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Adaptive Frame Methods for Elliptic Operator Equations: The Steepest Descent Approach*

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This paper is concerned with the development of adaptive numerical methods for elliptic operator equations. We are particularly interested in discretization schemes based on wavelet frames. We show that by using three basic subroutines an implementable, convergent scheme can be derived, which, moreover, has optimal computational complexity. The scheme is based on adaptive steepest descent iterations. We illustrate our findings by numerical results for the computation of solutions of the Poisson equation with limited Sobolev smoothness on intervals in 1D and on *L*-shaped domains in 2D.

Keywords: Operator equations, multiscale methods, adaptive algorithms, sparse matrices, Banach frames, norm equivalences.

1. Introduction

In recent years, wavelets have been very successfully applied to several tasks. In signal/image analysis/ compression, wavelet schemes are by now already well–accepted and convincingly compete with other methods. Moreover, wavelets have also been used in numerical analysis, especially for the treatment of elliptic operator equations. Current interest focuses on the development of *adaptive* discretization

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schemes. Based on the equivalence of Sobolev norms and weighted sequence norms of wavelet expansion coefficients, convergent adaptive wavelet schemes were designed for symmetric elliptic problems (see Cohen et al. (2001); Dahlke, Dahmen, Hochmuth and Schneider (1997); Gantumur et al. (2005)) as well as for nonsymmetric and stationary nonlinear problems (see Cohen et al. (2002, 2003); Gantumur (2006)).

Although quite convincing from the theoretical point of view, so far the potential of adaptive wavelet schemes has not been fully exploited in practice for the following reason. Usually, the operator under consideration is defined on a bounded domain or on a closed manifold, so that a construction of a suitable wavelet basis on this domain is needed. There exist by now several constructions such as those by, e.g., Canuto et al. (1999); Cohen and Masson (2000); Dahmen and Schneider (1999a,b); Harbrecht and Stevenson (2006); Stevenson (2006). Unfortunately, none of these bases seem to be fully satisfactory in the sense that, besides their relevant virtues, they lack reasonable quantitative stability properties. Moreover, the constructions in the aforementioned references are all based on *non-overlapping* domain decomposition techniques, most of them requiring certain matching conditions on the parametric mappings, which can be difficult to satisfy in practical situations.

One possible way to circumvent this bottleneck is to use a slightly weaker concept, i.e., to work with (wavelet) frames. In general, a sequence $\mathscr{F} = \{f_n\}_{n\in\mathscr{N}}$ in a Hilbert space $\mathscr{H}$ is a *frame* for the Hilbert space $\mathscr{H}$ if

$$A_{\mathscr{F}}\|f\|_{\mathscr{H}}^2 \leqslant \sum_{n\in\mathscr{N}} \left|\langle f, f_n\rangle_{\mathscr{H}}\right|^2 \leqslant B_{\mathscr{F}}\|f\|_{\mathscr{H}}^2, \quad \text{for all } f \in \mathscr{H},$$

for suitable constants $0 < A_{\mathscr{F}} \leqslant B_{\mathscr{F}} < \infty$ (see Christensen (2003); Daubechies (1992) for further details). Every element of $\mathscr{H}$ has an expansion with respect to the frame elements, but this expansion is not necessarily unique. On the one hand, this redundancy may cause problems in numerical applications since it gives rise to a singular stiffness matrix. On the other hand, it has turned out that the construction of suitable frames on domains and manifolds is a much simpler task compared to that of constructing stable multiscale bases (see Dahlke et al. (2004); Stevenson (2003)). The idea is to write the domain or manifold as an *overlapping* union of subdomains, each of them being the smooth parametric image of a reference domain. By lifting a wavelet basis on the reference domain to the subdomains, and taking the union of these lifted bases, a frame is obtained. Due to their nature, we refer to such frames as *aggregated wavelet frames.* In recent studies, it has been shown that, despite the singular stiffness matrix, a damped Richardson iteration can be generalized to the case of frames in a very natural way (cf. Dahlke et al. (2004); Stevenson (2003)). Then, by using the basic building blocks of the adaptive wavelet algorithms in Cohen et al. (2002), an implementable and asymptotically optimally convergent version of this scheme can be constructed.

This paper follows similar lines and can be interpreted as the continuation of the studies in Dahlke et al. (2004) and Stevenson (2003). Instead of using the classical Richardson iteration, here we are interested in the steepest descent method. As we will show, with this method again an asymptotically optimally convergent scheme can be derived. Its main advantage is that it releases the user from the task of providing a value of the damping parameter, as in Richardson's method, which is close to the optimal value. This, however, requires an accurate estimate of the largest and smallest non-zero eigenvalues of the stiffness matrix; in the case of a frame the smallest non-zero eigenvalue is hard to compute. Although the steepest descent method requires more computational effort per iteration, in our numerical experiments it is as efficient as the Richardson iteration. Moreover, in case the damping parameter in the Richardson iteration is not chosen optimally, then the steepest descent method can even outperform the Richardson iteration, see Section 4.

The optimal complexity of the algorithm presented in this paper relies on a technical assumption on the boundedness of the orthogonal projector onto the range of the discretization matrix in suitable Lorenz spaces. Although it has been verified in a special case, the proof of the boundedness property in its full generality is still an open problem. Nevertheless, the numerical experiments presented in Section 4 strongly indicate that this assumption is really valid.

The steepest descent method for the adaptive solution of infinite–dimensional systems has also been studied in Canuto and Urban (2005), however, there the results are restricted to the case of bases, whereas we are concerned with frames here.

In this paper, we confine the discussion to adaptive schemes based on wavelet frames. Nevertheless, let us mention that the recent studies by Dahlke et al. (2006) and Dahlke et al. (2004), Section 7, show that the adaptive strategies can also be carried over to biinfinite matrix equations in a Jaffard algebra. In this setting, the boundedness of the projector stated above can be rigorously proved. Matrices in a Jaffard class naturally arise in Gabor analysis, sampling theory, and in the discretization of specific pseudo–differential operators.

This paper is organized as follows. In Section 2, we discuss the range of problems we shall be concerned with and summarize the basic concepts of frame discretizations. Then, in Section 3, we introduce the adaptive steepest descent method and establish its convergence and optimality. Finally, in Section 4, we present numerical experiments for the special case of the Poisson equation on an interval in 1D and on an *L*–shaped domain in 2D. The results fully confirm the expected convergence and optimality for both the Richardson and the steepest descent iterations. A comparison of the two schemes is also discussed.

2. Preliminaries

In this section, we briefly describe the range of problems we shall be concerned with. Moreover, we recall the basic concepts of frame discretization schemes for operator equations.

We consider linear operator equations

$$\mathscr{L}u = f, \tag{2.1}$$

where we will assume $\mathscr{L}$ to be a boundedly invertible operator from some Hilbert space H into its normed dual H', i.e.,

$$\|\mathscr{L}u\|_{H'} \eqsim \|u\|_H, \quad u \in H. \tag{2.2}$$

Here '$a \eqsim b$' means that both quantities can be uniformly bounded by constant multiples of each other. Likewise, '$\lesssim$' indicates inequalities up to constant factors. We write out such constants explicitly only when their values matter. Since $\mathscr{L}$ is assumed to be boundedly invertible, (2.1) has a unique solution u for any $f \in H'$. In the sequel, we shall focus on the important special case where

$$a(v,w) := \langle \mathscr{L}v, w\rangle \tag{2.3}$$

defines a *symmetric* bilinear form on H, $\langle\cdot,\cdot\rangle$ corresponding to the duality pairing of H' and H. We will always assume that $a(\cdot,\cdot)$ is *elliptic* in the sense that

$$a(v,v) \eqsim \|v\|_H^2, \tag{2.4}$$

which is easily seen to imply (2.2).

Typical examples are variational formulations of second-order elliptic boundary value problems on a domain $\Omega \subset \mathbb{R}^d$, such as the Poisson equation

$$\begin{aligned} -\triangle u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{2.5}$$

In this case, $H = H_0^1(\Omega)$, $H' = H^{-1}(\Omega)$, and the corresponding bilinear form is given by

$$a(v,w) = \int_\Omega \nabla v \cdot \nabla w \mathrm{d}x. \tag{2.6}$$

Thus typically H is a Sobolev space. Therefore, from now on, we will always assume that H and H', together with $L_2(\Omega)$, form a *Gelfand triple*, i.e.,

$$H \subset L_2(\Omega) \subset H' \tag{2.7}$$

with continuous and dense embeddings.

The design of adaptive wavelet or frame schemes in the aforementioned setting starts with a reformulation of (2.1) as an equivalent discrete problem on some sequence space $\ell_2(\mathscr{N})$. However, to perform this transformation, it will not be sufficient to work with a simple frame in L_2, since the operator $\mathscr{L}$ acts between Sobolev spaces. Similarly to the case of classical wavelets, we need specific norm equivalences of Sobolev norms and weighted sequence norms of frame coefficients. These can be realized by the so–called Gelfand frames as introduced in Dahlke et al. (2004). Given a frame $\mathscr{F}$ in $\mathscr{H}$, one usually defines the corresponding operators of analysis and synthesis to be

$$F : \mathscr{H} \to \ell_2(\mathscr{N}), \quad f \mapsto \big(\langle f, f_n\rangle_{\mathscr{H}}\big)_{n\in\mathscr{N}}, \tag{2.8}$$

$$F^* : \ell_2(\mathscr{N}) \to \mathscr{H}, \quad \mathbf{c} \mapsto \sum_{n\in\mathscr{N}} c_n f_n. \tag{2.9}$$

The composition $S := F^*F$ is a boundedly invertible (positive and self–adjoint) operator, called the *frame operator*, and $\tilde{\mathscr{F}} := S^{-1}\mathscr{F}$ is again a frame for $\mathscr{H}$, the *canonical dual frame*. Then, a frame $\mathscr{F}$ for $\mathscr{H}$ is called a *Gelfand frame* for the Gelfand triple $(\mathscr{B}, \mathscr{H}, \mathscr{B}')$, if $\mathscr{F} \subset \mathscr{B}$, $\tilde{\mathscr{F}} \subset \mathscr{B}'$ and there exists a Gelfand triple $\big(\mathscr{B}_d, \ell_2(\mathscr{N}), \mathscr{B}_d'\big)$ of sequence spaces such that

$$F^* : \mathscr{B}_d \to \mathscr{B},\ F^*\mathbf{c} = \sum_{n\in\mathscr{N}} c_n f_n \quad \text{and} \quad \tilde{F} : \mathscr{B} \to \mathscr{B}_d,\ \tilde{F}f = \big(\langle f, \tilde{f}_n\rangle_{\mathscr{B}\times\mathscr{B}'}\big)_{n\in\mathscr{N}} \tag{2.10}$$

are bounded operators.

REMARK 2.1 i) For the applications we have in mind, clearly the case $(\mathscr{B}, \mathscr{H}, \mathscr{B}') = (H, L_2(\Omega), H')$, where H denotes some Sobolev space, is the most important one. Then, similarly to the case of a classical wavelet basis, the spaces $\mathscr{B}_d$ and $\mathscr{B}_d'$ are weighted $\ell_2(\mathscr{N})$–spaces (see Dahlke et al. (2004) for details).

ii) It can be shown that Gelfand frames are also Banach frames for the spaces $\mathscr{B}$ and $\mathscr{B}'$ in the sense of Gröchenig (1991) (see again Dahlke et al. (2004) for details).

iii) A natural way to construct wavelet Gelfand frames on domains and manifolds is by means of overlapping partitions of parametric images of unit cubes (see Section 4 and Dahlke et al. (2004); Stevenson (2003) for details). We call such frames aggregated wavelet frames.

For the transformation of (2.1) into a discrete problem on $\ell_2(\mathscr{N})$, we have to assume that there exists an isomorphism $D_{\mathscr{B}} : \mathscr{B}_d \to \ell_2(\mathscr{N})$, so that its $\ell_2(\mathscr{N})$–adjoint $D_{\mathscr{B}}^* : \ell_2(\mathscr{N}) \to \mathscr{B}_d'$ is also an isomorphism. Then, the following lemma holds (see Dahlke et al. (2004); Stevenson (2003)).

LEMMA 2.1 Under the aforementioned assumptions on the frame, as well as (2.3), (2.4) on $\mathscr{L}$, the operator

$$\mathbf{G} := (D_{\mathscr{B}}^*)^{-1} F \mathscr{L} F^* D_{\mathscr{B}}^{-1} \tag{2.11}$$

is a bounded operator from $\ell_2(\mathscr{N})$ to $\ell_2(\mathscr{N})$. Moreover $\mathbf{G} = \mathbf{G}^*$, and it is boundedly invertible on its range $\mathrm{ran}(\mathbf{G}) = \mathrm{ran}((D_{\mathscr{B}}^*)^{-1} F)$.

