle\in P_0}$.

Remark 4.9. For *constant* $\mathbf{A}=\mathbf{id}$, in [8] other values for the coefficients $\mu_{v,1},\ldots,\mu_{v,4}$ from (4.2) were proposed; generally, all four of these coefficients are non-zero. Although uniform refinements of an arbitrary initial partition are considered, just as outlined above, for admissible $P^a=P^a_n$ a subset of the wavelet basis for $\mathcal{S}_{P_n^*}$ spans $\mathcal{S}_{P^a}$. For some $\tilde{s}>0$ and $s\in(-\tilde{s},\frac{3}{2})$, the infinite collection of properly scaled wavelets from [8] is shown to generate a Riesz basis for $\mathcal{H}^s(\Omega)=\begin{cases}H^s_0(\Omega) & \text{when } s\geq 0,\\ (H^{-s}_0(\Omega))' & \text{when } s<0.\end{cases}$

On regular "type-I triangulations" of the whole of $\mathbb{R}^2$, the wavelet proposals from [18, §6] or [8] reduce to the so-called coarse-grid stabilized HB-systems from [13] with parameters $a=\frac{1}{8}$ or $a=-\frac{3}{16}$ respectively. There, it is shown that for these uniform triangulations the exact $H^s(\mathbb{R}^2)$ stability ranges are $s\in(0.022818,\frac{3}{2})$ and $s\in(-0.440765,\frac{3}{2})$ respectively (cf. also [9]).

Finally, numerical results ([13, Table 1.2]) show that both wavelet bases are also quantitatively well-conditioned ($\kappa_{\bar{\Psi}}$ is approximately 16 or 10, respectively, for $\mathbf{A}=\mathbf{id}$ and P_0 being the standard regular partition of the unit square into 8 triangles, so that $\#V_{P^0}=1$).

5. A coarsening algorithm. In [6], a coarsening step was introduced into an adaptive algorithm in the framework of a wavelet method. The idea of such a step, which has to be applied after each fixed number of iterations that produce increasingly more accurate approximations, is to remove a possibly large number of small terms in the current approximation that hardly contribute to its quality, but which, because of their number, spoil the complexity. With wavelet methods such 'small terms' stand for terms in a wavelet expansion with small coefficients, and in our finite element setting they correspond to a representation of the approximation as a piecewise linear function subordinate to a locally fine partition, whereas it is close to being linear on the union of these triangles.

Given some current approximation defined on some partition, in order to find a more efficient representation without increasing the error too much, one cannot simply join arbitrary collections of triangles since, generally, their union will not be a triangle. Instead one can only join groups of all siblings of one parent, that is, one has to respect the underlying tree structure. In view of this, in [3], for each triangle in the tree associated to the partition, an error functional was defined. It was shown that, for any subtree, the ℓ_2-norm of these error functionals over the leaves is bounded by some multiple of the error of the best continuous piecewise linear approximation subordinate to the partition defined by this subtree. Giving a tolerance that one allows to be added to the current error, a tree-coarsening algorithm from [5] was run, that, modulo some constant factor, yields the smallest subtree for which the above ℓ_2-norm is less than this tolerance. The ℓ_2-norm could not be shown to be equivalent to the $|\cdot|_1$-norm of the error in the best approximation. Therefore, this procedure had to be completed by a number of uniform refinement steps.

In this paper, for the different type of partitions we consider, based on ideas from [3, 5] an alternative coarsening procedure is developed, which we hope is more attractive for practical computations. Given a current approximation from $\mathcal{S}_P$, in case P is not admissible, we will first embed it into $\mathcal{S}_{P^a}$, where P^a is constructed using Algorithm 3.6. Next, we determine its finite set of wavelet coefficients. Now using the norm equivalence (4.6), an obvious coarsening procedure would be just to order these coefficients by their modulus, and then to remove coefficients, starting with the smallest one, until the tolerance is met. Yet, the task is not to find an approximation with a minimum number of wavelet coefficients, but to find an approximation from a finite element space subordinate to a partition that has, modulo some constant factor, a minimum number of triangles, and the suggested procedure will generally fail to deliver this. Therefore we will equip the infinite index set V_* of all wavelets with a tree structure, and run the algorithm from [5] to find a subtree approximation at a distance less than or equal to the tolerance, which has, modulo some constant factor, a minimum number of terms. Since the tree structure will be designed so that for any subtree $\tilde{V}$ we can construct a partition P such that $\mathrm{span}\{\psi^v : v \in \tilde{V}\} \subset \mathcal{S}_P$ where $\#P \lesssim \#\tilde{V}$, we will be able to conclude that we found an approximation subordinate to a partition that has, modulo some constant factor, a minimum number of triangles. An advantage of our coarsening procedure will be that it not only gives a (quasi-) optimal partition, but that, at the same time, it also yields a (quasi-) optimal continuous piecewise linear approximation subordinate to this partition.

The tree structure with which we equip V_* is defined as follows. The vertices from $V_{P_0^\star}$ are the roots of the tree. For $i = 1$ and $v \in V_{P_i^*} \backslash V_{P_{i-1}^*}$, we assume that at least one of the vertices $v_1(v), \ldots, v_4(v)$ is not on $\partial\Omega$, and we just pick one of them to be the parent of v. Now let $i > 1$ and $v \in V_{P_i^*} \backslash V_{P_{i-1}^*}$. At least one of $v_1(v), \ldots, v_4(v)$ is

in $V_{P^*_{i-1}} \backslash V_{P^*_{i-2}}$, and we may pick just one of them to be the parent of v. In the case of multiple choices, a deterministic rule to make the selection is given, for example, by the following: If one of $v_2(v)$ or $v_3(v)$ is in $\bar{V}_{P^*_{i-2}}$, then the other is in $V_{P^*_{i-1}} \backslash V_{P^*_{i-2}}$, and we define it to be the parent of v. Otherwise, if both $v_2(v), v_3(v) \in \bar{V}_{P^*_{i-1}} \backslash \bar{V}_{P^*_{i-2}}$, then one of the remaining $v_1(v), v_4(v)$ is in $\bar{V}_{P^*_{i-2}}$, and we call it $v_1(v)$, whereas the other, thus called $v_4(v)$, is in $\bar{V}_{P^*_{i-1}} \backslash \bar{V}_{P^*_{i-2}}$. After numbering $v_1(v), v_2(v), v_3(v)$ in clockwise direction, we select the first of $v_2(v), v_3(v), v_4(v) \in V_{P^*_{i-1}} \backslash V_{P^*_{i-2}}$ to be the parent of v. The number of children of any parent in this tree is uniformly bounded, and is only dependent on P_0.

For $w \in H^1_0(\Omega)$, let $w = \sum_{v \in V_*} c_v \bar{\psi}^v$ be its expansion with respect to (4.5). Obviously, for any $\tilde{V} \subset V_*$, its best approximation with respect to $\|\!|\cdot|\!\|_1$ from $\operatorname{span}\{\bar{\psi}^v : v \in \tilde{V}\}$ is $\sum_{v \in \tilde{V}} c_v \bar{\psi}^v$. The squared error $E(\tilde{V})$ of this approximation with respect to $\|\!|\cdot|\!\|_1$ is $E(\tilde{V}) = \sum_{v \in V_* \backslash \tilde{V}} |c_v|^2$.

We will call $\tilde{V} \subset V_*$ a *subtree* when it contains all roots $V_{P^*_0}$, and when, for any $v \in \tilde{V}$, all its ancestors and all its siblings, i.e., those $w \in V_*$ that have the same parent as v, are also in $\tilde{V}$. The set of leaves $\mathcal{L}(\tilde{V})$ is defined as the set of those $v \in \tilde{V}$ which have no children in $\tilde{V}$. Defining for $v \in V_*$ the error functional $e(v) := \sum_{\bar{v} \text{ a descendant of } v} |c_{\bar{v}}|^2$, we have $E(\tilde{V}) = \sum_{v \in \mathcal{L}(\tilde{V})} e(v)$.

Following [5], we define a modified error functional $\tilde{e}(v)$ for $v \in V_*$ as follows. For the roots $v \in V_{P^*_0}$, $\tilde{e}(v) := e(v)$. Assuming that $\tilde{e}(v)$ has been defined, then for all of its children $v_1, \ldots, v_m$,

$$\tilde{e}(v_j) := \frac{\sum_{i=1}^m e(v_i)}{e(v) + \tilde{e}(v)} \tilde{e}(v).$$

Now given a $w \in H^1_0(\Omega)$ and a tolerance $\varepsilon > 0$, the *thresholding second algorithm* from [5] for determining a quasi-optimal *subtree approximation* runs as follows.

Algorithm 5.1.

$\tilde{V} := V_{P^*_0}$
`while` $E(\tilde{V}) > \varepsilon^2$ `do`
    compute $\rho = \max_{v \in \mathcal{L}(\tilde{V})} \tilde{e}(v)$
    `forall` $v \in \mathcal{L}(\tilde{V})$ *with* $\tilde{e}(v) = \rho$ `do` *add all children of* v *to* $\tilde{V}$ `od`
`od`

Remark 5.2. During the evaluation of Algorithm 5.1, the values $\tilde{e}(v)$ for the current leaves should be stored as an ordered list. As a consequence, with $\tilde{V}$ being the subtree at termination, the operation count of Algorithm 5.1 will contain a term $\mathcal{O}(\#\tilde{V} \log((\#\tilde{V}))$ due the insertions of $\tilde{e}(v)$ for newly created leaves in this list. Since the other costs of the algorithm are $\mathcal{O}(\#\tilde{V})$, asymptotically the cost of these insertions will dominate. Although it seems unlikely that this will happen with practical problem sizes, for mathematical completeness we sketch here a modification with which the log-factor is avoided (see [1, 14, 20] for solutions of a similar problem in a wavelet context).

Noting that $\tilde{e}(v_j) \leq \sum_{i=1}^m e(v_i) \leq e(v) \leq \|\!|w|\!\|_1^2$, we may store the current leaves v in binary bins $V_0, \ldots, V_q$, where for $0 \leq i \leq q-1$, V_i contains those v with $\tilde{e}(v) \in (2^{-(i+1)} \|\!|w|\!\|_1^2, 2^{-i} \|\!|w|\!\|_1^2]$, and the remaining v, thus with $\tilde{e}(v) \leq 2^{-q} \|\!|w|\!\|_1^2$, are put into V_q. Instead of replacing all $v \in \mathcal{L}(\tilde{V})$ with maximal $\tilde{e}(v)$ by their children, in each iteration of the while-loop we replace just one v taken from the first non-empty bin by its children. Again, from $\sum_{i=1}^m e(v_i) \leq e(v)$ we have $\tilde{e}(v_j) \leq \tilde{e}(v)$ meaning that for any i, once $V_0, \ldots, V_i$ got empty they will remain empty.

If q is chosen so that during the iteration we only extract v from bins V_i with $i < q$, then the corresponding $\tilde{e}(v)$ will be at most a factor 2 smaller than the current maximal value of $\tilde{e}$. As a consequence, one may verify that, with the exception of the operation-counts, all results proven in [5] about Algorithm 5.1 are also valid for this modified version (making use of the property $\sum_{i=1}^{m} e(v_i) \leq e(v)$, which is stronger than the assumption made in [5]; only (5.13) of [5] has to be adapted).

With $\tilde{V}$ being the subtree at termination, the number of operations required by this modified Algorithm 5.1 is $\lesssim \#\tilde{V} + q$, where q is the maximum number of bins that have to be generated or inspected for containing leaves. Thinking of the situation when in the course of the iteration the maximum value of $\tilde{e}$ over the leaves varies largely in size, note that generally q cannot be bounded in terms of $\#\tilde{V}$.

We will apply the modified Algorithm 5.1 only in the situation when there exists a finite subtree $\bar{V} \subset V_*$ such that

$$e(v) = 0 \qquad (v \in \mathcal{L}(\bar{V})), \tag{5.1}$$

which allows us to make a suitable choice for q. Note that the subtree $\tilde{V}$ at termination satisfies $\tilde{V} \subset \bar{V}$. The definition of $\tilde{e}(v)$ shows that $\sum_{j=1}^{m} \frac{e(v_j)}{\tilde{e}(v_j)} = 1 + \frac{e(v)}{\tilde{e}(v)}$. A recursive application of this formula gives $\sum_{v\in\mathcal{L}(\tilde{V})} \frac{e(v)}{\tilde{e}(v)} = \#(\tilde{V}\backslash\mathcal{L}(\tilde{V})) + \#V_{P_0^*} \leq \#\tilde{V} \leq \#\bar{V}$, and so $E(\tilde{V}) \leq \#\bar{V} \max_{v\in\mathcal{L}(\tilde{V})} \tilde{e}(v)$. We conclude that the modified Algorithm 5.1 terminates before any leaf v with $\tilde{e}(v) \leq \varepsilon^2/\#\bar{V}$ is replaced by its children. Solving for the smallest $q \in \mathbb{N}_0$ with $2^{-q}|||w|||_1^2 = \varepsilon^2/\#\bar{V}$ yields $q = \max\{0, \lceil \log_2(\varepsilon^{-2}|||w|||_1^2\#\bar{V})\rceil\}$; such a q thus satisfies the assumption we made earlier.

