ON THE COMPRESSIBILITY OF OPERATORS IN WAVELET COORDINATES

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ABSTRACT. In [CDD02], Cohen, Dahmen and DeVore proposed an adaptive wavelet algorithm for solving operator equations. Assuming that the operator defines a boundedly invertible mapping between a Hilbert space and its dual, and that a Riesz basis of wavelet type for this Hilbert space is available, the operator equation can be transformed into an equivalent well-posed infinite matrix-vector system. This system is solved by an iterative method, where each application of the infinite stiffness matrix is replaced by an adaptive approximation. For a certain range of $s > 0$, determined by the compressibility of the stiffness matrix, i.e., by how well it can be approximated by sparse matrices, it was proven that if the errors of best linear combinations from the wavelet bases with N terms are $\mathcal{O}(N^{-s})$, then approximations yielded by the adaptive method with N terms also have errors of $\mathcal{O}(N^{-s})$, where their computation takes only $\mathcal{O}(N)$ operations. With the available estimates for both differential- and singular integral operators, the compressibility of the stiffness matrix appears to limit the rate of convergence of the adaptive method, in the sense that for solutions that have a sufficiently high (Besov) regularity, these best N-term approximations convergence with a better rate than can be shown for the approximations produced by the adaptive method. In this paper, considering piecewise smooth wavelets as spline or finite element wavelets, using modified sparse matrix approximations we derive improved results concerning compressibility. From these results it will follow that for the full range of s for which, under appropriate smoothness conditions, convergence of the best N-term approximations of $\mathcal{O}(N^{-s})$ can be shown, the adaptive method converges with that rate.

1. INTRODUCTION

As has been first observed in [BCR91], the stiffness matrix resulting from a Galerkin discretization of a singular integral operator, which using standard single scale bases is densely populated, turns out to be close to a sparse matrix when wavelet bases are exploited. Responsible for this phenomenon is that the kernel of such an operator is increasingly smooth away from the diagonal, and that the wavelets have vanishing moments. Quantitative analyses in [Sch98, DHS02, vPS97] show that for wavelets that have a sufficient number of

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vanishing moments, the stiffness matrix can be compressed to a sparse one, whose application requires only $\mathcal{O}(n)$ operations with n being the number of unknowns, whereas the order of convergence is maintained.

Compared to alternative approaches for compression as panel clustering ([HN89]) and multipole expansions ([GR87]), the wavelet approach has the additional advantage that properly scaled wavelets generate Riesz bases for a range of Sobolev spaces. So, in any case for strongly elliptic equations, if the operator defines a boundedly invertible mapping between a Sobolev space in this range and its dual, then the stiffness matrices with respect to the wavelet bases are well-conditioned uniformly in their sizes, allowing for a fast iterative solution. In summary, with suitable wavelets the discretization error accuracy can be realized in $\mathcal{O}(n)$ operations.

This Riesz basis property inspired Cohen, Dahmen en DeVore in [CDD01, CDD02] to go one step further. Instead of first discretizing the problem, i.e. replacing the underlying infinite dimensional space by some fixed finite dimensional one, and then solving the resulting finite dimensional system by some iterative method, they transformed the original problem into an equivalent well-posed infinite matrix-vector system. This system can be solved iteratively, where in each iteration the application of the infinite matrix has to be approximated. The main advantage of their approach is that in the course of the iteration the spaces in which the approximations are sought, which are always spanned by a finite linear combination of wavelets, will adapt to the solution in an optimal way. Because of this adaptivity, the method is attractive for both integral- and differential equations in variational form.

In the following we assume that the problem to be solved has the form

$$Lu = g,$$

where for some closed subspace $\mathcal{H} \subset H^t$, being a Sobolev space of order $t \in \mathbb{R}$, the linear operator $L : \mathcal{H} \to \mathcal{H}'$ is boundedly invertible, the right-hand side $g \in \mathcal{H}'$, and thus the unknown solution $u \in \mathcal{H}$. With $\Psi = \{\psi_\lambda : \lambda \in \Lambda\}$ being a Riesz basis for $\mathcal{H}$ of wavelet type, the equivalent infinite matrix-vector problem reads as

$$\mathbf{M}\mathbf{u} = \mathbf{g}, \tag{1.1}$$

where $\mathbf{M} := \langle \Psi, L\Psi \rangle : \ell_2(\Lambda) \to \ell_2(\Lambda)$ is boundedly invertible, $\mathbf{g} := \langle \Psi, g \rangle \in \ell_2(\Lambda)$, with $\langle \, , \, \rangle$ denoting the duality product on $\mathcal{H} \times \mathcal{H}'$, and $u = \mathbf{u}^T \Psi$.

In [CDD01, CDD02], the quality of the proposed iterative method is assessed by comparing the $\ell_2(\Lambda)$-error of the obtained approximation with say N coefficients with that of a *best N-term approximation* for $\mathbf{u}$, i.e., a vector $\mathbf{u}_N$ with at most N non-zero coefficients that has distance to $\mathbf{u}$ less or equal to that of any vector with a support of that size. Recall that since Ψ is a Riesz basis, the sizes of the error measured in $\ell_2(\Lambda)$ or H^t-metric differ at most a constant factor.

In any case for wavelets that are sufficiently smooth, the theory of non-linear approximation ([DeV98, Coh00]) learns that if *both*

$$0 < s < \tfrac{d-t}{n},$$

where n is the space dimension and d is the order of the wavelets, *and* u is in the Besov space $B_\tau^{sn+t}(L_\tau)$ with $\tau = (\frac{1}{2}+s)^{-1}$, then

$$\sup_{N\in\mathbb{N}} N^s \|\mathbf{u}-\mathbf{u}_N\| < \infty. \tag{1.2}$$

Here with the order of the wavelets we mean the order of the primal multiresolution analysis, or equivalently, the number of vanishing moments of the dual wavelets.

The attractive feature of these best N-term approximations is the fact that the condition involving Besov regularity is much milder that the corresponding condition $u \in H^{sn+t}$ involving Sobolev regularity that would be needed to guarantee the same rate of convergence with linear approximation in the spaces spanned by N wavelets on the coarsest scales. Indeed, assuming a sufficiently smooth right-hand side, for several boundary value problems it has been proven that the solution has a much higher Besov- than Sobolev regularity ([DD97, Dah99]). Note that with wavelets of order d, the maximum rate that can be expected by only imposing appropriate smoothness conditions on the solution is $\frac{d-t}{n}$.

Returning to the adaptive wavelet algorithm from [CDD02], besides a clean-up step that is applied after every K iterations to remove small coefficients in order to control the vector length, the other crucial ingredient is the adaptive way in which in each iteration the application of the infinite matrix $\mathbf{M}$ to a finitely supported vector is approximated. Given such a vector, before multiplication each column of $\mathbf{M}$ that corresponds to a non-zero entry in this vector is replaced by a finitely supported approximation within a tolerance that decreases as function of the size of this coefficient. To be able to prove results about complexity, information is needed about the number of entries that is necessary to approximate any column within some given tolerance. We recall the following definition from [CDD02]:

Definition 1.1. $\mathbf{M}$ is called s^*-*compressible*, when for each $j \in \mathbb{N}$ there exists an infinite matrix $\tilde{\mathbf{M}}_j$ with at most $\alpha_j 2^j$ non-zero entries in each row and column with $\sum_{j\in\mathbb{N}} \alpha_j < \infty$, such that for any $s < s^*$, $\sum_{j\in\mathbb{N}} 2^{js}\|\mathbf{M}-\tilde{\mathbf{M}}_j\| < \infty$.

An equivalent definition is obtained by requiring that for any $s < s^*$ and any $N \in \mathbb{N}$, there exists a matrix on distance of order N^{-s} having at most N non-zeros in each row and column.

The main theorem from [CDD02] now says that if (1.2) is valid for some s, and $\mathbf{M}$ is s^*-compressible with $s^* > s$, then the number of arithmetic operations and storage locations used by the adaptive wavelet algorithm for computing an approximation for $\mathbf{u}$ within tolerance ϵ is of the order $\epsilon^{-1/s}$. Since in view of (1.2) the same order of operations is already needed to approximate $\mathbf{u}$ within this tolerance using best N-term approximations assuming these would be available, this result shows that this solution method has *optimal computational complexity*.

It remains to determine the value of s^*. First of all, note that even for a differential operator, $\mathbf{M}$ itself is not sparse. Indeed, any two wavelets ψ_λ, $\psi_{\lambda'}$ from the infinite collection with $\mathrm{vol}(\mathrm{supp}\,\psi_\lambda \cap \mathrm{supp}\,\psi_\lambda) > 0$ give rise to a generally non-zero entry. Furthermore, in contrast to the non-adaptive setting, here we do not have the possibility for the matrix-vector multiplication to switch to a single-scale representation, which for differential operators

would be sparse. On the other hand, it can be shown that for wavelets that both have vanishing moments and have some global smoothness, the modulus of an entry decreases with increasing distance in scale of the involved wavelets. Assuming that for some $\sigma > 0$, L and its adjoint L' be bounded from $H^{t+\sigma} \to H^{-t+\sigma}$, by substituting the estimates [Dah97, (9.4.5), (9.4.8)] into [CDD01, Proposition 6.6.2] we infer that $\mathbf{M}$ is s^*-compressible with

$$s^* = \frac{\min\{t+\tilde{d}, \sigma, \gamma - t\}}{n} - \frac{1}{2}, \tag{1.3}$$

where $\tilde{d}$ is the order of the dual wavelets, and $\gamma = \sup_s\{\Psi \subset H^s\}$ (here we used that the condition $\sigma < t+\tilde{\gamma}$ imposed for [Dah97, (9.4.8)] can actually be relaxed to $\sigma \leq t+\tilde{d}$). This result holds true for differential operators as well as for singular integral operators. Note that in contrast to the non-adaptive setting discussed at the beginning of this introduction, global smoothness of the wavelets is required.

The result (1.3) however is not satisfactory. Indeed, since $\gamma < d$ and so $s^* \leq \frac{\gamma-t}{n} - \frac{1}{2} < \frac{d-t}{n}$, on basis of (1.3) optimal computational complexity of the adaptive wavelet method can be concluded only for solutions u that have limited Besov regularity. Indeed, when $u \in B^{sn+t}_\tau(L_\tau)$ with $\tau = (\frac{1}{2}+s)^{-1}$ and $s > s^*$, then the best N-term approximations converge at a faster rate than can be shown for the approximations yielded by the adaptive wavelet method.

For the special case of L being the Laplace operator and spline wavelets, in [BBC$^+$01, DDU02] it has been proved that $s^* = \frac{\gamma-t}{n}$, which however is still less than $\frac{d-t}{n}$.

In this paper we consider piecewise smooth wavelets as spline or finite element wavelets. For both differential as singular integral operators, and for wavelets that have a sufficient global smoothness and have cancellation properties of a sufficiently high order, we will prove that $\mathbf{M} = \langle \Psi, L\Psi\rangle$ is actually s^*-compressible with $s^* > \frac{d-t}{n}$. This result shows that if for some arbitrary s from the full range $(0, \frac{d-t}{n}]$ the best N-term approximations convergence with errors of $\mathcal{O}(N^{-s})$, then so do the approximations yielded by the adaptive method. The key to obtain this improved result is that we use slightly different sparse approximations and that on essential places we estimate directly norms of blocks of the matrix $\mathbf{M}$, instead of deriving such estimates in terms of the sizes of the individual entries via the Schur lemma.

This paper is organized as follows: In §2 we prove s^*-compressibility with $s^* > \frac{d-t}{n}$ for differential operators on a domain.

In §3 we prove this result for a class of singular integral operators on a sufficiently regular manifold, which includes operators resulting from applying the boundary integral method. Since the regularity of the manifold imposes a limit to both the smoothness of the wavelets and the continuity properties of the singular integral operators, depending on the other parameters it may restrict the compressibility. In §3.3, we give a general proof of a decay estimate for entries corresponding to wavelets with disjoint supports, which so far was only shown in specific situations. In §3.4, relying on techniques developed in [Sch98, DHS02], we prove a new decay estimate for entries corresponding to wavelets, or more generally, corresponding to a linear combination of wavelets and another wavelet,

that may have overlapping supports, but for which the support of the linear combination has empty intersection with the singular support of the other wavelet. In contrast to the bounds from [Sch98, DHS02], this estimate benefits from global smoothness of the wavelets. Apart from its use in the analysis of adaptive schemes, this estimate also results in quantitatively better compression rates when applied in the analysis of non-adaptive schemes.

At the end of this introduction, we fix some notations. We always think of the space L_2 of all measurable square integrable functions on a domain Ω or manifold Γ as being equipped with the *standard* scalar product $\langle\,,\,\rangle$ and corresponding norm $\|\,\|$, defined by $\langle u,v\rangle = \int_\Omega u(x)v(x)dx$ or $\langle u,v\rangle = \int_\Gamma u(x)v(x)d\mu(x)$, with $d\mu$ being the induced Lebesgue measure.

For H being a Hilbert space embedded in L_2 and any $u \in L_2$, the mapping $v \mapsto \langle v,u\rangle$ is continuous on H. This procedure defines an embedding of L_2 into H', or equivalently, it fixes an interpretation of a function in L_2 as a functional in H'. Different scalar products on L_2, defining equivalent norms on L_2, give rise to different embeddings of L_2 into H', that may lead to non-equivalent H'-norms of L_2-functions (cf. [NS03, §4]). This observation, together with the fact that on a few places in the wavelet literature non-standard L_2-scalar are applied is the reason here to emphasize our choice of the L_2-scalar product.

If H is dense in L_2, then the above embedding of L_2 into H' is dense, meaning that the L_2-scalar product restricted to $H \times L_2$ has an unique extension to the duality product on $H \times H'$. For such an H, without risk of confusion, we may use $\langle\,,\,\rangle$ to denote either product.

For any countable index set Λ, the notations $\langle\,,\,\rangle$ and $\|\,\|$ will also be used to denote the standard scalar product and norm, as well as the resulting operator norm, on the space $\ell_2(\Lambda)$ of square summable scalar sequences.