With

$$\mathbf{f} := (D_{\mathscr{B}}^*)^{-1} F f, \tag{2.12}$$

we are therefore left with the task of solving the following problem:

$$\mathbf{G}\mathbf{u} = \mathbf{f}. \tag{2.13}$$

A natural way to solve (2.13) would be to use a damped Richardson iteration. Indeed, the following theorem can be shown (Dahlke et al. (2004); Stevenson (2003)).

THEOREM 2.1 Let $\mathscr{L}$ satisfy (2.3) and (2.4). Then, with $\mathbf{G}$ and $\mathbf{f}$ as in (2.11) and (2.12), respectively, the solution u of (2.1) can be computed as

$$u = F^* D_{\mathscr{B}}^{-1} \mathbf{u} \tag{2.14}$$

with $\mathbf{u}$ given by

$$\mathbf{u} = \left(\alpha \sum_{n=0}^{\infty} (\mathbf{I} - \alpha \mathbf{G})^{\mathbf{n}} \right) \mathbf{f}, \tag{2.15}$$

with $0 < \alpha < 2/\|\mathbf{G}\|_{\ell_2(\mathscr{N}) \to \ell_2(\mathscr{N})}$.

Observe that (2.15) is just an infinite damped Richardson iteration

$$\mathbf{u}^{(i+1)} = \mathbf{u}^{(i)} + \alpha(\mathbf{f} - \mathbf{G}\mathbf{u}^{(i)}), \quad i = 0, 1, \ldots, \tag{2.16}$$

starting with $\mathbf{u}^{(0)} = \mathbf{0}$. This scheme has been analyzed in Dahlke et al. (2004) and Stevenson (2003). In this paper, we use a different approach and work with a version of the *steepest descent scheme* which we describe in the following section.

3. The Steepest Descent Scheme

In this section, we introduce and analyze a steepest descent scheme for the solution of (2.13). In Subsection 3.1, we explain the basic setting, and we prove a perturbation theorem for this scheme. Then, in Subsection 3.2, we derive an implementable version and show its asymptotically optimal convergence.

3.1 *Basic Setting*

The first step is to introduce a natural energy (semi)–norm on $\ell_2(\mathscr{N})$. In the following, we write $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ for $\|\cdot\|_{\ell_2(\mathscr{N})}$, or $\|\cdot\|_{\ell_2(\mathscr{N}) \to \ell_2(\mathscr{N})}$, and $\langle \cdot, \cdot \rangle_{\ell_2(\mathscr{N})}$, respectively. We set $\langle\langle \cdot, \cdot \rangle\rangle := \langle \mathbf{G} \cdot, \cdot \rangle$ and the semi–norm $|||\cdot||| := \langle\langle \cdot, \cdot \rangle\rangle^{\frac{1}{2}}$. With $\mathbf{G}^\dagger$ being the Moore-Penrose pseudo inverse of $\mathbf{G}$, and

$$\mathbf{Q} : \ell_2(\mathscr{N}) \to \mathrm{ran}\, \mathbf{G}$$

being the orthogonal projector onto the range of $\mathbf{G}$, for any $\mathbf{v} \in \ell_2(\mathcal{N})$ we have

$$\|\mathbf{G}^\dagger\|^{-\frac{1}{2}}\|\mathbf{Qv}\| \leqslant \|\!|\mathbf{v}|\!\| \leqslant \|\mathbf{G}\|^{\frac{1}{2}}\|\mathbf{Qv}\|, \quad \|\mathbf{G}^\dagger\|^{-\frac{1}{2}}\|\!|\mathbf{v}|\!\| \leqslant \|\mathbf{Gv}\| \leqslant \|\mathbf{G}\|^{\frac{1}{2}}\|\!|\mathbf{v}|\!\|. \tag{3.1}$$

Then, the steepest descent scheme and its error-reduction in one iterative step read as follows.

PROPOSITION 3.1 Let $\mathbf{w}$ be an approximation for $\mathbf{u}$ with $\mathbf{r} := \mathbf{f} - \mathbf{Gw} \neq 0$. Then, with $\kappa(\mathbf{G}) := \|\mathbf{G}\|\|\mathbf{G}^\dagger\|$, for

$$\tilde{\mathbf{w}} := \mathbf{w} + \frac{\langle \mathbf{r}, \mathbf{r}\rangle}{\langle \mathbf{Gr}, \mathbf{r}\rangle}\mathbf{r} \tag{3.2}$$

we have

$$\|\!|\mathbf{u} - \tilde{\mathbf{w}}|\!\| \leqslant \frac{\kappa(\mathbf{G}) - 1}{\kappa(\mathbf{G}) + 1}\|\!|\mathbf{u} - \mathbf{w}|\!\|.$$

The proof is a standard argument on the convergence of iterative descent methods. In the following, we will often use $\mathbf{r}$ as a shorthand notation for the residual $\mathbf{f} - \mathbf{Gw}$.

It is clear that (3.2) cannot be implemented directly since infinite sequences and biinfinite matrices are involved. Therefore the challenging task is to transform (3.2) into an implementable version. This will be done in the next section. One has to replace the infinite sequences by finite ones without destroying the overall convergence of the scheme. The basic tool for this is the following perturbation result.

PROPOSITION 3.2 For any $\lambda \in (\frac{\kappa(\mathbf{G})-1}{\kappa(\mathbf{G})+1}, 1)$, there exists a $\delta = \delta(\lambda) > 0$ small enough, such that if $\|\tilde{\mathbf{r}} - \mathbf{r}\| \leqslant \delta\|\tilde{\mathbf{r}}\|$ and $\|\mathbf{z} - \mathbf{G}\tilde{\mathbf{r}}\| \leqslant \delta\|\tilde{\mathbf{r}}\|$, then with

$$\tilde{\mathbf{w}} := \mathbf{w} + \frac{\langle \tilde{\mathbf{r}}, \tilde{\mathbf{r}}\rangle}{\langle \mathbf{z}, \tilde{\mathbf{r}}\rangle}\tilde{\mathbf{r}},$$

we have

$$\|\!|\mathbf{u} - \tilde{\mathbf{w}}|\!\| \leqslant \lambda\|\!|\mathbf{u} - \mathbf{w}|\!\|,$$

and $|\frac{\langle \tilde{\mathbf{r}}, \tilde{\mathbf{r}}\rangle}{\langle \mathbf{z}, \tilde{\mathbf{r}}\rangle}| \lesssim 1$. If, for some $\eta > 0$, in addition $\|\tilde{\mathbf{r}} - \mathbf{r}\| \leqslant \eta$, then $\|(\mathbf{I} - \mathbf{Q})(\tilde{\mathbf{w}} - \mathbf{w})\| \leqslant C_3\eta$, with some absolute constant $C_3 > 0$.

Proof. Eq. (3.1) implies that $\langle \mathbf{Gr}, \mathbf{r}\rangle \eqsim \|\mathbf{r}\|^2$. The first step is to show that, for a sufficiently small $\bar{\delta}$ and any $0 < \delta \leqslant \bar{\delta}$,

$$\langle \mathbf{z}, \tilde{\mathbf{r}}\rangle \eqsim \|\mathbf{r}\|^2 \qquad \text{and} \qquad \|\tilde{\mathbf{r}}\| \eqsim \|\mathbf{r}\| \tag{3.3}$$

hold. We have

$$\langle \mathbf{z}, \tilde{\mathbf{r}}\rangle = \langle \mathbf{z} - \mathbf{G}\tilde{\mathbf{r}} + \mathbf{G}\tilde{\mathbf{r}}, \tilde{\mathbf{r}}\rangle \leqslant \|\mathbf{z} - \mathbf{G}\tilde{\mathbf{r}}\|\|\tilde{\mathbf{r}}\| + \|\mathbf{G}\tilde{\mathbf{r}}\|\|\tilde{\mathbf{r}}\| \leqslant (\delta + \|\mathbf{G}\|)\|\tilde{\mathbf{r}}\|^2$$

and

$$\|\tilde{\mathbf{r}}\|^2 \eqsim \langle \mathbf{G}\tilde{\mathbf{r}}, \tilde{\mathbf{r}}\rangle = \langle \mathbf{G}\tilde{\mathbf{r}} - \mathbf{z} + \mathbf{z}, \tilde{\mathbf{r}}\rangle \leqslant \|\mathbf{G}\tilde{\mathbf{r}} - \mathbf{z}\|\|\tilde{\mathbf{r}}\| + \langle \mathbf{z}, \tilde{\mathbf{r}}\rangle \leqslant \delta\|\tilde{\mathbf{r}}\|^2 + \langle \mathbf{z}, \tilde{\mathbf{r}}\rangle,$$

which implies the first equivalence in (3.3). The second one can be proved in a similar fashion. From (3.3), we infer that

$$\left|\frac{\langle \tilde{\mathbf{r}}, \tilde{\mathbf{r}}\rangle}{\langle \mathbf{z}, \tilde{\mathbf{r}}\rangle} - \frac{\langle \mathbf{r}, \mathbf{r}\rangle}{\langle \mathbf{Gr}, \mathbf{r}\rangle}\right| \lesssim \delta,$$

since

$$\begin{aligned}\frac{\langle\tilde{\mathbf{r}},\tilde{\mathbf{r}}\rangle}{\langle\mathbf{z},\tilde{\mathbf{r}}\rangle}-\frac{\langle\mathbf{r},\mathbf{r}\rangle}{\langle\mathbf{G}\mathbf{r},\mathbf{r}\rangle}&=\frac{\langle\tilde{\mathbf{r}},\tilde{\mathbf{r}}\rangle-\langle\mathbf{r},\mathbf{r}\rangle}{\langle\mathbf{z},\tilde{\mathbf{r}}\rangle}+\langle\mathbf{r},\mathbf{r}\rangle\Big[\frac{1}{\langle\mathbf{z},\tilde{\mathbf{r}}\rangle}-\frac{1}{\langle\mathbf{G}\mathbf{r},\mathbf{r}\rangle}\Big]\\&=\frac{\langle\tilde{\mathbf{r}},\tilde{\mathbf{r}}\rangle-\langle\mathbf{r},\mathbf{r}\rangle}{\langle\mathbf{z},\tilde{\mathbf{r}}\rangle}+\frac{\langle\mathbf{r},\mathbf{r}\rangle}{\langle\mathbf{z},\tilde{\mathbf{r}}\rangle\langle\mathbf{G}\mathbf{r},\mathbf{r}\rangle}[\langle\mathbf{G}\mathbf{r},\mathbf{r}\rangle-\langle\mathbf{z},\tilde{\mathbf{r}}\rangle]\\&=\frac{\langle 2(\tilde{\mathbf{r}}-\mathbf{r}),\mathbf{r}\rangle+\|\tilde{\mathbf{r}}-\mathbf{r}\|^2}{\langle\mathbf{z},\tilde{\mathbf{r}}\rangle}+\frac{\langle\mathbf{r},\mathbf{r}\rangle}{\langle\mathbf{z},\tilde{\mathbf{r}}\rangle\langle\mathbf{G}\mathbf{r},\mathbf{r}\rangle}[\langle\mathbf{G}\mathbf{r},\mathbf{r}-\tilde{\mathbf{r}}\rangle+\langle\mathbf{G}\tilde{\mathbf{r}}-\mathbf{z},\tilde{\mathbf{r}}\rangle+\langle\mathbf{G}(\mathbf{r}-\tilde{\mathbf{r}}),\tilde{\mathbf{r}}\rangle].\end{aligned}$$

Writing

$$\frac{\langle\tilde{\mathbf{r}},\tilde{\mathbf{r}}\rangle}{\langle\mathbf{z},\tilde{\mathbf{r}}\rangle}\tilde{\mathbf{r}}-\frac{\langle\mathbf{r},\mathbf{r}\rangle}{\langle\mathbf{G}\mathbf{r},\mathbf{r}\rangle}\mathbf{r}=\Big[\frac{\langle\tilde{\mathbf{r}},\tilde{\mathbf{r}}\rangle}{\langle\mathbf{z},\tilde{\mathbf{r}}\rangle}-\frac{\langle\mathbf{r},\mathbf{r}\rangle}{\langle\mathbf{G}\mathbf{r},\mathbf{r}\rangle}\Big]\mathbf{r}+\frac{\langle\tilde{\mathbf{r}},\tilde{\mathbf{r}}\rangle}{\langle\mathbf{z},\tilde{\mathbf{r}}\rangle}[\tilde{\mathbf{r}}-\mathbf{r}], \tag{3.4}$$

we find that $\|\frac{\langle\tilde{\mathbf{r}},\tilde{\mathbf{r}}\rangle}{\langle\mathbf{z},\tilde{\mathbf{r}}\rangle}\tilde{\mathbf{r}}-\frac{\langle\mathbf{r},\mathbf{r}\rangle}{\langle\mathbf{G}\mathbf{r},\mathbf{r}\rangle}\mathbf{r}\|\lesssim\delta\|\mathbf{r}\|\lesssim\delta|\!|\!|\mathbf{u}-\mathbf{w}|\!|\!|$, which, together with Proposition 3.1, completes the proof of the first statement.