The analysis of Algorithm 5.1 from [5], together with the additions from the above remark concerning our slight modification, yields the following result.

PROPOSITION 5.3 ([5, Corollary 5.3]). *The subtree $\tilde{V}$ yielded by (the modified) Algorithm 5.1 satisfies $E(\tilde{V}) \leq \varepsilon^2$. Moreover, there exists absolute constants $t_1, T_2 > 0$, necessarily with $t_1 \leq 1 \leq T_2$, such that if $\hat{V}$ is a subtree with $E(\hat{V}) \leq t_1\varepsilon^2$, then $\#\tilde{V} \leq T_2\#\hat{V}$. The number of evaluations of e and the number of additional arithmetic operations required by the modified Algorithm 5.1 are $\lesssim \#\tilde{V} + \max\{0, \log(\varepsilon^{-2}|||w|||_1^2\#\bar{V})\}$, with $\bar{V}$ being any subtree satisfying* (5.1).

We are now almost ready to define our coarsening routine. Inside this routine, for some admissible partition P^a we will apply the (modified) Algorithm 5.1 to a $w_{P^a} \in \mathcal{S}_{P^a}$. Such a w_{P^a} has an expansion $w_{P^a} = \sum_{v\in V_{P^a}} c_v\bar{\psi}^v$, i.e., $c_v = 0$ for $v \in V_*\backslash V_{P^a}$. Although V_{P^a} is nearly a subtree since it contains the roots V_{P_0} as well as all ancestors of any $v \in V_{P^a}$, it may contain $v \in V_{P^a}$ with siblings outside V_{P^a}. As a consequence, the (modified) Algorithm 5.1 may output a subtree $\tilde{V}$ containing such siblings. Yet, since the corresponding wavelet coefficients are zero, these siblings do not contribute to the approximation and can therefore be discarded. An efficient way of implementing this is to call the algorithm pretending that V_* is V_{P^a} so that siblings outside V_{P^a} will never be created. The set $\bar{V}$ from (5.1) can be taken to be equal to V_{P^a}.

COARSE$[P, w_P, \varepsilon] \to [\tilde{P}^a, w_{\tilde{P}^a}]$:
% $P = P_n$ is some admissible partition, $w_P \in \mathcal{S}_P$ is given by its values $(w_P(v))_{v\in V_P}$
% and $\varepsilon > 0$. The output $\tilde{P}^a$ is an admissible partition, and $w_{\tilde{P}^a} \in \mathcal{S}_{\tilde{P}^a}$ is given by
% its values $(w_{\tilde{P}^a}(v))_{v\in V_{\tilde{P}^a}}$.

(i) *When P is admissible, let $P^a = P$; otherwise compute an admissible refinement P^a of P by applying Algorithm 3.6. Compute the values $(w_{P^a}(v))_{v\in V_{P^a}}$ of $w_{P^a} := w_P$.*

(ii) *Compute the wavelet coefficients* $(c_v)_{v\in V_{P^a}}$ *of* w_{P^a} *using Algorithm 4.4.*

(iii) *Compute recursively* $(e(v))_{v\in V_{P^a}}$ *starting with each* $v\in V_{P^0}$ *as follows:*

`if` *v has no children in* V_{P^a} `then` $e(v):=0$

`else` $e(v):=\sum_{\{\tilde v\in V_{P^a}:\tilde v \text{ is a child of } v\}} e(\tilde v)+c_{\tilde v}^2$ `fi`

(iv) *Apply the modified Algorithm 5.1, with* $q:=\max\{0,\lceil\log_2(\varepsilon^{-2}\|w_{P^a}\|_1^2\#V_{P^a})\rceil\}$, *yielding a set* $\tilde V\subset V_{P^a}$.

(v) *Determine a partition* $\tilde P$ *as follows:*

$\tilde P_0:=P_0$, $i:=1$

`while` $\tilde V\cap(V_{P_i^*}\backslash V_{P_{i-1}^*})\neq\emptyset$ `do`

construct $\tilde P_i$ *from* $\tilde P_{i-1}$ *by refining those* $\triangle\in\tilde P_{i-1}$ *that contain a* $v\in\tilde V\cap(V_{P_i^*}\backslash V_{P_{i-1}^*})$, *i:=i+1*

`od`

(vi) *Apply Algorithm 3.6 to determine an admissible refinement* $\tilde P^a$ *of* $\tilde P$. *Note that* $\tilde V\subset V_{\tilde P}\subset V_{\tilde P^a}$, *and thus that* $w_{\tilde P^a}:=\sum_{v\in\tilde V}c_v\bar\psi^v\in\mathcal{S}_{\tilde P^a}$.

(vii) *Apply Algorithm 4.5 to compute* $(w_{\tilde P^a}(v))_{v\in V_{\tilde P^a}}$.

THEOREM 5.4. *(a)* $[\tilde P^a,w_{\tilde P^a}]:=\mathbf{COARSE}[P,w_P,\varepsilon]$ *satisfies* $\|w_P-w_{\tilde P^a}\|_1\le\varepsilon$. *There exists an absolute constant* $D>0$, *such that for any partition* $\hat P$ *for which there exists a* $w_{\hat P}\in\mathcal{S}_{\hat P}$ *with* $\|w_P-w_{\hat P}\|_1\le t_1^{\frac12}\varepsilon$, *we have that* $\#\tilde P^a\le D\#\hat P$.

(b) The call requires $\lesssim\#P+\max\{0,\log(\varepsilon^{-1}\|w_P\|_1)\}$ *arithmetic operations.*

Proof. (a) The first statement follows by construction. Let $(c_v)_{v\in V_*}$ be the wavelet coefficients in the expansion $w_P=\sum_{v\in V_*}c_v\bar\psi_v$. Let $\hat P$, $w_{\hat P}\in\mathcal{S}_{\hat P}$ with $\|w_P-w_{\hat P}\|_1\le t_1^{\frac12}\varepsilon$. Let $\hat P^a$ be the admissible refinement of $\hat P$ constructed by applying Algorithm 3.6, and let $(\hat c_v)_{v\in V_{\hat P^a}}$ be the wavelet coefficients of $w_{\hat P}\in\mathcal{S}_{\hat P}\subset\mathcal{S}_{\hat P^a}$. Let $\hat V$ be the enlargement of $V_{\hat P^a}$ by adding all siblings of all $v\in V_{\hat P^a}$ to this set. Then $\hat V$ is a subtree with $\#\hat V\lesssim\#V_{\hat P^a}\lesssim\#\hat P^a\lesssim\#\hat P$. Because of

$$E(\hat V)=\sum_{v\in V_*\backslash\hat V}|c_v|^2\le\sum_{v\in V_*}|c_v-\hat c_v|^2=\|w_P-\hat w_P\|_1^2\le t_1\varepsilon^2,$$

an application of Proposition 5.3 shows that $\tilde V$ constructed in (iv) satisfies $\#\tilde V\le T_2\#\hat V$. The proof is completed by noting that the partitions $\tilde P$ and $\tilde P^a$ constructed in (v) and (vi) satisfy $\#\tilde P^a\lesssim\#\tilde P\lesssim\#\tilde V$.

(b) The number of arithmetic operations required by (i)-(iii) is $\lesssim\#P^a\lesssim\#P$. Since $\#V_{P^a}\lesssim\#P^a\lesssim\#P$, Proposition 5.3 shows that (iv) requires $\lesssim\#\tilde V+\max\{0,\log(\varepsilon^{-2}\|w_P\|_1^2\#V_{P^a})\}\lesssim\#P+\max\{0,\log(\varepsilon^{-1}\|w_P\|_1)\}$ arithmetic operations. The number of arithmetic operations required by (v)-(vii) is $\lesssim\#\tilde V\lesssim\#P$. □

Had the unmodified Algorithm 5.1 been applied inside **COARSE**, the required number of arithmetic operations would have been $\lesssim\#P\log(\#P)$. In contrast with such a log-factor, for our application it will turn out that the log-*term* from Theorem 5.4(b) is completely harmless. The next corollary shows that if for $u\in H_0^1(\Omega)$ and $s>0$ the errors of the best approximations from any $\mathcal{S}_P$ with $\#P\le n$ are $\mathcal{O}(n^{-s})$, then given any $\varepsilon>0$, a partition P and a $w_P\in\mathcal{S}_P$ with $|u-w_P|_1\le\varepsilon$, by allowing the tolerance to increase by some suitable, sufficiently large constant factor, the coarsening procedure yields an (admissible) partition $\tilde P^a$ and a $w_{\tilde P^a}\in\mathcal{S}_{\tilde P^a}$ with $|u-w_{\tilde P^a}|_1\lesssim\varepsilon$ and $\#\tilde P^a\lesssim\varepsilon^{-1/s}$, which, in view of the assumption, is the smallest size, modulo some constant factor, one can generally expect for an approximation

with this accuracy. The short proof of this corollary is based on an argument taken from [3, proof of Theorem 4.9].

COROLLARY 5.5. *Let* $\gamma > t_1^{-\frac{1}{2}}$. *Then, for any* $\varepsilon > 0$, $u \in H_0^1(\Omega)$, *a partition* P, $w_P \in \mathcal{S}_P$ *with* $\|\!|u - w_P|\!\|_1 \leq \varepsilon$, *for* $[\tilde{P}^a, w_{\tilde{P}^a}] := \mathbf{COARSE}[P, w_P, \gamma\varepsilon]$ *we have that* $\|\!|u - w_{\tilde{P}^a}|\!\|_1 \leq (1+\gamma)\varepsilon$, *and*

$$\tilde{P}^a \leq D\#\hat{P}$$

for any partition $\hat{P}$ *with* $\inf_{w_{\hat{P}} \in \mathcal{S}_{\hat{P}}} \|\!|u - w_{\hat{P}}|\!\|_1 \leq (t_1^{\frac{1}{2}}\gamma - 1)\varepsilon$.

Proof. The first statement is an obvious consequence of Theorem 5.4. The second one also follows from this theorem using that

$$\inf_{w_{\hat{P}} \in \mathcal{S}_{\hat{P}}} \|\!|w_P - w_{\hat{P}}|\!\|_1 \leq \|\!|u - w_P|\!\|_1 + \inf_{w_{\hat{P}} \in \mathcal{S}_{\hat{P}}} \|\!|w_{\hat{P}} - u|\!\|_1 \leq \varepsilon + (t_1^{\frac{1}{2}}\gamma - 1)\varepsilon = t_1^{\frac{1}{2}}\gamma\varepsilon.$$

□

6. A convergent adaptive refinement strategy. We derive an a posteriori estimate of the error in the Galerkin approximation (4.1) of the solution of the boundary value problem (2.1) where we temporarily assume that the right-hand side $f \in L_2(\Omega)$. Secondly, under the assumption that f is piecewise constant with respect to the current partition, we derive a refinement strategy which guarantees that the difference between the Galerkin solutions on the new and old partition is greater than some fixed multiple of the error estimate for the solution on the old partition. Exploiting Galerkin orthogonality we can therefore conclude that the error in the new solution is less than some absolute constant times the error in the old solution.

This section is largely based on ideas from [15] by Morin, Nochetto and Siebert on the construction of an adaptive finite element method that can be proven to converge. New aspects are that:

- we consider a different type of partitions (non-conforming ones generated by red-refinements, vs. conforming ones generated by newest vertex bisection).
- Under Assumption 3.8 our results are valid uniformly in the size of jumps of $\rho = \rho(\mathbf{A})$.
- We use approximations of the right-hand side for setting up the discrete systems, with which the application of quadrature can be modeled, and, as in [3], we solve these systems inexactly. We evaluate the a posteriori error estimator using the inexact discrete solution and the approximate right-hand side. Because of the latter we can allow right-hand sides in $H^{-1}(\Omega)$ that lie outside $L_2(\Omega)$, at least when they can be sufficiently well approximated by piecewise constants.

We start by introducing some notation. We call e *an edge of a partition* P, when e is an edge of some $\triangle \in P$ and e connects two vertices from $\bar{V}_P$. Note that since we allow non-conforming partitions, not all edges of $\triangle \in P$ are edges of P. With $\bar{E}_P$ or E_P respectively, we denote the set of all edges of P or all edges of P which are not part of $\partial\Omega$. Note that for an admissible partition P^a, any edge $e \in E_{P^a}$ is either the common edge of $\triangle_1, \triangle_2 \in P^a$, or it is the common edge of $\triangle_1, \hat{\triangle}$ where $\triangle_1 \in P^a$ and $\hat{\triangle}$ is the parent of four triangles in P^a.