Finally, in order to avoid the repeated use of generic but unspecified constants, by $C \lesssim D$ we mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

2. Compressibility of differential operators

For some domain $\Omega \subset \mathbb{R}^n$, $t \in \mathbb{N}_0$ and $\Gamma_D \subset \partial\Omega$, possibly with $\Gamma_D = \emptyset$, let

$$H^t_{0,\Gamma^D}(\Omega) = \mathrm{clos}_{H^t(\Omega)}\{u \in H^t(\Omega) \cap C^\infty(\Omega) : \mathrm{supp}\, u \cap \Gamma^D = \emptyset\},$$

and let $L : H^t_{0,\Gamma^D}(\Omega) \to (H^t_{0,\Gamma^D}(\Omega))'$ be defined by

$$\langle u, Lv\rangle = \sum_{|\alpha|,|\beta|\le t} \langle \partial^\alpha u, a_{\alpha\beta}\partial^\beta v\rangle,$$

where $a_{\alpha\beta} \in L_\infty(\Omega)$ so that L is bounded. Obviously L has an extension, that we will also denote by L, as a bounded operator from $H^t(\Omega) \to H^{-t}(\Omega)$. For completeness, $H^s(\Omega)$ for $s < 0$ denotes the dual of $H^{-s}(\Omega)$.

We assume that there exists a $\sigma > 0$, such that

$$L, L' : H^{t+\sigma}(\Omega) \to H^{t-\sigma}(\Omega) \quad \text{are bounded.} \tag{2.1}$$

Sufficient is that for arbitrary $\varepsilon > 0$, and all α, β with $\min\{|\alpha|, |\beta|\} > m - \sigma$, it holds that $a_{\alpha\beta} \in \begin{cases} W_\infty^{\sigma-m+\min\{|\alpha|,|\beta|\}}(\Omega) & \text{when } \sigma \in \mathbb{N}, \\ C^{\sigma-m+\min\{|\alpha|,|\beta|\}+\varepsilon}(\Omega) & \text{when } \sigma \notin \mathbb{N}. \end{cases}$

Let

$$\Psi = \{\psi_\lambda : \lambda \in \Lambda\}$$

be a *Riesz basis for* $H^t_{0,\Gamma^D}(\Omega)$ of wavelet type. The index λ encodes both the level, denoted by $|\lambda| \in \mathbb{N}_0$, and the location of the wavelet ψ_λ.

We assume that the wavelets are *local*, in the sense that

$$\operatorname{diam}(\operatorname{supp}\psi_\lambda) \lesssim 2^{-|\lambda|} \quad \text{and} \quad \sup_{x\in\Omega, \ell\in\mathbb{N}_0} \#\{|\lambda| = \ell : B(x; 2^{-\ell}) \cap \operatorname{supp}\psi_\lambda\} < \infty,$$

and that they are *piecewise smooth*, with which we mean that $\operatorname{supp}\psi_\lambda \backslash \operatorname{sing}\operatorname{supp}\psi_\lambda$ is the disjoint union of m open "uniformly Lipschitz" domains $\Xi_{\lambda,1}, \dots, \Xi_{\lambda,m}$, with $\cup_{i=1}^m \overline{\Xi_{\lambda,i}} = \operatorname{supp}\psi_\lambda$, and that $\psi_\lambda|_{\Xi_{\lambda,i}}$ is smooth with for any $\beta \in \mathbb{N}_0^n$,

$$\sup_{x\in\Xi_{\lambda,i}} |\partial^\beta \psi_\lambda(x)| \lesssim 2^{(|\beta|+\frac{n}{2}-t)|\lambda|}. \tag{2.2}$$

Examples of piecewise smooth wavelets are tensor products of univariate spline wavelets, or finite element wavelets subordinate to a subdivision of the domain into n-simplices.

Remark 2.1. Precisely, we will call a collection of domains $\{A_\nu\} \subset \mathbb{R}^n$ to be uniformly Lipschitz domains, when there exist affine mappings B_ν with $|DB_\nu| \lesssim \operatorname{vol}(A_\nu)^{-1}$ and $|(DB_\nu)^{-1}| \lesssim \operatorname{vol}(A_\nu)$, such that the sets $B_\nu(A_\nu)$ satisfy the condition of "minimal smoothness" ([Ste70, Ch.VI, §3]) with uniform parameters "ε", "N" and "M". Examples are given by, the interiors of, "non-degenerate" polygons.

For minimally smooth domains it is known ([Ste70]) that there exist universal extension operators, with Sobolev norms only dependent on the aforementioned parameters. So in particular, by first transforming $\psi_\lambda|_{\Xi_{\lambda,i}}$ using such an affine mapping to a function on a minimally smooth domain with volume one, then making the extension, and finally applying the inverse transformation, by (2.2) we conclude that there exists a smooth function $\varphi_{\lambda,i}$ on $\mathbb{R}^n$, equal to ψ_λ on $\Xi_{\lambda,i}$, with for any $s \geq 0$, $p \in [1, \infty]$,

$$\|\varphi_{\lambda,i}\|_{W_p^s(\mathbb{R}^n)} \lesssim 2^{(s-t+\frac{n}{2}-\frac{n}{p})|\lambda|}.$$

Furthermore, we assume that there exist $\gamma > t$, $\tilde{d} > -t$ such that for $r \in [-\tilde{d}, \gamma)$, $s < \gamma$,

$$\|\cdot\|_{H^r(\Omega)} \lesssim 2^{\ell(r-s)} \|\cdot\|_{H^s(\Omega)}, \qquad \text{on } W_\ell := \operatorname{span}\{\psi_\lambda : |\lambda| = \ell\}. \tag{2.3}$$

Remark 2.2. It is known that the above wavelet assumptions are satisfied by biorthogonal wavelets when the primal and dual spaces have regularity indices $\gamma > \max\{0, t\}$, $\tilde{\gamma} > \max\{0, -t\}$ and orders $d > \gamma$, $\tilde{d} > \tilde{\gamma}$ respectively (cf. [Dah96, DS99c]), the primal spaces consist of "piecewise" smooth functions, and finally, no boundary conditions are imposed

on the dual spaces ("complementary boundary conditions", see [DS98]). In particular, (2.3) for $r \in [-\tilde{d}, -\tilde{\gamma}]$ can be deduced from the lines following (A.2) in [DS99c]. In Remark 2.5 we will comment on the case when the dual spaces satisfy the same boundary conditions as the primal ones.

Let us consider two wavelets $\psi_{\lambda'}$, ψ_λ with $\mathrm{vol}(\operatorname{supp}\psi_{\lambda'} \cap \operatorname{supp}\psi_\lambda) > 0$ and $|\lambda'| \gg |\lambda|$. Then on the scale of the size of $\operatorname{supp}\psi_{\lambda'}$, ψ_λ and so by the continuity assumption (2.1), $L\psi_\lambda$ are smooth, and therefore the latter function can be well approximated by a polynomial that integrated against $\psi_{\lambda'}$ vanishes because of its vanishing moments, showing that the entry $\langle \psi_{\lambda'}, L\psi_\lambda\rangle$ is small. The quantification of the smoothness of ψ_λ is governed by the parameter γ, whereas the number of vanishing moments of $\psi_{\lambda'}$ is given by $\tilde{d}$. Since $t > 0$, usually $t + \tilde{d} \geq \gamma - t$, with which γ becomes the restricting factor in compression results. To relax the restriction imposed by γ, in the following theorem we will make use of the fact that most of these $\psi_{\lambda'}$ will have their support inside some patch $\Xi_{\lambda,i}$ on which ψ_λ is infinitely smooth, and that only for $\psi_{\lambda'}$ with supports that intersect the $(n-1)$-dimensional singular support of ψ_λ, the estimate of the corresponding entry has to rely on the global smoothness parameter γ. Furthermore, to obtain sharp results, instead of estimating individual entries, we will directly estimate norms of blocks of the infinite matrix containing all entries that will be dropped.

Theorem 2.3. *Let $\mathbf{M} = \langle \Psi, L\Psi\rangle$. For $j \in \mathbb{N}$, and with*

$$k(j,n) := \frac{j}{n-1} \quad \text{when } n > 1,$$

and $j \leq k(j,1) \leq 2^j$, $k(j,1) > j\frac{\min\{t+\tilde{d},\sigma\}}{\gamma-t}$, we define the infinite matrix $\mathbf{M}_j$ by replacing all entries $\mathbf{M}_{\lambda,\lambda'} = \langle \psi_\lambda, L\psi_{\lambda'}\rangle$ by zeros when

$$\big||\lambda| - |\lambda'|\big| > k(j,n), \quad \text{or} \tag{2.4}$$

$$\big||\lambda| - |\lambda'|\big| > \tfrac{j}{n} \quad \text{and for some } 1 \leq i \leq m, \begin{cases} \operatorname{supp}\psi_\lambda \subset \overline{\Xi_{\lambda',i}} & \text{when } |\lambda| > |\lambda'|, \\ \operatorname{supp}\psi_{\lambda'} \subset \overline{\Xi_{\lambda,i}} & \text{when } |\lambda| < |\lambda'|. \end{cases} \tag{2.5}$$

Then the number of non-zero entries in each row and column of $\mathbf{M}_j$ is of order 2^j, and for any

$$s \leq \min\{\tfrac{t+\tilde{d}}{n}, \tfrac{\sigma}{n}\}, \text{ with } s < \tfrac{\gamma-t}{n-1} \text{ when } n > 1,$$

it holds that $\|\mathbf{M} - \mathbf{M}_j\| \lesssim 2^{-sj}$.

Remark 2.4. In view of Definition 1.1, by taking $\tilde{\mathbf{M}}_j = \mathbf{M}_{\lceil j+\log(\alpha_j)\rceil}$, with for example $\alpha_j = j^{-(1+\varepsilon)}$ for some $\varepsilon > 0$, we infer that $\mathbf{M}$ is s^*-compressible with

$$s^* = \min\{\tfrac{t+\tilde{d}}{n}, \tfrac{\sigma}{n}, \tfrac{\gamma-t}{n-1}\}$$

($s^* = \min\{t+\tilde{d}, \sigma\}$ when $n = 1$). So, with d being the order of the wavelets, if $\tilde{d} > d - 2t$, $\sigma > d - t$ and when $n > 1$, $\frac{\gamma-t}{n-1} > \frac{d-t}{n}$, then indeed $s^* > \frac{d-t}{n}$. The condition involving γ when $n > 1$ is satisfied for instance when $\frac{d-t}{n} > \frac{1}{2}$ and $\gamma = d - \frac{1}{2}$ (spline wavelets).

Proof of Theorem 2.3. Let λ be some given index. By the locality of the wavelets, the number of indices λ' with fixed $|\lambda'|$ for which $\mathrm{vol}(\operatorname{supp}\psi_{\lambda'}\cap\operatorname{supp}\psi_\lambda)>0$ is of order $\max\{1,2^{(|\lambda'|-|\lambda|)n}\}$. By using in addition the piecewise smoothness of the wavelets, the number of indices λ' with fixed $|\lambda'|>|\lambda|$ for which $\mathrm{vol}(\operatorname{supp}\psi_{\lambda'}\cap\operatorname{supp}\psi_\lambda)>0$ and $\operatorname{supp}\psi_{\lambda'}$ is not contained in some $\overline{\Xi_{\lambda,i}}$ is of order $2^{(|\lambda'|-|\lambda|)(n-1)}$. We conclude that the number of non-zero entries in the λth row and column of $\mathbf{M}_j$ is of order

$$\sum_{||\lambda'|-|\lambda||\le\frac{j}{n}}\max\{1,2^{(|\lambda'|-|\lambda|)n}\}+\sum_{\frac{j}{n}<||\lambda'|-|\lambda||\le k(j,n)}\max\{1,2^{(|\lambda'|-|\lambda|)(n-1)}\}\eqsim 2^j.$$

Let $\hat{\mathbf{M}}_j$ be defined by $(\mathbf{M}-\hat{\mathbf{M}}_j)_{\lambda,\lambda'}=\begin{cases}\mathbf{M}_{\lambda,\lambda'} & \text{when } \big||\lambda|-|\lambda'|\big|>k(j,n),\\ 0 & \text{otherwise.}\end{cases}$ The continuity assumptions on L,L', together with (2.3) show that for

$$r\in(0,t+\tilde{d}]\cap(0,\sigma]\cap(0,\gamma-t),$$

and $w_\ell\in W_\ell$, $w_{\ell'}\in W_{\ell'}$,

$$\begin{aligned}|\langle w_\ell,Lw_{\ell'}\rangle|&\lesssim\|w_\ell\|_{H^{t-r}(\Omega)}\|Lw_{\ell'}\|_{H^{-t+r}(\Omega)}\\ &\lesssim\|w_\ell\|_{H^{t-r}(\Omega)}\|w_{\ell'}\|_{H^{t+r}(\Omega)}\lesssim 2^{r(\ell'-\ell)}\|w_\ell\|_{H^t(\Omega)}\|w_{\ell'}\|_{H^t(\Omega)},\end{aligned}\tag{2.6}$$

and analogously, $|\langle w_\ell,Lw_{\ell'}\rangle|=|\langle L'w_\ell,w_{\ell'}\rangle|\lesssim 2^{r(\ell-\ell')}\|w_\ell\|_{H^t(\Omega)}\|w_{\ell'}\|_{H^t(\Omega)}$. So for arbitrary $\mathbf{c},\mathbf{d}\in\ell_2(\Lambda)$, we have

$$\begin{aligned}|\langle\mathbf{c},(\mathbf{M}-\hat{\mathbf{M}}_j)\mathbf{d}\rangle|&=|\sum_{|\ell-\ell'|>k(j,n)}\langle\sum_{|\lambda|=\ell}\mathbf{c}_\lambda\psi_\lambda,L\sum_{|\lambda'|=\ell'}\mathbf{d}_{\lambda'}\psi_{\lambda'}\rangle|\\ &\lesssim\sum_{|\ell-\ell'|>k(j,n)}2^{-r|\ell-\ell'|}\|\sum_{|\lambda|=\ell}\mathbf{c}_\lambda\psi_\lambda\|_{H^t(\Omega)}\|\sum_{|\lambda'|=\ell'}\mathbf{d}_{\lambda'}\psi_{\lambda'}\|_{H^t(\Omega)}\\ &\lesssim 2^{-k(j,n)r}\sqrt{\sum_\ell\|\sum_{|\lambda|=\ell}\mathbf{c}_\lambda\psi_\lambda\|^2_{H^t(\Omega)}}\sqrt{\sum_{\ell'}\|\sum_{|\lambda'|=\ell'}\mathbf{d}_{\lambda'}\psi_{\lambda'}\|^2_{H^t(\Omega)}}\\ &\eqsim 2^{-k(j,n)r}\|\mathbf{c}\|\|\mathbf{d}\|,\end{aligned}$$

or $\|\mathbf{M}-\hat{\mathbf{M}}_j\|\lesssim 2^{-k(j,n)r}$.