From (3.4) and $(\mathbf{I}-\mathbf{Q})\mathbf{r}=0$, we have $(\mathbf{I}-\mathbf{Q})(\tilde{\mathbf{w}}-\mathbf{w})=\frac{\langle\tilde{\mathbf{r}},\tilde{\mathbf{r}}\rangle}{\langle\mathbf{z},\tilde{\mathbf{r}}\rangle}(\mathbf{I}-\mathbf{Q})(\tilde{\mathbf{r}}-\mathbf{r})$, which by $|\frac{\langle\tilde{\mathbf{r}},\tilde{\mathbf{r}}\rangle}{\langle\mathbf{z},\tilde{\mathbf{r}}\rangle}|\lesssim 1$ and $\|\mathbf{I}-\mathbf{Q}\|\leqslant 1$ completes the proof of the second statement. □

3.2 *Numerical Realization*

Obviously, the steepest descent scheme in Proposition 3.1 cannot be implemented since neither infinite sequences nor biinfinite matrices can be handled computationally. Therefore our aim is to replace the scheme (3.2) by an implementable one. The guideline given by Proposition 3.2 is to approximate the infinite expressions by finite ones within a certain, sufficiently small, relative tolerance.

In the sequel, we shall make the following basic assumptions. Let Σ_N denote the (nonlinear) subspace of $\ell_2(\mathscr{N})$ consisting of all vectors with at most N nonzero coordinates. Given $\mathbf{v}\in\ell_2(\mathscr{N})$, we introduce the approximation error

$$\sigma_N(\mathbf{v}):=\inf_{\mathbf{w}\in\Sigma_N}\|\mathbf{v}-\mathbf{w}\|. \tag{3.5}$$

Clearly this infimum is attained for $\mathbf{w}$ being a best N-term approximation for $\mathbf{v}$, i.e., a vector from Σ_N that agrees with $\mathbf{v}$ in those coordinates on which $\mathbf{v}$ takes its N largest values in modulus. Such a best N-term approximation for $\mathbf{v}$ will be denoted as $\mathbf{v}_N$. Note that it is not necessarily unique.

For some $s>0$, we assume that

$$\sup_{N\in\mathbb{N}}N^s\sigma_N(\mathbf{u})<\infty. \tag{3.6}$$

Eq. (3.6) describes how well the solution $\mathbf{u}$ to (2.13) can be approximated by the elements of Σ_N. Essentially, (3.6) is a regularity assumption on the exact solution u to (2.1). Indeed, in the case of a wavelet basis, it is well–known that the convergence order of the best N–term approximation is determined by the maximum of the polynomial order and a specific Besov regularity of the object that we want to approximate (cf. DeVore (1998)). For aggregated wavelet frames the same holds true (see Stevenson (2003)). Specifically, when H is a Sobolev space of order t over an n-dimensional domain, and the aggregated wavelet frame has order d, then $s=\frac{d-t}{n}$ if not limited by the Besov regularity. Fortunately, recent studies indicate that for the solution of elliptic operator equations this Besov regularity index is quite large (see, e.g., Dahlke (1999); Dahlke, Dahmen and DeVore (1997); Dahlke and DeVore (1997)), and, moreover, that in many cases it is much larger than the Sobolev regularity index that governs the convergence rate of non-adaptive schemes.

The concept of best N–term approximation is closely related to the *weak ℓ_τ–spaces* $\ell_\tau^w(\mathcal{N})$. Given some $0<\tau<2$, $\ell_\tau^w(\mathcal{N})$ is defined as

$$\ell_\tau^w(\mathcal{N}) := \{\mathbf{c}\in \ell_2(\mathcal{N}) :\ |\mathbf{c}|_{\ell_\tau^w} := \sup_{n\in\mathbb{N}} n^{1/\tau}|\gamma_n(\mathbf{c})| < \infty\}, \tag{3.7}$$

where $\gamma_n(\mathbf{c})$ is the nth largest coefficient in modulus of $\mathbf{c}$. Then, for each $s>0$,

$$\sup_N N^s \sigma_N(\mathbf{v}) \eqsim |\mathbf{v}|_{\ell_\tau^w}, \tag{3.8}$$

where here, and for the remainder of this paper, s and τ are always related according to

$$\tau = \left(\frac{1}{2}+s\right)^{-1}.$$

The expression $|\mathbf{v}|_{\ell_\tau^w}$ defines only a quasi-norm since it does not necessarily satisfy the triangle inequality. Yet, for each τ in the range $0<\tau<2$, there exists a $C_1(\tau)>0$ with

$$|\mathbf{v}+\mathbf{w}|_{\ell_\tau^w} \leqslant C_1(\tau)\left(|\mathbf{v}|_{\ell_\tau^w}+|\mathbf{w}|_{\ell_\tau^w}\right) \qquad (\mathbf{v},\mathbf{w}\in \ell_\tau^w(\mathcal{N})). \tag{3.9}$$

We refer to Cohen et al. (2001) and DeVore (1998) for further details on the quasi–Banach spaces $\ell_\tau^w(\mathcal{N})$.

For some s^* larger than any s for which (3.6) can be expected (i.e., $s>\frac{d-t}{n}$), we assume the existence of the following three subroutines:

- **APPLY**$[\mathbf{w},\varepsilon]\to\mathbf{z}_\varepsilon$. Determines, for $\varepsilon>0$ and a finitely supported $\mathbf{w}$, a finitely supported $\mathbf{z}_\varepsilon$ with

  $$\|\mathbf{G}\mathbf{w}-\mathbf{z}_\varepsilon\| \leqslant \varepsilon. \tag{3.10}$$

  Moreover, for any $s<s^*$, $\#\operatorname{supp}\mathbf{z}_\varepsilon \lesssim \varepsilon^{-1/s}|\mathbf{w}|_{\ell_\tau^w}^{1/s}$, where the number of arithmetic operations and storage locations used by this call is bounded by some absolute multiple of $\varepsilon^{-1/s}|\mathbf{w}|_{\ell_\tau^w}^{1/s} + \#\operatorname{supp}\mathbf{w}+1$.

- **RHS**$[\varepsilon]\to\mathbf{f}_\varepsilon$. Determines, for $\varepsilon>0$, a finitely supported $\mathbf{f}_\varepsilon$ with $\|\mathbf{f}-\mathbf{f}_\varepsilon\|\leqslant\varepsilon$. Moreover, for any $s<s^*$, if $\mathbf{u}\in\ell_\tau^w(\mathcal{N})$, then $\#\operatorname{supp}\mathbf{f}_\varepsilon \lesssim \varepsilon^{-1/s}|\mathbf{u}|_{\ell_\tau^w}^{1/s}$, where the number of arithmetic operations and storage locations used by the call is bounded by some absolute multiple of $\varepsilon^{-1/s}|\mathbf{u}|_{\ell_\tau^w}^{1/s}+1$.

- **COARSE**$[\mathbf{w},\varepsilon]\to\mathbf{w}_\varepsilon$. Determines, for a finitely supported $\mathbf{w}$, a finitely supported $\mathbf{w}_\varepsilon$, such that

  $$\|\mathbf{w}-\mathbf{w}_\varepsilon\| \leqslant \varepsilon. \tag{3.11}$$

  Moreover, $\#\operatorname{supp}\mathbf{w}_\varepsilon \lesssim \inf\{N : \sigma_N(\mathbf{w})\leqslant\varepsilon\}$, and **COARSE** can be arranged to take a number of arithmetic operations and storage locations that is bounded by an absolute multiple of $\#\operatorname{supp}\mathbf{w} + \max\{\log(\varepsilon^{-1}\|\mathbf{w}\|),1\}$.

Using that $\mathbf{G}:\ell_2(\mathcal{N})\to\ell_2(\mathcal{N})$ is bounded, the properties of **APPLY** and **RHS** imply the following:

PROPOSITION 3.3 For any $s\in(0,s^*)$, $\mathbf{G}:\ell_\tau^w(\mathcal{N})\to\ell_\tau^w(\mathcal{N})$ is bounded. For $\mathbf{z}_\varepsilon := \mathbf{APPLY}[\mathbf{w},\varepsilon]$ and $\mathbf{f}_\varepsilon := \mathbf{RHS}[\varepsilon]$, we have $|\mathbf{z}_\varepsilon|_{\ell_\tau^w}\lesssim|\mathbf{w}|_{\ell_\tau^w}$ and $|\mathbf{f}_\varepsilon|_{\ell_\tau^w}\lesssim|\mathbf{u}|_{\ell_\tau^w}$, uniformly over $\varepsilon>0$ and all finitely supported $\mathbf{w}$.

Proof. Since the proof in Stevenson (2003) is incomplete, we include a proof here. We first show that for $s \in (0, s^*)$, $\mathbf{G} : \ell_\tau^w(\mathcal{N}) \to \ell_\tau^w(\mathcal{N})$ is bounded. Let $C > 0$ be a constant such that for $\mathbf{z}_\varepsilon := \mathbf{APPLY}[\mathbf{w}, \varepsilon]$, $\operatorname{supp} \mathbf{z}_\varepsilon \leqslant C\varepsilon^{-1/s}|\mathbf{w}|_{\ell_\tau^w}^{1/s}$. Let $\mathbf{v} \in \ell_\tau^w(\mathcal{N})$ and $N \in \mathbb{N}$ be given. For $\bar{\varepsilon} := C^s|\mathbf{v}_N|_{\ell_\tau^w} N^{-s}$, let $\mathbf{z}_{\bar{\varepsilon}} := \mathbf{APPLY}[\mathbf{v}_N, \bar{\varepsilon}]$. Then, by (3.8),

$$\begin{aligned}\|\mathbf{G}\mathbf{v} - \mathbf{z}_{\bar{\varepsilon}}\| &\leqslant \|\mathbf{G}\mathbf{v}_N - \mathbf{z}_{\bar{\varepsilon}}\| + \|\mathbf{G}\|\|\mathbf{v} - \mathbf{v}_N\| \\ &\lesssim C^s|\mathbf{v}_N|_{\ell_\tau^w} N^{-s} + \|\mathbf{G}\| N^{-s}|\mathbf{v}|_{\ell_\tau^w} \lesssim N^{-s}|\mathbf{v}|_{\ell_\tau^w}.\end{aligned}$$

Since $\#\operatorname{supp} \mathbf{z}_\varepsilon \leqslant N$, from (3.8) again we infer that $|\mathbf{G}\mathbf{v}|_{\ell_\tau^w} \lesssim |\mathbf{v}|_{\ell_\tau^w}$.

By using that for any $\mathbf{v} \in \ell_\tau^w(\mathcal{N})$, and finitely supported $\mathbf{z}$, we have

$$|\mathbf{z}|_{\ell_\tau^w} \lesssim |\mathbf{v}|_{\ell_\tau^w} + (\#\operatorname{supp} \mathbf{z})^s \|\mathbf{v} - \mathbf{z}\| \tag{3.12}$$

(Cohen et al.; 2001, Lemma 4.11), for finitely supported $\mathbf{w}$, $\varepsilon > 0$, and with $\mathbf{z}_\varepsilon := \mathbf{APPLY}[\mathbf{w}, \varepsilon]$, we have $|\mathbf{z}_\varepsilon|_{\ell_\tau^w} \lesssim |\mathbf{G}\mathbf{w}|_{\ell_\tau^w} + (\#\operatorname{supp} \mathbf{z}_\varepsilon)^s \varepsilon \leqslant |\mathbf{G}\mathbf{w}|_{\ell_\tau^w} + C^s|\mathbf{w}|_{\ell_\tau^w} \lesssim |\mathbf{w}|_{\ell_\tau^w}$. Similarly, for $\mathbf{f}_\varepsilon := \mathbf{RHS}[\varepsilon]$, we have $|\mathbf{f}_\varepsilon|_{\ell_\tau^w} \lesssim |\mathbf{G}\mathbf{u}|_{\ell_\tau^w} + (\#\operatorname{supp} \mathbf{f}_\varepsilon)^s \varepsilon \leqslant |\mathbf{u}|_{\ell_\tau^w}$. ☐

Thanks to the properties of **COARSE** we have the following result.