For $e \in E_P$ and $u \in \mathcal{S}_P$, we set

$$\eta_e(u) := \frac{\operatorname{diam}(e)}{\max\{\rho|_{e^-}, \rho|_{e^+}\}} \|[\mathbf{A}\nabla u]_e \cdot \mathbf{n}_e\|^2_{L_2(e)},$$

where $\mathbf{n}_e$ is an unit vector orthogonal to e, $[\mathbf{A}\nabla u]_e$ denotes the jump of $\mathbf{A}\nabla u$ in the direction of $\mathbf{n}_e$, and $\rho|_{e^\pm} = \rho(x \pm \delta \mathbf{n}_e)$ for arbitrary $x \in e$ and $\delta > 0$ small enough. For $\triangle \in P$ and $f \in L_2(\Omega)$, we set

$$\zeta_\triangle(f) := \frac{\mathrm{diam}(\triangle)^2}{\rho|_\triangle} \|f\|^2_{L_2(\triangle)},$$

and finally,

$$\mathcal{E}(P, f, u) := [\sum_{\triangle \in P} \zeta_\triangle(f) + \sum_{e \in E_P} \eta_e(u)]^{\frac{1}{2}}.$$

The a posteriori error estimate given in the following theorem extends results from [2, 17] for conforming partitions to admissible non-conforming partitions.

THEOREM 6.1. *There exists an absolute constant C_1, such that for any $f \in L_2(\Omega)$ and an admissible partition P^a, with $u := L^{-1}f$ and $u_{P^a} := L^{-1}_{P^a} f$, we have that*

$$|u - u_{P^a}|_1 \leq C_1 \mathcal{E}(P^a, f, u_{P^a}).$$

Proof. For any $w \in H^1_0(\Omega)$, $w_{P^a} \in \mathcal{S}_{P^a}$, by the Galerkin orthogonality and because $\mathbf{A}$ is piecewise constant, integration by parts shows that

$$\begin{aligned} a(u - u_{P^a}, w) = a(u - u_{P^a}, w - w_{P^a}) &= \int_\Omega f(w - w_{P^a}) - a(u_{P^a}, w - w_{P^a}) \\ &= \sum_{\triangle \in P^a} \int_\triangle f(w - w_{P^a}) - \sum_{e \in E_{P^a}} \int_e ([\mathbf{A}\nabla u_{P^a}]_e \cdot \mathbf{n}_e)(w - w_{P^a}), \quad (6.1) \end{aligned}$$

where $\mathbf{n}_\triangle$ denotes the unit exterior normal to $\triangle$.

It was shown in [16, p. 17-18] that for any triangle $\triangle$ and any of its vertices v, there exists a $\varphi(\triangle, v) \in L_\infty(\triangle)$ such that $\int_\triangle \varphi(\triangle, v)p = p(v)$ for all polynomials p of degree 1, and $\|\varphi(\triangle, v)\|_{L_\infty} \lesssim \mathrm{meas}(\triangle)^{-1}$ independently of $\triangle$. For each $v \in V_{P^a}$ we now select $\triangle_v = \mathrm{argmax}\{\rho|_\triangle : \triangle \in P^a, v \in \triangle\}$, and define $w_{P^a} \in \mathcal{S}_{P^a}$ by $w_{P^a}(v) = \int_{\triangle_v} \varphi(\triangle_v, v)w$. For $v \in \bar{V}_{P^a} \backslash V_{P^a}$, we set $\triangle_v = \emptyset$.

We start by estimating the first sum from (6.1). Let $\triangle \in P^a$, say $\triangle \in P^a_i$. The triangle $\triangle$ has between 0 and 3 hanging vertices. Since P^a is admissible, by Proposition 3.4 each of these hanging vertices is the midpoint of an edge connecting two vertices from $\bar{V}_{P^a_{i-1}}$. In the following we consider the case when $\triangle$ has one hanging vertex, and so in particular $i > 0$, but the other cases can be treated similarly.

Let $\tilde{\triangle} \in P^a_{i-1}$ be the parent of $\triangle$, v_1, v_2 the vertices of $\tilde{\triangle}$ such that the hanging vertex of $\triangle$ is the midpoint of the edge connecting v_1 and v_2, and let v_3 and v_4 be the non-hanging vertices of $\triangle$; see Figure 6.1. The linear function $w_{P^a}|_\triangle$ is the solution of the elementary interpolation problem with data $w_{P^a}(v_i)$ $(1 \leq i \leq 4)$, and it is easily seen that $\|w_{P^a}\|_{L_2(\triangle)} \lesssim \mathrm{diam}(\triangle) \max_{1 \leq i \leq 4} |w_{P^a}(v_i)|$. By construction, $|w_{P^a}(v_i)| \lesssim \mathrm{diam}(\triangle_{v_i})^{-1} \|w\|_{L_2(\triangle_{v_i})}$, or, when $v_i \in \partial\Omega$, $w_{P^a}(v_i) = 0$.

Since ρ is quasi-monotone (see Assumption 3.8), there exists a uniformly bounded collection $P^a_\triangle \subset P^a$ such that $\triangle, \triangle_{v_1}, \ldots, \triangle_{v_4} \subset P^a_\triangle$, $\Omega_\triangle := \cup_{\tilde{\triangle} \subset P^a_\triangle} \tilde{\triangle}$ is a simply connected uniformly Lipschitz domain, $\rho|_\triangle \lesssim \inf_{\tilde{\triangle} \subset P^a_\triangle} \rho|_{\tilde{\triangle}}$, and, when one of $v_1, \ldots, v_4$ is on $\partial\Omega$, $P^a_\triangle$ contains a $\tilde{\triangle}$ having an edge on $\partial\Omega$. From $\|w - w_{P^a}\|_{L_2(\triangle)} \lesssim \|w\|_{L_2(\Omega_\triangle)}$, a

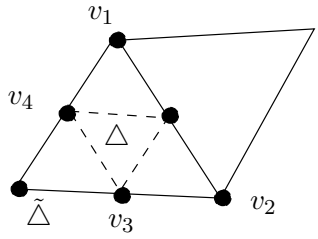

FIG. 6.1. *Illustration with the proof of Theorem 6.1.*

homogeneity argument, and either the fact that our interpolation operator reproduces any polynomial of first degree, and, in particular, any constant, together with the Bramble-Hilbert lemma, or, when one of $v_1, \ldots, v_4$ is on $\partial\Omega$, the Poincaré-Friedrichs inequality, we infer that

$$\|w - w_{P^a}\|^2_{L_2(\triangle)} \lesssim \operatorname{diam}(\Omega_\triangle)^2 |w|^2_{H^1(\Omega_\triangle)} \lesssim \frac{\operatorname{diam}(\triangle)^2}{\rho|_\triangle} \sum_{\tilde{\triangle} \in P^a_\triangle} \rho|_{\tilde{\triangle}} |w|^2_{H^1(\triangle)}.$$

By applying the Cauchy-Schwarz inequality, we deduce that

$$|\sum_{\triangle \in P^a} \int_\triangle f(w - w_{P^a})|^2 \lesssim \sum_{\triangle \in P^a} \frac{\operatorname{diam}(\triangle)^2}{\rho|_\triangle} \|f\|^2_{L_2(\triangle)} \sum_{\triangle \in P^a} \rho|_\triangle |w|^2_{H^1(\triangle)}. \tag{6.2}$$

Now we estimate the second sum from (6.1). Let $P^a = P^a_n$ and $e \in E_{P^a}$, then $e = \triangle_1 \cap \triangle_2$ for some $\triangle_1, \triangle_2 \in \cup_{i=0}^n P^a_i$. Let us assume that $\rho|_{\triangle_1} \geq \rho|_{\triangle_2}$. Note that either $\triangle_1 \in P^a$ or it is the parent of four triangles from P^a. From the trace theorem we have $\|w\|_{L_2(e)} \lesssim \operatorname{diam}(e)^{-\frac{1}{2}} \|w\|_{L_2(\triangle_1)} + \operatorname{diam}(e)^{\frac{1}{2}} |w|_{H^1(\triangle_1)}$. With $v_1, v_2 \in \bar{V}_{P^a}$ being the endpoints of e, we have $\|w_{P^a}\|_{L_2(e)} \leq \operatorname{diam}(e)^{\frac{1}{2}} \max\{|w_{P^a}(v_1)|, |w_{P^a}(v_2)|\}$. From $|w_{P^a}(v_i)| \lesssim \operatorname{diam}(\triangle_{v_i})^{-1} \|w\|_{L_2(\triangle_{v_i})}$, or, when $v_i \in \partial\Omega$, $w_{P^a}(v_i) = 0$, we find that $\|w - w_{P^a}\|_{L_2(e)} \lesssim \operatorname{diam}(e)^{-\frac{1}{2}} \|w\|_{L_2(\triangle_1 \cup \triangle_{v_1} \cup \triangle_{v_2})} + \operatorname{diam}(e)^{\frac{1}{2}} |w|_{H^1(\triangle \cup \triangle_1 \cup \triangle_2)}$. As above, we can extend $\triangle_1, \triangle_{v_1}, \triangle_{v_2}$ to a uniformly bounded collection $P^a_e \subset P^a$, such that $\Omega_e := \cup_{\tilde{\triangle} \subset P^a_e} \tilde{\triangle}$ is a simply connected uniformly Lipschitz domain, $\rho|_{\triangle_1} \lesssim \inf_{\tilde{\triangle} \subset P^a_e} \rho|_{\tilde{\triangle}}$, and, when v_1 or v_2 is on $\partial\Omega$, P^a_e contains a $\tilde{\triangle}$ having an edge on $\partial\Omega$. Using the same arguments as above, we infer that

$$\|w - w_{P^a}\|^2_{L_2(e)} \lesssim \operatorname{diam}(e) |w|^2_{H^1(\Omega_e)} \lesssim \frac{\operatorname{diam}(e)}{\max\{\rho|_{e^-}, \rho|_{e^+}\}} \sum_{\tilde{\triangle} \in P^a_e} \rho|_{\tilde{\triangle}} |w|^2_{H^1(\tilde{\triangle})}.$$

By applying the Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned} |\sum_{e \in E_{P^a}} \int_e ([\mathbf{A} \nabla u_{P^a}]_e \cdot \mathbf{n}_e)(w - w_{P^a})|^2 \\ \lesssim \sum_{e \in E_{P^a}} \frac{\operatorname{diam}(e)}{\max\{\rho|_{e^-}, \rho|_{e^+}\}} \|[\mathbf{A} \nabla u_{P^a}]_e \cdot \mathbf{n}_e\|^2_{L_2(e)} \sum_{\triangle \in P^a} \rho|_\triangle |w|^2_{H^1(\triangle)}. \end{aligned} \tag{6.3}$$

On noting that $\sum_{\triangle \in P^a} \rho|_\triangle |w|^2_{H^1(\triangle)} \eqsim |w|^2_1$ and by substituting $w = u - u_{P^a}$, the proof follows from (6.1), (6.2), (6.3). ∎

The next lemma, which is based on [15, Lemma 4.2], gives local lower bounds on the difference between the Galerkin solution on some partition and a refinement of this partition. The differences with [15] are that we consider a different class of partitions, and that our results hold uniformly in the size of jumps of ρ. Furthermore, we simply assume that the right-hand side f is piecewise constant with respect to the first partition; for the moment we postpone the analysis in the case of more general f.