Finally, we analyze the error as a consequence of dropping entries with indices that satisfy criterion (2.5). For $\lambda\in\Lambda$, $1\le i\le m$ and $\ell>|\lambda'|$, let

$$A_{\ell,\lambda',i}:=\{|\lambda|=\ell:\operatorname{supp}\psi_\lambda\subset\overline{\Xi_{\lambda',i}}\}.$$

For $w_{\ell,\lambda',i}\in\operatorname{span}\{\psi_\lambda:\lambda\in A_{\ell,\lambda',i}\}\subset W_\ell$ and

$$q\in(0,t+\tilde{d}]\cap(0,\sigma],$$

we will prove that

$$|\langle w_{\ell,\lambda',i},L\psi_{\lambda'}\rangle|\lesssim 2^{q(|\lambda'|-\ell)}\|w_{\ell,\lambda',i}\|_{H^t(\Omega)}.\tag{2.7}$$

Because of (2.6), it is sufficient to consider $q \geq \gamma - t \geq -t$. From Remark 2.1, recall that $\varphi_{\lambda',i}$ is the extension of $\psi_{\lambda'}|_{\Xi_{\lambda',i}}$ to a smooth function on $\mathbb{R}^n$ with for $s \geq 0$, $\|\varphi_{\lambda',i}\|_{H^s(\mathbb{R}^n)} \lesssim 2^{(s-t)|\lambda'|}$. From the locality and continuity of L, we conclude that

$$\begin{aligned}|\langle w_{\ell,\lambda',i}, L\psi_{\lambda'}\rangle| = |\langle w_{\ell,\lambda',i}, L\varphi_{\lambda',i}\rangle| &\lesssim \|w_{\ell,\lambda',i}\|_{H^{t-q}(\Omega)}\|L\varphi_{\lambda',i}\|_{H^{-t+q}(\Omega)}\\ &\lesssim \|w_{\ell,\lambda',i}\|_{H^{t-q}(\Omega)}\|\varphi_{\lambda',i}\|_{H^{t+q}(\Omega)} \lesssim 2^{-q(\ell-|\lambda'|)}\|w_{\ell,\lambda',i}\|_{H^t(\Omega)}.\end{aligned}$$

For any $\mathbf{c}, \mathbf{d} \in \ell_2(\Lambda)$ and $q \in (0, t+\tilde{d}] \cap (0, \sigma]$, from (2.7) we have

$$\begin{aligned}&|\sum_{\frac{j}{n}<\ell-\ell'\leq k(j,n)}\sum_{|\lambda'|=\ell'}\mathbf{d}_{\lambda'}\langle\sum_{i=1}^m\sum_{\lambda\in A_{\ell,\lambda',i}}\mathbf{c}_\lambda\psi_\lambda, L\psi_{\lambda'}\rangle|\\
&\lesssim \sum_{\frac{j}{n}<\ell-\ell'\leq k(j,n)}\sum_{|\lambda'|=\ell'}|\mathbf{d}_{\lambda'}|2^{-q(\ell-\ell')}\sum_{i=1}^m\|\sum_{\lambda\in A_{\ell,\lambda',i}}\mathbf{c}_\lambda\psi_\lambda\|_{H^t(\Omega)}\\
&\lesssim \sum_{\frac{j}{n}<\ell-\ell'\leq k(j,n)}2^{-q(\ell-\ell')}\sqrt{\sum_{|\lambda'|=\ell'}|\mathbf{d}_{\lambda'}|^2}\sqrt{\sum_{|\lambda'|=\ell'}\left(\sum_{i=1}^m\sqrt{\sum_{\lambda\in A_{\ell,\lambda',i}}|\mathbf{c}_\lambda|^2}\right)^2}\\
&\lesssim \sum_{\frac{j}{n}<\ell-\ell'\leq k(j,n)}2^{-q(\ell-\ell')}\sqrt{\sum_{|\lambda'|=\ell'}|\mathbf{d}_{\lambda'}|^2}\sqrt{\sum_{|\lambda|=\ell}|\mathbf{c}_\lambda|^2} \lesssim 2^{-\frac{j}{n}q}\|\mathbf{d}\|\|\mathbf{c}\|,\end{aligned}$$

where for the last line we have used that for fixed $|\lambda'|$, each λ is contained in at most a uniformly bounded number of sets $A_{|\lambda|,\lambda',i}$.

Since, analogous to (2.7), $|\langle\psi_\lambda, Lw_{\ell',\lambda,i}\rangle| = |\langle L'\psi_\lambda, w_{\ell',\lambda,i}\rangle| \lesssim 2^{-q(\ell'-|\lambda|)}\|w_{\ell',\lambda,i}\|_{H^t(\Omega)}$ when $w_{\ell',\lambda,i} \in \operatorname{span}\{\psi_{\lambda'} : \lambda' \in A_{\ell',\lambda,i}\}$, and so

$$|\sum_{\frac{j}{n}<\ell'-\ell\leq k(j,n)}\sum_{|\lambda|=\ell}\mathbf{c}_\lambda\langle\psi_\lambda, L\sum_{i=1}^m\sum_{\lambda'\in A_{\ell',\lambda,i}}\mathbf{d}_{\lambda'}\psi_{\lambda'}\rangle| \lesssim 2^{-\frac{j}{n}q}\|\mathbf{c}\|\|\mathbf{d}\|,$$

we conclude that $\|\hat{\mathbf{M}}_j - \mathbf{M}_j\| \lesssim 2^{-\frac{j}{n}q}$.

A combination of the estimates for $\mathbf{M}-\hat{\mathbf{M}}_j$ and $\hat{\mathbf{M}}_j-\mathbf{M}_j$ shows that for $s \leq \min\{\frac{t+\tilde{d}}{n}, \frac{\sigma}{n}\}$, with $s < \frac{\gamma-t}{n-1}$ when $n > 1$, it holds that $\|\mathbf{M} - \mathbf{M}_j\| \lesssim 2^{-sj}$. □

Remark 2.5. Let us consider the situation that $t \in \mathbb{N}$, $\Gamma_D \neq \emptyset$, and that Ψ is a biorthogonal basis for $H^t_{0,\Gamma_D}(\Omega)$, where now also the dual spaces satisfy homogeneous Dirichlet boundary conditions on Γ_D. Then since for $s \geq 0$, $H^s(\Omega) \cap H^t_{0,\Gamma_D}(\Omega)$ is only dense in $H^s(\Omega)$ when $s < \frac{1}{2}$, we can expect (2.3) only for $r \in [-\tilde{d}, \gamma)$, $s < \gamma$ with $r > -\frac{1}{2}$. As a consequence, in the proof of Theorem 2.3, the range of r for which (2.6) holds is restricted to $r \in (0, t+\tilde{d}] \cap (0, \sigma] \cap (0, \gamma - t) \cap (0, t + \frac{1}{2})$.

The same problems are encountered for proving (2.7). However, instead of restricting the range of q, here another solution is possible. The homogeneous Dirichlet boundary

conditions on the dual spaces only affect wavelets with supports near Γ_D. More precisely, one can expect that there exists a constant $\theta > 0$, such that for any $r \in [-\tilde{d}, \gamma)$, $s < \gamma$,

$$\|\cdot\|_{H^r(\Omega)} \lesssim 2^{\ell(r-s)} \|\cdot\|_{H^s(\Omega)} \quad \text{on } \operatorname{span}\{\psi_\lambda : |\lambda| = \ell,\ \operatorname{dist}(\operatorname{supp}\psi_\lambda, \Gamma_D) \geq \theta 2^{-|\lambda|}\}. \tag{2.8}$$

Let us now add to the dropping criterion (2.5) the condition that $\operatorname{dist}(\operatorname{supp}\psi_\lambda, \Gamma_D) \geq \theta 2^{-|\lambda|}$ when $|\lambda| > |\lambda'|$, or $\operatorname{dist}(\operatorname{supp}\psi_{\lambda'}, \Gamma_D) \geq \theta 2^{-|\lambda'|}$ when $|\lambda| < |\lambda'|$. Then it is easily verified that the resulting $\mathbf{M}_j$, although a little bit less sparse, still has at most order 2^j non-zero entries in each row and column. On the other hand, changing the definition of $A_{\ell,\lambda',i}$ into

$$A_{\ell,\lambda',i} := \{|\lambda| = \ell : \operatorname{supp}\psi_\lambda \subset \overline{\Xi_{\lambda',i}},\ \operatorname{dist}(\operatorname{supp}\psi_\lambda, \Gamma_D) \geq \theta 2^{-|\lambda|}\},$$

for $w_{\ell,\lambda',i} \in \operatorname{span}\{\psi_\lambda : \lambda \in A_{\ell,\lambda',i}\}$ and $q \in (0, t + \tilde{d}] \cap (0, \sigma)$, using (2.8) we can prove that $|\langle w_{\ell,\lambda',i}, L\psi_{\lambda'}\rangle| \lesssim 2^{-q(\ell-|\lambda'|)} \|w_{\ell,\lambda',i}\|_{H^t(\Omega)}$. By copying the remainder of the proof, and for $k(j,1) > j\frac{\min\{t+\tilde{d},\sigma\}}{\min\{\gamma-t,t+\frac{1}{2}\}}$, we conclude that for

$$s \leq \min\{\tfrac{t+\tilde{d}}{n}, \tfrac{\sigma}{n}\}, \text{ with } s < \min\{\tfrac{\gamma-t}{n-1}, \tfrac{t+\frac{1}{2}}{n-1}\} \text{ when } n > 1,$$

it holds that $\|\mathbf{M} - \mathbf{M}_j\| \lesssim 2^{-sj}$.

Remark 2.6. As follows from Remark 2.4, to show s^*-compressibility with $s^* > \frac{d-t}{n}$, it will be necessary that both $\tilde{d}$ and γ increase linearly as function of d. To benefit from an often much higher regularity of the solution in Besov than in Sobolev scale, we are mainly interested to apply the adaptive method with a relatively large value of $d - t$. Indeed, for small $d - t$, the adaptive method can give at most a small improvement in the order of convergence compared to non-adaptive methods, which in practice might not compensate for the overhead it requires.

On general, non tensor product domains, smooth wavelets, i.e., with large values of γ, are not easy to construct. The approach from [DS99b] based on a non-overlapping domain decomposition yields wavelet bases that in principal for any d satisfy all requirements concerning smoothness and cancellation properties to obtain $s^* > \frac{d-t}{n}$. Other approaches based on non-overlapping domain decomposition ([DS99a, CTU99, CM00]) yield wavelets which over the interfaces between subdomains are only continuous. Note that although for non-adaptive wavelet methods the fact that wavelets along some lower-dimensional interface are less smooth or have reduced cancellation properties might not influence the overall complexity-accuracy balance, for adaptive methods it generally does. Indeed, it might happen that the solution is smooth everywhere except exactly along that interface, meaning that the adaptive method mainly produces coefficients corresponding to wavelets with degenerated properties. Also finite element wavelets as constructed in [DS99c, CES00, Ste00] are only continuous. For example for $t = 1$ and $n = 2$, with continuous wavelets only for orders $d \leq 2$, $s^* \geq \frac{d-t}{n}$ can be shown.

Still thinking of a non-overlapping decomposition of the domain into a number of subdomains or patches $\Omega_1, \ldots, \Omega_M$, as an alternative it seems not too difficult, in any case if one refrains from having local dual wavelets which are not needed here anyway, to construct wavelets of some given order d, which restricted to each patch are again wavelets

characterized by parameters $\gamma \leq d - \frac{1}{2}$ and $\tilde{d}$ ('*patchwise*' smoothness and cancellation properties) that can be chosen at ones convenience. If in addition, these wavelets have a sufficient global smoothness such that they generate a Riesz basis for $H^t_{0,\Gamma_D}(\Omega)$, then $\langle \Psi, L\Psi\rangle = \sum_q \langle \Psi|_{\Omega_q}, L\Psi|_{\Omega_q}\rangle$. Theorem 2.3 now directly applies to the matrices in the sum with conditions in terms of the 'local' γ and $\tilde{d}$, and so when these are sufficiently large, $s^* > \frac{d-t}{n}$ follows.

In [Ste02] we generalized the adaptive wavelet method from [CDD02] to the case that Ψ is a *frame* for $H^t_{0,\Gamma_D}(\Omega)$ instead of a Riesz basis. Writing the domain as an *overlapping* union of subdomains Ω_q, a suitable frame Ψ is given by $\cup_{q=1}^M \omega_q \Psi_q$, where Ψ_q is a Riesz basis of wavelet type order d for a Sobolev space of order t on Ω_q, and ω_q a smooth weight function that vanishes at the internal boundary of Ω_q. Since both the presence of smooth weight functions and the fact that for $q \neq q'$, Ψ_q and $\Psi_{q'}$ are different are harmless, Theorem 2.3 can be applied to verify s^*-compressibility of any of the matrices $\langle \omega_q \Psi_q, L\omega_{q'}\Psi_{q'}\rangle$ and with that of $\mathbf{M} := \langle \Psi, L\Psi\rangle = \sum_{q,q'} \langle \omega_q\Psi_q, L\omega_{q'}\Psi_{q'}\rangle$. Indeed, note that in the proof of this theorem it was not used that the wavelets are piecewise smooth with respect to partitions that are nested as function of the level. The advantage of this frame approach is that smoothness requirements on the wavelets Ψ_q are easily satisfied, since this construction requires no linking of functions from different subdomains over interfaces, and so that $s^* > \frac{d-t}{n}$ can easily be realized.

3. Compressibility of boundary integral operators

In this section, we will generalize the compression result Theorem 2.3 to a certain class of non-local operators L. Since in particular we think of integral operators resulting from the boundary integral method, in addition we replace the underlying domain Ω by a manifold Γ.

