COMPOSITE WAVELET BASES WITH EXTENDED STABILITY AND CANCELLATION PROPERTIES*

ROB STEVENSON†

Abstract. The efficient solution of operator equations using wavelets requires that they generate a Riesz basis for the underlying Sobolev space, and that they have cancellation properties of a sufficiently high order. Suitable biorthogonal wavelets were constructed on reference domains as the n-cube. Via a domain decomposition approach, these bases have been used as building blocks to construct biorthogonal wavelets on general domains or manifolds, where, in order to end up with local wavelets, biorthogonality was realized with respect to a modified L_2-scalar product. The use of this modified scalar product restricts the application of these so-called composite wavelets to problems of orders strictly larger than -1. Moreover, those wavelets with supports that extend to more than one patch generally have no cancellation properties. In this paper, we construct local, composite wavelets that are close to being biorthogonal with respect to the standard L_2-scalar product. As a consequence, they generate Riesz bases for the Sobolev spaces H^s for *full range of* s allowed by the continuous gluing of functions over the patch interfaces, the properties of the primal and dual approximation spaces on the reference domain, and, in the manifold case, by the regularity of the manifold. Moreover, all these wavelets have *cancellation properties of the full order* induced by the approximation properties of the dual spaces on the reference domain. We illustrate our findings by a concrete realization of wavelets on a perturbed sphere.

Key words. Wavelets, Riesz bases, cancellation properties, domain decomposition, boundary integral equations.

AMS subject classifications. 46B15, 46E35, 65N55, 65T60.

1. Introduction.

The use of wavelet bases for solving operator equations, as *partial differential equations* or *(boundary) integral equations*, has a number of advantages, cf. [9, 3]. Let us assume that the operator is symmetric, and, for H being some Hilbert space, H-bounded and H-coercive, and that the infinite collection of, properly scaled, wavelets generates a *Riesz basis* for H. Then the stiffness matrix in wavelet coordinates resulting from a Ritz-Galerkin discretization is well-conditioned uniformly in its size, guaranteeing a uniform rate of convergence of an iterative method. In case of a differential operator, this stiffness matrix is not truly sparse, but has the well-known "finger structure". For multiplying with this matrix, however, one may switch to single-scale basis, with respect to which the stiffness matrix is sparse.

For integral operators, the stiffness matrix with respect to both single-scale and wavelet basis is densely populated. Here the second important property of wavelets can be exploited of having vanishing moments or, more generally, *cancellation properties*, meaning that the integral of a wavelet against a smooth function vanishes with a certain order of the length scale of the wavelet. If, depending on the order of the operator and the order of approximation, this order of the cancellation properties is sufficiently large, then the stiffness matrix with respect to the wavelet basis can be a priorily compressed to a sparse one without reducing the order of convergence. With this a method of linear complexity is obtained for solving integral equations ([18, 10]).

Instead of projecting the operator equation onto a fixed finite dimensional space, and then solving the resulting matrix-vector problem with an iterative method, the availability of a *Riesz basis* for H opens an attractive alternative for approximating

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†Department of Mathematics, Utrecht University, P.O. Box 80.010, NL-3508 TA Utrecht, The Netherlands (stevenson@math.uu.nl).

the solution by adaptive wavelet methods ([4, 5]). By writing this unknown solution in terms of this basis and testing the equation for all basis functions, one obtains an infinite dimensional matrix vector problem. This problem is *equivalent* to the operator equation, and it is well-posed in ℓ_2-metric, meaning that it can be solved using an iterative method. In each iteration of such a method, the application of the infinite stiffness matrix to the current approximation vector has to be approximated. Here the concept of *adaptivity* enters; the accuracy with which a column is approximated grows with the modulus of the corresponding entry of the vector. The resulting method, extended with a so-called coarsening routine to remove small entries from the approximation vector, can be proven to be optimal in the following sense. Whenever, for a certain range of s, the solution is in a class of functions for which the error of the best N-term approximations from the wavelet basis decays like N^{-s}, the sequence of approximations produced by this adaptive method has the same rate of convergence, whereas the computational cost is equivalent to their support sizes. A necessary condition for this statement to be true is that the stiffness matrix is sufficiently close to a sparse matrix, which depends on the smoothness of the wavelets and, again, on the *order of the cancellation properties* ([21]). Recently, it has been shown that an optimal adaptive wavelet method can even be obtained without coarsening ([15]).