LEMMA 6.2. *Let P^a be an admissible partition and $\tilde{P}$ a refinement of P^a. Let $f_{P^a} \in L_2(\Omega)$ be* piecewise constant with respect to P^a, *i.e., $f_{P^a} \in \mathcal{S}^0_{P^a}$, and let $u_{P^a} = L^{-1}_{P^a} f_{P^a}, u_{\tilde{P}} = L^{-1}_{\tilde{P}} f_{P^a}$ be the corresponding Galerkin solutions.*

(a) Let $\triangle_1, \triangle_2 \in P^a$ such that $e := \triangle_1 \cap \triangle_2 \in E_{P^a}$. Assume that $V_{\tilde{P}}$ contains points interior to $\triangle_1$, $\triangle_2$ and e, see Figure 6.2. Then

$$|u_{\tilde{P}} - u_{P^a}|^2_{1,\triangle_1\cup\triangle_2} \gtrsim \eta_e(u_{P^a}) + \sum_{i=1}^{2} \zeta_{\triangle_i}(f_{P^a}).$$

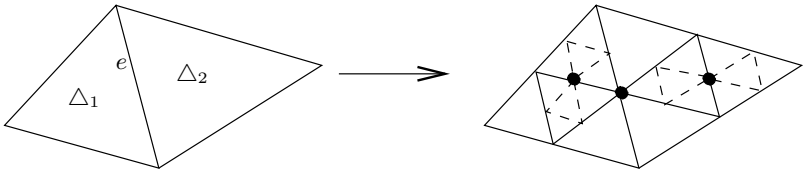

FIG. 6.2. *Illustration of Lemma 6.2(a).*

(b) Let $\triangle_1, \hat{\triangle}$ such that $e := \triangle_1 \cap \hat{\triangle} \in E_{P^a}$, $\triangle_1 \in P^a$ and $\hat{\triangle}$ is the parent of four triangles $\triangle_2, \ldots, \triangle_5 \in P^a$, numbered such that $\triangle_2$, $\triangle_3$ have an edge e_2, e_3 on e. Assume that $V_{\tilde{P}}$ contains points interior to $\triangle_1$, $\triangle_2$, $\triangle_3$, e_2 and e_3, see Figure 6.3. Then

$$|u_{\tilde{P}} - u_{P^a}|^2_{1,\triangle_1\cup\triangle_2\cup\triangle_3} \gtrsim \eta_e(u_{P^a}) + \sum_{i=1}^{3} \zeta_{\triangle_i}(f_{P^a}).$$

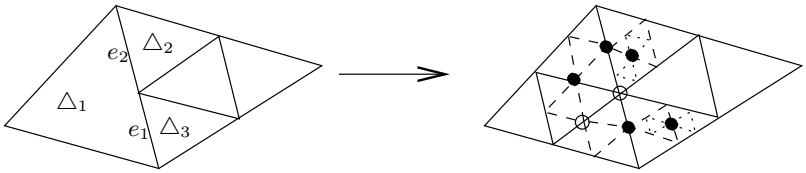

FIG. 6.3. *Illustration of Lemma 6.2(b). Open circles correspond to degrees of freedom that are not used.*

(c) Assume that $V_{\tilde{P}}$ contains a point interior to $\triangle \in P^a$. Then

$$|u_{\tilde{P}} - u_{P^a}|^2_{1,\triangle} \gtrsim \zeta_{\triangle}(f_{P^a}).$$

Proof. (a) By assumption, there exist $\varphi_1, \varphi_2, \varphi_3 \in H^1_0(\triangle_1 \cup \triangle_2) \cap \mathcal{S}_{\tilde{P}}$ with $|\varphi_i|_1 = 1$, and such that for $i \in \{1,2\}$,

$$\mathrm{supp}(\varphi_i) \subset \triangle_i,\ \int_{\triangle_i} \varphi_i \eqsim \frac{\mathrm{meas}(\triangle_i)}{\rho|^{\frac{1}{2}}_{\triangle_i}},\ \int_{\triangle_i} \varphi_3 \eqsim \frac{\mathrm{meas}(\triangle_i)}{\max\{\rho|_{e^-}, \rho|_{e^+}\}^{\frac{1}{2}}},\ \int_e \varphi_3 \eqsim \frac{\mathrm{meas}(e)}{\max\{\rho|_{e^-}, \rho|_{e^+}\}^{\frac{1}{2}}}.$$

For any $\varphi = \sum_{i=1}^3 \mathbf{c}_i \varphi_i$, integration by parts shows that

$$\begin{aligned} \int_{\triangle_1 \cup \triangle_2} f_{P^a}\varphi - \int_e ([\mathbf{A}\nabla u_{P^a}]_e \cdot \mathbf{n}_e)\varphi &= a(u_{\tilde{P}} - u_{P^a}, \varphi) \\ &\le |u_{\tilde{P}} - u_{P^a}|_{1,\triangle_1\cup\triangle_2}|\varphi|_1 \lesssim |u_{\tilde{P}} - u_{P^a}|_{1,\triangle_1\cup\triangle_2}\|\mathbf{c}\|. \end{aligned} \tag{6.4}$$

Let $g_j \in H^1_0(\triangle_1 \cup \triangle_2)'$ be defined by $g_j(\varphi) = \frac{\rho|^{\frac{1}{2}}_{\triangle_j}}{\mathrm{diam}(\triangle_j)\mathrm{meas}(\triangle_j)^{\frac{1}{2}}} \int_{\triangle_j} \varphi$ when $j = 1$ or 2, and $g_3(\varphi) = \frac{\max\{\rho|_{e^-}, \rho|_{e^+}\}^{\frac{1}{2}}}{(\mathrm{diam}(e)\mathrm{meas}(e))^{\frac{1}{2}}} \int_e \varphi$, and let $\mathbf{B}_{ij} := g_j(\varphi_i)$. Then with $\mathbf{d}_j = \frac{\mathrm{diam}(\triangle_j)}{\rho|^{\frac{1}{2}}_{\triangle_j}} \|f_{P^a}\|_{L_2(\triangle_j)}$ when $j = 1$ or 2, and $\mathbf{d}_3 = \frac{\mathrm{diam}(e)^{\frac{1}{2}}}{\max\{\rho|_{e^-}, \rho|_{e^+}\}^{\frac{1}{2}}} \|[\mathbf{A}\nabla u]_e \cdot \mathbf{n}_e\|_{L_2(e)}$, the left-hand side of (6.4) is $\sum_{i,j=1}^3 \mathbf{d}_j \mathbf{c}_i \mathbf{B}_{ij}$. Using (6.4), the statement of the lemma reduces to the question as to whether $\sup_{\mathbf{c}} \frac{\langle \mathbf{Bd}, \mathbf{c}\rangle}{\|\mathbf{c}\|} \gtrsim \|\mathbf{d}\|$ or $\|\mathbf{B}^{-1}\| \lesssim 1$. Since $\mathbf{B}_{ii} \eqsim 1$, and $\mathbf{B}_{31}, \mathbf{B}_{32} \lesssim 1$, whereas the other coefficients of $\mathbf{B}$ are zero, the proof of (a) is completed.

Part (b) can be proven similarly to (a). Note that, generally, $[\mathbf{A}\nabla u_{P^a}]_e$ has different values on e_2 and e_3. The proof of (c) poses no additional difficulties. □

As an immediate consequence we have the following result.

COROLLARY 6.3. *Let P^a be an admissible partition, $f_{P^a} \in \mathcal{S}^0_{P^a}$, and let $\tilde{P}$ be a refinement of P^a, such that for some $G \subset P^a$, $F \subset E_{P^a}$, for all $\triangle \in G$, $V_{\tilde{P}}$ satisfies the conditions from Lemma 6.2(c), and for all $e \in F$, $V_{\tilde{P}}$ satisfies the conditions from either Lemma 6.2(a) or Lemma 6.2(b) (it is sufficient that $\tilde{P}$ contains all grandchildren of all $\triangle \in P^a$ which either are in G or have an edge on an $e \in F$). Then, with $u_{P^a} = L^{-1}_{P^a} f_{P^a}, u_{\tilde{P}} = L^{-1}_{\tilde{P}} f_{P^a}$ denoting the corresponding Galerkin solutions, we have that*

$$|u_{\tilde{P}} - u_{P^a}|^2_1 \ge c_2^2 \{ \sum_{\triangle \in F} \zeta_\triangle(f_{P^a}) + \sum_{e \in G} \eta_e(u_{P^a}) \},$$

for some absolute constant $c_2 > 0$.

The way to obtain a convergent adaptive refinement strategy is to select the sets F and G such that $\sum_{\triangle\in F} \zeta_\triangle(f_{P^a}) + \sum_{e\in G} \eta_e(u_{P^a})$ is bounded from below by some multiple of $\sum_{\triangle\in P^a} \zeta_\triangle(f_{P^a}) + \sum_{e\in E_{P^a}} \eta_e(u_{P^a}) = \mathcal{E}(P^a, f_{P^a}, u_{P^a})^2$. Then, convergence follows by combining of Theorem 6.1 and Corollary 6.3.

Since Lemma 6.2 also applies when $u_{\tilde{P}} = L^{-1}_{\tilde{P}} f_{P^a}$ is replaced by $u = L^{-1} f_{P^a}$, for later use we state the following result.

COROLLARY 6.4. *Let P^a be an admissible partition and let $f_{P^a} \in \mathcal{S}^0_{P^a}$. With $u := L^{-1} f_{P^a}$, $u_{P^a} := L^{-1}_{P^a} f_{P^a}$ and c_2 the constant from Corollary 6.3, we have that*

$$|u - u_{P^a}|_1 \ge c_2 \mathcal{E}(P^a, f_{P^a}, u_{P^a}).$$

In Corollary 6.3 it was assumed that the right-hand side f is piecewise constant with respect to the current partition P^a, and that the discrete system is solved exactly. In the remainder of this section we will relax both of these assumptions.

LEMMA 6.5. *There exists an absolute constant $C_3 > 0$ such that for any partition P, $f \in L_2(\Omega)$, $u_P, \tilde{u}_P \in \mathcal{S}_P$,*

$$|\mathcal{E}(P,f,u_P) - \mathcal{E}(P,f,\tilde{u}_P)| \leq C_3 |u_P - \tilde{u}_P|_1.$$

Proof. We have that

$$\begin{aligned}|\mathcal{E}(P,f,u_P) - \mathcal{E}(P,f,\tilde{u}_P)| &= [\sum_{\triangle\in P} \zeta_\triangle(f) + \sum_{e\in E_P} \eta_e(u_P)]^{\frac12} - [\sum_{\triangle\in P} \zeta_\triangle(f) + \sum_{e\in E_P} \eta_e(\tilde{u}_P)]^{\frac12} \\ &\leq [\sum_{e\in E_P} (\eta_e(u_P)^{\frac12} - \eta_e(\tilde{u})^{\frac12})^2]^{\frac12}.\end{aligned}$$

For any $e \in E_P$,

$$\begin{aligned}|\eta_e(u_P)^{\frac12} - \eta_e(\tilde{u}_P)^{\frac12}| &= \frac{\mathrm{diam}(e)^{\frac12}}{\max\{\rho|_{e^-}, \rho|_{e^+}\}^{\frac12}} \big| \|[\mathbf{A}\nabla u_P]_e \cdot \mathbf{n}_e\|_{L_2(e)} - \|[\mathbf{A}\nabla \tilde{u}_P]_e \cdot \mathbf{n}_e\|_{L_2(e)} \big| \\ &\leq \frac{\mathrm{diam}(e)^{\frac12}}{\max\{\rho|_{e^-}, \rho|_{e^+}\}^{\frac12}} \|[\mathbf{A}\nabla(u_P - \tilde{u}_P)]_e \cdot \mathbf{n}_e\|_{L_2(e)}.\end{aligned}$$

The proof is completed by the observation that, for any edge e_i of a $\triangle \in P$ and any $w \in P_1(\triangle)$ and unit vector $\mathbf{n}$, by a homogeneity argument we have that

$$\mathrm{diam}(e_i)^{\frac12} \|\mathbf{A}\nabla w \cdot \mathbf{n}\|_{L_2(e_i)} \lesssim \rho|_\triangle |w|_{H^1(\triangle)} \eqsim \rho|_\triangle^{\frac12} |w|_{1,\triangle}.$$

□

For some *fixed constant* $\theta \in (0,1]$, we consider the following refinement procedure.

REFINE$[P^a, f_{P^a}, w_{P^a}] \to \tilde{P}$
% P^a is an admissible partition, $f_{P^a} \in \mathcal{S}^0_{P^a}$, and $w_{P^a} \in \mathcal{S}_{P^a}$.
Select $F \subset P^a$, $G \subset E_{P^a}$ such that

$$\sum_{\triangle\in F} \zeta_\triangle(f_{P^a}) + \sum_{e\in G} \eta_e(w_{P^a}) \geq \theta^2\, \mathcal{E}(P^a, f_{P^a}, w_{P^a})^2.$$

Determine a refinement $\tilde{P}$ of P^a such that for all $\triangle \in F$, $V_{\tilde{P}}$ satisfies the conditions from Lemma 6.2(c), and for all $e \in G$, $V_{\tilde{P}}$ satisfies the conditions from either Lemma 6.2(a) or Lemma 6.2(b), where at the same time each $\tilde{\triangle} \in \tilde{P}$ is either in P^a or it is a child or a grandchild of a $\triangle \in P^a$.

As long as the selection of F and G is organized so that it does not involve the exact ordering of all $\zeta_\triangle(f_{P^a})$ and $\eta_e(w_{P^a})$ by their moduli (cf. discussion from Remark 5.2), we have the following result.

PROPOSITION 6.6. *The call* **REFINE**$[P^a, f_{P^a}, w_{P^a}]$ *requires a number of arithmetic operations $\lesssim \#P^a$.*

As stated before, we would like to consider $f \in H^{-1}(\Omega)$, possibly outside $L_2(\Omega)$, and we will use approximate right-hand sides both for setting up the discrete systems as well as for the evaluation of the a posteriori error estimator. In fact, since the terms $\zeta_\triangle(f)$ are only defined for $f \in L_2(\Omega)$, generally we will need approximate right-hand sides different from the exact one. The following theorem shows that if **REFINE**

is called with a sufficiently accurate piecewise constant approximation for the right-hand side, and a sufficiently accurate approximation of the discrete solution, then the solution on the new partition has an error that is less than the error in the solution on the previous partition. Together with the coarsening routine from §5, in the next two subsections this theorem will be the basis for constructing adaptive finite element methods that converge with optimal rates.