3.1. **Definitions and main result.** For some $\mu \in \mathbb{N}$, let Γ be a patchwise smooth, compact n-dimensional, globally $C^{\mu-1,1}$ manifold in $\mathbb{R}^{n+1}$. Following [DS99b], we assume that $\Gamma = \cup_{q=1}^M \overline{\Gamma_q}$, with $\Gamma_q \cap \Gamma_{q'} = \emptyset$ when $q \neq q'$, and that for each $1 \leq q \leq M$, there exists

- a domain $\Omega_q \subset \mathbb{R}^n$ and a C^∞-parametrization $\kappa_q : \mathbb{R}^n \to \mathbb{R}^{n+1}$ with $\mathrm{Im}(\kappa_q|_{\Omega_q}) = \Gamma_q$,
- a domain $\mathbb{R}^n \supset \hat{\Omega}_q \supset\supset \Omega_q$, and an extension of $\kappa_q|_{\Omega_q}$ to a $C^{\mu-1,1}$ parametrization $\hat{\kappa}_q : \hat{\Omega}_q \to \mathrm{Im}(\hat{\kappa}_q) \subset \Gamma$.

For $|s| \leq \mu$, the Sobolev spaces $H^s(\Gamma)$ are well-defined, where for $s < 0$, $H^s(\Gamma)$ is the dual of $H^{-s}(\Gamma)$. For some $|t| \leq \mu$, let L be a bounded operator from $H^t(\Gamma) \to H^{-t}(\Gamma)$, where we have in mind a singular integral operator of order $2t$. Let

$$\Psi = \{\psi_\lambda : \lambda \in \Lambda\}$$

be a *Riesz basis for* $H^t(\Gamma)$ of wavelet type.

We assume that the wavelets are *local*, in the sense that

$$\operatorname{diam}(\operatorname{supp}\psi_\lambda)\lesssim 2^{-|\lambda|}\quad\text{and}\quad\sup_{x\in\Gamma,\ell\in\mathbb{N}_0}\#\{|\lambda|=\ell : B(x;2^{-\ell})\cap\operatorname{supp}\psi_\lambda\}<\infty,$$

where for $A\subset\Gamma$ now $B(A;\varepsilon):=\{y\in\mathbb{R}^{n+1} : \operatorname{dist}(A,y)<\varepsilon\}$, and that they are *piecewise smooth*, with which we mean that $\operatorname{supp}\psi_\lambda\backslash\operatorname{sing}\operatorname{supp}\psi_\lambda$ is the disjoint union of sets $\Xi_{\lambda,1},\dots,\Xi_{\lambda,m}$, with $\cup_{i=1}^m\overline{\Xi_{\lambda,i}}=\operatorname{supp}\psi_\lambda$, such that each $\Xi_{\lambda,i}$ is contained in some Γ_q, $\kappa_q^{-1}(\Xi_{\lambda,i})$ are uniformly Lipschitz domains, and $(\psi_\lambda\circ\kappa_q)|_{\kappa_q^{-1}(\Xi_{\lambda,i})}$ is smooth with for any $\beta\in N_0^n$,

$$\sup_{\xi\in\kappa_q^{-1}(\Xi_{\lambda,i})}|\partial^\beta(\psi_\lambda\circ\kappa_q)(\xi)|\lesssim 2^{(|\beta|+\frac{n}{2}-t)|\lambda|}.\tag{3.1}$$

Implicitly via (2.3), in §2 we assumed that the wavelets have $\tilde d$ vanishing moments. A rough translation of this property to manifolds is that a wavelet integrated against a function vanishes when on each patch the pre-image of this function is a polynomial of degree less than $\tilde d$. More precisely, we assume that the wavelets have the so-called *cancellation property of order* $\tilde d\in\mathbb{N}$, saying that there exists a constant $\eta>0$, such that for any $p\in[1,\infty]$, for all continuous, patchwise smooth functions v and $\lambda\in\Lambda$,

$$|\langle v,\psi_\lambda\rangle|\lesssim 2^{-|\lambda|(\frac{n}{2}-\frac{n}{p}+t+\tilde d)}\max_{1\le q\le M}|v|_{W_p^{\tilde d}(B(\operatorname{supp}\psi_\lambda;2^{-|\lambda|}\eta)\cap\Gamma_q)}.\tag{3.2}$$

Remark 3.1. Proofs of (3.2) for $p=\infty$ given in [DS99c, Prop. 4.7], [NS03, Prop. 3.4] can easily be generalized to yield (3.2) for any $p\in[1,\infty]$.

Furthermore, for some $k\in\mathbb{N}_0\cup\{-1\}$, with $k<\mu$ and

$$\gamma:=k+\tfrac{3}{2}>t,\tag{3.3}$$

we assume that all $\psi_\lambda\in C^k(\Gamma)$, where $k=-1$ means no global continuity condition, and, similar to (2.3), that for all $r\in[-\tilde d,\gamma)$, $s<\gamma$, necessarily with $|s|,|r|\le\mu$,

$$\|\cdot\|_{H^r(\Gamma)}\lesssim 2^{\ell(r-s)}\|\cdot\|_{H^s(\Gamma)}\qquad\text{on } W_\ell:=\operatorname{span}\{\psi_\lambda : |\lambda|=\ell\}.\tag{3.4}$$

Inside a patch, a similar property can be required for larger ranges: For all $1\le q\le M$, and $r\in[-\tilde d,\gamma)$, $s<\gamma$, we assume that

$$\|\cdot\|_{H^r(\Gamma_q)}\lesssim 2^{\ell(r-s)}\|\cdot\|_{H^s(\Gamma_q)}\quad\text{on }\operatorname{span}\{\psi_\lambda : |\lambda|=\ell, B(\operatorname{supp}\psi_\lambda;2^{-\ell}\eta)\subset\overline{\Gamma_q}\}.\tag{3.5}$$

Remark 3.2. For the case that each parametrization κ_q has a constant Jacobian, in [DS99c] a simple construction is given of continuous finite element wavelets, i.e., $k=0$ and so $\gamma=\frac{3}{2}$, that in principal for any $\tilde d$ and order d satisfies the above assumptions. This restriction on the Jacobians was removed in [NS03], however here wavelets with supports that extend to more than one patch have the cancellation property of only order 1. In a forthcoming paper, we will remove this inconvenience for the application in adaptive methods, and construct wavelets that all have the cancellation property of order $\tilde d$.

As in the domain case, wavelets that satisfy the assumptions for in principal any d, $\tilde d$ and smoothness permitted by both d and the regularity of the manifold were constructed in

[DS99b]. As will appear from Theorem 3.3, via several parameters this regularity however seems to impose a principal barrier to the compressibility. In case of differential operators, we could reduce conditions concerning smoothness and cancellation properties to corresponding patchwise conditions. Yet, the arguments used for that do not carry over to the case of non-local, integral operators.

With the constructions from [DS99a, CTU99, CM00], biorthogonality was realized with respect to a modified $L_2(\Gamma)$-scalar product. As a consequence, with the interpretation of functions as functionals via the Riesz mapping with respect to the standard $L_2(\Gamma)$ scalar product, for negative t the wavelets only generate a Riesz basis for $H^t(\Gamma)$ when $t > -\frac{1}{2}$ (cf. [NS03, §4]), and likewise wavelets with supports that extend to more than one patch generally have no cancellation properties in the sense of (3.2).

The frame approach discussed in the previous section seems to be even more attractive in the compact manifold case. Because of the absence of a boundary, all wavelet bases on the overlapping patches may satisfy periodic boundary conditions. One may verify that Theorem 3.3 given below formulated for the Riesz basis case, as well as the verification of the estimates (3.6) and (3.7), extend to any of the matrices $\langle \omega_q \Psi_q, L\omega_{q'}\Psi_{q'}\rangle$. It is easy to construct collections Ψ_q with any smoothness permitted by the regularity of the manifold.

For the differential operator case, we made use of the fact that $\langle \psi_\lambda, L\psi_{\lambda'}\rangle = 0$ whenever $\mathrm{vol}(\operatorname{supp}\psi_\lambda \cap \operatorname{supp}\psi_{\lambda'}) = 0$. As an replacement, for the next theorem we will assume estimate (3.6) saying that the size of $|\langle \psi_\lambda, L\psi_{\lambda'}\rangle|$ decreases rapidly with increasing distance between the supports of ψ_λ and $\psi_{\lambda'}$. In §3.3, we will verify this estimate for an important class of singular integral operators that have Schwarz kernels that decay rapidly away from the diagonal, and which therefore act mainly local. For ψ_λ and $\psi_{\lambda'}$ with $\mathrm{vol}(\operatorname{supp}\psi_\lambda \cap \operatorname{supp}\psi_{\lambda'}) > 0$, in the differential operator case we made use of the faster decay of $|\langle \psi_\lambda, L\psi_{\lambda'}\rangle|$ for $|\lambda| - |\lambda'| \to \infty$ whenever $\operatorname{supp}\psi_\lambda$ does not intersect the singular support of $\psi_{\lambda'}$. For the aforementioned class of singular integral operators, as a replacement in §3.4 we will prove estimate (3.7) showing that $|\langle \psi_\lambda, L\psi_{\lambda'}\rangle|$ decreases for increasing $|\lambda| - |\lambda'|$ and increasing distance between $\operatorname{supp}\psi_\lambda$ and $\operatorname{sing\,supp}\psi_\lambda$. To obtain (3.7) we will need the additional condition that $\tilde{d} > \gamma - 2t$, which will be needed anyway in Remark 3.4 to conclude s^*-compressibility with $s^* > \frac{d-t}{n}$.

Theorem 3.3. *Let $L : H^t(\Gamma) \to H^{-t}(\Gamma)$ be bounded, and for some $\sigma \in (0, \mu - |t|]$, let both L and its adjoint L' be bounded from $H^{t+\sigma}(\Gamma) \to H^{-t+\sigma}(\Gamma)$. For Ψ being a Riesz basis for $H^t(\Gamma)$ as described above with $t + \tilde{d} > 0$, let $\mathbf{M} = \langle \Psi, L\Psi\rangle$.*

For any $\lambda, \lambda' \in \Lambda$, let

$$|\langle \psi_\lambda, L\psi_{\lambda'}\rangle| \lesssim \left(\frac{2^{-||\lambda|-|\lambda'||/2}}{\delta(\lambda,\lambda')}\right)^{n+2t+2\tilde{d}} \quad \text{when } \delta(\lambda,\lambda') \geq 3\eta, \tag{3.6}$$

where

$$\delta(\lambda,\lambda') := 2^{\min\{|\lambda|,|\lambda'|\}}\mathrm{dist}(\operatorname{supp}\psi_\lambda, \operatorname{supp}\psi_{\lambda'}),$$

and η is from (3.2).

For some $\tau \geq \sigma$, and all $\lambda' \in \Lambda$, $\ell > |\lambda'|$, $A_\ell \subset \{\lambda \in \Lambda : |\lambda| = \ell\}$ with

$$1 \gtrsim \tilde{\delta} := 2^{|\lambda'|}\mathrm{dist}(\cup_{\lambda\in A_\ell}\mathrm{supp}\,\psi_\lambda, \mathrm{sing\,supp}\,\psi_{\lambda'}) \geq 2\eta 2^{|\lambda'|-\ell},$$
$$\mathrm{diam}(\cup_{\lambda\in A_\ell}\mathrm{supp}\,\psi_\lambda) \lesssim 2^{-|\lambda'|},$$

let

$$\left.\begin{array}{l}|\langle w_\ell, L\psi_{\lambda'}\rangle| \\ |\langle L'\psi_{\lambda'}, w_\ell\rangle|\end{array}\right\} \lesssim \|w_\ell\|_{H^t(\Gamma)} \max\left\{\frac{2^{(|\lambda'|-\ell)(t+\tilde{d})}}{\tilde{\delta}^{2t+\tilde{d}-\gamma}}, 2^{(|\lambda'|-\ell)\tau}\right\} \tag{3.7}$$

for any $w_\ell \in \mathrm{span}\{\psi_\lambda : \lambda \in A_\ell\}$.

Let $\alpha \in (\frac{1}{2}, 1)$ and $b_i := (1+i)^{-1-\varepsilon}$ for some $\varepsilon > 0$. Then for $j \in \mathbb{N}$, and with

$$k(j,n) := \frac{j}{n-1} \quad \text{when } n > 1,$$

and $j\frac{\min\{t+\tilde{d},\tau\}}{\min\{t+\mu,t+\tilde{d},\sigma\}} \leq k(j,1) \leq 2^j$, $k(j,1) > j\frac{\min\{t+\tilde{d},\tau\}}{\gamma-t}$, we define the infinite matrix $\mathbf{M}_j$ by replacing all entries $\mathbf{M}_{\lambda,\lambda'} = \langle\psi_\lambda, L\psi_{\lambda'}\rangle$ by zeros when

$$\big||\lambda| - |\lambda'|\big| > k(j,n), \quad \text{or} \tag{3.8}$$

$$\big||\lambda| - |\lambda'|\big| \leq \tfrac{j}{n} \quad \text{and} \quad \delta(\lambda,\lambda') \geq \max\{3\eta, 2^{\alpha(\frac{j}{n}-||\lambda|-|\lambda'||)}\}, \quad \text{or} \tag{3.9}$$

$$\big||\lambda| - |\lambda'|\big| > \tfrac{j}{n} \quad \text{and} \quad \tilde{\delta}(\lambda,\lambda') \geq \max\{2^{n(\frac{j}{n}-||\lambda|-|\lambda'||)}b_{||\lambda|-|\lambda'||-\frac{j}{n}}, 2\eta 2^{-||\lambda|-|\lambda'||}\}, \tag{3.10}$$

where

$$\tilde{\delta}(\lambda,\lambda') := 2^{\min\{|\lambda|,|\lambda'|\}} \times \left\{\begin{array}{ll}\mathrm{dist}(\mathrm{supp}\,\psi_\lambda, \mathrm{sing\,supp}\,\psi_{\lambda'}) & \text{when} |\lambda| > |\lambda'|, \\ \mathrm{dist}(\mathrm{sing\,supp}\,\psi_\lambda, \mathrm{supp}\,\psi_{\lambda'}) & \text{when} |\lambda| < |\lambda'|.\end{array}\right.$$

Then the number of non-zero entries in each row and column of $\mathbf{M}_j$ is of order 2^j, and for any

$$s \leq \min\left\{\tfrac{t+\tilde{d}}{n}, \tfrac{\tau}{n}\right\}, \text{ with } s < \tfrac{\gamma-t}{n-1},\ s \leq \tfrac{\sigma}{n-1} \text{ and } s \leq \tfrac{t+\mu}{n-1} \text{ when } n > 1,$$

it holds that $\|\mathbf{M} - \mathbf{M}_j\| \lesssim 2^{-sj}$.