Aiming at the aforementioned applications, this paper deals with *the construction on general n-dimensional domains or manifolds of wavelets that, properly scaled, generate Riesz bases for a range of Sobolev spaces, and satisfy cancellation properties of any required order*. To be able to choose this order independently from the order of approximation, we will consider biorthogonal wavelets. Their construction starts with two nested sequences of approximation spaces that both satisfy Jackson and Bernstein estimates ("multiresolution analyses"). Then the primal and dual wavelets are sought as bases of the biorthogonal complements of successive approximation spaces at primal and dual side, respectively. In case the primal and dual approximation spaces can be equipped with bases of local, biorthogonal scaling functions, local primal wavelets are found by applying the biorthogonal projector onto a local basis of some complement space of two successive primal approximation spaces. In this case, under some mild additional condition, also the corresponding dual wavelets are local. Actually, for constructing only local primal wavelets, a reduced set of assumptions already suffices, which for simplicity we will ignore in this introduction. Note that in algorithms for solving operator equations, usually dual wavelets do not play any role.

Biorthogonal scaling functions have been constructed on the real line ([6]), and as adaptations of these, on the interval ([11]). By taking tensor products, one obtains biorthogonal scaling functions on the n-dimensional unit cube. To construct biorthogonal scaling functions and wavelets on general domains and manifolds, a domain decomposition approach has been developed by Dahmen and Schneider in [12] (see [1, 7] for related approaches). The domain or manifold of interest is written as a disjoint union of smooth parametric images of the unit cube. The biorthogonal scaling functions on the cube are lifted to the patches, and, assuming that the decomposition satisfies some matching condition, they are continuously connected over the interfaces. With respect to a *modified L_2-scalar product*, defined by ignoring the Jacobian determinants of the parametrizations in the definition of the canonical L_2-scalar product, the resulting collections of scaling functions are biorthogonal. Wavelets, in this setting called *composite wavelets*, can now be constructed using the biorthogonal projector. There are, however, two principal limitations related to the realization of biorthogonality with respect to the modified L_2-scalar product. First of all, wavelets

with supports that extend to more than one patch, generally have *no cancellation properties* with respect to the canonical L_2-scalar product. So results concerning matrix compression do not apply to entries involving such wavelets. Secondly, with respect to the interpretation of a wavelet as a functional using the duality pairing in terms of the canonical L_2-scalar product, generally the resulting wavelets *cannot* generate a *Riesz basis for* H^s *for* $s \leq -\frac{1}{2}$. So for operators of order $2s \leq -1$, like the single-layer potential operator, neither optimal preconditioning results are valid, nor the adaptive wavelet method can be applied.

These limitations were already recognized by the authors in [12]. In [13], they developed an elegant approach to construct wavelets on general domains or manifolds that, properly scaled, generate Riesz bases for H^s for in principal any s, and that have cancellation properties of any required order. Unfortunately, so far with this approach it seems not easy to construct wavelets that have competitive quantitative properties. A recent investigation of this approach was made in [16].

In this paper, we reconsider the approach from [12], except that, in view of the aforementioned limitations, we make use of the canonical L_2-scalar product. Although, generally, the lifted and connected scaling functions are not biorthogonal with respect to this scalar product, we can derive a general formula for the corresponding biorthogonal wavelets. Since this formula, however, involves the inverse of the matrix consisting of the L_2-scalar products between all primal and dual scaling functions, which matrix is thus generally not diagonal, these wavelet have global supports. On the other hand, this matrix is nearly diagonal, so that its inverse can be well approximated by sparse matrices which gives rise to local, approximate wavelets. We derive general conditions under which, properly scaled, such approximate wavelets generate a Riesz basis for H^s for the *full range of* s allowed by the continuous gluing of the scaling functions over the interfaces, the properties of the primal and dual approximation spaces on the cube, and, in the manifold case, by the regularity of the manifold. We give three possibilities for the construction of approximate wavelets that are local, generate Riesz bases for H^s for the aforementioned full range of s, and *all* have *cancellation properties of the full order* induced by the approximation properties of the dual spaces on the unit cube. Firstly, we show that the approximation of the inverse of the matrix of L_2-scalar products of primal and dual scaling functions by a suitable, fixed number of Jacobi iterations yields such approximate wavelets. In view of the relatively large supports of these wavelets, secondly, we show that away from the patch interfaces they can be replaced by the wavelets one gets by ignoring the Jacobian determinants, which are the wavelets from [12]. Thirdly, we show that also along the patch interfaces suitable approximate wavelets with smaller supports can be constructed, which, however, will involve solving some local systems. Although several proofs will be quite involved, we emphasize that the implementation of the approximate wavelets is relatively straightforward.