THEOREM 6.7. *Let $f \in H^{-1}(\Omega)$, $u = L^{-1}f$, let P^a be an admissible partition, $f_{P^a} \in \mathcal{S}^0_{P^a}$, $u_{P^a} = L^{-1}_{P^a} f_{P^a}$, $\bar{u}_{P^a} \in \mathcal{S}_{P^a}$, $\tilde{P} = \mathbf{REFINE}[P^a, f_{P^a}, \bar{u}_{P^a}]$ or a refinement of it, $f_{\tilde{P}} \in H^{-1}(\Omega)$ and $u_{\tilde{P}} = L^{-1}_{\tilde{P}} f_{\tilde{P}}$. Then*

$$|u - u_{\tilde{P}}|_1 \leq [1 - \tfrac{1}{2}(\tfrac{c_2\theta}{C_1})^2]^{\frac{1}{2}} |u - u_{P^a}|_1 + 2c_2C_3|u_{P^a} - \bar{u}_{P^a}|_1 + 3\|f - f_{P^a}\|_{-1} + \|f - f_{\tilde{P}}\|_{-1}. \tag{6.5}$$

Proof. Let $\hat{u} = L^{-1}f_{P^a}$ and $\hat{u}_{\tilde{P}} = L^{-1}_{\tilde{P}} f_{P^a}$. With $F \subset P^a$, $G \subset E_{P^a}$ as determined in the call $\mathbf{REFINE}[P^a, f_{P^a}, \bar{u}_{P^a}]$, Corollary 6.3, two applications of Lemma 6.5, and Theorem 6.1 show that

$$\begin{aligned}
|\hat{u}_{\tilde{P}} - u_{P^a}|_1 &\geq c_2[\{\sum_{\triangle \in F} \zeta_{\triangle}(f_{P^a}) + \sum_{e \in G} \eta_e(u_{P^a})\}]^{\frac{1}{2}} \\
&\geq c_2[\{\sum_{\triangle \in F} \zeta_{\triangle}(f_{P^a}) + \sum_{e \in G} \eta_e(\bar{u}_{P^a})\}^{\frac{1}{2}} - C_3|u_{P^a} - \bar{u}_{P^a}|_1] \\
&\geq c_2[\theta\mathcal{E}(P^a, f_{P^a}, \bar{u}_{P^a}) - C_3|u_{P^a} - \bar{u}_{P^a}|_1] \\
&\geq c_2[\theta\mathcal{E}(P^a, f_{P^a}, u_{P^a}) - 2C_3|u_{P^a} - \bar{u}_{P^a}|_1] \\
&\geq c_2[\tfrac{\theta}{C_1}|\hat{u} - u_{P^a}|_1 - 2C_3|u_{P^a} - \bar{u}_{P^a}|_1].
\end{aligned}$$

Since for any scalars a, b, $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$, we infer that

$$|\hat{u}_{\tilde{P}} - u_{P^a}|_1^2 \geq \tfrac{1}{2}(\tfrac{c_2\theta}{C_1})^2 |\hat{u} - u_{P^a}|_1^2 - 4c_2^2C_3^2|u_{P^a} - \bar{u}_{P^a}|_1^2.$$

Since $\hat{u}_{\tilde{P}} \in \mathcal{S}_{\tilde{P}}$ is the Galerkin approximation of $\hat{u}$ from $\mathcal{S}_{\tilde{P}}$ and $u_{P^a} - \hat{u}_{\tilde{P}} \in \mathcal{S}_{\tilde{P}}$, we have

$$\begin{aligned}
|\hat{u} - \hat{u}_{\tilde{P}}|_1^2 &= |\hat{u} - u_{P^a}|_1^2 - |u_{P^a} - \hat{u}_{\tilde{P}}|_1^2 \\
&\leq [1 - \tfrac{1}{2}(\tfrac{c_2\theta}{C_1})^2]|\hat{u} - u_{P^a}|_1^2 + 4c_2^2C_3^2|u_{P^a} - \bar{u}_{P^a}|_1^2 \\
&\leq \{[1 - \tfrac{1}{2}(\tfrac{c_2\theta}{C_1})^2]^{\frac{1}{2}}|\hat{u} - u_{P^a}|_1 + 2c_2C_3|u_{P^a} - \bar{u}_{P^a}|_1\}^2,
\end{aligned}$$

where we have used that $c_2 \leq C_1$ and thus that $1 - \frac{1}{2}(\frac{c_2\theta}{C_1})^2 > 0$. The proof is completed by observing that $|u - \hat{u}|_1 = \|f - f_{P^a}\|_{-1}$ and $|u_{\tilde{P}} - \hat{u}_{\tilde{P}}|_1 \leq \|f_{\tilde{P}} - f_{P^a}\|_{-1} \leq \|f_{\tilde{P}} - f\|_{-1} + \|f - f_{P^a}\|_{-1}$. ∎

7. A first optimal adaptive finite element method. We start with a corollary that is an easy consequence of Theorem 6.7. It shows that the reduction of the error in the exact discrete solutions as a result of refinement extends to a similar reduction of the error in sufficiently accurate approximations of these discrete solutions.

COROLLARY 7.1. *For any $\mu \in ([1 - \frac{1}{2}(\frac{c_2\theta}{C_1})^2]^{\frac{1}{2}}, 1)$, there exists a sufficiently small constant $\delta > 0$ such that if for $f \in H^{-1}(\Omega)$, an admissible partition P^a, $\bar{u}_{P^a} \in \mathcal{S}_{P^a}$,*

$f_{P^a} \in \mathcal{S}^0_{P^a}$, $\tilde{P} = \mathbf{REFINE}[P^a, f_{P^a}, \bar{u}_{P^a}]$ or a refinement of it, $\bar{u}_{\tilde{P}} \in \mathcal{S}_{\tilde{P}}$, $f_{\tilde{P}} \in H^{-1}(\Omega)$ and $\varepsilon > 0$, with $u = L^{-1}f$, $u_{P^a} = L^{-1}_{P^a} f_{P^a}$ and $u_{\tilde{P}} = L^{-1}_{\tilde{P}} f_{\tilde{P}}$, we have that $|u - \bar{u}_{P^a}|_1 \leq \varepsilon$ and

$$|u_{P^a} - \bar{u}_{P^a}|_1 + \|f - f_{P^a}\|_{-1} + |u_{\tilde{P}} - \bar{u}_{\tilde{P}}|_1 + \|f - f_{\tilde{P}}\|_{-1} \leq 2(1+\mu)\delta\varepsilon$$

then $|u - \bar{u}_{\tilde{P}}|_1 \leq \mu\varepsilon$.

Proof. The proof is an easy consequence of Theorem 6.7, $|u - \bar{u}_{\tilde{P}}|_1 \leq |u - u_{\tilde{P}}|_1 + |u_{\tilde{P}} - \bar{u}_{\tilde{P}}|_1$ and $|u - u_{P^a}|_1 \leq |u - \bar{u}_{P^a}|_1 + |u_{P^a} - \bar{u}_{P^a}|_1$. □

We assume the availability of the following routine.

GALSOLVE$[P^a, f_{P^a}, u^{(0)}_{P^a}, \varepsilon] \to \bar{u}_{P^a}$
% P^a is an admissible partition, $f_{P^a} \in (\mathcal{S}_{P^a})'$, and $u^{(0)}_{P^a} \in \mathcal{S}_{P^a}$. With $u_{P^a} := L^{-1}_{P^a} f_{P^a}$,
% the output $\bar{u}_{P^a} \in \mathcal{S}_{P^a}$ satisfies

$$|u_{P^a} - \bar{u}_{P^a}|_1 \leq \varepsilon.$$

% The call requires $\lesssim \max\{1, \log(\varepsilon^{-1}|u_{P^a} - u^{(0)}_{P^a}|_1|)\}\#P^a$ arithmetic operations.

Thus we not only assume that we have an iterative solver at our disposal that converges with a rate independent of the problem size, but in accordance with the idea of an adaptive solver, we additionally assume that we have an efficient and reliable control of the algebraic error. As a consequence the number of iterations to be performed does not to depend on a possibly pessimistic a priori bound on the initial error. Two possible realizations of **GALSOLVE** are discussed in the next remark.

Remark 7.2. One can apply Conjugate Gradients, starting with $u^0_{P^a}$, to the representation of $L_{P^a} u_{P^a} = f_{P^a}$ with respect to $\{\bar{\psi}^v : v \in V_{P^a}\}$. In each iteration, that takes $\eqsim \#P^a$ operations, the $|\cdot|_1$-norm of the error is multiplied by a factor less than or equal to some constant $\tau < 1$ only dependent on $\kappa_{\bar{\Psi}}$, meaning that after $\lceil \log_\tau(\varepsilon |u_{P^a} - u^{(0)}_{P^a}|_1^{-1}) \rceil$ iterations the $|\cdot|_1$-norm of the error is $\leq \varepsilon$. The $|\cdot|_1$-norm of the error in an approximation for u_{P^a} from $\mathcal{S}_{P^a}$ is less (respectively, greater) than or equal to $\lambda_{\bar{\Psi}}^{-\frac{1}{2}}$ (respectively, $\Lambda_{\bar{\Psi}}^{-\frac{1}{2}}$) times the Euclidean norm of the corresponding residual vector. So if one stops the iteration as soon as the latter norm is $\leq \lambda_{\bar{\Psi}}^{\frac{1}{2}} \varepsilon$, then the $|\cdot|_1$-norm of the error is $\leq \varepsilon$, whereas the number of iterations is bounded by

$$\lceil \log_\tau(\kappa_{\bar{\Psi}}^{-\frac{1}{2}} \varepsilon |u_{P^a} - u^{(0)}_{P^a}|_1^{-1}) \rceil \lesssim \max\{1, \log(\varepsilon^{-1}|u_{P^a} - u^{(0)}_{P^a}|_1)\},$$

showing that this approach results in a valid routine **GALSOLVE**.

Alternatively, one may apply the Conjugate Gradient method to the representation of $L_{P^a} u_{P^a} = f_{P^a}$ with respect to the nodal basis $\{\phi^v_{P^a} : v \in V_{P^a}\}$ using a BPX preconditioner where, similarly as above, the Euclidean norm of the residual of the preconditioned system may serve to develop a stopping criterion. Indeed, when $P^a = P^a_n$, for $0 \leq i \leq n$ one can select $\tilde{V}_{P^a_i} \subset V_{P^a_i}$ such that both $\mathrm{span}\{\bar{\psi}^v : v \in V_{P^a_i} \backslash V_{P^a_{i-1}}\} \subset \mathrm{span}\{\phi^v_{P^a_i} : v \in \tilde{V}_{P^a_i}\}$ and $\#\tilde{V}_{P^a_i} \lesssim \#(P^a_i \backslash P^a_{i-1})$. Using (4.6), it can then be proven that on $\mathcal{S}_{P^a}$, $\inf_{u = \sum_{i=0}^n \sum_{v \in \tilde{V}_{P^a_i}} c^v_i \phi^v_{P^a_i}} |c^v_i|^2 |\phi^v_{P^a_i}|_1^2 \eqsim |u|_1^2$ showing that the resulting BPX preconditioner or, more precisely, in view of possible jumps of ρ the MDS preconditioner (cf. [16]) gives rise to uniformly well-conditioned systems, whereas it can be implemented in $\lesssim \#P^a$ operations.

Before continuing let us first explain what we mean by an optimal method for solving the boundary value problem (2.1). Following [3], we say that a method is

optimal, if whenever the solution u is such that for some $s > 0$ the error of the best approximation from any $\mathcal{S}_P$ with $\#P \leq n$ is $\lesssim n^{-s}$, then for any $\varepsilon > 0$, the method yields a partition P and a $w_P \in \mathcal{S}_P$ with $|u - w_P|_1 \leq \varepsilon$ taking only $\lesssim \#P$ operations where $\#P \lesssim \varepsilon^{-1/s}$. Indeed, note that in view of the assumption on u, the smallest partition P for which there exists such a $w_P \in \mathcal{S}_P$ generally has cardinality $\eqsim \varepsilon^{-1/s}$. A definition of the class of functions $u \in H^1_0(\Omega)$ for which for some $s > 0$ the errors of the best approximations decay as indicated above is given by

$$\mathcal{A}^s = \{u \in H^1_0(\Omega) : \sup_{n\geq 0} n^s \inf_{\#P\leq n} \inf_{u_P\in\mathcal{S}_P} |u - u_P|_1 < \infty\},$$

where P is any partition of the type we consider.