Remark 3.4. As in Remark 2.4, we infer that $\mathbf{M}$ is s^*-compressible with

$$s^* = \min\{\tfrac{t+\tilde{d}}{n}, \tfrac{\tau}{n}, \tfrac{\gamma-t}{n-1}, \tfrac{\sigma}{n-1}, \tfrac{t+\mu}{n-1}\}$$

($s^* = \min\{t+\tilde{d}, \tau\}$ when $n = 1$). So, if $\tilde{d} > d - 2t$, $\tau > d - t$, and when $n > 1$, $\frac{\min\{\gamma-t,\sigma,t+\mu\}}{n-1} > \frac{d-t}{n}$, then $s^* > \frac{d-t}{n}$.

Proof. Let λ be some given index. Since Γ is a Lipschitz manifold, by the locality of the wavelets, the number of indices λ' with fixed $|\lambda'| \geq |\lambda|$ and $\mathrm{dist}(\mathrm{supp}\,\psi_\lambda, \mathrm{supp}\,\psi_{\lambda'}) \leq R$ is of order $(2^{|\lambda'|}(2^{-|\lambda|} + R))^n$. By using in addition the piecewise smoothness of the wavelets, the number of indices λ' with fixed $|\lambda'| > |\lambda|$ and $\mathrm{dist}(\mathrm{sing\,supp}\,\psi_\lambda, \mathrm{supp}\,\psi_{\lambda'}) \leq R$, where

$2^{-|\lambda'|} \lesssim R \lesssim 2^{-|\lambda|}$ is of order $2^{(|\lambda'|-|\lambda|)(n-1)}2^{|\lambda'|}R$. From this, one may infer that the number of non-zero entries in the λth row or column of $\mathbf{M}_j$ is of order

$$\sum_{-k(j,n)\le|\lambda'|-|\lambda|<0} 1 \;+\; \sum_{0\le|\lambda'|-|\lambda|\le\frac{j}{n}} (2^{|\lambda'|}(2^{-|\lambda|}+2^{-|\lambda|}\max\{3\eta, 2^{\alpha(\frac{j}{n}-|\lambda'|+|\lambda|}\}))^n \;+$$
$$\sum_{\frac{j}{n}<|\lambda'|-|\lambda|\le k(j,n)} 2^{(|\lambda'|-|\lambda|)(n-1)}2^{|\lambda'|}2^{-|\lambda|}\max\{2^{n(\frac{j}{n}-|\lambda'|+|\lambda|)}b_{|\lambda'|-|\lambda|-\frac{j}{n}}, 2\eta 2^{|\lambda|-|\lambda'|}\} \;\eqsim\; 2^j,$$

because of $\alpha<1$ and $\sum_i b_i<\infty$.

Let $\hat{\mathbf{M}}_j$ be defined by $(\mathbf{M}-\hat{\mathbf{M}}_j)_{\lambda,\lambda'} = \begin{cases} \mathbf{M}_{\lambda,\lambda'} & \text{when } \big||\lambda|-|\lambda'|\big|>k(j,n), \\ 0 & \text{otherwise.}\end{cases}$ The continuity assumptions on L, L', together with (3.4) show that for

$$r\in(0,t+\tilde{d}]\cap(0,t+\mu]\cap(0,\sigma]\cap(0,\gamma-t),$$

and $w_\ell\in W_\ell$, $w_{\ell'}\in W_{\ell'}$,

$$\begin{aligned}|\langle w_\ell, Lw_{\ell'}\rangle| &\lesssim \|w_\ell\|_{H^{t-r}(\Gamma)}\|Lw_{\ell'}\|_{H^{-t+r}(\Gamma)}\\ &\lesssim \|w_\ell\|_{H^{t-r}(\Gamma)}\|w_{\ell'}\|_{H^{t+r}(\Gamma)} \lesssim 2^{r(\ell'-\ell)}\|w_\ell\|_{H^t(\Gamma)}\|w_{\ell'}\|_{H^t(\Gamma)},\end{aligned}$$

and analogously, $|\langle w_\ell, Lw_{\ell'}\rangle| = |\langle L'w_\ell, w_{\ell'}\rangle| \lesssim 2^{r(\ell-\ell')}\|w_\ell\|_{H^t(\Gamma)}\|w_{\ell'}\|_{H^t(\Gamma)}$. As in the proof of Theorem 2.3, we conclude that $\|\mathbf{M}-\hat{\mathbf{M}}_j\|\lesssim 2^{-k(j,n)r}$.

As a second step, let $\tilde{\mathbf{M}}_j$ be defined by

$$(\hat{\mathbf{M}}_j-\tilde{\mathbf{M}}_j)_{\lambda,\lambda'} = \begin{cases} \mathbf{M}_{\lambda,\lambda'} & \text{when } \delta(\lambda,\lambda')\ge\max\{1,3\eta,2^{\alpha(\frac{j}{n}-||\lambda|-|\lambda'||)}\} \text{ and } \big||\lambda|-|\lambda'|\big|\le k(j,n), \\ 0 & \text{otherwise.}\end{cases}$$

Let us recall the Schur lemma: If, for some positive scalars ω_λ, $\sum_{\lambda'}\frac{\omega_\lambda|\mathbf{b}_{\lambda,\lambda'}|}{\omega_{\lambda'}}\le c$, and $\sum_\lambda\frac{\omega_{\lambda'}|\mathbf{b}_{\lambda,\lambda'}|}{\omega_\lambda}\le c$, then $\|(\mathbf{b}_{\lambda,\lambda'})_{\lambda,\lambda'\in\Lambda}\|\le c$. We apply this lemma onto $\hat{\mathbf{M}}_j-\tilde{\mathbf{M}}_j$ with $\omega_\lambda = 2^{|\lambda|\frac{n}{2}}$. By the locality of the wavelets, for each λ, ℓ', $R\gtrsim 1$ and $\beta>n$, one has

$$\sum_{\{\lambda':|\lambda'|=\ell',\,\delta(\lambda,\lambda')>R\}}\delta(\lambda,\lambda')^{-\beta}\lesssim R^{-\beta+n}2^{n\max\{0,\ell'-|\lambda|\}}.$$

Because of $t+\tilde{d}>0$, from the decay estimate (3.6) we obtain that

$$\begin{aligned}&\sum_{\lambda'}\frac{\omega_\lambda|(\hat{\mathbf{M}}_j-\tilde{\mathbf{M}}_j)_{\lambda,\lambda'}|}{\omega_{\lambda'}}\\ &\quad\lesssim\sum_{|\lambda'|}2^{(|\lambda|-|\lambda'|)\frac{n}{2}}2^{-||\lambda|-|\lambda'||(\frac{n}{2}+t+\tilde{d})}\Big(\max\{1,2^{\alpha(\frac{j}{n}-||\lambda|-|\lambda'||)}\}\Big)^{-(2\tilde{d}+2t)}2^{n\max\{0,|\lambda'|-|\lambda|\}}\\ &\quad\eqsim 2^{-\frac{j}{n}(t+\tilde{d})},\end{aligned}$$

by $\alpha > \frac{1}{2}$. By the symmetry of the right-hand side of (3.6) in λ, λ', analogously we have $\sum_\lambda \frac{\omega_\lambda |(\hat{\mathbf{M}}_j - \tilde{\mathbf{M}}_j)_{\lambda,\lambda'}|}{\omega_{\lambda'}} \lesssim 2^{-\frac{j}{n}(t+\tilde{d})}$, and so $\|\hat{\mathbf{M}}_j - \tilde{\mathbf{M}}_j\| \lesssim 2^{-\frac{j}{n}(t+\tilde{d})}$.

Given λ' and $\ell > |\lambda'|$, let

$$A_{\ell,\lambda'} = \{|\lambda| = \ell : \tilde{\delta}(\lambda,\lambda') \geq \max\{2^{n(\frac{j}{n}-\ell+|\lambda'|)} b_{\ell-|\lambda'|-\frac{j}{n}}, 2\eta 2^{|\lambda'|-\ell}\},\ \delta(\lambda,\lambda') < \max\{1,3\eta\}\}.$$

Since for $\big||\lambda| - |\lambda'|\big| > \frac{j}{n}$, entries $\mathbf{M}_{\lambda,\lambda'}$ with $\delta(\lambda,\lambda') \geq \max\{1,3\eta\}$ were already been removed from $\hat{\mathbf{M}}_j$, we have

$$(\hat{\mathbf{M}}_j - \mathbf{M}_j)_{\lambda,\lambda'} = \begin{cases} \mathbf{M}_{\lambda,\lambda'} & \begin{cases} \text{when} & \frac{j}{n} < |\lambda| - |\lambda'| \leq k(j,n) & \text{and} & \lambda \in A_{|\lambda|,\lambda'}, \\ \text{or} & \frac{j}{n} < |\lambda'| - |\lambda| \leq k(j,n) & \text{and} & \lambda' \in A_{|\lambda'|,\lambda}, \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

So from the decay estimate (3.7), for any $\mathbf{c}, \mathbf{d} \in \ell_2(\Lambda)$ we have

$$\begin{aligned}
&|\sum_{\frac{j}{n}<\ell-\ell'\leq k(j,n)} \sum_{|\lambda'|=\ell'} \mathbf{d}_{\lambda'} \langle \sum_{\lambda\in A_{\ell,\lambda'}} \mathbf{c}_\lambda \psi_\lambda, L\psi_{\lambda'}\rangle| \\
&\lesssim \sum_{\frac{j}{n}<\ell-\ell'\leq k(j,n)} \sum_{|\lambda'|=\ell'} |\mathbf{d}_{\lambda'}| \max\{\frac{2^{(\ell'-\ell)(t+\tilde{d})}}{(2^{n(\frac{j}{n}-\ell+\ell')} b_{\ell-\ell'-\frac{j}{n}})^{2t+\tilde{d}-\gamma}}, 2^{(\ell'-\ell)\tau}\} \|\sum_{\lambda\in A_{\ell,\lambda'}} \mathbf{c}_\lambda \psi_\lambda\|_{H^t(\Gamma)} \\
&\lesssim \sum_{\frac{j}{n}<\ell-\ell'\leq k(j,n)} \max\{\frac{2^{(\ell'-\ell)(t+\tilde{d})}}{(2^{n(\frac{j}{n}-\ell+\ell')} b_{\ell-\ell'-\frac{j}{n}})^{2t+\tilde{d}-\gamma}}, 2^{(\ell'-\ell)\tau}\} \sqrt{\sum_{|\lambda'|=\ell'} |\mathbf{d}_{\lambda'}|^2} \sqrt{\sum_{|\lambda'|=\ell'} \sum_{\lambda\in A_{\ell,\lambda'}} |\mathbf{c}_\lambda|^2} \\
&\lesssim \sum_{\frac{j}{n}<\ell-\ell'\leq k(j,n)} \max\{\frac{2^{(\ell'-\ell)(t+\tilde{d})}}{(2^{n(\frac{j}{n}-\ell+\ell')} b_{\ell-\ell'-\frac{j}{n}})^{2t+\tilde{d}-\gamma}}, 2^{(\ell'-\ell)\tau}\} \sqrt{\sum_{|\lambda'|=\ell'} |\mathbf{d}_{\lambda'}|^2} \sqrt{\sum_{|\lambda|=\ell} |\mathbf{c}_\lambda|^2},
\end{aligned}$$

where for the last line we have used that for fixed $|\lambda'|$, each λ is contained in at most a uniformly bounded number of sets $A_{|\lambda|,\lambda'}$. Since the analogous estimate is valid with interchanged roles of ℓ and ℓ', and for $s \leq \min\{\frac{t+\tilde{d}}{n}, \frac{\tau}{n}\}$, with $s < \frac{\gamma-t}{n-1}$ when $n > 1$,

$$\sum_{m=1}^{k(j,n)-\frac{j}{n}} \max\{2^{-(m+\frac{j}{n})(t+\tilde{d})} \left(2^{-mn} b_m\right)^{\gamma-2t-\tilde{d}}, 2^{-(m+\frac{j}{n})\tau}\} \lesssim 2^{-sj}, \tag{3.11}$$

we conclude that for such s, $\|\tilde{\mathbf{M}}_j - \mathbf{M}_j\| \lesssim 2^{-sj}$.

A combination of the estimates for $\mathbf{M} - \hat{\mathbf{M}}_j$, $\hat{\mathbf{M}}_j - \tilde{\mathbf{M}}_j$ and $\tilde{\mathbf{M}}_j - \mathbf{M}_j$ shows that for $s \leq \min\{\frac{t+\tilde{d}}{n}, \frac{\tau}{n}\}$, with for $n > 1$, $s < \frac{\gamma-t}{n-1}$, $s \leq \frac{\sigma}{n-1}$ and $s \leq \frac{t+\mu}{n-1}$, it holds that $\|\mathbf{M} - \mathbf{M}_j\| \lesssim 2^{-sj}$. □

3.2. **Singular integral operators.** In §3.3 and 3.4, we verify the decay estimates (3.6) and (3.7) for operators

$$Lu(x) = \int_\Gamma K(x,y) u(y) d\mu(y), \qquad (x \in \Gamma),$$

with kernels that satisfy for all $1 \le q, q' \le M$, $\xi \in \Omega_q$, $\eta \in \Omega_{q'}$,

$$|\partial_\xi^\alpha \partial_\eta^\beta K(\kappa_q(\xi), \kappa_{q'}(\eta))| \lesssim \operatorname{dist}(\kappa_q(\xi), \kappa_{q'}(\eta)))^{-(n+2t+|\alpha|+|\beta|)}, \quad (n+2t+|\alpha|+|\beta| > 0), \tag{3.12}$$

(cf. [DHS02, Def. 2.1]). Following [DHS02], we emphasize that (3.12) requires patchwise smoothness but no global smoothness of Γ. Only assuming global Lipschitz continuity of Γ, the kernel of a boundary integral operator of order $2t$ can be shown to satisfy (3.12).