In [14, 20], we constructed wavelet bases for Lagrange finite element spaces based on a subdivision of polygonal domains into n-simplices. In this paper, we include the option that these finite element wavelets, or more precisely the underlying scaling functions are used as building blocks for wavelets on general (non-polygonal) domains or manifolds, where then the unit n-cube as reference domain should be replaced by some reference n-simplex.

This paper is organized as follows. In the remainder of this section we fix a few notations. In Section 2, we specify the type of domains and manifolds and their parametrizations that we will consider. We recall the definition of the Sobolev spaces,

that may involve zero order Dirichlet boundary conditions, which we are going to equip with Riesz bases. In Section 3, we collect all assumptions on the multiresolution analyses on the reference domain. The induced, continuous multiresolution analyses on the target domain or manifold are defined in Section 4. Although put here into a more general framework, the main construction principles from Sections 2-4 originate from [12]. Biorthogonal space decompositions and the, generally, globally supported biorthogonal wavelets are constructed in Section 5. Sections 6 and 7, which form the main part of this paper, are devoted to the construction of local, approximate wavelets. Finally, in Section 8 we show examples of approximate wavelets on a perturbed sphere, and give some numerically computed condition numbers.

In order to limit the size of this paper, for some technical (parts of) proofs we will refer to the extended preprint version [22].

In order to avoid the repeated use of generic but unspecified constants, in this paper by $C \lesssim D$ we mean that C can be bounded by a multiple of D, independently of parameters on which C and D may depend. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

Let H be a separable Hilbert space with with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. For a countable collection Σ of functions in H, which we formally view as a (column) vector, and for $\mathbf{c} = (c_\sigma)_{\sigma\in\Sigma}$ a vector of scalars, with $\mathbf{c}^T\Sigma$ we will mean the expansion $\sum_{\sigma\in\Sigma} c_\sigma\sigma$. The span of Σ will be denoted as $\mathcal{S}(\Sigma)$. For $x \in H$, with $\langle\Sigma, x\rangle$ and $\langle x, \Sigma\rangle$ we will mean the column- and row-vectors with coefficients $\langle\sigma, x\rangle$ and $\langle x, \sigma\rangle$, $\sigma \in \Sigma$. When $\tilde\Sigma$ is another countable collection in H, with $\langle\Sigma, \tilde\Sigma\rangle$ we denote the matrix $(\langle\sigma, \tilde\sigma\rangle)_{\sigma\in\Sigma,\tilde\sigma\in\tilde\Sigma}$. For $V \subset H$ being a dense, continuously embedded Banach space, as usual we will use $\langle\cdot,\cdot\rangle$ sometimes also to denote the duality pairing $\langle\cdot,\cdot\rangle_{V\times V'}$, which, with the aforementioned meaning, can also be applied to collections from V and/or V'.

On the spaces of (possibly infinite) scalar vectors or matrices, we will exclusively use the ℓ_2-scalar product, ℓ_2-norm or the resulting operator norm, that we therefore simply denote by $\langle\cdot,\cdot\rangle$ or $\|\cdot\|$, respectively. A collection Σ is called a *Riesz system* when $\|\mathbf{c}^T\Sigma\| \eqsim \|\mathbf{c}\|$, i.e., when $\langle\Sigma,\Sigma\rangle$ is boundedly invertible, and Σ is called a *Riesz basis* when it is in addition a basis for H. When Σ depends on a parameter, we will speak about *uniform Riesz systems* (or bases) when the above equivalence holds uniformly over the values this parameter may attain. We set $\|\Sigma\| = \|\langle\Sigma,\Sigma\rangle\|^{\frac12}$, and collect a few properties related to this definition:

Proposition 1.1.
(i) $\sup_{\mathbf{c}\neq0} \frac{\|\mathbf{c}^T\Sigma\|}{\|\mathbf{c}\|} = \|\Sigma\|$,
(ii) $\|\langle\Sigma,\tilde\Sigma\rangle\| \le \|\Sigma\|\|\tilde\Sigma\|$,
(iii) $\|\Sigma+\tilde\Sigma\| \le \|\Sigma\| + \|\tilde\Sigma\|$,
(iv) *For a matrix* $\mathbf{A}$, $\|\mathbf{A}\Sigma\| \le \|\mathbf{A}\|\|\Sigma\|$.