It is well-known that for $s \leq \frac{1}{2}$, $H^1_0(\Omega) \cap H^{1+2s}(\Omega) \subset \mathcal{A}^s$. Indeed, for functions in $H^1_0(\Omega) \cap H^{1+2s}(\Omega)$, the sequence of errors of the best continuous piecewise linear approximations subordinate to the sequence of uniform refinements P^*_i of P_0 already exhibits a decay of $\lesssim (\#P^*_i)^{-s}$. Obviously, the class $\mathcal{A}^s$ contains many more functions than only those in $H^1_0(\Omega)\cap H^{1+2s}(\Omega)$, which is the reason to consider adaptive methods anyway. Although the class $\mathcal{A}^s$ is non-trivial for any $s > 0$, as it contains all $u \in \mathcal{S}_P$ for any partition P, since we are approximating with piecewise linears only for $s \leq \frac{1}{2}$ membership of $\mathcal{A}^s$ can be guaranteed by imposing suitable smoothness conditions. For partitions generated by the so-called newest vertex bisection, a characterization of $\mathcal{A}^s$ for $s \leq \frac{1}{2}$ in terms of Besov spaces can be found in [4]. We expect the same results to be valid for the type of partitions considered here. The characterization via Besov spaces, together with regularity results, as in [10], allows one to obtain a priori knowledge about the class $\mathcal{A}^s$ to which the solution u of the boundary value problem (2.1) belongs.

With the adaptive refinement strategy from Theorem 6.7 and Corollary 7.1, convergence can only be guaranteed when sufficiently accurate piecewise constant approximations to the right-hand side f are available. We assume the availability of the following routine **RHS** that, as the routine **REFINE**, may involve a refinement of the current partition.

RHS$[P, f, \varepsilon] \to [P^a, f_{P^a}]$
% P is a partition, $f \in H^{-1}(\Omega)$ and $\varepsilon > 0$. The output consists of an admissible
% refinement P^a of P, and an $f_{P^a} \in \mathcal{S}^0_{P^a}$ with $\|f - f_{P^a}\|_{-1} \leq \varepsilon$.

Assuming that the solution $u \in \mathcal{A}^s$ for some $s > 0$, the cost of approximating the right-hand side f using a routine **RHS** will generally not dominate the other costs of our adaptive method only if there is some constant c_f such that for any $\varepsilon > 0$ and any partition P, for $[P^a, f_{P^a}] := \mathbf{RHS}[P, f, \varepsilon]$ both $\#P^a$ and the number of arithmetic operations required by this call are $\lesssim \#P + c_f^{1/s}\varepsilon^{-1/s}$. We will call such a pair $(f, \mathbf{RHS})$ to be *s-optimal*. Obviously, given s, such a pair only exists when $f \in \bar{\mathcal{A}}^s$, defined by

$$\bar{\mathcal{A}}^s = \{f \in H^{-1}(\Omega) : \sup_{n\geq 0} n^s \inf_{\#P\leq n} \inf_{f_P\in\mathcal{S}^0_P} \|f - f_P\|_{-1} < \infty\},$$

where P is any partition of the type we consider.

When $f \in L_2(\Omega)$ with $\|\rho^{-\frac{1}{2}} f\|_{L_2(\Omega)} \lesssim 1$, the routine **RHS** can be based on uniform refinements. Indeed, for some integer i to be determined below, let $\tilde{P}$ denote the smallest common refinement of the given P and P^*_i, let $\tilde{P}^a$ be its admissible refinement as a result of applying Algorithm 3.6, and with $Q^0_{\tilde{P}^a} : L_2(\Omega) \to \mathcal{S}^0_{\tilde{P}^a}$

denoting the $L_2(\Omega)$-orthogonal projector onto $\mathcal{S}^0_{\tilde{P}^a}$, let $f_{\tilde{P}^a} = Q^0_{\tilde{P}^a} f$. For any $w \in H^1_0(\Omega)$, we have

$$|\int_\Omega (f - f_{\tilde{P}^a})w| = |\int_\Omega f(w - Q^0_{\tilde{P}^a} w)| \leq C \sum_{\triangle \in \tilde{P}^a} \|f\|_{L_2(\triangle)} \mathrm{diam}(\triangle) \|\nabla w\|_{L_2(\triangle)}$$
$$\leq C 2^{-i} \|\rho^{-\frac{1}{2}} f\|_{L_2(\Omega)} |w|_1,$$

where $C > 0$ is some absolute constant. By taking i to be the smallest integer such that $2^{-i} C \|\rho^{-\frac{1}{2}} f\|_{L_2(\Omega)} \leq \varepsilon$, we have $\|f - f_{\tilde{P}^a}\|_{-1} \leq \varepsilon$, and

$$\#\tilde{P}^a \lesssim \#P + \#P^*_i \lesssim \#P + (2^i)^2 \lesssim \#P + \varepsilon^{-2} \|\rho^{-\frac{1}{2}} f\|^2_{L_2(\Omega)}.$$

Thus, when for any $\triangle \in \cup_{j \geq 0} P^*_j$ the evaluation of $\int_\triangle f$ takes $\mathcal{O}(1)$ operations, we may conclude that for **RHS** based on this procedure $(f, \mathbf{RHS})$ is s-optimal with $s = \frac{1}{2}$, which, as we have seen, covers the range of main interest. Alternatively, instead of assuming the exact evaluation of $\int_\triangle f$, one easily infers that it also suffices to approximate the integral with an error $\lesssim \rho^{\frac{1}{2}} \mathrm{diam}(\triangle)$, which, in any case, is possible to accomplish in $\mathcal{O}(1)$ operations when $\rho^{-\frac{1}{2}} f$ has some piecewise smoothness with respect to P_0.

Obviously, the class $\bar{\mathcal{A}}^s$ is much larger than $L_2(\Omega)$. Yet, for $f \notin L_2(\Omega)$ the realization of a suitable routine **RHS** has to depend on the right-hand side at hand. We give one example.

Example 7.3. Let $\rho = 1$, and for a sufficiently smooth curve K in Ω, let $f \in H^{-1}(\Omega)$ be defined by $f(w) = \int_K w$. We define $\mathbf{RHS}[P, f, \varepsilon]$ by the following steps. Recursively refine all $\triangle \in P$ that have non-empty intersection with K, until all those $\triangle$ satisfy $\mathrm{diam}(\triangle) \leq C\varepsilon^2$ for some constant $C > 0$. Let P^a be the admissible refinement of the obtained partition. Define $f_{P^a} \in \mathcal{S}^0_{P^a}$ by $f_{P^a}|_\triangle = \frac{\mathrm{length}(K \cap \triangle)}{\mathrm{vol}(\triangle)}$ $(\triangle \in P^a)$. Then one may verify that by choosing C suitably, we have that $\|f - f_{P^a}\|_{-1} \leq \varepsilon$, and that both the number of arithmetic operations required by this call of **RHS** and $\#P^a$ are $\lesssim \#P + \varepsilon^{-2}$, showing that this $(f, \mathbf{RHS})$ pair is s-optimal with $s = \frac{1}{2}$. Note that although, depending on the curve, the exact evaluation of f applied to any $v_P \in \mathcal{S}_P$ might not pose any problems, nevertheless the piecewise constant approximations are required because of the evaluation of the a posteriori error estimator.

We are ready to give our first adaptive finite element method for solving (2.1). As expected, it is based on the repeated application of the triple **REFINE**, **RHS** and **GALSOLVE**, the latter two with suitable tolerances, which by Corollary 7.1 give rise to linearly convergent approximations. To obtain an optimal work-accuracy balance, **COARSE** is applied after every M iterations of the above triple, with M being some fixed constant.

SOLVE1$[f, \varepsilon, \bar{u}_{P_0}, \varepsilon_0] \to [P^a, \bar{u}_{P^a}]$:
% The following constants are fixed: $\delta < \frac{1}{3}$ *and it is small enough so that it corresponds % to a* $\mu < 1$ *as in Corollary 7.1;* $\gamma > t_1^{-\frac{1}{2}}$ *with* t_1 *as in Proposition 5.3;* $M \in \mathbb{N}$ *% such that* $\frac{(1+\gamma)\kappa_{\bar{\Psi}}^{\frac{1}{2}} \mu^M}{1-3\delta} < 1$.
% The input must satisfy $f \in H^{-1}(\Omega)$ *such that a valid routine* **RHS** *is available, %* $\varepsilon > 0$, $\bar{u}_{P_0} \in \mathcal{S}_{P_0}$ *and* $\varepsilon_0 \geq |u - \bar{u}_{P_0}|_1$.

$P^a := P_0$, $f_{P^a} := f_{P_0}$, $\bar{u}_{P^a} := \bar{u}_{P_0}$
`if` $\varepsilon \geq \varepsilon_0$ `then` $N := 0$
`else` $N \geq 1$ *is the smallest integer with* $(\frac{\mu^M}{1-3\delta})^N((1+\gamma)\kappa_{\bar{\Psi}}^{\frac{1}{2}})^{N-1}\varepsilon_0 \leq \varepsilon$ `fi`
`for` $i = 1, \ldots, N$ `do`
  `if` $i = 1$ `then` $\varepsilon_1 := \frac{\varepsilon_0}{1-3\delta}$
  `else` $[P^a, \bar{u}_{P^a}] := \mathbf{COARSE}[P^a, \bar{u}_{P^a}, \gamma\lambda_{\bar{\Psi}}^{-\frac{1}{2}}\mu^M\varepsilon_{i-1}]$, $\varepsilon_i := \frac{(1+\gamma)\kappa_{\bar{\Psi}}^{\frac{1}{2}}\mu^M}{1-3\delta}\varepsilon_{i-1}$ `fi`
  $[P^a, f_{P^a}] := \mathbf{RHS}[P^a, f, \delta\varepsilon_i]$
  $\bar{u}_{P^a} := \mathbf{GALSOLVE}[P^a, f_{P^a}, \bar{u}_{P^a}, \delta\varepsilon_i]$
  `for` $j = 1, \ldots, M$ `do`
    $P := \mathbf{REFINE}[P^a, f_{P^a}, \bar{u}_{P^a}]$
    $[P^a, f_{P^a}] := \mathbf{RHS}[P, f, \delta\mu^j\varepsilon_i]$
    $\bar{u}_{P^a} := \mathbf{GALSOLVE}[P^a, f_{P^a}, \bar{u}_{P^a}, \delta\mu^j\varepsilon_i]$
  `od`
`od`

The next theorem shows that **SOLVE1** is an optimal method, whenever this is allowed by the $(f, \mathbf{RHS})$ pair.

THEOREM 7.4. $[P^a, \bar{u}_{P^a}] := \mathbf{SOLVE1}[f, \varepsilon, \bar{u}_{P_0}, \varepsilon_0]$ *satisfies* $|u - \bar{u}_{P^a}|_1 \leq \varepsilon$. *Assuming* $\varepsilon_0 \lesssim |u|_1$, *if for some* $s > 0$, $u \in \mathcal{A}^s$ *and* $(f, \mathbf{RHS})$ *is* s*-optimal, then both* $\#P^a$ *and the number of arithmetic operations required by this call are* $\lesssim \max\{1, \varepsilon^{-1/s}(c_f^{1/s} + |u|_{\mathcal{A}^s}^{1/s})\}$.

Proof. For $\varepsilon \geq \varepsilon_0$ there is nothing to prove, so let us assume that $\varepsilon < \varepsilon_0$. By induction on i we prove that at termination of the if-then-else-fi clause inside the loop over i,

$$|u - \bar{u}_{P^a}|_1 \leq (1 - 3\delta)\varepsilon_i. \tag{7.1}$$

For $i = 1$, this follows from the input condition on ε_0. Let us now assume (7.1) for some $i \geq 1$. Then, after the call $[P^a, f_{P^a}] := \mathbf{RHS}[P^a, f, \delta\varepsilon_i]$, by definition

$$\|f - f_{P^a}\|_{-1} \leq \delta\varepsilon_i. \tag{7.2}$$

So, for $u_{P^a} := L_{P^a}^{-1} f_{P^a}$ we have

$$\begin{aligned} |u - u_{P^a}|_1 &\leq |u - L^{-1}f_{P^a}|_1 + |L^{-1}f_{P^a} - u_{P^a}|_1 \leq |u - L^{-1}f_{P^a}|_1 + |L^{-1}f_{P^a} - \bar{u}_{P^a}|_1 \\ &\leq 2|u - L^{-1}f_{P^a}|_1 + |u - \bar{u}_{P^a}|_1 = 2\|f - f_{P^a}\|_{-1} + |u - \bar{u}_{P^a}|_1 \\ &\leq 2\delta\varepsilon_i + (1 - 3\delta)\varepsilon_i = (1 - \delta)\varepsilon_i, \end{aligned} \tag{7.3}$$

where for the second inequality we have used that $\bar{u}_{P^a} \in \mathcal{S}_{P^a}$ and that u_{P^a} is the best approximation with respect to $|\cdot|_1$ of $L^{-1}f_{P^a}$ from $\mathcal{S}_{P^a}$. We conclude that after the update of $\bar{u}_{P^a}$ by the call of **GALSOLVE**,

$$|u_{P^a} - \bar{u}_{P^a}|_1 \leq \delta\varepsilon_i \text{ and so } |u - \bar{u}_{P^a}|_1 \leq |u - u_{P^a}|_1 + |u_{P^a} - \bar{u}_{P^a}|_1 \leq \varepsilon_i. \tag{7.4}$$