If Γ is a C^∞-manifold, then these boundary integral operators are known to be pseudo-differential operators, meaning that for any $\sigma \in \mathbb{R}$ they define bounded mappings from $H^{t+\sigma}(\Gamma) \to H^{-t+\sigma}(\Gamma)$. In this case we may conclude that $\mathbf{M}$ is s^*-compressibility with $s^* = \min\{\frac{t+\tilde{d}}{n}, \frac{\gamma-t}{n-1}\}$. For Γ being only Lipschitz continuous, for the classical boundary integral equations it is known that $L : H^{t+\sigma}(\Gamma) \to H^{-t+\sigma}(\Gamma)$ is bounded for the in this case maximum possible value $\sigma = 1 - |t|$ (cf. [Cos88]). With increasing smoothness of Γ one may expect this boundedness for larger values of σ. Few results in this direction seem yet available.

3.3. **Decay estimate** (3.6)**.** This estimate for singular integral operators with wavelets that satisfy the cancellation property (3.2) of order $\tilde{d}$ has first been proved in [Sch98] for C^∞-manifolds. In [DS99c], it has been shown for Lipschitz manifolds for a specific wavelet construction. For convenience, in this subsection we recall the arguments used there, and show that they also apply to the general setting discussed in this paper.

With $\eta > 0$ from (3.2), let $\lambda, \lambda' \in \Lambda$ with $\delta(\lambda, \lambda') \ge 3\eta$. Then with $\Gamma_{\lambda,\eta} := B(\operatorname{supp}\psi_\lambda; 2^{-|\lambda|}\eta)$, it holds that

$$2^{\min\{|\lambda|,|\lambda'|\}} \operatorname{dist}(\Gamma_{\lambda,\eta}, \Gamma_{\lambda',\eta}) \ge \tfrac{1}{3}\delta(\lambda, \lambda') > 0.$$

Because of

$$n + 2t + 2\tilde{d} > 0,$$

from (3.2) and (3.12), we infer that

$$\begin{aligned}
|\langle \psi_\lambda, L\psi_{\lambda'}\rangle| &\lesssim 2^{-|\lambda|(\frac{n}{2}+t+\tilde{d})} \sup_{\substack{|\alpha|=\tilde{d}, 1\le q\le M,\\ \xi\in\kappa_q^{-1}(\Gam_{\lambda,\eta}\cap\Gamma_q)}} \Big| \int_\Gamma \partial_\xi^\alpha K(\kappa_q(\xi), y)\psi_{\lambda'}(y)d\mu(y)\Big| \\
&\lesssim 2^{-(|\lambda|+|\lambda'|)(\frac{n}{2}+t+\tilde{d})} \sup_{\substack{|\alpha|=\tilde{d}, 1\le q\le M,\\ \xi\in\kappa_q^{-1}(\Gamma_{\lambda,\eta}\cap\Gamma_q)}} \sup_{\substack{|\beta|=\tilde{d}, 1\le q'\le M,\\ \zeta\in\kappa_{q'}^{-1}(\Gamma_{\lambda',\eta}\cap\Gamma_{q'})}} |\partial_\xi^\alpha \partial_\zeta^\beta K(\kappa_q(\xi), \kappa_{q'}(\zeta))| \\
&\lesssim 2^{-(|\lambda|+|\lambda'|)(\frac{n}{2}+t+\tilde{d})} (2^{-\min\{|\lambda|,|\lambda'|\}}\delta(\lambda, \lambda'))^{-(n+2t+2\tilde{d})} \\
&= \left(\frac{2^{-||\lambda|-|\lambda'||/2}}{\delta(\lambda, \lambda')}\right)^{n+2t+2\tilde{d}}.
\end{aligned}$$

3.4. **Decay estimate** (3.7)**.** Let $\Gamma = \cup_{q=1}^M \overline{\Gamma_q}$ be a compact n-dimensional, globally $C^{\mu-1,1}$-manifold in $\mathbb{R}^{n+1}$, where the Γ_q are C^∞-manifolds as described in §3.1. For some $|t| \le \mu$, let L be a singular integral operator of order $2t$ as described in §3.2, which is bounded from $H^t(\Gamma) \to H^{-t}(\Gamma)$, and for which there exists a $\sigma > 0$ such that $L, L' : H^{t+\sigma}(\Gamma) \to H^{-t+\sigma}(\Gamma)$

are bounded. Let Ψ be a Riesz basis for $H^t(\Gamma)$ as described in §3.1, consisting of local and piecewise smooth $C^k(\Gamma)$ wavelets, that have cancellation properties of order $\tilde{d}$, where $k \in \mathbb{N}_0 \cup \{-1\}$, $k < \mu$ and $\gamma := k + \frac{3}{2} > t$.

In addition, in this subsection we assume that

$$\tilde{d} > \gamma - 2t. \tag{3.13}$$

Furthermore, with $\tilde{H}^s(\Gamma_q) := \begin{cases} H^s(\Gamma_q) & \text{when } s \geq 0, \\ (H_0^{-s}(\Gamma_q))' & \text{when } s < 0, \end{cases}$ we assume that there exists a $\tau \in (0, \mu - |t|]$ such that for all $1 \leq q \leq M$,

$$L : H^{t+\tau}(\Gamma) \to \tilde{H}^{-t+\tau}(\Gamma_q) \quad \text{is bounded.} \tag{3.14}$$

Remark 3.5. Since for any $|s| \leq \mu$, the restriction of functions on Γ to Γ_q is a bounded mapping from $H^s(\Gamma)$ to $\tilde{H}^s(\Gamma_q)$, from the boundedness of $L : H^{t+\sigma}(\Gamma) \to H^{-t+\sigma}(\Gamma)$, it follows that in any case (3.14) is valid for $\tau = \sigma$. So for example for Γ being a C^∞-manifold, (3.14) is valid for any $\tau \in \mathbb{R}$. Yet, in particular when $t < 0$, for Γ being less smooth it might happen that (3.14) is valid for a τ that is strictly larger that any σ for which $L : H^{t+\sigma}(\Gamma) \to H^{-t+\sigma}(\Gamma)$ is bounded.

Proposition 3.6. *In the above setting, for all $\lambda' \in \Lambda$, $\ell > |\lambda'|$ and $A_\ell \subset \{\lambda \in \Lambda : |\lambda| = \ell\}$ with $\tilde{\delta} := 2^{|\lambda'|}\mathrm{dist}(\cup_{\lambda \in A_\ell} \mathrm{supp}\, \psi_\lambda, \mathrm{sing\,supp}\, \psi_{\lambda'}) \geq 2\eta 2^{|\lambda'|-\ell}$ and $\rho := 2^{|\lambda'|}\mathrm{diam}(\cup_{\lambda \in A_\ell} \mathrm{supp}\, \psi_\lambda)$, it holds that*

$$\left.\begin{matrix} |\langle w_\ell, L\psi_{\lambda'}\rangle| \\ |\langle L'\psi_{\lambda'}, w_\ell\rangle| \end{matrix}\right\} \lesssim$$
$$\|w_\ell\|_{H^t(\Gamma)} \max\left\{2^{(|\lambda'|-\ell)(t+\tilde{d})}\tilde{\delta}^{-2t-\tilde{d}+k+1}\rho^{\frac{n-1}{2}} \min\{\tilde{\delta}, \rho\}^{\frac{1}{2}}, 2^{(|\lambda'|-\ell)\min\{\tau, t+\tilde{d}\}}\right\},$$

for any $w_\ell \in \mathrm{span}\{\psi_\lambda : \lambda \in A_\ell\}$. By substituting $\tilde{\delta}, \rho \lesssim 1$ and by using that $k + \frac{3}{2} = \gamma$, the decay estimate (3.7) *is obtained.*

The proof of Proposition 3.6 will largely rely on techniques developed in [Sch98, DHS02]. We start with a lemma that will be used to estimate contributions from pairs of functions with disjoint supports.

Lemma 3.7. *For $\lambda' \in \Lambda$, $J \subset \{1, \ldots, m\}$, let either $E = \cup_{i \in J} \overline{\Xi_{\lambda',i}}$ or $E = \overline{\Gamma \setminus \cup_{i \in J} \overline{\Xi_{\lambda',i}}}$, and for $\ell > |\lambda'|$, let $A_\ell \subset \{\lambda \in \Lambda : |\lambda| = \ell\}$ with $\bar{\delta} := 2^{|\lambda'|}\mathrm{dist}(\cup_{\lambda \in A_\ell} \mathrm{supp}\, \psi_\lambda, E) \geq 2\eta 2^{|\lambda'|-\ell}$ and $\rho := 2^{|\lambda'|}\mathrm{diam}(\cup_{\lambda \in A_\ell} \mathrm{supp}\, \psi_\lambda)$. Then for any $w_\ell \in \mathrm{span}\{\psi_\lambda : \lambda \in A_\ell\}$, $v \in L_\infty(\Gamma)$ with $\mathrm{supp}\, v \subset E$ and*

$$|v(y)| \lesssim 2^{(k+1+\frac{n}{2}-t)|\lambda'|}\mathrm{dist}(y, \partial E)^{k+1},$$

it holds that

$$|\langle w_\ell, Lv\rangle| \lesssim \|w_\ell\|_{H^t(\Gamma)} 2^{(|\lambda'|-\ell)(t+\tilde{d})}\bar{\delta}^{-2t-\tilde{d}+k+1}\rho^{\frac{n-1}{2}} \min\{\tilde{\delta}, \rho\}^{\frac{1}{2}}.$$

Proof. By (3.2), for any $\lambda \in A_\ell$, it holds that

$$|\langle \psi_\lambda, Lv\rangle| \lesssim 2^{-\ell(\frac{n}{2}+t+\tilde{d})} \max_{1 \leq q \leq M} |Lv|_{W_\infty^{\tilde{d}}(B(\mathrm{supp}\, \psi_\lambda; 2^{-\ell}\eta) \cap \Gamma_q)}.$$

For any $1 \le q \le M$, $x = \kappa_q(\xi) \in B(\operatorname{supp}\psi_\lambda; 2^{-\ell}\eta) \cap \Gamma_q$ and $|\alpha| = \tilde{d}$, by (3.12) it holds that

$$\begin{aligned}
|\partial_\xi^\alpha((Lv)\circ\kappa_q)(\xi)| &\lesssim \int_E |\partial_\xi^\alpha K(\kappa_q(\xi), y)| 2^{(k+1+\frac{n}{2}-t)|\lambda'|} \operatorname{dist}(y, \partial E)^{k+1} d\mu(y) \\
&\lesssim 2^{(k+1+\frac{n}{2}-t)|\lambda'|} \int_E |x-y|^{-(n+2t+\tilde{d})} |x-y|^{k+1} d\mu(y) \\
&\lesssim 2^{(k+1+\frac{n}{2}-t)|\lambda'|} \int_{z\in\mathbb{R}^n, |z|\ge \operatorname{dist}(x,E)} |z|^{k+1-(n+2t+\tilde{d})} dz \\
&\eqsim 2^{(k+1+\frac{n}{2}-t)|\lambda'|} \operatorname{dist}(x,E)^{-2t-\tilde{d}+k+1},
\end{aligned}$$

because of $-2t - \tilde{d} + k + 1 < 0$ by (3.13).

Writing $w_\ell = \sum_{\lambda\in A_\ell} c_\lambda \psi_\lambda$, by the locality of the wavelets and because of $\bar{\delta} \ge 2\eta 2^{|\lambda'|-\ell}$ we find that

$$\begin{aligned}
|\langle w_\ell, Lv\rangle| \lesssim& \Big(\sum_{\lambda\in A_\ell} |c_\lambda|^2\Big)^{\frac12} 2^{-\ell(\frac{n}{2}+t+\tilde{d})} 2^{(k+1+\frac{n}{2}-t)|\lambda'|} \\
&\times \Big(\sum_{\lambda\in A_\ell} \sup_{x\in B(\operatorname{supp}\psi_\lambda; 2^{-\ell}\eta)} \operatorname{dist}(x,E)^{-4t-2\tilde{d}+2k+2}\Big)^{\frac12} \\
\eqsim& \|w_\ell\|_{H^t(\Gamma)} 2^{-\ell(\frac{n}{2}+t+\tilde{d})} 2^{(k+1+\frac{n}{2}-t)|\lambda'|} \\
&\times \Big(2^{\ell n} \int_{\cup_{\lambda\in A_\ell}\operatorname{supp}\psi_\lambda} \operatorname{dist}(x,E)^{-4t-2\tilde{d}+2k+2} d\mu(x)\Big)^{\frac12}. \qquad (3.15)
\end{aligned}$$

Since $-4t - 2\tilde{d} + 2k + 2 < -1$ by (3.13), because of the geometry of E we may estimate

$$\begin{aligned}
&\int_{\cup_{\lambda\in A_\ell}\operatorname{supp}\psi_\lambda} \operatorname{dist}(x,E)^{-4t-2\tilde{d}+2k+2} d\mu(x) \\
&\quad\lesssim (2^{-|\lambda'|}\rho)^{n-1} \int_{2^{-|\lambda'|}\bar{\delta}}^{2^{-|\lambda'|}\bar{\delta}+2^{-|\lambda'|}\rho} z^{-4t-2\tilde{d}+2k+2} dz \\
&\quad\eqsim (2^{-|\lambda'|}\rho)^{n-1} [(2^{-|\lambda'|}\bar{\delta})^{-4t-2\tilde{d}+2k+3} - (2^{-|\lambda'|}(\bar{\delta}+\rho))^{-4t-2\tilde{d}+2k+3}] \\
&\quad\eqsim 2^{|\lambda'|(4t+2\tilde{d}-2k-2-n)} \rho^{n-1} \bar{\delta}^{-4t-2\tilde{d}+2k+2} \min\{\bar{\delta}, \rho\}.
\end{aligned}$$

By substituting this result into (3.15) the proof is completed. □

Proof of Proposition 3.6. **(I)**. Let $\lambda' \in \Lambda$, $\ell > |\lambda'|$ and $A_\ell \subset \{\lambda \in \Lambda : |\lambda| = \ell\}$ with $\tilde{\delta} := 2^{|\lambda'|}\operatorname{dist}(\cup_{\lambda\in A_\ell}\operatorname{supp}\psi_\lambda, \operatorname{sing\,supp}\psi_{\lambda'}) \ge 2\eta 2^{|\lambda'|-\ell}$ and $\rho := 2^{|\lambda'|}\operatorname{diam}(\cup_{\lambda\in A_\ell}\operatorname{supp}\psi_\lambda)$ and let $w_\ell \in \operatorname{span}\{\psi_\lambda : \lambda \in A_\ell\}$. It suffices to prove the bound for $\langle w_\ell, L\psi_{\lambda'}\rangle$, since the proof for $\langle L'\psi_{\lambda'}, w_\ell\rangle$ is similar.