PROPOSITION 3.4 Let $\mu > 1$ and $s > 0$. Then, for any $\varepsilon > 0$, $\mathbf{v} \in \ell_\tau^w(\mathcal{N})$, and finitely supported $\mathbf{w}$ with

$$\|\mathbf{v} - \mathbf{w}\| \leqslant \varepsilon,$$

for $\overline{\mathbf{w}} := \mathbf{COARSE}[\mu\varepsilon, \mathbf{w}]$ we have that

$$\#\operatorname{supp} \overline{\mathbf{w}} \lesssim \varepsilon^{-1/s}|\mathbf{v}|_{\ell_\tau^w}^{1/s}.$$

Obviously $\|\mathbf{v} - \overline{\mathbf{w}}\| \leqslant (1+\mu)\varepsilon$, and

$$|\overline{\mathbf{w}}|_{\ell_\tau^w} \lesssim |\mathbf{v}|_{\ell_\tau^w}.$$

Proof. Let N be the smallest integer such that $\|\mathbf{v}_N - \mathbf{v}\| \leqslant (\mu - 1)\varepsilon$ for a best N–term approximation $\mathbf{v}_N$ of v. Then $\#\operatorname{supp} \mathbf{v}_N \lesssim \varepsilon^{-1/s}|\mathbf{v}|_{\ell_\tau^w}^{1/s}$. Furthermore $\|\mathbf{v}_N - \mathbf{w}\| \leqslant \|\mathbf{v}_N - \mathbf{v}\| + \|\mathbf{v} - \mathbf{w}\| \leqslant (\mu - 1 + 1)\varepsilon = \mu\varepsilon$, and so $\#\operatorname{supp} \overline{\mathbf{w}} \lesssim \#\operatorname{supp} \mathbf{v}_N$. The last statement follows from an application of (3.12). ☐

Let us briefly discuss the assumptions we made on **APPLY**, **RHS** and **COARSE**. The approximate matrix-vector product **APPLY** can be implemented in the way introduced in (Cohen et al.; 2001, §6.4). Then the question whether **APPLY** has the assumed properties reduces to the question as to how well **G** can be approximated by sparse matrices constructed by dropping small entries. This can be quantified by the concept of s^*-compressibility, meaning that if **G** is s^*-compressible, then **APPLY** has the assumed properties with that value of s^*. For the case of a basis, for both differential operators and singular integral operators, and for sufficiently smooth wavelets with sufficiently many vanishing moments in relation to their approximation order, it was shown in Stevenson (2004) that **G** is s^*-compressible with s^* larger than any $s = 1/\tau - 1/2$ for which $\mathbf{u} \in \ell_\tau^w(\mathcal{N})$ can be expected. This result extends to aggregated wavelet frames (see Stevenson (2003) for details).

Above, we implicitly assumed that the remaining entries from the sparse approximations for **G** are exactly available. Generally, however, these entries have to be approximated by numerical quadrature. For the case of a basis, in Gantumur and Stevenson (2006a,b) it was verified that these remaining entries can be approximated within a sufficiently small tolerance by quadrature rules that, on average over each row and column, take only $\mathcal{O}(1)$ operations per entry, showing that the "fully discrete" version of **APPLY** has the required properties. The development of suitable numerical quadrature is more complicated in the case of an aggregated wavelet frame, since in overlapping regions, pairs of frame elements can be piecewise smooth with respect to uncorrelated partitions. Despite of this, in a forthcoming paper we will show that relatively easy implementable quadrature schemes exist that realize the above $\mathscr{O}(1)$ condition in the case of an aggregated wavelet frame as well. In the nice setting of the numerical examples in this paper, all entries of $\mathbf{G}$ are exactly available at unit cost, so the question of numerical quadrature does not play a role.

Concerning **RHS**, for some $s < s^*$, let $\mathbf{u} \in \ell_\tau^w(\mathscr{N})$. Then Proposition 3.3 shows that $\mathbf{f} = \mathbf{G}\mathbf{u} \in \ell_\tau^w(\mathscr{N})$ with $|\mathbf{f}|_{\ell_\tau^w} \lesssim |\mathbf{u}|_{\ell_\tau^w}$. So (3.8) shows that for any $\varepsilon > 0$, there exists an $\mathbf{f}_\varepsilon$ with $\|\mathbf{f} - \mathbf{f}_\varepsilon\| \leqslant \varepsilon$ and $\#\mathrm{supp}\,\mathbf{f}_\varepsilon \lesssim \varepsilon^{-1/s}|\mathbf{u}|_{\ell_\tau^w}^{1/s}$. The question how to construct such an $\mathbf{f}_\varepsilon$ in $\mathscr{O}(\varepsilon^{-1/s}|\mathbf{u}|_{\ell_\tau^w}^{1/s} + 1)$ operations cannot be answered in general, as it depends on the right-hand side at hand.

Finally, the routine **COARSE** with the aforementioned properties can be based on binary binning (see Barinka (2005); Stevenson (2003) for details).

We are going to solve $\mathbf{G}\mathbf{u} = \mathbf{f}$ with an approximate steepest descent method. Unless $\mathscr{F}$ is a basis, $\mathbf{G}$ has a non-trivial kernel, meaning that, as with any iterative method, a component of the error in a current approximation $\mathbf{w}$ that is in $\ker(\mathbf{G})$ will never be reduced in subsequent iterations. Although such components do not influence the resulting approximation $w := F^* D_{\mathscr{B}}^{-1}\mathbf{w}$ because $\ker(F^* D_{\mathscr{B}}^{-1}) = \ker(\mathbf{G})$, in principle they may cause an unbounded increase of $|\mathbf{w}|_{\ell_\tau^w}$ as the iteration proceeds, making the cost of calls of **APPLY** possibly uncontrollable. Under the assumption given below, we will nevertheless be able to control this cost, which allows us to show optimality of the method.

Assumption 3.5 For any $s \in (0, s^*)$, $\mathbf{Q}$ is bounded on $\ell_\tau^w(\mathscr{N})$.

This assumption has been verified for the special case that $H = L_2(\Omega)$, the wavelet bases making up the aggregated frame are $L_2(\Omega)$-orthonormal, and some damping is applied to the wavelets near the interior boundaries, see Stevenson (2003) for details.

The general proof of the boundedness assumption 3.5 on $\mathbf{Q}$ is a difficult open problem. Its validity can be indirectly verified by numerical experiments, like those in Section 4. According to (Stevenson; 2003, Remark 3.13), the boundedness of $\mathbf{Q}$ on $\ell_\tau^w(\mathscr{N})$ for all $s \in (0, s^*)$ is (almost) a necessary requirement for the scheme to behave optimally.
Moreover, not restricting our analysis to wavelet frames and to differential equations, there exist other frames, for example time–frequency localized Gabor frames (and more generally all intrinsically polynomially localized frames, cf. Dahlke et al. (2004); Fornasier and Gröchenig (2005)), for which the boundedness of the corresponding $\mathbf{Q}$ has been proven rigorously (see (Dahlke et al.; 2004, Theorem 7.1 in Section 7)). Therefore, for specific operator equations, optimality of the adaptive algorithm introduced below based on, e.g., Gabor frame discretizations, is justified theoretically.

REMARK 3.1 For cases in which Assumption 3.5 might not be valid, one can apply a modified algorithm that contains a recurrent inexact application of a projector to reduce components in $\ker(\mathbf{G})$, similar to the algorithm **modSOLVE** in Stevenson (2003) based on Richardson iteration. Although for this algorithm optimal computational complexity can also be shown, even with a simpler proof, we focus on the algorithm without this projector, since we expect it to have better quantitative properties.

Now we are in the position to formulate our inexact steepest descent scheme. The first step is to establish a routine that computes an approximate residual of the current approximation $\mathbf{w}$ for $\mathbf{u}$ within a sufficiently small tolerance ζ such that either, in view of Proposition 3.2, the relative error in this approximate residual is below some prescribed tolerance δ, or the residual itself, being a measure of the error in $\mathbf{w}$, is below some other prescribed tolerance ε. In view of controlling the components of the approximations in $\ker(\mathbf{G})$, the tolerance ζ should be in any way below some third input parameter ξ.

$\mathbf{RES}[\mathbf{w},\xi,\delta,\varepsilon] \to [\tilde{\mathbf{r}},\nu]$:

$\zeta := 2\xi$
do $\zeta := \zeta/2$
  $\tilde{\mathbf{r}} := \mathbf{RHS}[\zeta/2] - \mathbf{APPLY}[\mathbf{w},\zeta/2]$
until $\nu := \|\tilde{\mathbf{r}}\| + \zeta \leqslant \varepsilon$ or $\zeta \leqslant \delta\|\tilde{\mathbf{r}}\|$

THEOREM 3.6 The routine **RES** has the following properties.

i) $[\tilde{\mathbf{r}},\nu] = \mathbf{RES}[\mathbf{w},\xi,\delta,\varepsilon]$ terminates with $\nu \geqslant \|\mathbf{r}\|$, $\nu \gtrsim \min\{\xi,\varepsilon\}$ and $\|\mathbf{r}-\tilde{\mathbf{r}}\| \leqslant \xi$.

ii) If, for $s \leqslant \breve{s} < s^*$, with, as always, $\tau = (\frac{1}{2}+s)^{-1}$ and $\breve{\tau} = (\frac{1}{2}+\breve{s})^{-1}$, $\mathbf{u} \in \ell_\tau^w(\mathcal{N})$, then

$$\begin{aligned} \#\operatorname{supp}\tilde{\mathbf{r}} &\lesssim \min\{\xi,\nu\}^{-1/s}|\mathbf{u}|_{\ell_\tau^w}^{1/s} + \min\{\xi,\nu\}^{-1/\breve{s}}|\mathbf{w}|_{\ell_{\breve{\tau}}^w}^{1/\breve{s}}, &(3.13)\\ \min\{\xi,\nu\}^{(\breve{s}/s)-1}|\tilde{\mathbf{r}}|_{\ell_{\breve{\tau}}^w} &\lesssim |\mathbf{u}|_{\ell_\tau^w}^{\breve{s}/s} + \min\{\xi,\nu\}^{(\breve{s}/s)-1}|\mathbf{w}|_{\ell_{\breve{\tau}}^w}, &(3.14)\end{aligned}$$

and the number of arithmetic operations and storage locations required by the call is bounded by some absolute multiple of

$$\min\{\xi,\nu\}^{-1/s}|\mathbf{u}|_{\ell_\tau^w}^{1/s} + \min\{\xi,\nu\}^{-1/\breve{s}}[|\mathbf{w}|_{\ell_{\breve{\tau}}^w}^{1/\breve{s}} + \xi^{1/\breve{s}}(\#\operatorname{supp}\mathbf{w}+1)].$$

iii) In addition, if **RES** terminates with $\nu > \varepsilon$, then $\|\mathbf{r}-\tilde{\mathbf{r}}\| \leqslant \delta\|\tilde{\mathbf{r}}\|$, $\nu \leqslant (1+\delta)\|\tilde{\mathbf{r}}\|$, and $\nu \leqslant \frac{1+\delta}{1-\delta}\|\mathbf{r}\|$.

Proof. Let us start by proving i). If at evaluation of the until-case, $\zeta > \delta\|\tilde{\mathbf{r}}\|$, then $\|\tilde{\mathbf{r}}\| + \zeta < (\delta^{-1}+1)\zeta$. Since ζ is halved in each iteration, we infer that, if not by $\zeta \leqslant \delta\|\tilde{\mathbf{r}}\|$, **RES** will terminate by $\|\tilde{\mathbf{r}}\| + \zeta \leqslant \varepsilon$.

Since after any evaluation of $\tilde{\mathbf{r}}$ inside the algorithm, $\|\tilde{\mathbf{r}}-\mathbf{r}\| \leqslant \zeta$, any value of ν determined inside the algorithm is an upper bound on $\|\mathbf{r}\|$.