After the first calls of **REFINE**, **RHS** and **GALSOLVE** in the inner loop, i.e., when $j = 1$, for the new P^a, f_{P^a}, $\bar{u}_{P^a}$, $u_{P^a} := L_{P^a}^{-1} f_{P^a}$ we have $\|f - f_{P^a}\|_{-1} \leq \delta\mu\varepsilon_i$ and $|u_{P^a} - \bar{u}_{P^a}|_1 \leq \delta\mu\varepsilon_i$, and so by (7.2), (7.4), Corollary 7.1 shows that $|u - \bar{u}_{P^a}|_1 \leq \mu\varepsilon_i$. Repeating this argument for $j = 2, \ldots, M$ shows that at termination of the inner loop over j, we have that $|u - \bar{u}_{P^a}|_1 \leq \mu^M\varepsilon_i$. In particular, when $i = N$, we have that

$|u-\bar u_{P^a}|_1 \le \mu^M \varepsilon_N = (\frac{\mu^M}{1-3\delta})^N((1+\gamma)\kappa_{\bar\Psi}^{\frac12})^{N-1}\varepsilon_0 \le \varepsilon$ by definition of N. Otherwise, if $i<N$, then in the next iteration, thus after increasing i by one, just before the call of **COARSE**, we have that $\|\!|u-\bar u_{P^a}|\!\|_1 \le \lambda_{\bar\Psi}^{-\frac12}|u-\bar u_{P^a}|_1 \le \lambda_{\bar\Psi}^{-\frac12}\mu^M\varepsilon_{i-1}$. By Corollary 5.5, after this call we have

$$|u-\bar u_{P^a}|_1 \le \Lambda_{\bar\Psi}^{\frac12}\|\!|u-\bar u_{P^a}|\!\|_1 \le \Lambda_{\bar\Psi}^{\frac12}(1+\gamma)\lambda_{\bar\Psi}^{-\frac12}\mu^M\varepsilon_{i-1} = (1-3\delta)\varepsilon_i, \tag{7.5}$$

which completes the proof of (7.1), and thus that of $|u-\bar u_{P^a}|_1 \le \varepsilon$ at termination of **SOLVE1**.

Now we will prove that for any $i=1,\dots,N$, both $\#P^a$ at the end of the outer cycle for this i and the cost of this cycle excluding, for $i>1$, the cost of the **COARSE**, but including, for $i<N$, the cost of the **COARSE** in the next cycle, are $\lesssim \varepsilon_i^{-1/s}(c_f^{1/s}+|u|_{\mathcal{A}^s}^{1/s})$. Because of $\sum_{i=1}^N \varepsilon_i^{-1/s} \lesssim \varepsilon_N^{-1/s} \le \varepsilon^{-1/s}$ this will prove the statement about the complexity.

At the start of the outer cycle for $i=1$, we have $\#P^a \lesssim \varepsilon_i^{-1/s}|u|_{\mathcal{A}^s}^{1/s}$, which follows from $\#P^a = \#P_0 \lesssim 1$ and the assumption that $\varepsilon_0 \lesssim |u|_1$. Since, as we have seen, for $i>1$ before the call of **COARSE**, we have that $\|\!|u-\bar u_{P^a}|\!\|_1 \le \lambda_{\bar\Psi}^{-\frac12}\mu^M\varepsilon_{i-1}$, Corollary 5.5 shows that after this call, $\#P^a \le D\#\hat P$ for any partition $\hat P$ with $\inf_{u_{\hat P}\in\mathcal{S}_{\hat P}}\|\!|u-u_{\hat P}|\!\|_1 \le (t_1^{\frac12}\gamma-1)\lambda_{\bar\Psi}^{-\frac12}\mu^M\varepsilon_{i-1}$. From $\|\!|\cdot|\!\|_1 \le \lambda_{\bar\Psi}^{-\frac12}|\cdot|_1$ and $u\in\mathcal{A}^s$, we find that

$$\#P^a \le D(\#P_0 + [(t_1^{\frac12}\gamma-1)\mu^M\varepsilon_{i-1}]^{-1/s}|u|_{\mathcal{A}^s}^{1/s}) \lesssim \varepsilon_i^{-1/s}|u|_{\mathcal{A}^s}^{1/s}.$$

Since $(f,\mathbf{RHS})$ is s-optimal and M is a fixed constant, from the properties of **RHS** and **REFINE** we conclude that for any i, at the end of the outer cycle

$$\#P^a \lesssim \varepsilon_i^{-1/s}(c_f^{1/s}+|u|_{\mathcal{A}^s}^{1/s}), \tag{7.6}$$

whereas the cost of all calls of **RHS** and **REFINE** inside this cycle are also $\lesssim \varepsilon_i^{-1/s}(c_f^{1/s}+|u|_{\mathcal{A}^s}^{1/s})$. Furthermore, for $i<N-1$, Theorem 5.4(b) shows that the cost of **COARSE** in the next iteration is $\lesssim \#P^a + \max\{0,\log(\varepsilon_i^{-1}\|\!|\bar u_{P^a}|\!\|_1)\}$. From $\log(\varepsilon_i^{-1}\|\!|\bar u_{P^a}|\!\|_1) \le \varepsilon_i^{-1/s}\|\!|\bar u_{P^a}|\!\|_1^{1/s}$, $\|\!|\bar u_{P^a}|\!\|_1 \lesssim |\bar u_{P^a}|_1 \le |u|_1+\varepsilon_0 \lesssim |u|_1 \le |u|_{\mathcal{A}^s}$ and (7.6), we conclude that also these cost are $\lesssim \varepsilon_i^{-1/s}(c_f^{1/s}+|u|_{\mathcal{A}^s}^{1/s})$.

What is left is to bound the cost of the applications of **GALSOLVE**. As we have seen, just before the call **GALSOLVE**$[P^a, f_{P^a}, \bar u_{P^a}, \delta\varepsilon_i]$ outside the inner loop over j, we have that $|u-\bar u_{P^a}|_1 \le (1-3\delta)\varepsilon_i$, and with $u_{P^a} := L_{P^a}^{-1}f_{P^a}$, $|u-u_{P^a}|_1 \le (1-\delta)\varepsilon_i$, and so $|u_{P^a}-\bar u_{P^a}|_1 \le 2(1-2\delta)\varepsilon_i$. Since $\frac{2(1-2\delta)\varepsilon_i}{\delta\varepsilon_i}$ is a constant, we conclude that the cost of this call is $\lesssim \#P^a$.

Let us now consider a call **GALSOLVE**$[P^a, f_{P^a}, \bar u_{P^a}, \delta\mu^j\varepsilon_i]$ inside the loop over j. Just before this call we have that $|u-\bar u_{P^a}|_1 \le \mu^{j-1}\varepsilon_i$ and $\|f-f_{P^a}\|_{-1} \le \delta\mu^j\varepsilon_i$. As in (7.3), for $u_{P^a} := L_{P^a}^{-1}f_{P^a}$ we have

$$|u-u_{P^a}|_1 \le 2\|f-f_{P^a}\|_{-1} + |u-\bar u_{P^a}|_1 \le (2\delta\mu^j+\mu^{j-1})\varepsilon_i,$$

and so $|u_{P^a}-\bar u_{P^a}|_1 \le 2(\delta\mu^j+\mu^{j-1})\varepsilon_i$. Since $\frac{(2\delta\mu^j+\mu^{j-1})\varepsilon_i}{\delta\mu^j\varepsilon_i}$ is a constant, we conclude that the cost of this call is $\lesssim \#P^a$, which completes the proof. □

8. An optimal adaptive finite element method with a posteriori error control. It follows from the proof of Theorem 7.4 that the approximations $\bar{u}_{P^a}$ on the sequence of partitions produced by **SOLVE1** converge with an asymptotic rate $\leq (\frac{(1+\gamma)\kappa_{\bar{\Psi}}^{\frac{1}{2}}\mu^M}{1-3\delta})^{1/(M+1)}$, which is close to μ when M is not too small. Because of the application of a coarsening, the asymptotic rate is generally even equal to the above number. Indeed, after the evaluation of $[P^a, \bar{u}_{P^a}] := \mathbf{COARSE}[P^a, `\bar{u}_{P^a}^{old}', \gamma\lambda_{\bar{\Psi}}^{-\frac{1}{2}}\mu^M\varepsilon_{i-1}]$, we have that

$$|u-\bar{u}_{P^a}|_1 \geq \lambda_{\bar{\Psi}}^{\frac{1}{2}}\|\!|u-\bar{u}_{P^a}|\!\|_1 \geq \lambda_{\bar{\Psi}}^{\frac{1}{2}}\left|\|\!|\bar{u}_{P^a}-\bar{u}_{P^a}^{old}|\!\|_1 - \|\!|u-\bar{u}_{P^a}^{old}|\!\|_1\right| \geq ((\gamma-\eta)-1)\mu^M\varepsilon_{i-1}$$

where η can be arbitrary small, so that this lower bound is only by a constant factor smaller than the upper bound for $|u-\bar{u}_{P^a}|_1$ from (7.5). The value μ has to be supplied by the user. It should be large enough to ensure that indeed $\lambda_{\bar{\Psi}}^{-\frac{1}{2}}\mu^M\varepsilon_{i-1}$ is an upper bound for $\|\!|u-\bar{u}_{P^a}^{old}|\!\|_1$, so that the quasi-optimality of the partition after **COARSE** is guaranteed by Corollary 5.5. A safe choice of μ will be the result of a worst-case analysis, and so likely it will be unnecessarily close to 1, resulting in a quantitatively less attractive algorithm. All adaptive finite element or wavelet methods based on coarsening introduced so far share this drawback that a judicious choice of such a parameter μ has to be made.

In this final subsection, we develop a modified routine **SOLVE2** in which the tolerances used in the routines **COARSE**, **RHS** and **GALSOLVE** will depend on an a posteriori estimate of the error, instead of an a priori one. Moreover, as is also suggested in [7], instead of performing a fixed number of iterations of the **REFINE**, **RHS**, **GALSOLVE** triple between two applications of **COARSE**, in **SOLVE2** the iteration is stopped as soon as sufficient reduction of the error has taken place as indicated by the a posteriori error estimator. We will use our estimator $\mathcal{E}$, extended with some terms to incorporate the error in the right-hand side and that as a consequence of the inexact solution of the discrete system, cf. (8.1). One problem is that a monotone decrease of the a posteriori error estimates cannot be guaranteed. In Proposition 8.1 however, we show that the error estimates are equivalent to quantities that do decrease linearly, cf. (8.2) and (8.4).

PROPOSITION 8.1. *Let $C_4 := 1 + C_1C_3$, where $C_1, C_3 > 0$ are the constants from Theorem 6.1 and Lemma 6.5. For $f \in H^{-1}(\Omega)$, any admissible partition P^a, $\bar{u}_{P^a} \in \mathcal{S}_{P^a}$, and $f_{P^a} \in L_2(\Omega)$, with $u = L^{-1}f$, $u_{P^a} = L_{P^a}^{-1}f_{P^a}$ we have*

$$|u-\bar{u}_{P^a}|_1 \leq C_1\mathcal{E}(P^a, f_{P^a}, \bar{u}_{P^a}) + \|f-f_{P^a}\|_{-1} + C_4|u_{P^a}-\bar{u}_{P^a}|_1. \tag{8.1}$$

With $f_{P^a} \in \mathcal{S}^0_{P^a}$, and ζ_{P^a} being an upper bound for $\|f-f_{P^a}\|_{-1}+C_4|u_{P^a}-\bar{u}_{P^a}|_1$, we have that

$$C_1\mathcal{E}(P^a, f_{P^a}, \bar{u}_{P^a}) + \zeta_{P^a} \eqsim |u-\bar{u}_{P^a}|_1 + \zeta_{P^a}. \tag{8.2}$$

Finally, for any $\mu \in ([1-\frac{1}{2}(\frac{c_2\theta}{C_1})^2]^{\frac{1}{2}}, 1)$, there exists $\bar{\delta} > 0$ small enough and $C_5 > 0$ large enough, such that if $\tilde{P} = \mathbf{REFINE}[P^a, f_{P^a}, \bar{u}_{P^a}]$ or a refinement of it, $\bar{u}_{\tilde{P}} \in \mathcal{S}_{\tilde{P}}$, $f_{\tilde{P}} \in H^{-1}(\Omega)$, $u_{\tilde{P}} := L_{\tilde{P}}^{-1}f_{\tilde{P}}$, and

$$\zeta_{\tilde{P}} \leq \bar{\delta}(1+C_4)[C_1\mathcal{E}(P^a, f_{P^a}, \bar{u}_{P^a}) + \zeta_{P^a}], \tag{8.3}$$

where thus $\zeta_{\tilde{P}}$ denotes an upper bound on $\|f-f_{\tilde{P}}\|_{-1} + C_4|u_{\tilde{P}}-\bar{u}_{\tilde{P}}|_1$, then

$$|u-\bar{u}_{\tilde{P}}|_1 + C_5\zeta_{\tilde{P}} \leq \mu[|u-\bar{u}_{P^a}|_1 + C_5\zeta_{P^a}]. \tag{8.4}$$

Proof. With $\hat{u} := L^{-1} f_{P^a}$, (8.1) follows from Theorem 6.1 and Lemma 6.5 by

$$\begin{aligned}|u - \bar{u}_{P^a}|_1 &\leq |u - \hat{u}|_1 + |\hat{u} - u_{P^a}|_1 + |u_{P^a} - \bar{u}_{P^a}|_1 \\ &\leq \|f - f_{P^a}\|_{-1} + C_1 \mathcal{E}(P^a, f_{P^a}, u_{P^a}) + |u_{P^a} - \bar{u}_{P^a}|_1 \\ &\leq \|f - f_{P^a}\|_{-1} + C_1 \mathcal{E}(P^a, f_{P^a}, \bar{u}_{P^a}) + (1 + C_1 C_3)|u_{P^a} - \bar{u}_{P^a}|_1.\end{aligned}$$

In one direction, (8.2) follows immediately from (8.1), whereas in the other direction it is a consequence of Corollary 6.4 in combination with Lemma 6.5.