For $1 \le i \le m$, we define $w_\ell^{(i)}$ by

$$w_\ell^{(i)}(x) = \begin{cases} w_\ell(x) & \text{when } x \in \Xi_{\lambda',i}, \\ 0 & \text{elsewhere,} \end{cases}$$

and set $w_\ell^{(0)} = w_\ell - \sum_{i=1}^m w_\ell^{(i)}$, meaning that $\operatorname{supp} w_\ell^{(0)} \cap \operatorname{supp}\psi_{\lambda'} = \emptyset$.

We assume that $\operatorname{dist}(\operatorname{supp}\psi_\lambda, \operatorname{sing\,supp}\psi_{\lambda'}) \geq 2\eta 2^{-|\lambda|}$ implies that either $\operatorname{supp}\psi_\lambda \subset \overline{\Xi_{\lambda',i}}$ for some $1 \leq i \leq m$, or $\operatorname{supp}\psi_\lambda \cap \operatorname{supp}\psi_{\lambda'} = \emptyset$. In the very unlikely situation that this does not hold "automatically", we can always increase the parameter η such that this is true, since $\operatorname{diam}(\operatorname{supp}\psi_\lambda) \lesssim 2^{-|\lambda|}$. Under this assumption, for all i we have $w_\ell^{(i)} \in W_\ell$ and so $\|w_\ell^{(i)}\|_{H^t(\Gamma)} \lesssim \|w_\ell\|_{H^t(\Gamma)}$.

(II). We consider $\langle w_\ell^{(0)}, L\psi_{\lambda'}\rangle$. Let $E = \operatorname{supp}\psi_{\lambda'}$. If i is such that $\overline{\Xi_{\lambda',i}} \cap \partial E \neq \emptyset$, then because of $\psi_{\lambda'} \in C^k(\Gamma)$ and (3.1) it follows that

$$|\psi_{\lambda'}(y)| \lesssim 2^{(k+1+\frac{n}{2}-t)|\lambda'|}\operatorname{dist}(y,\partial E)^{k+1}, \qquad (y \in \Xi_{\lambda',i}). \tag{3.16}$$

If $\overline{\Xi_{\lambda',i}} \cap \partial E = \emptyset$, then by the "shape regularity" of all sets $\Xi_{\lambda',i'}$ for $i' \neq i$, we have $\operatorname{dist}(\Xi_{\lambda',i}, E) \gtrsim 2^{-|\lambda'|}$, and so (3.16) follows from $|\psi_{\lambda'}(y)| \lesssim 2^{(\frac{n}{2}-t)|\lambda'|}$. From an application of Lemma 3.7 with "w"$= w_\ell^{(0)}$ and "v"$= \psi_{\lambda'}$, we conclude that

$$|\langle w_\ell^{(0)}, L\psi_{\lambda'}\rangle| \lesssim \|w_\ell\|_{H^t(\Gamma)} 2^{(|\lambda'|-\ell)(t+\tilde{d})}\tilde{\delta}^{-2t-\tilde{d}+k+1}\rho^{\frac{n-1}{2}}\min\{\tilde{\delta},\rho\}^{\frac{1}{2}}. \tag{3.17}$$

(III). Let $1 \leq i \leq m$, and let $1 \leq q \leq M$ such that $\Xi_{\lambda',i} \subset \Gamma_q$. Note that $\operatorname{supp} w_\ell^{(i)} \subset \operatorname{supp}\psi_\lambda$. From (3.5), for $r \in [-\tilde{d}, \gamma)$, $s < \gamma$, we have $\|w_\ell^{(i)}\|_{H^r(\Gamma_q)} \lesssim 2^{\ell(r-s)}\|w_\ell^{(i)}\|_{H^s(\Gamma_q)}$. We have to handle the less interesting case $\tau < \gamma - t$ separately.

(a). If $\tau < \gamma - t$, and so $\tau \leq t + \tilde{d}$ by (3.13), then by $\tau \leq \mu - t$, the continuity of L as stated in (3.14), and when $t - \tau > 0$ additionally by $w_\ell^{(i)} \in H_0^{t-\tau}(\Gamma_q)$, we have

$$\begin{aligned}|\langle w_\ell^{(i)}, L\psi_{\lambda'}\rangle| &\lesssim \|w_\ell^{(i)}\|_{H^{t-\tau}(\Gamma_q)}\|L\psi_{\lambda'}\|_{\tilde{H}^{\tau-t}(\Gamma_q)}\\ &\lesssim 2^{(t-\tau)\ell}\|w_\ell^{(i)}\|\|\psi_{\lambda'}\|_{H^{\tau+t}(\Gamma)} \lesssim 2^{(|\lambda'|-\ell)\tau}\|w_\ell\|_{H^t(\Gamma)},\end{aligned}$$

which completes the proof in this case.

(b). Let now $\tau + t \geq \gamma \geq 0$. By assumption, $\psi_{\lambda'} \circ \kappa_q$ is smooth on $\kappa_q^{-1}(\Xi_{\lambda',i})$, which is a uniformly Lipschitz domain. From (3.1) and Remark 2.1, we learn that $(\psi_{\lambda'} \circ \kappa_q)|_{\kappa_q^{-1}(\Xi_{\lambda',i})}$ has an extension to a smooth function $\varphi_{\lambda',i}$, with for $s \geq 0$ and $p \in [1,\infty]$, $\|\varphi_{\lambda',i}\|_{W_p^s(\mathbb{R}^n)} \lesssim 2^{(s-t+\frac{n}{2}-\frac{n}{p})|\lambda'|}$. By multiplying $\varphi_{\lambda',i}$ by a smooth function that is one on Ω_q, and has support inside the "extended" domain $\hat{\Omega}_q$, we may assume that $\operatorname{supp}\varphi_{\lambda',i} \subset \hat{\Omega}_q$, so that $\varphi_{\lambda',i} \circ \hat{\kappa}_q^{-1} \in H^\mu(\Gamma)$ with for $s \in [0,\mu]$, $\|\varphi_{\lambda',i} \circ \hat{\kappa}_q^{-1}\|_{H^s(\Gamma)} \lesssim 2^{(s-t)|\lambda'|}$. With $\max\{0,-t\} \leq s := \min\{\tau, t+\tilde{d}\} \leq \mu - t$, the same arguments as applied in **(a)** show that

$$\begin{aligned}|\langle w_\ell^{(i)}, L(\varphi_{\lambda',i} \circ \hat{\kappa}_q^{-1})\rangle| &\lesssim \|w_\ell^{(i)}\|_{H^{t-s}(\Gamma_q)}\|L(\varphi_{\lambda',i} \circ \hat{\kappa}_q^{-1})\|_{\tilde{H}^{s-t}(\Gamma_q)}\\ &\lesssim 2^{(t-s)\ell}\|w_\ell^{(i)}\|\|\varphi_{\lambda',i} \circ \hat{\kappa}_q^{-1}\|_{H^{s+t}(\Gamma)} \lesssim 2^{(|\lambda'|-\ell)s}\|w_\ell\|_{H^t(\Gamma)}.\end{aligned} \tag{3.18}$$

There remains to estimate $|\langle w_\ell^{(i)}, L(\psi_{\lambda'} - \varphi_{\lambda',i} \circ \hat{\kappa}_q^{-1})\rangle|$. Recall that $\operatorname{supp} w_\ell^{(i)} \subset \Xi_{\lambda',i}$, whereas $\psi_{\lambda'} - \varphi_{\lambda',i} \circ \hat{\kappa}_q^{-1}$ vanishes on $\Xi_{\lambda',i}$. The global smoothness of $\psi_{\lambda'}$ will ensure that directly outside $\Xi_{\lambda',i}$, $\psi_{\lambda'}$ is sufficiently close to the smooth extension $\varphi_{\lambda',i} \circ \hat{\kappa}_q^{-1}$ of $\psi_{\lambda'}|_{\Xi_{\lambda',i}}$ so that an application of Lemma 3.7 will lead to the desired bound. We have to distinguish between a number of cases.

Suppose $i' \neq i$ with $\overline{\Xi_{\lambda',i'}} \cap \overline{\Xi_{\lambda',i}} \neq \emptyset$, and let q' such that $\Xi_{\lambda',i'} \subset \Gamma_{q'}$. Then from

(a) $\sup_{\xi\in\kappa_{q'}^{-1}(\Xi_{\lambda',i'})}|\partial^\beta(\psi_{\lambda'}\circ\kappa_{q'})(\xi)|\lesssim 2^{(|\beta|+\frac{n}{2}-t)|\lambda'|}$, $(\beta\in I\!N_0)$ ((3.1)),

(b) $\kappa_{q'}^{-1}\circ\hat\kappa_q\in C^{\mu-1,1}(\hat\kappa_q^{-1}(\Gamma_{q'}\cap\operatorname{Im}\hat\kappa_q))$, where $\mu>k$,

(c) $\sup_{\xi\in\hat\Omega_q}|\partial^\beta\varphi_{\lambda',i}(\xi)|\lesssim 2^{(|\beta|+\frac{n}{2}-t)|\lambda'|}$, $(\beta\in I\!N_0)$,

(d) $\psi_{\lambda'}\circ\hat\kappa_q-\varphi_{\lambda',i}\in C^k(\hat\Omega_q)$, where it vanishes on $\hat\kappa_q^{-1}(\Xi_{\lambda',i})$,

one infers that $|(\psi_{\lambda'}\circ\hat\kappa_q-\varphi_{\lambda',i})(\xi)|\lesssim 2^{(k+1+\frac{n}{2}-t)|\lambda'|}\operatorname{dist}(\xi,\hat\kappa_q^{-1}(\Xi_{\lambda',i}))^{k+1}$ when $\xi\in\hat\kappa_q^{-1}(\Xi_{\lambda',i'}\cap\operatorname{Im}\hat\kappa_q)$, so that for $y\in\Xi_{\lambda',i'}\cap\operatorname{Im}\hat\kappa_q$,

$$|(\psi_{\lambda'}-\varphi_{\lambda',i}\circ\hat\kappa_q^{-1})(y)|\lesssim 2^{(k+1+\frac{n}{2}-t)|\lambda'|}\operatorname{dist}(y,\Xi_{\lambda',i})^{k+1}. \tag{3.19}$$

If $\overline{\Gamma\backslash\operatorname{supp}\psi_{\lambda',i}}\cap\overline{\Xi_{\lambda',i}}\neq\emptyset$, then (c), (d) show that (3.19) is also valid for $y\in(\Gamma\backslash\operatorname{supp}\psi_{\lambda',i})\cap\operatorname{Im}\hat\kappa_q$. For the remaining cases that either $y\in\Xi_{\lambda',i'}$ with $\overline{\Xi_{\lambda',i'}}\cap\overline{\Xi_{\lambda',i}}=\emptyset$, or $y\in(\Gamma\backslash\operatorname{supp}\psi_{\lambda',i})\cap\operatorname{Im}\hat\kappa_q$ when $\overline{\Gamma\backslash\operatorname{supp}\psi_{\lambda',i}}\cap\overline{\Xi_{\lambda',i}}=\emptyset$, or $y\notin\operatorname{Im}\hat\kappa_q$, the "shape regularity" of all sets $\Xi_{\lambda',i}$ show that $\operatorname{dist}(y,\Xi_{\lambda',i})\gtrsim 2^{-|\lambda'|}$, and so from $|(\psi_{\lambda'}-\varphi_{\lambda',i}\circ\hat\kappa_q^{-1})(y)|\lesssim 2^{(\frac{n}{2}-t)|\lambda'|}$, we conclude that (3.19) is valid for *all* $y\in\Gamma\backslash\Xi_{\lambda',i}$. An application of Lemma 3.7 with $E=\overline{\Gamma\backslash\Xi_{\lambda',i}}$, "$w$"$=w_\ell^{(i)}$ and "v"$=\psi_{\lambda'}-\varphi_{\lambda',i}\circ\hat\kappa_q^{-1}$ now shows that

$$|\langle w_\ell^{(i)},L(\psi_{\lambda'}-\varphi_{\lambda',i}\circ\hat\kappa_q^{-1})\rangle|\lesssim\|w_\ell\|_{H^t(\Gamma)}2^{(|\lambda'|-\ell)(t+\tilde d)}\tilde\delta^{-2t-\tilde d+k+1}\rho^{\frac{n-1}{2}}\min\{\tilde\delta,\rho\}^{\frac12},$$

which together with (3.17) and (3.18) completes the proof for the case that $\tau+t\geq\gamma$. □

Below we discuss some extensions of Proposition 3.6.

In case $\operatorname{supp}w_\ell\cap\operatorname{supp}\psi_{\lambda'}=\emptyset$, then step **(II)** of the proof shows that

$$\left.\begin{matrix}|\langle w_\ell,L\psi_{\lambda'}\rangle|\\ |\langle L'\psi_{\lambda'},w_\ell\rangle|\end{matrix}\right\}\lesssim\|w_\ell\|_{H^t(\Gamma)}2^{(|\lambda'|-\ell)(t+\tilde d)}\tilde\delta^{-2t-\tilde d+k+1}\rho^{\frac{n-1}{2}}\min\{\tilde\delta,\rho\}^{\frac12}. \tag{3.20}$$

In case $\operatorname{supp}\psi_{\lambda'}$ is contained in one patch $\overline{\Gamma_q}$, then

$$\left.\begin{matrix}|\langle w_\ell,L\psi_{\lambda'}\rangle|\\ |\langle L'\psi_{\lambda'},w_\ell\rangle|\end{matrix}\right\}\lesssim$$

$$\|w_\ell\|_{H^t(\Gamma)}\max\left\{2^{(|\lambda'|-\ell)(t+\tilde d)}\tilde\delta^{-2t-\tilde d+k+1}\rho^{\frac{n-1}{2}}\min\{\tilde\delta,\rho\}^{\frac12},2^{(|\lambda'|-\ell)(t+\tilde d)}\right\}. \tag{3.21}$$

To see this, consider $\langle w_\ell^{(i)},L\psi_{\lambda'}\rangle$ as in step **(III)** of the proof. The functions $w_\ell^{(i)}$ and $\psi_{\lambda'}$ both live on the same patch $\overline{\Gamma_q}$, and so $\langle w_\ell^{(i)},L\psi_{\lambda'}\rangle$ is independent of $\Gamma\backslash\Gamma_q$. That is, $\langle w_\ell^{(i)},L\psi_{\lambda'}\rangle$ gets the same value if Γ is replaced by some extension of Γ_q to a compact C^∞-manifold. If we now follow the proof of Proposition 3.6 using this smooth manifold, then $\tau\geq\sigma$ can be any positive number, with which we conclude (3.21).