If the do–loop terminates in the first iteration, then $\nu \geqslant \xi$. In the other case, let $\tilde{\mathbf{r}}^{\text{old}} := \mathbf{RHS}[\zeta] - \mathbf{APPLY}[\mathbf{w},\zeta]$. We have $\|\tilde{\mathbf{r}}^{\text{old}}\| + 2\zeta > \varepsilon$ and $2\zeta > \delta\|\tilde{\mathbf{r}}^{\text{old}}\|$, so that

$$\nu \geqslant \zeta > (2\delta^{-1}+2)^{-1}(\|\tilde{\mathbf{r}}^{\text{old}}\| + 2\zeta) > \frac{\delta\varepsilon}{2+2\delta},$$

and i) is shown.

The next step is to establish part ii). For any finitely supported $\mathbf{v}$, we have

$$|\mathbf{v}|_{\ell_{\breve{\tau}}^w} \leqslant (\#\operatorname{supp}\mathbf{v})^{\breve{s}-s}|\mathbf{v}|_{\ell_\tau^w}. \tag{3.15}$$

So for $\mathbf{g} := \mathbf{RHS}[\zeta]$, from $\#\operatorname{supp}\mathbf{g} \lesssim \zeta^{-1/s}|\mathbf{u}|_{\ell_\tau^w}^{1/s}$ and $|\mathbf{g}|_{\ell_\tau^w} \lesssim |\mathbf{u}|_{\ell_\tau^w}$, we have $\zeta^{(\breve{s}/s)-1}|\mathbf{g}|_{\ell_{\breve{\tau}}^w} \lesssim |\mathbf{u}|_{\ell_\tau^w}^{\breve{s}/s}$. With ζ, $\tilde{\mathbf{r}}$, and ν having their values at termination, the properties of **APPLY**, cf. Proposition 3.3, now show that

$$\#\operatorname{supp}\tilde{\mathbf{r}} \lesssim \zeta^{-1/s}|\mathbf{u}|_{\ell_\tau^w}^{1/s} + \zeta^{-1/\breve{s}}|\mathbf{w}|_{\ell_{\breve{\tau}}^w}^{1/\breve{s}},$$

and

$$\zeta^{(\breve{s}/s)-1}|\tilde{\mathbf{r}}|_{\ell_{\breve{\tau}}^w} \lesssim |\mathbf{u}|_{\ell_\tau^w}^{\breve{s}/s} + \zeta^{(\breve{s}/s)-1}|\mathbf{w}|_{\ell_{\breve{\tau}}^w}.$$

Therefore, (3.13) and (3.14) follow from these expressions once we have shown that $\zeta \gtrsim \min\{\xi, \nu\}$. When the do–loop terminates in the first iteration, we have $\zeta \gtrsim \xi$. In the other case, with $\tilde{\mathbf{r}}^{\text{old}}$ as above, we have $\delta\|\tilde{\mathbf{r}}^{\text{old}}\| < 2\zeta$, and so from $\|\tilde{\mathbf{r}} - \tilde{\mathbf{r}}^{\text{old}}\| \leqslant \zeta + 2\zeta$, we infer $\|\tilde{\mathbf{r}}\| \leqslant \|\tilde{\mathbf{r}}^{\text{old}}\| + 3\zeta < (2\delta^{-1}+3)\zeta$, so that $\nu < (2\delta^{-1}+4)\zeta$.

To complete the proof of ii), it remains to estimate the number or arithmetic operations. Again the properties of **APPLY** and that of **RHS** together with the geometric decrease of ζ inside the algorithm, imply that the total cost can be bounded by some multiple of $\zeta^{-1/s}|\mathbf{u}|_{\ell^w_\tau}^{1/s} + \zeta^{-1/\breve{s}}|\mathbf{w}|_{\ell^w_{\breve{\tau}}}^{1/\breve{s}} + K(\#\operatorname{supp}\mathbf{w}+1)$, with K being the number of calls of **APPLY** that were made. Taking into account its initial value, and the geometric decrease of ζ inside the algorithm, we have $K(\#\operatorname{supp}\mathbf{w}+1) = K\xi^{-1/\breve{s}}\xi^{1/\breve{s}}(\#\operatorname{supp}\mathbf{w}+1) \lesssim \zeta^{-1/\breve{s}}\xi^{1/\breve{s}}(\#\operatorname{supp}\mathbf{w}+1)$. Since we have already shown that $\zeta \gtrsim \min\{\xi, \nu\}$, this completes the proof of ii).

Finally, let us check iii). Suppose that **RES** terminates with $\nu > \varepsilon$, and thus with $\zeta \leqslant \delta\|\tilde{\mathbf{r}}\|$. Then obviously $\|\mathbf{r} - \tilde{\mathbf{r}}\| \leqslant \delta\|\tilde{\mathbf{r}}\|$.

From $\|\tilde{\mathbf{r}}\| \leqslant \|\mathbf{r} - \tilde{\mathbf{r}}\| + \|\mathbf{r}\| \leqslant \delta\|\tilde{\mathbf{r}}\| + \|\mathbf{r}\|$, we have $\|\tilde{\mathbf{r}}\| \leqslant \frac{\|\mathbf{r}\|}{1-\delta}$, and so we arrive at $\nu = \|\tilde{\mathbf{r}}\| + \zeta \leqslant (1+\delta)\|\tilde{\mathbf{r}}\| \leqslant \frac{1+\delta}{1-\delta}\|\mathbf{r}\|$. □

The routine **RES** is the basic building block for our fundamental algorithm which reads as follows.

Algorithm 1 $\mathbf{SOLVE}[\omega, \varepsilon] \to \mathbf{w}$*:*
% Input should satisfy $\omega \geqslant \|\mathbf{Q}\mathbf{u}\|$.
% Let λ *and* $\delta = \delta(\lambda)$ *be constants as in Proposition 3.2.*
% Fix some constants $\mu > 1$, $\beta \in (0,1)$.
% Let K, M be the smallest integers with $\beta^K\omega \leqslant \varepsilon$, $\lambda^M \leqslant \frac{1-\delta}{1+\delta}\frac{\beta}{(1+3\mu)\kappa(\mathbf{G})}$, *respectively.*

$\mathbf{w}_0 := 0$; $\omega_0 := \omega$
`for` $i := 1$ `to` K `do`
  $\bar{\mathbf{w}}_i := \mathbf{w}_{i-1}$; $\omega_i := \beta\omega_{i-1}$; $\xi_i := \frac{\omega_i}{(1+3\mu)C_3M}$ *% C_3 from Proposition 3.2*
  `while` *with* $[\tilde{\mathbf{r}}_i, \nu_i] := \mathbf{RES}[\bar{\mathbf{w}}_i, \xi_i, \delta, \frac{\omega_i}{(1+3\mu)\|\mathbf{G}^\dagger\|}]$, $\nu_i > \frac{\omega_i}{(1+3\mu)\|\mathbf{G}^\dagger\|}$ `do`
    $\mathbf{z}_i := \mathbf{APPLY}[\tilde{\mathbf{r}}_i, \delta\|\tilde{\mathbf{r}}_i\|]$
    $\bar{\mathbf{w}}_i := \bar{\mathbf{w}}_i + \frac{\langle\tilde{\mathbf{r}}_i, \tilde{\mathbf{r}}_i\rangle}{\langle\mathbf{z}_i, \tilde{\mathbf{r}}_i\rangle}\tilde{\mathbf{r}}_i$
  `enddo`
  $\mathbf{w}_i := \mathbf{COARSE}[\bar{\mathbf{w}}_i, \frac{3\mu\omega_i}{1+3\mu}]$
`endfor`

It turns out that Algorithm 1 indeed converges with the optimal order. This is confirmed by the following theorem which is the main result of this paper.

THEOREM 3.7 i) If $\omega \geqslant \|\mathbf{Q}\mathbf{u}\|$, then $\mathbf{w} := \mathbf{SOLVE}[\omega, \varepsilon]$ terminates with $\|\mathbf{Q}(\mathbf{u}-\mathbf{w})\| \leqslant \varepsilon$.

ii) For any $\eta \in (0, s^*)$, let $\breve{s} = s^* - \frac{\eta}{2}$, $\breve{\tau} = (\frac{1}{2} + \breve{s})^{-1}$, and let the constant β inside **SOLVE** satisfy

$$\beta < \min\{1, [C_1(\breve{\tau})C_2(\breve{\tau})|\mathbf{I}-\mathbf{Q}|_{\ell^w_{\breve{\tau}} \to \ell^w_{\breve{\tau}}}]^{2(s^*-\eta)/\eta}\}.$$

Then, if for some $s \in (0, s^*-\eta]$, $\mathbf{u} \in \ell^w_\tau(\mathcal{N})$, then $\#\operatorname{supp}\mathbf{w} \lesssim \varepsilon^{-1/s}|\mathbf{u}|_{\ell^w_\tau}^{1/s}$ and, when $\varepsilon \lesssim \omega \lesssim \|\mathbf{u}\|$, the number of arithmetic operations and storage locations required by the call is bounded by some absolute multiple of the same expression.

Proof. The first step is to prove i). Let us consider the ith iteration of the for-loop. Assume that

$$\|\mathbf{Q}(\mathbf{u}-\mathbf{w}_{i-1})\| \leqslant \omega_{i-1}, \tag{3.16}$$

which holds by assumption for $i=1$. The inner loop terminates after not more than $M+1$ calls of **RES**. Indeed, suppose that this is not the case, then the first $M+1$ calls of **RES** do not terminate because the first condition in the until-clause is satisfied, and so Theorem 3.6 iii), Proposition 3.2, (3.1) and assumption (3.16) show that the $(M+1)$th call outputs a ν_i with

$$\begin{aligned}
\nu_i &\leqslant \frac{1+\delta}{1-\delta}\|\mathbf{f}-\mathbf{G}\bar{\mathbf{w}}_i\| = \frac{1+\delta}{1-\delta}\|\mathbf{G}(\mathbf{u}-\bar{\mathbf{w}}_i)\| \leqslant \frac{1+\delta}{1-\delta}\|\mathbf{G}\|^{\frac{1}{2}}|||\mathbf{u}-\bar{\mathbf{w}}_i||| \\
&\leqslant \frac{1+\delta}{1-\delta}\|\mathbf{G}\|^{\frac{1}{2}}\lambda^M|||\mathbf{u}-\mathbf{w}_{i-1}||| \leqslant \frac{1+\delta}{1-\delta}\|\mathbf{G}\|^{\frac{1}{2}}\lambda^M\|\mathbf{G}\|^{\frac{1}{2}}\|\mathbf{Q}(\mathbf{u}-\mathbf{w}_{i-1})\| \\
&\leqslant \frac{\omega_i}{(1+3\mu)\|\mathbf{G}^\dagger\|}
\end{aligned}$$

by definition of M, which gives a contradiction.

With $\hat{\mathbf{w}}_i$ denoting $\bar{\mathbf{w}}_i$ at termination of the inner loop, we have by (3.1) and the properties of **RES**

$$\|\mathbf{Q}(\mathbf{u}-\hat{\mathbf{w}}_i)\| \leqslant \|\mathbf{G}^\dagger\|^{\frac{1}{2}}|||\mathbf{u}-\hat{\mathbf{w}}_i||| \leqslant \|\mathbf{G}^\dagger\|\|\mathbf{G}(\mathbf{u}-\hat{\mathbf{w}}_i)\| \leqslant \|\mathbf{G}^\dagger\|\nu_i \leqslant \frac{\omega_i}{1+3\mu}, \tag{3.17}$$

so that, by the properties of **COARSE**,

$$\|\mathbf{Q}(\mathbf{u}-\mathbf{w}_i)\| \leqslant \tfrac{\omega_i}{1+3\mu} + \tfrac{3\mu\omega_i}{1+3\mu} = \omega_i,$$

showing convergence, and by definition of K completes the proof of the first statement.