Theorem 6.7 shows that with $C_6 := \max\{\frac{1+2c_2C_3}{C_4}, 3\}$,

$$|u - \bar{u}_{\tilde{P}}|_1 + C_5 \zeta_{\tilde{P}} \leq [1 - \tfrac{1}{2}(\tfrac{c_2\theta}{C_1})^2]^{\frac{1}{2}} |u - \bar{u}_{P^a}|_1 + C_6(\zeta_{P^a} + \zeta_{\tilde{P}}) + C_5 \zeta_{\tilde{P}},$$

which, given a $\mu \in ([1 - \frac{1}{2}(\frac{c_2\theta}{C_1})^2]^{\frac{1}{2}}, 1)$, is less than or equal to $\mu[|u - \bar{u}_{P^a}|_1 + C_5 \zeta_{P^a}]$ if and only if

$$(C_6 + C_5)\zeta_{\tilde{P}} \leq (\mu - [1 - \tfrac{1}{2}(\tfrac{c_2\theta}{C_1})^2]^{\frac{1}{2}})|u - \bar{u}_{P^a}|_1 + (\mu C_5 - C_6)\zeta_{P^a}.$$

Hence, by selecting the constant $C_5 > \frac{C_6}{\mu}$, the proof of (8.4) is completed by observing that for $\bar{\delta}$ small enough,

$$\bar{\delta}(1+C_4)[C_1 \mathcal{E}(P^a, f_{P^a}, \bar{u}_{P^a}) + \zeta_{P^a}] \leq \frac{(\mu - [1 - \frac{1}{2}(\frac{c_2\theta}{C_1})^2]^{\frac{1}{2}})|u - \bar{u}_{P^a}|_1 + (\mu C_5 - C_6)\zeta_{P^a}}{C_6 + C_5},$$

which is a consequence of (8.2). □

We are ready to formulate the adaptive finite element method **SOLVE2** in which the tolerances are controlled by the a posteriori error estimator. Any faster convergence of the approximations produced by the **REFINE**, **RHS**, **GALSOLVE** triple than appears from a priori estimates can be expected to lead to better quantitative properties for **SOLVE2** than for **SOLVE1**.

SOLVE2$[f, \varepsilon, \bar{u}_{P_0}, \varepsilon_0] \to [P^a, \bar{u}_{P^a}]$:
% The following constants are fixed: $\bar{\delta}$ is small enough so that it correspond to a
% $\mu < 1$ as in Proposition 8.1; $\gamma > t_1^{-\frac{1}{2}}$ with t_1 as in Proposition 5.3; and $\sigma \in (0,1)$.
% The input must satisfy $f \in H^{-1}(\Omega)$ such that a valid routine **RHS** *is available,*
% $\varepsilon > 0$, $\bar{u}_{P_0} \in \mathcal{S}_{P_0}$ and $\varepsilon_0 \geq |u - \bar{u}_{P_0}|_1$.
$P^a := P_0$, $f_{P^a} := f_{P_0}$, $\bar{u}_{P^a} := \bar{u}_{P_0}$, $\bar{e} := \tilde{e} := \varepsilon_0$
`while` $\bar{e} > \varepsilon$ `do`
  `if` *not first iteration* `then`
    $[P^a, \bar{u}_{P^a}] := \mathbf{COARSE}[P^a, \bar{u}_{P^a}, \gamma \lambda_{\bar{\Psi}}^{-\frac{1}{2}} \bar{e}]$, $\tilde{e} := (1+\gamma)\kappa_{\bar{\Psi}}^{\frac{1}{2}} \bar{e}$
  `fi`
  $[P^a, f_{P^a}] := \mathbf{RHS}[P^a, f, \bar{\delta}\tilde{e}]$
  $\bar{u}_{P^a} := \mathbf{GALSOLVE}[P^a, f_{P^a}, \bar{u}_{P^a}, \bar{\delta}\tilde{e}]$
  $\tilde{e} := C_1 \mathcal{E}(P^a, f_{P^a}, \bar{u}_{P^a}) + (1 + C_4)\bar{\delta}\tilde{e}$
  `while` $\tilde{e} > \sigma \bar{e}$ `do`
    $P := \mathbf{REFINE}[P^a, f_{P^a}, \bar{u}_{P^a}]$
    $[P^a, f_{P^a}] := \mathbf{RHS}[P, f, \bar{\delta}\tilde{e}]$
    $\bar{u}_{P^a} := \mathbf{GALSOLVE}[P^a, f_{P^a}, \bar{u}_{P^a}, \bar{\delta}\tilde{e}]$
    $\tilde{e} := C_1 \mathcal{E}(P^a, f_{P^a}, \bar{u}_{P^a}) + (1 + C_4)\bar{\delta}\tilde{e}$
  `od`

$\bar{e} := \tilde{e}$

`od`

The next theorem shows that **SOLVE2** is an optimal method, whenever this is allowed by the $(f, \mathbf{RHS})$ pair.

THEOREM 8.2. *$[P^a, \bar{u}_{P^a}] := \mathbf{SOLVE1}[f, \varepsilon, \bar{u}_{P_0}, \varepsilon_0]$ satisfies $|u - \bar{u}_{P^a}|_1 \leq \varepsilon$. Assuming $\varepsilon_0 \lesssim |u|_1$, if for some $s > 0$, $u \in \mathcal{A}^s$ and $(f, \mathbf{RHS})$ is s-optimal, then both $\#P^a$ and the number of arithmetic operations required by this call are $\lesssim \max\{1, \varepsilon^{-1/s}(c_f^{1/s} + |u|_{\mathcal{A}^s}^{1/s})\}$.*

Proof. At the beginning of a cycle of the outer while-loop, we have that $|u - \bar{u}_{P^a}|_1 \leq \bar{e}$, which for a cycle other than the first one is a consequence of Proposition 8.1. In particular, when the outer loop terminates, we have $|u - \bar{u}_{P^a}|_1 \leq \varepsilon$.

After the if-then-fi clause, we have that $|u - \bar{u}_{P^a}|_1 \leq \tilde{e}$, where $\tilde{e} = \bar{e}$ for the first iteration, and $\tilde{e} = (1+\gamma)\kappa_\Psi^{\frac{1}{2}}\bar{e}$ otherwise (apply Corollary 5.5 and (4.6)). After the call of **RHS**, by definition we have $\|f - f_{P^a}\|_{-1} \leq \bar{\delta}\tilde{e}$, and so as in (7.3), for $u_{P^a} := L_{P^a}^{-1} f_{P^a}$ we have $|u - u_{P^a}|_1 \leq 2\|f - f_{P^a}\|_{-1} + |u - \bar{u}_{P^a}|_1 \leq (2\bar{\delta} + 1)\tilde{e}$. After the call of **GALSOLVE**, by definition we have $|u_{P^a} - \bar{u}_{P^a}|_1 \leq \bar{\delta}\tilde{e}$, and so $|u - \bar{u}_{P^a}|_1 \leq (3\bar{\delta} + 1)\tilde{e}$. By applying (8.2), these estimates show that just before starting the inner while-loop, the new $\tilde{e}$ satisfies

$$\tilde{e} \leq C\bar{e}, \tag{8.5}$$

for some absolute constant $C > 0$.

Let us now consider any newly computed $\bar{u}_{P^a}$ in the inner while-loop, and let us denote by τ_1, τ_2 the tolerances that were used in the corresponding calls of **RHS** and **GALSOLVE** and let $\zeta_{P^a} = \tau_1 + C_4\tau_2$. We have that ζ_{P^a} is equal to $(1 + C_4)\bar{\delta}[C_1\mathcal{E}(P^a, f_{P^a}, \bar{u}_{P^a}) + \zeta_{P^a}]$ where in the latter expression P^a, f_{P^a}, $\bar{u}_{P^a}$ and ζ_{P^a} refer to the previous partition, right-hand side, approximate solution and ζ_{P^a}, respectively. Since by assumption $\bar{\delta}$ corresponds to a $\mu < 1$ as in Proposition 8.1, formula (8.4) shows that in each iteration of the inner loop $|u - \bar{u}_{P^a}|_1 + C_5\zeta_{P^a}$ is multiplied by a factor $\leq \mu$. Since by (8.2), $C_1\mathcal{E}(P^a, f_{P^a}, \bar{u}_{P^a}) + \zeta_{P^a} \eqsim |u - \bar{u}_{P^a}|_1 + C_5\zeta_{P^a}$, the geometric decrease of $|u - \bar{u}_{P^a}|_1 + C_5\zeta_{P^a}$ together with (8.5) shows that the inner while-loop terminates within an (absolute) constant number of iterations. After termination of the inner while-loop, the new $\bar{e}$ will be less than or equal to $\sigma < 1$ times the previous $\bar{e}$ showing that **SOLVE2** terminates, with, as we have seen, $|u - \bar{u}_{P^a}|_1 \leq \varepsilon$.

Let us consider any cycle of the outer loop with $\bar{e} > \varepsilon$ being the value at the beginning of this cycle. After the if-then-fi clause, we have that $\#P^a \lesssim \bar{e}^{-1/s}|u|_{\mathcal{A}^s}^{1/s}$, which for the first iteration follows from $\#P_0 \lesssim 1$ and $\varepsilon_0 \lesssim |u|_1$ by assumption, and which for any other cycle follows from Corollary 5.5 analogously as in the proof of Theorem 7.4. The properties of **RHS** and **REFINE** and the fact that the inner while-loop terminates within a fixed number of iterations show that at termination of this outer cycle, $\#P^a \lesssim \bar{e}^{-1/s}(c_f^{1/s} + |u|_{\mathcal{A}^s}^{1/s})$, which in particular proves the statement about $\#P^a$ at termination of **SOLVE2**. Furthermore, the cost of all calls of **RHS** and **REFINE** as well as the cost of **COARSE** in the possibly next cycle are $\lesssim \bar{e}^{-1/s}(c_f^{1/s} + |u|_{\mathcal{A}^s}^{1/s})$. Assuming, for the moment, that the cost of any application of **GALSOLVE** in **SOLVE2** on a partition P^a is $\lesssim \#P^a$, by the geometric decrease of the values of $\bar{e}$ at the beginning of the outer while-loop we conclude that the total cost of **SOLVE2** is $\lesssim \varepsilon^{-1/s}(c_f^{1/s} + |u|_{\mathcal{A}^s}^{1/s})$.

As we have seen, just before the evaluation of $\mathbf{GALSOLVE}[P^a, f_{P^a}, \bar{u}_{P^a}, \bar{\delta}\tilde{e}]$ outside the inner while-loop, we have $|u - \bar{u}_{P^a}|_1 \leq \tilde{e}$, and with $u_{P^a} := L_{P^a}^{-1} f_{P^a}$, $|u - u_{P^a}|_1 \leq (2\bar{\delta}+1)\tilde{e}$ and so $|u_{P^a} - \bar{u}_{P^a}|_1 \leq 2(\bar{\delta}+1)\tilde{e}$. We conclude that the cost of this call is $\lesssim \#P^a$.

Analogously, just before an evaluation of $\mathbf{GALSOLVE}[P^a, f_{P^a}, \bar{u}_{P^a}, \bar{\delta}\tilde{e}]$ inside the inner while-loop, we have $|u - \bar{u}_{P^a}|_1 \leq \tilde{e}$, $\|f - f_{P^a}\|_{-1} \leq \bar{\delta}\tilde{e}$, and so, as in (7.3), with $u_{P^a} := L_{P^a}^{-1} f_{P^a}$, $|u - u_{P^a}|_1 \leq 2\bar{\delta}\tilde{e} + \tilde{e}$ and so $|u_{P^a} - \bar{u}_{P^a}|_1 \leq 2(\bar{\delta}\tilde{e} + \tilde{e})$. We conclude that the cost of such a call is also $\lesssim \#P^a$. □

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