The estimate (3.21) is also valid when w_ℓ vanishes on all $\Xi_{\lambda',i}$ that do not have distance $\gtrsim 2^{-|\lambda'|}$ to an interface between different patches. Indeed, for the other $\Xi_{\lambda',i}$ the smooth extension of $\psi_\lambda|_{\Xi_{\lambda',i}}$ can always be chosen to be supported in one patch. So only in the remaining situation that $\operatorname{supp}\psi_{\lambda'}$ is not contained in one patch $\overline{\Gamma_q}$ *and* w_ℓ does not vanish on all $\Xi_{\lambda',i}$ adjacent to a patch interface, the parameter τ from (3.14) enters the upperbound for $|\langle w_\ell,L\psi_{\lambda'}\rangle|$ and $|\langle L'\psi_{\lambda'},w_\ell\rangle|$. Nevertheless, unfortunately the value of τ does enter the estimate for s^*-compressibility, and although this can be expected also for less smooth

manifolds, only for C^∞-manifolds we have shown that the value of s^* is never limited by τ.

As we have said, our proof of Proposition 3.6 is a modification of the approach from [DHS02]. A direct application of the technique from that paper yields the estimate

$$\left.\begin{array}{l}|\langle w_\ell, L\psi_{\lambda'}\rangle| \\ |\langle L'\psi_{\lambda'}, w_\ell\rangle|\end{array}\right\} \lesssim$$

$$\|w_\ell\|_{H^t(\Gamma)} \max\left\{2^{(|\lambda'|-\ell)(t+\tilde{d})}\tilde{\delta}^{-2t-\tilde{d}}\rho^{\frac{n-1}{2}}\min\{\tilde{\delta},\rho\}^{\frac{1}{2}}, 2^{(|\lambda'|-\ell)(t+\tilde{d})}\right\}, \tag{3.22}$$

which is less sharp when $k \geq 0$, except possibly, dependent on the value of τ from (3.14), in cases where (3.20) or (3.21) do not apply. If instead of Proposition 3.6 we apply (3.22) in the proof of Theorem 3.3, then by only adapting (3.11) we find that $\mathbf{M}$ is s^*-compressible with $s^* = \min\{\frac{t+\tilde{d}}{n}, \frac{1/2-t}{n-1}, \frac{\sigma}{n-1}, \frac{t+\mu}{n-1}\}$, and $s^* = t+\tilde{d}$ when $n = 1$.

The idea to obtain (3.22) is to write for $1 \leq i \leq m$,

$$\langle w_\ell^{(i)}, L\psi_{\lambda'}\rangle = \langle w_\ell^{(i)}, L(\psi_{\lambda'}|_{\Xi_{\lambda',i}})\rangle + \langle w_\ell^{(i)}, L(\psi_{\lambda'} - \psi_{\lambda'}|_{\Xi_{\lambda',i}})\rangle.$$

Since $\psi_{\lambda'} - \psi_{\lambda'}|_{\Xi_{\lambda',i}}$ is generally discontinuous over $\partial\Xi_{\lambda',i}$, the second term can only be bounded by $2^{(|\lambda'|-\ell)(t+\tilde{d})}\tilde{\delta}^{-2t-\tilde{d}}\rho^{\frac{n-1}{2}}\min\{\tilde{\delta},\rho\}^{\frac{1}{2}}$. On the other hand, $w_\ell^{(i)}$ and $\psi_{\lambda'}|_{\Xi_{\lambda',i}}$ both live on the same patch $\overline{\Gamma_q}$, so for estimating the first term we can always think of Γ as being a C^∞-manifold, so that for any $\sigma > 0$, $L : H^{t+\sigma}(\Gamma) \to H^{-t+\sigma}(\Gamma)$ is bounded. Writing $\psi_{\lambda'}|_{\Xi_{\lambda',i}}$ as the sum of a smooth function on this manifold plus their difference which is zero on $\Xi_{\lambda',i}$, but which generally is discontinuous over $\partial\Xi_{\lambda',i}$, the estimate (3.22) can be obtained.

The upperbounds for $|\langle w_\ell, L\psi_{\lambda'}\rangle|$ and $|\langle L'\psi_{\lambda'}, w_\ell\rangle|$ from Proposition 3.6, (3.21) and (3.22) are all given as a minimum of two terms. The first term decreases when $\rho := 2^{|\lambda'|}\mathrm{diam}(\cup_{\lambda\in A_\ell}\mathrm{supp}\,\psi_\lambda)$ decreases. The second term is independent of ρ. Responsible for this term is the part of w_ℓ spanned by those ψ_λ with $\lambda \in \tilde{A}_\ell := \{\lambda \in A_\ell : \mathrm{supp}\,\psi_\lambda \subset \mathrm{supp}\,\psi_{\lambda'}\}$. Since for our application of Proposition 3.6 in Theorem 3.3 it held that $1 \gtrsim \rho \geq \tilde{\rho} := 2^{|\lambda'|}\mathrm{diam}(\cup_{\lambda\in\tilde{A}_\ell}\mathrm{supp}\,\psi_\lambda) \gtrsim 1$, there was no need to construct an estimate that improves with decreasing diameter. However, for example to give sharp estimates of the individual entries of the infinite stiffness matrix $\langle\Psi, L\Psi\rangle$, such a need does exist.

For simplicity, let us consider a situation where we may pretend that Γ is a compact C^∞-manifold, as for (3.21) and (3.22). By the Sobolev embedding theorem, boundedness for any $\sigma \geq 0$ of $L : H^{t+\sigma}(\Gamma) \to H^{-t+\sigma}(\Gamma)$ implies boundedness for any $\sigma \geq 0$ and $p \in [2,\infty)$ of $L : H^{t+\sigma}(\Gamma) \to W_p^{-t+\sigma-n(\frac{1}{2}-\frac{1}{p})}(\Gamma)$. With $\tilde{A}_\ell^{(i)} := \{\lambda \in A_\ell : \mathrm{supp}\,\psi_\lambda \subset \Xi_{\lambda',i}\}$ and $\frac{1}{p'} := 1 - \frac{1}{p}$,

using (3.2) and the locality of the wavelets, for $w_\ell = \sum_{\lambda\in A_\ell} c_\lambda\psi_\lambda$ we can replace (3.18) by

$$\begin{aligned}
&|\langle \sum_{\lambda\in\tilde A_\ell^{(i)}} c_\lambda\psi_\lambda, L(\varphi_{\lambda',i}\circ\hat\kappa_q^{-1})\rangle| \\
&\quad\lesssim 2^{-\ell(\frac{n}{2}-\frac{n}{p}+t+\tilde d)}\Big(\sum_{\lambda\in\tilde A_\ell^{(i)}}|c_\lambda|^{p'}\Big)^{\frac{1}{p'}}\Big(\sum_{\lambda\in\tilde A_\ell^{(i)}}|L(\varphi_{\lambda',i}\circ\hat\kappa_q^{-1})|^p_{W_p^{\tilde d}(B(\operatorname{supp}\psi_\lambda;2^{-|\lambda|}\eta))}\Big)^{\frac{1}{p}} \\
&\quad\lesssim 2^{-\ell(\frac{n}{2}-\frac{n}{p}+t+\tilde d)}(\#\tilde A_\ell^{(i)})^{\frac12-\frac1p}\Big(\sum_{\lambda\in\tilde A_\ell^{(i)}}|c_\lambda|^2\Big)^{\frac12}|L(\varphi_{\lambda',i}\circ\hat\kappa_q^{-1})|_{W_p^{\tilde d}(\Gamma_q)} \\
&\quad\lesssim 2^{-\ell(\frac{n}{2}-\frac{n}{p}+t+\tilde d)}(2^{\ell n}\mathrm{vol}(\cup_{\lambda\in\tilde A_\ell^{(i)}}\operatorname{supp}\psi_\lambda))^{\frac12-\frac1p}\|w_\ell\|_{H^t(\Gamma)}\|\varphi_{\lambda',i}\circ\hat\kappa_q^{-1}\|_{H^{\tilde d+2t+n(\frac12-\frac1p)}(\Gamma)} \\
&\quad\lesssim 2^{-\ell(\frac{n}{2}-\frac{n}{p}+t+\tilde d)}2^{(\ell-|\lambda'|)n(\frac12-\frac1p)}\tilde\rho^{n(\frac12-\frac1p)}\|w_\ell\|_{H^t(\Gamma)}2^{(t+\tilde d+n(\frac12-\frac1p))|\lambda'|} \\
&\quad\le \rho^{n(\frac12-\frac1p)}\|w_\ell\|_{H^t(\Gamma)}2^{(|\lambda'|-\ell)(t+\tilde d)}.
\end{aligned}$$

We conclude that for any $p\in[2,\infty)$, we may replace (3.21) by

$$\left.\begin{matrix}|\langle w_\ell, L\psi_{\lambda'}\rangle| \\ |\langle L'\psi_{\lambda'}, w_\ell\rangle|\end{matrix}\right\}\lesssim$$

$$\|w_\ell\|_{H^t(\Gamma)}\max\Big\{2^{(|\lambda'|-\ell)(t+\tilde d)}\tilde\delta^{-2t-\tilde d+k+1}\rho^{\frac{n-1}{2}}\min\{\tilde\delta,\rho\}^{\frac12}, 2^{(|\lambda'|-\ell)(t+\tilde d)}\rho^{n(\frac12-\frac1p)}\Big\}. \tag{3.23}$$

and (3.22) by

$$\left.\begin{matrix}|\langle w_\ell, L\psi_{\lambda'}\rangle| \\ |\langle L'\psi_{\lambda'}, w_\ell\rangle|\end{matrix}\right\}\lesssim$$

$$\|w_\ell\|_{H^t(\Gamma)}\max\Big\{2^{(|\lambda'|-\ell)(t+\tilde d)}\tilde\delta^{-2t-\tilde d}\rho^{\frac{n-1}{2}}\min\{\tilde\delta,\rho\}^{\frac12}, 2^{(|\lambda'|-\ell)(t+\tilde d)}\rho^{n(\frac12-\frac1p)}\Big\}. \tag{3.24}$$

With $w_\ell=\psi_\lambda$ for some $|\lambda|=\ell$ and thus $\tilde\delta\gtrsim\rho\eqsim 2^{|\lambda'|-\ell}$, the estimate (3.24) can already be found in [DHS02] (however with $p=\infty$ which seems to be a mistake).

The above technique can also be applied in the general setting of Proposition 3.6, thus in situations where (3.20) or (3.21) are not necessarily valid. However, in that case the resulting estimate will depend on continuity properties of L defined on a $C^{\mu-1,1}$ manifold with respect to $W_p^s(\Gamma)$-norms for $p\neq 2$.

Remark 3.8. In contrast to the derivation of the decay estimate from §3.3, in the present subsection we never used that $\psi_{\lambda'}$ is a wavelet, i.e., that it has cancellation properties. In other words, all results derived in this subsection are also valid when the wavelet on the lowest of the two involved levels is replaced by an arbitrary 'scaling function' on that level, assuming that its $H^t(\Gamma)$-norm is of order 1.

To illustrate (3.23) and (3.24) we end with an

Example 3.9. Let Γ be the boundary of the unit square $[0,1]^2$, and let L be the single-layer operator, i.e., $(Lu)(x)=\int_\Gamma\log|x-y|u(y)d\mu(y)$. Let $\phi_{\lambda'}$ be the standard continuous, piecewise linear 'hat'-function with respect to the mesh $\Gamma\cap 2^{-|\lambda'|}\mathbb{N}_0^2$, attaining

its maximum in the point denoted by $m(\lambda')$. Let ψ_λ be a continuous piecewise linear function with respect to $\Gamma \cap 2^{-|\lambda|}\mathbb{N}_0^2$, with $\operatorname{diam}(\operatorname{supp}\psi_\lambda) \eqsim 2^{-|\lambda|}$, $\operatorname{supp}\psi_\lambda \subset \operatorname{supp}\phi_{\lambda'}$, $\tilde\delta := 2^{|\lambda'|}\operatorname{dist}(\operatorname{supp}\psi_\lambda, \operatorname{sing\,supp}\phi_{\lambda'}) = 2^{|\lambda'|}\operatorname{dist}(\operatorname{supp}\psi_\lambda, m(\lambda'))$, having the cancellation property of order 3, and which support is contained in one of the edges of $[0,1]^2$. Note that $n=1$, $\mu=1$, $t=-\frac{1}{2}$, $k=0$, $d=2$ and $\tilde d = 3$.

Using Maple, we computed $e := \frac{|\langle\psi_\lambda, L\phi_{\lambda'}\rangle|}{\|\phi_{\lambda'}\|_{H^t(\Gamma)}\|\psi_\lambda\|_{H^t(\Gamma)}}$. Regardless of the position of $m(\lambda')$, for $\tilde\delta \eqsim 1$ we found $e \eqsim 2^{3(|\lambda'|-|\lambda|)}$, which is in correspondence with (3.24) if we could take $p=\infty$. For $\tilde\delta = 2^{|\lambda'|-|\lambda|}$ and $m(\lambda')$ being a corner of Γ we found $e \eqsim 2^{(|\lambda'|-|\lambda|)}$ as predicted by (3.24). For $\tilde\delta = 2^{|\lambda'|-|\lambda|}$ and $m(\lambda')$ not being a corner, and so $\operatorname{supp}\phi_{\lambda'}$ contained in of the edges, we found $e \eqsim 2^{2(|\lambda'|-|\lambda|)}$ which matches our new estimate (3.23), and which is an order better than predicted by (3.24).

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