The proof of ii) follows the lines of the proof of (Stevenson; 2003, Theorem 3.12). In our case where $\mathbf{G}$ has possibly a non-trivial kernel, generally, due to the errors in ran$(\mathbf{I}-\mathbf{Q})$, we have no convergence of $\hat{\mathbf{w}}_i$ to $\mathbf{u}$ for $i\to\infty$, and as a consequence, we are not able to bound $|\mathbf{w}_i|_{\ell^w_\tau}$ by some absolute multiple of $|\mathbf{u}|_{\ell^w_\tau}$. Instead we prove a weaker result (3.21), that, however, suffices to conclude optimal computational complexity. By part i) of Theorem 3.6, Proposition 3.2 and the definition of the ξ_i,

$$\|(\mathbf{I}-\mathbf{Q})(\hat{\mathbf{w}}_i-\mathbf{w}_{i-1})\| \leqslant C_3 M\xi_i = \tfrac{\omega_i}{1+3\mu}. \tag{3.18}$$

Since $\mathbf{Q}$ is bounded on ℓ_2, and by Assumption 3.5, it is bounded on $\ell^w_{\check\tau}$, an interpolation argument (cf. (DeVore; 1998, (4.24))) shows that it is bounded on ℓ^w_τ, uniformly in $\tau\in[\check\tau,2]$. Let N_i be the smallest integer such that

$$\|\mathbf{Qu}-(\mathbf{Qu})_{N_i}\| \leqslant \tfrac{\omega_i}{1+3\mu}, \tag{3.19}$$

where $(\mathbf{Qu})_N$ denotes the best N-term approximation for $\mathbf{Qu}$. Then, using the assumption $\mathbf{u}\in\ell^w_\tau(\mathscr{N})$, (3.8) shows that

$$N_i \lesssim \omega_i^{-1/s}|\mathbf{Qu}|^{1/s}_{\ell^w_\tau} \lesssim \omega_i^{-1/s}|\mathbf{u}|^{1/s}_{\ell^w_\tau},$$

and so, using (3.15),

$$\omega_i^{(\check s/s)-1}|(\mathbf{Qu})_{N_i}|_{\ell^w_{\check\tau}} \lesssim |\mathbf{u}|^{(\check s/s)-1}_{\ell^w_\tau}|(\mathbf{Qu})_{N_i}|_{\ell^w_\tau} \lesssim |\mathbf{u}|^{(\check s/s)-1}_{\ell^w_\tau}|\mathbf{Qu}|_{\ell^w_\tau} \lesssim |\mathbf{u}|^{\check s/s}_{\ell^w_\tau}. \tag{3.20}$$

From (3.17), (3.18) and (3.19), we get

$$\|(\mathbf{Qu})_{N_i}+(\mathbf{I}-\mathbf{Q})\mathbf{w}_{i-1}-\hat{\mathbf{w}}_i\| \leqslant \tfrac{3\omega_i}{1+3\mu}.$$

From Proposition 3.4, with $\mathbf{v}$ reading as $(\mathbf{Qu})_{N_i} + (\mathbf{I}-\mathbf{Q})\mathbf{w}_{i-1}$ and by using that $\mu > 1$, it follows that $\mathbf{w}_i := \mathbf{COARSE}[\hat{\mathbf{w}}_i, \frac{3\mu\omega_i}{1+3\mu}]$ satisfies

$$\begin{aligned}|\mathbf{w}_i|_{\ell^w_{\check\tau}} &\leqslant C_2(\check\tau)|(\mathbf{Qu})_{N_i} + (\mathbf{I}-\mathbf{Q})\mathbf{w}_{i-1}|_{\ell^w_{\check\tau}} \\ &\leqslant C_1(\check\tau)C_2(\check\tau)|(\mathbf{Qu})_{N_i}|_{\ell^w_{\check\tau}} + C_1(\check\tau)C_2(\check\tau)|(\mathbf{I}-\mathbf{Q})|_{\ell^w_{\check\tau}\leftarrow\ell^w_{\check\tau}}|\mathbf{w}_{i-1}|_{\ell^w_{\check\tau}}\end{aligned}$$

by (3.9), and so by (3.20),

$$\omega_i^{(\check s/s)-1}|\mathbf{w}_i|_{\ell^w_{\check\tau}} \leqslant C|\mathbf{u}|^{\check s/s}_{\ell^w_\tau} + C_1(\check\tau)C_2(\check\tau)|(\mathbf{I}-\mathbf{Q})|_{\ell^w_{\check\tau}\leftarrow\ell^w_{\check\tau}}\beta^{(\check s/s)-1}\omega_{i-1}^{(\check s/s)-1}|\mathbf{w}_{i-1}|_{\ell^w_{\check\tau}}$$

for some absolute constant $C > 0$. The assumption on β made in the theorem shows that

$$C_1(\check\tau)C_2(\check\tau)|(\mathbf{I}-\mathbf{Q})|_{\ell^w_{\check\tau}\leftarrow\ell^w_{\check\tau}}\beta^{(\check s/s)-1} < 1,$$

from which we conclude by a geometric series argument that

$$\omega_i^{(\check s/s)-1}|\mathbf{w}_i|_{\ell^w_{\check\tau}} \lesssim |\mathbf{u}|^{\check s/s}_{\ell^w_\tau}, \tag{3.21}$$

which, as we emphasize here, holds uniformly in i. Moreover, knowing this, Proposition 3.4 and (3.20) show that

$$\begin{aligned}\#\operatorname{supp}\mathbf{w}_i &\lesssim \omega_i^{-1/\check s}|(\mathbf{Qu})_{N_i} + (\mathbf{I}-\mathbf{Q})\mathbf{w}_{i-1}|^{1/\check s}_{\ell^w_{\check\tau}} \\ &\lesssim \omega_i^{-1/s}\big(\omega_i^{(\check s/s)-1}\big[|(\mathbf{Qu})_{N_i}|_{\ell^w_{\check\tau}} + |\mathbf{I}-\mathbf{Q}|_{\ell^w_{\check\tau}\to\ell^w_{\check\tau}}|\mathbf{w}_{i-1}|_{\ell^w_{\check\tau}}\big]\big)^{1/\check s} \\ &\lesssim \omega_i^{-1/s}|\mathbf{u}|^{1/s}_{\ell^w_\tau}, \end{aligned}\tag{3.22}$$

again uniformly in i.

For any computed ν_i in the inner loop, Theorem 3.6 i) shows that $\frac{\omega_i}{(1+3\mu)\|\mathbf{G}^\dagger\|} \lesssim \nu_i$. At termination of the inner loop we have $\nu_i \lesssim \omega_i$, whereas for any evaluation of **RES** that does not lead to termination, Theorem 3.6 iii) and Proposition 3.2 show that

$$\nu_i \leqslant \frac{1+\delta}{1-\delta}\|\mathbf{f}-\mathbf{G}\bar{\mathbf{w}}_i\| \lesssim |||\mathbf{u}-\bar{\mathbf{w}}_i||| \leqslant |||\mathbf{u}-\mathbf{w}_{i-1}||| \lesssim \omega_{i-1}.$$

We conclude that

$$\nu_i \eqsim \omega_i,$$

uniformly in i and over all computations of ν_i in the inner loop.

Inside the body of the inner loop, we have that the tolerance for the call of **APPLY** satisfies $\delta\|\tilde{\mathbf{r}}_i\| \geqslant \frac{\delta\nu_i}{1+\delta}$ by Theorem 3.6 iii) and, by Proposition 3.2, that $|\frac{\langle\tilde{\mathbf{r}}_i,\tilde{\mathbf{r}}_i\rangle}{\langle\mathbf{z}_i,\tilde{\mathbf{r}}_i\rangle}| \lesssim 1$. By (3.21) and the fact that the number of iterations of the inner loop is uniformly bounded, Theorem 3.6 ii) shows that

$$\omega_i^{(\check s/s)-1}|\tilde{\mathbf{r}}_i|_{\ell^w_{\check\tau}} \lesssim |\mathbf{u}|^{\check s/s}_{\ell^w_\tau}, \quad \omega_i^{(\check s/s)-1}|\bar{\mathbf{w}}_i|_{\ell^w_{\check\tau}} \lesssim |\mathbf{u}|^{\check s/s}_{\ell^w_\tau}.$$

With this result and (3.22), Theorem 3.6 ii) and the properties of **APPLY** (with s reading as $\check s$) show that

$$\#\operatorname{supp}\tilde{\mathbf{r}}_i \lesssim \omega_i^{-1/s}|\mathbf{u}|^{1/s}_{\ell^w_\tau}, \quad \#\operatorname{supp}\mathbf{z}_i \lesssim \omega_i^{-1/s}|\mathbf{u}|^{1/s}_{\ell^w_\tau}, \quad \#\operatorname{supp}\bar{\mathbf{w}}_i \lesssim \omega_i^{-1/s}|\mathbf{u}|^{1/s}_{\ell^w_\tau}.$$

By using these results concerning the lengths of the supports and the ℓ^w_τ-norms, again Theorem 3.6 ii) and the properties of **APPLY** and **COARSE** show that the number of arithmetic operations and storage locations required for the computation of $\mathbf{w}_i$ starting from $\mathbf{w}_{i-1}$ is bounded by an absolute multiple of $\omega_i^{-1/s}|\mathbf{u}|^{1/s}_{\ell^w_\tau} + \max\{\log(\omega_i^{-1}\|\hat{\mathbf{w}}_i\|), 1\}$. From $\log(\omega_i^{-1}\|\hat{\mathbf{w}}_i\|) \lesssim \omega_i^{-1/\breve{s}}\|\hat{\mathbf{w}}_i\|^{1/\breve{s}} \lesssim \omega_i^{-1/\breve{s}}|\hat{\mathbf{w}}_i|^{1/\breve{s}}_{\ell^w_\tau} \lesssim \omega_i^{-1/s}|\mathbf{u}|^{1/s}_{\ell^w_\tau}$, as well as $1 \lesssim \omega^{-1/s}\|\mathbf{u}\|^{1/s} \lesssim \omega_i^{-1/s}|\mathbf{u}|^{1/s}_{\ell^w_\tau}$ by assumption, the geometric decrease of the ω_i, and $\omega_K \gtrsim \varepsilon$, which, in case $K = 0$, is an assumption, the proof is completed. □

REMARK 3.2 In principle, all the building blocks we have developed so far can also be combined to obtain an adaptive frame version of the conjugate gradient algorithm. However, our first numerical experiments indicate that such a method does not really pay off compared to the simpler steepest descent scheme. A similar observation has also been made in Dahmen et al. (2002) for the case of a basis.

4. Numerical Experiments

After the construction of a convergent and asymptotically optimal steepest descent algorithm, we now investigate the practical applicability of the scheme. Moreover we want to compare it with the adaptive scheme based on the damped Richardson iteration. The version of the latter scheme appearing in Dahlke et al. (2004); Stevenson (2003) has been proven to converge and, under Assumption 3.5, to be also asymptotically optimal. Unfortunately its concrete implementation has shown it to be rather inefficient and therefore not well suited for comparisons. For this reason, we will compare the results of our adaptive frame algorithm **SOLVE** with those obtained with the Richardson iteration based method from Cohen et al. (2002). This scheme can be shown to converge also in the case of a wavelet frame discretization, but a proof of its optimality has not been achieved yet. Nevertheless we will see that it is in fact optimal in practice. Again the routines **RHS**, **APPLY**, and **COARSE** are the basic building blocks for its implementation which reads as follows.

Algorithm 2 CDD2SOLVE$[\eta, \varepsilon] \to \mathbf{w}$*:*
% Input should satisfy $\eta \geqslant \|\mathbf{Qu}\|$.
% Define the parameters $\alpha_{opt} := \frac{2}{\|\mathbf{G}\|+\|\mathbf{G}^\dagger\|^{-1}}$ *and* $\rho := \frac{\kappa(\mathbf{G})-1}{\kappa(\mathbf{G})+1}$.
% Let θ *and* K *be constants with* $2\rho^K < \theta < 1/2$.

$\mathbf{w} := 0$;
`while` $\eta > \varepsilon$ `do`
  `for` $j := 1$ `to` K `do`
    $\mathbf{w} := \mathbf{w} + \alpha_{opt}\left(\mathbf{RHS}[\frac{\rho^j\eta}{2\alpha K}] - \mathbf{APPLY}[\mathbf{w}, \frac{\rho^j\eta}{2\alpha K}]\right)$;
  `endfor`
  $\eta := 2\rho^K\eta/\theta$;
  $\mathbf{w} := \mathbf{COARSE}[\mathbf{w}, (1-\theta)\eta]$;
`enddo`

For the discretization we use aggregated wavelet frames on suitable overlapping domain decompositions, as the union of local wavelet bases lifted to the subdomains. As such local bases we use piecewise linear and piecewise quadratic wavelets with complementary boundary conditions from Dahmen and Schneider (1998), with order of polynomial exactness $d = 2$ or $d = 3$ and with $\tilde{d} = 2$ or $\tilde{d} = 5$ vanishing moments respectively. In particular, we impose here homogenous boundary conditions on the primal wavelets and free boundary conditions on the duals. We will test the algorithms on both 1D and 2D Poisson problems.