Proof. For (i), use $\|\mathbf{c}^T\Sigma\|^2 = \langle\langle\Sigma,\Sigma\rangle\mathbf{c},\mathbf{c}\rangle$. Part (ii) follows from $|\langle\langle\Sigma,\tilde\Sigma\rangle\mathbf{c},\tilde{\mathbf{c}}\rangle| = |\langle\mathbf{c}^T\Sigma, \tilde{\mathbf{c}}^T\tilde\Sigma\rangle| \le \|\mathbf{c}\|\|\tilde{\mathbf{c}}\|\|\Sigma\|\|\tilde\Sigma\|$ because of (i). Part (iii) follows easily from (ii). For (iv), use $\langle\mathbf{A}\Sigma,\mathbf{A}\Sigma\rangle = \mathbf{A}\langle\Sigma,\Sigma\rangle\mathbf{A}^*$. ∎

2. Domains and function spaces. For some $n' \ge n \ge 1$, let Γ be an n-dimensional bounded manifold in $\mathbb{R}^{n'}$, with or without a boundary. For $\square$ denoting the interior of either the n-cube $[0,1]^n$ or, despite of its notation, of some reference n-simplex, we assume that Γ is given as

$$\overline{\Gamma} = \cup_{q=1}^M \overline{\Gamma_q}, \text{ with } \Gamma_q \cap \Gamma_{q'} = \emptyset \text{ when } q \neq q', \text{ and } \Gamma_q = \kappa_q(\square),$$

where $\kappa_q : \mathbb{R}^n \to \mathbb{R}^{n'}$ be some smooth, regular parametrizations. With Π we will denote the collection of all affine mappings from $\square$ onto $\square$. So in case $\square$ is the interior of an n-simplex, this collection consists of the permutations of the $n+1$ barycentric coordinates, and otherwise it consists of the compositions of any permutation of the n Cartesian coordinates and reflections of type $y \mapsto (y_1, \ldots, y_{i-1}, 1-y_i, y_{i+1}, \ldots, y_n)$ $(1 \le i \le n)$. We assume that the splitting of Γ into the patches Γ_q is conforming in the sense for any $q \neq q'$, $\overline{\Gamma_q} \cap \overline{\Gamma_{q'}}$ is either empty, or

$$\kappa_q^{-1}(\overline{\Gamma_q} \cap \overline{\Gamma_{q'}}) \text{ is a face of } \square\,.$$

In addition, we assume that the parametrizations can be chosen such that the following matching condition is satisfied: There exists a $\pi \in \Pi$ with

$$\kappa_{q'} \circ \pi \circ \kappa_q^{-1} = Id \quad \text{on } \overline{\Gamma_q} \cap \overline{\Gamma_{q'}}. \tag{$\mathcal{M}$}$$

Here and in the remainder of this paper, with a "face" of $\square$, we mean a (complete, closed) face of any dimension $0 \le k \le n-1$, i.e., for $n=3$, it is either a vertex, an edge or a facet. Note that our setting allows Γ to be a bounded domain in $\mathbb{R}^n$, as well as an open or closed bounded manifold in $\mathbb{R}^{n'}$ for some $n' > n$.

We include the possibility that homogeneous, zero order Dirichlet boundary conditions are prescribed on some part $\partial\Gamma_D \subset \overline{\Gamma}\backslash\Gamma$, for which, for all $1 \le q \le M$,

$$\kappa_q^{-1}(\partial\Gamma_D \cap \overline{\Gamma}_q) \text{ is a, possibly empty, union of faces of } \square, \tag{2.1}$$

see Figure 2.1.

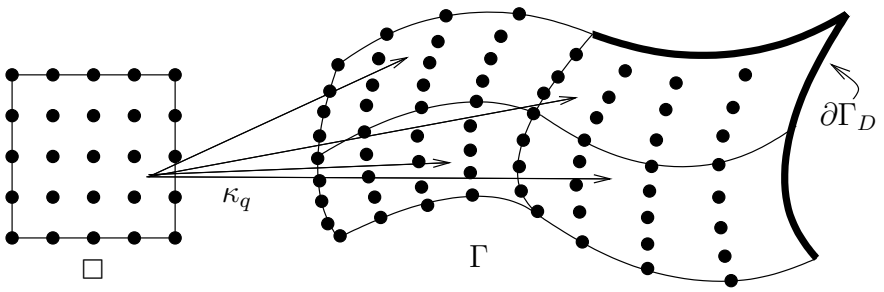

FIG. 2.1. *Illustration of the domain decomposition approach*

For some $s_\Gamma > 0$, we assume that, globally,

$$\Gamma \in C^{s_\Gamma} \text{ when } s_\Gamma \notin \mathbb{N}, \text{ or } \Gamma \in C^{s_\Gamma - 1,1} \text{ when } s_\Gamma \in \mathbb{N}.$$

This means that for $0 \le s < s_\Gamma \notin \mathbb{N}$, or $0 \le s \le s_\Gamma \in \mathbb{N}$, the Sobolev spaces

$$\mathcal{H}^s(\Gamma) := \begin{cases} H^s_{0,\partial\Gamma_D}(\Gamma) & \text{when } s \le 1, \\ H^s(\Gamma) \cap H^1_{0,\partial\Gamma_D}(\Gamma) & \text{when } s > 1 \end{cases}$$

can be defined in the usual way using a partition of unity relative to some atlas. For $s > 0$ in the above range, $\mathcal{H}^{-s}(\Gamma)$ will be understood as being the dual of $\mathcal{H}^s(\Gamma)$.