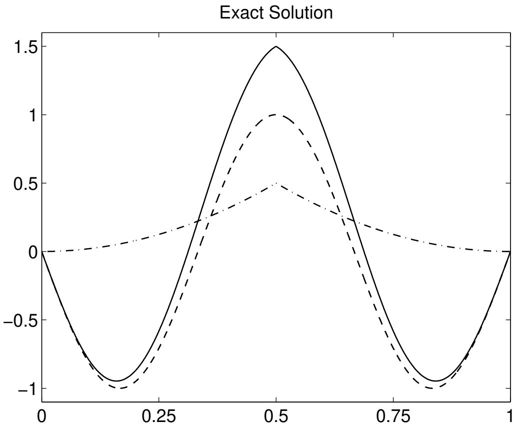

FIG. 1. Exact solution (solid line) for the one–dimensional example being the sum of the dashed and dash–dotted functions.

4.1 *Poisson Equation on the Interval*

We consider the variational formulation of the following problem of order $2t = 2$ on the interval $\Omega = (0,1)$, i.e., $n = 1$, with homogenous boundary conditions

$$-u'' = f \quad \text{on } \Omega, \quad u(0) = u(1) = 0. \tag{4.1}$$

The right-hand side f is given as the functional defined by $f(v) := 4v(\frac{1}{2}) + \int_0^1 g(x)v(x)dx$, where

$$g(x) = -9\pi^2 \sin(3\pi x) - 4.$$

The solution is consequently given by

$$u(x) = -\sin(3\pi x) + \begin{cases} 2x^2 & , \ x \in [0, \frac{1}{2}) \\ 2(1-x)^2, & x \in [\frac{1}{2}, 1] \end{cases} ,$$

see Figure 1. As an overlapping domain decomposition we choose $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 = (0, 0.7)$ and $\Omega_2 = (0.3, 1)$. Associated to this decomposition we construct our aggregrated wavelet frames just as the union of the local bases. It is shown in Dahlke et al. (2004); Stevenson (2003) that such a system is a (Gelfand) frame for $H_0^t(\Omega)$ and that it can provide a suitable characterization of Besov spaces in terms of wavelet coefficients. On the one hand, the solution u is contained in $H_0^{s+1}(\Omega)$ only for $s < \frac{1}{2}$. This means that linear methods can only converge with limited order. On the other hand, it can be shown that $u \in B_\tau^s(L_\tau(\Omega))$ for any $s > 0$, $1/\tau = s - 1/2$, so that the wavelet frame coefficients $\mathbf{u}$ associated with u define a sequence in ℓ_τ^w for any $s < \frac{d-t}{n}$ (see DeVore (1998); Stevenson (2003)). This ensures that the choice of wavelets with suitable order d can allow for any order of convergence in adaptive schemes like that presented in this paper, in the sense that the error is $O(N^{-s})$ where N is the number of unknowns. Due to our choice of piecewise linear wavelets with order $d = 2$, the optimal rate of convergence is expected to be $s = \frac{d-t}{n} = 1$. We will show that the numerical experiments confirm this expected rate.

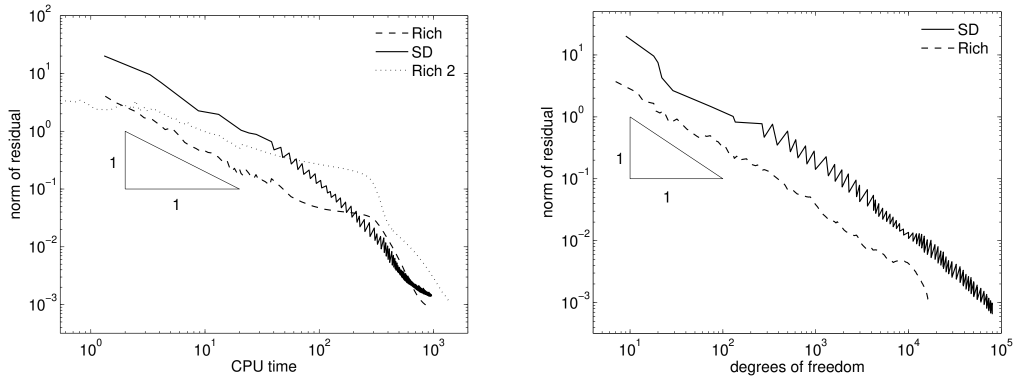

FIG. 2. *Left:* Convergence histories of **SOLVE** and **CDD2SOLVE** with respect to CPU time. Two tests for **CDD2SOLVE** with different fixed damping parameters are shown. *Right:* Convergence histories with respect to the support size of the iterands.

We have tested the adaptive wavelet algorithms **CDD2SOLVE** with parameters $\alpha_{opt} \approx 0.52$, $\theta = 2/7$, $K = 83$, and with initial $\eta = 64.8861$, and **SOLVE** with parameters $\delta = 1$, $\mu = 1.0001$, $\beta = 0.9$, $M = C_3 = 1$, $K = 134$, $\omega_0 = 64.8861$. The parameters M, C_3 have been chosen so as to produce an optimal response of the numerical results. The numerical results in Figure 2 illustrate the optimal computational complexity of the two schemes. In particular, we show that for a suboptimal choice of the damping parameter ($\alpha^* = 0.2 \leqslant \alpha_{opt} \approx 0.52$ in this specific test) **SOLVE** outperforms **CDD2SOLVE**. In practice, the wrong guess of the damping parameter can even spoil convergence and/or optimality.

In order to reduce computational time, we have implemented a caching strategy for the entries of the stiffness matrix involved. Due to limited memory resources one is then forced to fix in advance a certain maximal number of frame elements which can be taken into account during the iteration process, which means that we are solving a truncated problem. Thus, for the computation of the residuals, only wavelets up to a fixed scale are used. For small accuracies, the actually computed residuals may then deviate from the true ones. This effect shows up in the CPU time histories displayed in Figure 2 for small error tolerances. However, we observe that the influence of the truncation is almost negligible in the 1D–case, since here the finest refinement level can be chosen very high.

Finally, Figure 3 illustrates the distribution of the active wavelet frame elements used by the steepest descent scheme, each of them corresponding to a coloured rectangle. The two overlapping subintervals are treated separately. For both patches one observes that the adaptive scheme detects the singularity of the solution. The chosen frame elements arrange in a tree–like structure with large coefficients around the singularity, while on the smooth parts the coefficients are uniformly distributed, and along a fixed level they are of similar size here.

4.2 *Poisson Equation on the L-shaped Domain*

We consider the model problem of the variational formulation of Poisson's equation in two spatial dimensions:

$$-\Delta u = f \text{ in } \Omega, \quad u_{|\partial\Omega} = 0. \tag{4.2}$$

The problem will be chosen in such a way that the application of *adaptive* algorithms pays off most, as

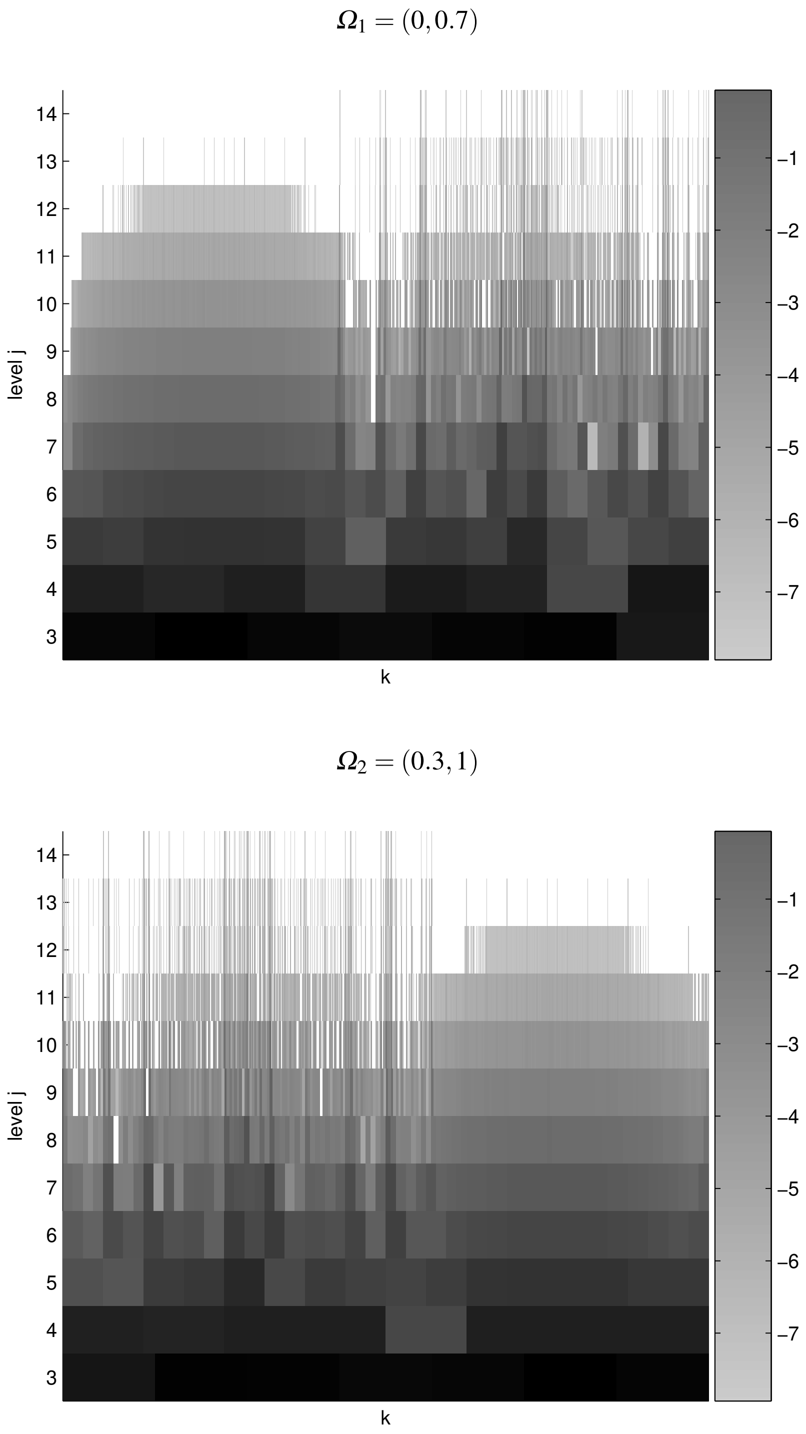

FIG. 3. Distribution of active wavelet frame elements in Ω_1 and Ω_2.

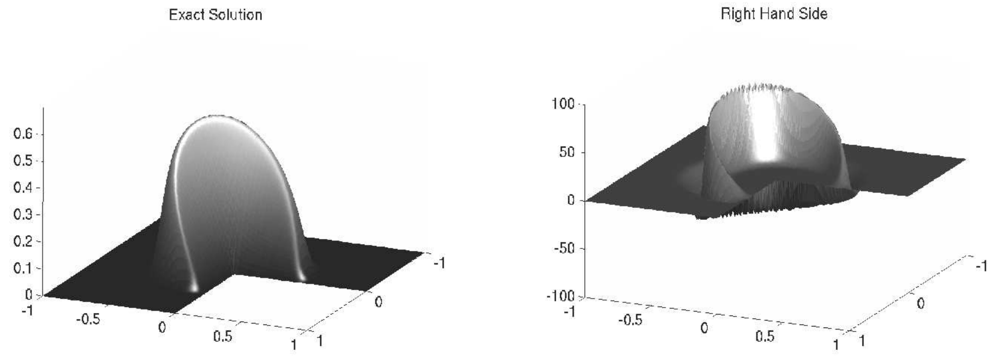

FIG. 4. Exact solution (left) and right–hand side for the two–dimensional Poisson equation in an *L*–shaped domain.

is the case for domains with reentrant corners. Here, the reentrant corners themselves lead to singular parts in the solutions, forcing them to have a limited Sobolev regularity, even for smooth right–hand sides f. For instance, considering the *L*-shaped domain $\Omega = (-1,1)^2 \backslash [0,1) \times [0,1)$, and $f \in L_2(\Omega)$, the solution u is known to be of the form

$$u = \kappa \mathscr{S} + \bar{u},$$

where $\bar{u} \in H^2(\Omega) \cap H_0^1(\Omega)$, κ is a generally non-zero constant, and, with respect to polar coordinates (r,θ) related to the reentrant corner,

$$\mathscr{S}(r,\theta) := \zeta(r) r^{2/3} \sin(\frac{2}{3}\theta),$$

where $\zeta \in C^\infty(\Omega)$ is a cut-off function. We use $\mathscr{S}$ as exact solution, which is shown together with the corresponding right–hand side in Figure 4. It is well-known that $\mathscr{S} \in H^s(\Omega)$ for $s < 5/3$ only, but it is contained in every Besov space $B^s_\tau(L_\tau(\Omega))$, where $s > 0$, $1/\tau = (s-1)/2 + 1/2$ (see Dahlke (1999)). As has been previously noted, the convergence rate of a uniform refinement strategy is determined by the Sobolev regularity of the solution, while in the context of adaptive schemes it depends on the Besov regularity (cf. Dahlke, Dahmen and DeVore (1997)). In particular, considering piecewise quadratic approximation, the best possible convergence rate in the $H^1(\Omega)$-norm for uniform refinement strategies is $\mathscr{O}(N^{-(\frac{5}{3}-1)/2})$, with N being the number of unknowns, whereas our adaptive frame scheme gives the optimal rate $\mathscr{O}(N^{-1})$. The latter can be shown, in view of our assumptions on the **APPLY** routine, provided that the wavelets used in the construction of the aggregated frame are smooth enough and have sufficiently many vanishing moments (see Stevenson (2003) for a detailed discussion of this relation).

More generally, assuming f is sufficiently smooth, with piecewise polynomial approximation of order d, a further expansion of u into more singularity functions associated to the corners of the domain shows that with the adaptive scheme the optimal rate $\mathscr{O}(N^{-(d-1)/2})$ is reached, whereas with uniform refinement strategies the rate is always restricted to $\mathscr{O}(N^{-(\frac{5}{3}-1)/2})$.

For our numerical experiments, we will use an aggregated wavelet frame. With $\Omega_1 = (-1,0) \times (-1,1)$, $\Omega_2 = (-1,1) \times (-1,0)$, and $\square = (0,1)^2$, let κ_i be affine bijections between $\square$ and Ω_i $(i = 1,2)$. For $\Psi^\square$ being a piecewise quadratic wavelet basis for $H_0^1(\square)$, where $d = 3$ and $\tilde{d} = 5$, we set $\mathscr{F} = \cup_{i=1}^2 \kappa_i(\Psi^\square)$. Although this construction is in the spirit of that from Dahlke et al. (2004) and Stevenson

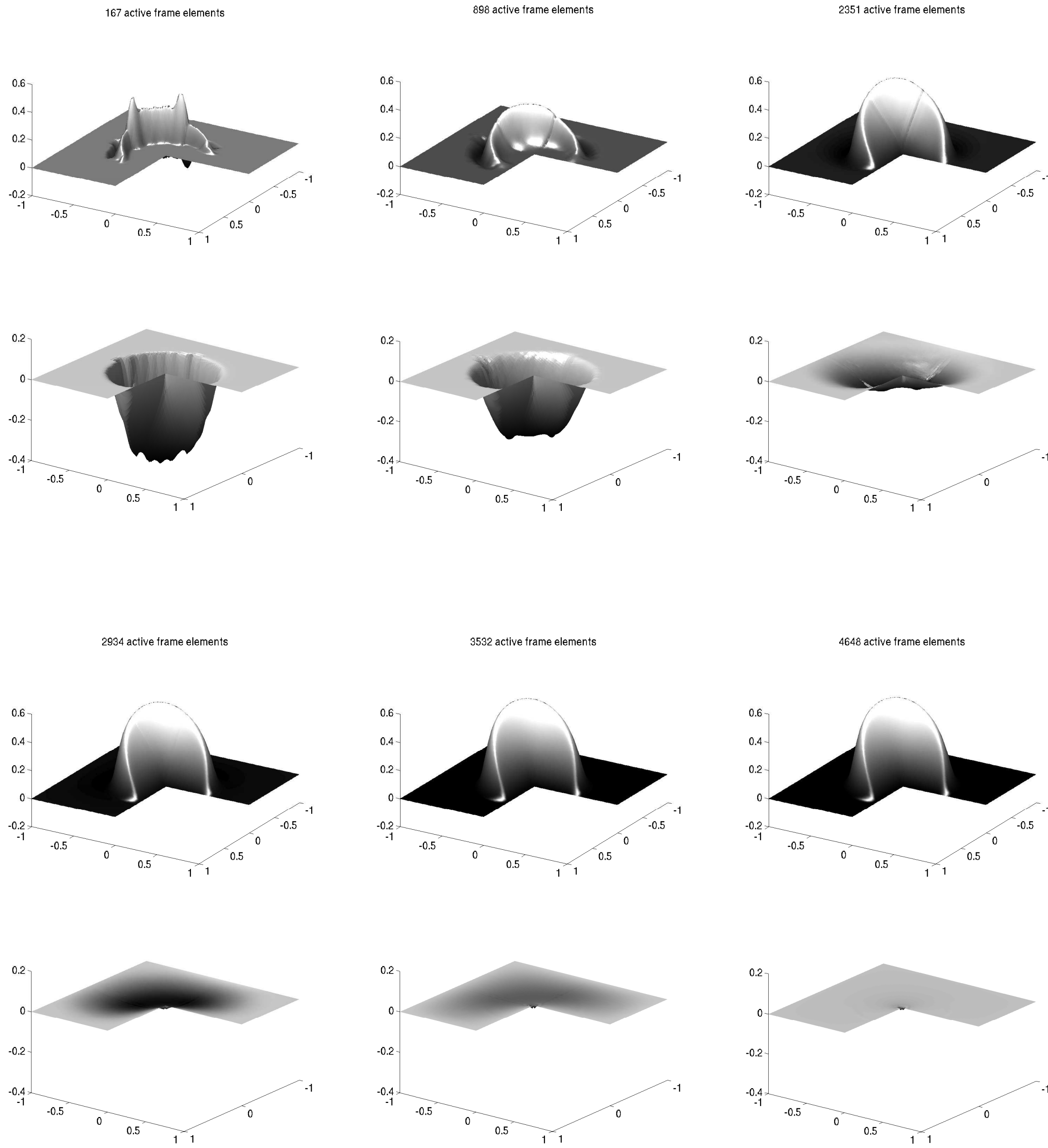

FIG. 5. Approximations and corresponding pointwise errors produced by the adaptive steepest descent algorithm, using piecewise quadratic frame elements. *Upper part:* Approximations with 167, 898, and 2351 frame elements. *Lower part:* Approximations with 2934, 3532, and 4648 frame elements.

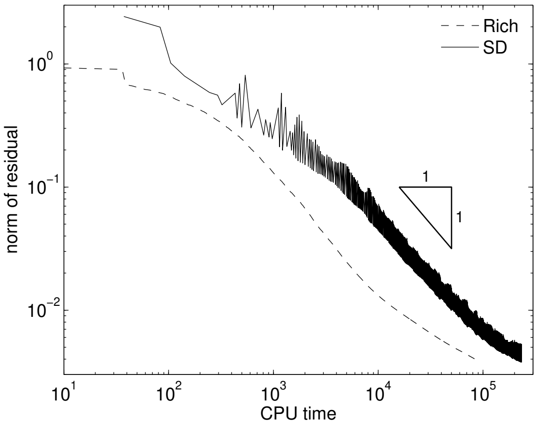

FIG. 6. Convergence histories of **SOLVE** and **CDD2SOLVE** with respect to CPU time using piecewise quadratic frame elements ($d = 3$, $\tilde{d} = 5$).

(2003), we cannot conclude from the theory developed there that $\mathscr{F}$ is actually a frame. The difficulty is that there does not exist a partition of unity with respect to the open covering $\Omega_1 \cup \Omega_2$ of Ω. The non-overlapping parts of Ω_1 and Ω_2 are infinitely close at the reentrant corner. We give a direct proof that nevertheless $\mathscr{F}$ is a frame.

Using that $\kappa_i(\Psi^\square)$ are frames (even bases) for $H_0^1(\Omega_i)$, it is sufficient to show that

$$\|u\|^2_{H^1(\Omega)} \eqsim \inf_{u_1 \in H_0^1(\Omega_1),\, u_2 \in H_0^1(\Omega_2),\, u = u_1 + u_2} \|u_1\|^2_{H^1(\Omega_1)} + \|u_2\|^2_{H^1(\Omega_2)},$$

uniformly in $u \in H_0^1(\Omega)$. Let $\phi : [0, \frac{3\pi}{2}] \to \mathbb{R}_{\geqslant 0}$ be a smooth function with $\phi(\theta) = 1$ for $\theta \leqslant \frac{\pi}{2}$ and $\phi(\theta) = 0$ for $\theta \geqslant \pi$. Writing $u = u_1 + u_2$ where $u_2(x,y) = u(x,y)\phi(\theta(x,y))$ with $(r(x,y), \theta(x,y))$ being the polar coordinates of (x,y) with respect to the reentrant corner, we have that $u_i \in H_0^1(\Omega_i)$ $(i = 1,2)$. Since Ω is a Lipschitz domain, with $\delta(x,y)$ denoting the distance of $(x,y) \in \Omega$ to the boundary, we know that for $u \in H_0^1(\Omega)$, $\delta^{-1}u \in L^2(\Omega)$ with $\|\delta^{-1}u\|_{L_2(\Omega)} \lesssim \|u\|_{H^1(\Omega)}$, uniformly in u (see (Grisvard; 1985, Theorem 1.4.4.4)). Since furthermore

$$\nabla(u\phi) = \phi\nabla u + \frac{u}{r}\left(-\sin(\theta)\frac{\partial\phi}{\partial\theta}, \cos(\theta)\frac{\partial\phi}{\partial\theta}\right)^T,$$

and $r(x,y) \geqslant \delta(x,y)$, we conclude that $\|u_i\|_{H^1(\Omega)} \lesssim \|u\|_{H^1(\Omega)}$ uniformly in u, which completes the proof of $\mathscr{F}$ being a frame for $H_0^1(\Omega)$.

We have tested the adaptive wavelet algorithms **CDD2SOLVE** with parameters $\alpha_{opt} \approx 0.15$, $\theta = 2/7$, $K = 80$, and with initial $\eta = 70.1$, and **SOLVE** with parameters $\delta = 1$, $\mu = 1.0001$, $\beta = 0.9$, $M = C_3 = 1$, $K = 150$, $\omega_0 = 70.1$.

In Figure 5 we show some of the approximations and the corresponding pointwise differences to the exact solution produced by our steepest descent scheme using piecewise quadratic frame elements. The numerical results in Figure 6 illustrate the optimal convergence of the two schemes.

REMARK 4.1 For the damped Richardson and the steepest descent schemes, optimality has been theoretically proven only under Assumption 3.5. As was previously mentioned, according to (Stevenson;

2003, Remark 3.13), the boundedness of $\mathbf{Q}$ on $\ell_\tau^w(\mathscr{N})$ for all $s \in (0, s^*)$ is (almost) a necessary requirement for the scheme to behave optimally. So our numerical results can also be seen as a possible indirect confirmation of such boundedness.

REMARK 4.2 In order to improve the approximation properties of the frame algorithms considered, one may think of imposing a tree structure onto the set of active frame elements. For the case of Riesz bases, it has been shown in Cohen et al. (2003) how to modify the main procedures **COARSE** and **APPLY** to end up with a numerical scheme which realizes the tree approximation rate N^{-s} under slightly stronger regularity assumptions on u. However, the generalization of these ideas to the setting of frames is beyond the scope of this paper.

5. Conclusion

In this paper we have presented a new optimally convergent adaptive scheme for the numerical solution of elliptic operator equations, based on redundant frame discretizations. The scheme is based on approximated iterations of steepest descent type. We have shown that the search of the damping parameter can be executed adaptively at each iteration, allowing for better practical usability compared to the damped Richarson iteration. There, the optimal damping parameter can often only be guessed, since the estimation of the lowest non–zero eigenvalue of the stiffness matrix is difficult in the case of frame discretizations. The use of frames instead of Riesz bases does not spoil the optimal convergence of the scheme that can be theoretically proved and numerically verified. Moreover, the construction of wavelet systems on domains with complicated geometry is extremely simplified by considering frames instead of Riesz bases. The numerical implementation is also significantly simplified.

The results included in Oswald (1997a,b) illustrate that frames can be naturally used for domain decomposition methods, where the overlapping patches induce a Schwarz alternating iteration. Together with adaptive schemes and the implementation of well-conditioned high order bases (Bittner (2006); Gori et al. (2004); Primbs (2006)), we expect that this line of research will produce a significant breakthrough for numerical schemes based on frame decompositions.

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