With μ being the induced Lebesgue measure on Γ, the inner product on $L_2(\Gamma)$ is given by

$$\langle u, v\rangle_{L_2(\Gamma)} = \int_\Gamma u\overline{v}d\mu = \sum_{q=1}^{M} \langle u \circ \kappa_q, v \circ \kappa_q\rangle_{L_2(\square),|\partial\kappa_q|}. \tag{2.2}$$

Here, for $w \in L^\infty(\square)$ with $w > 0$ a.e., $\langle f, g\rangle_{L_2(\square),w} := \int_\square f(y)\overline{g(y)}w(y)dy$, and $|\partial\kappa_q| : z \mapsto |\partial\kappa_q(z)|$ are the Jacobian determinants of the parametrizations. We will also make use of a modified inner product

$$\langle\!\langle u, v\rangle\!\rangle_0 := \sum_{q=1}^M \langle u\circ\kappa_q, v\circ\kappa_q\rangle_{L_2(\square)}, \tag{2.3}$$

which is the inner product one gets by ignoring the Jacobian determinants. It is equivalent to $\langle\,,\,\rangle_{L_2(\Gamma)}$ in the sense that $|\!|\!|\cdot|\!|\!|_0 := \langle\!\langle\cdot,\cdot\rangle\!\rangle_0^{\frac12} \eqsim \|\cdot\|_{L_2(\Gamma)}$. More generally, for any $s \geq 0$, we define

$$\langle\!\langle u, v\rangle\!\rangle_s = \sum_{q=1}^M \langle u\circ\kappa_q, v\circ\kappa_q\rangle_{H^s(\square)},$$

and let $\mathcal{H}_s(\Gamma)$ denote the closure with respect to $|\!|\!|\cdot|\!|\!|_s := \langle\!\langle\cdot,\cdot\rangle\!\rangle_s^{\frac12}$ of the set all *globally continuous*, and with respect to the subdivision $\overline{\Gamma} = \cup_{q=1}^M \overline{\Gamma_q}$, *piecewise* C^∞ functions on Γ that are *zero on* $\partial\Gamma_D$. For $s > 0$, we define $\mathcal{H}_{-s}(\Gamma) = (\mathcal{H}_s(\Gamma))'$. For $0 \leq s < s_\Gamma \notin \mathbb{N}$ or $0 \leq s \leq s_\Gamma \in \mathbb{N}$, it holds that $\|\cdot\|_{H^s(\Gamma)} \eqsim |\!|\!|\cdot|\!|\!|_s$ on $H^s(\Gamma)$. Furthermore, if $s < \frac32$, then the functions in the aforementioned set generate a dense subset in $\mathcal{H}^s(\Gamma)$. Using in addition duality, we infer that

$$\mathcal{H}^s(\Gamma) \asymp \mathcal{H}_s(\Gamma) \quad (|s| < \tfrac32 \text{ with } |s| < s_\Gamma \notin \mathbb{N} \text{ or } |s| \leq s_\Gamma \in \mathbb{N}), \tag{2.4}$$

meaning that both spaces agree as sets and have equivalent norms. The spaces $\mathcal{H}_s(\Gamma)$ will only serve as auxiliary spaces to be able to prove that the wavelets we are going to construct generate, properly scaled, a Riesz basis for $\mathcal{H}^s$ for the full range of s, in case this range is limited by the regularity of Γ to a *closed* range $[-s_\Gamma, s_\Gamma]$.

3. Multiresolution analyses on the reference domain. On the reference domain, we will need two nested sequences of approximation spaces (multiresolution analyses) that satisfy Jackson and Bernstein estimates. We will assume that these spaces are equipped with single-scale bases, that satisfy certain conditions concerning their supports and symmetry (cf. assumptions ($\mathcal{L}$), ($\mathcal{V}$), ($\mathcal{S}$)), so that after their lifting to the patches, they can be continuously connected over the interfaces. Furthermore, we will assume that the rate of best approximation from these sequences is realized by some concrete projector (cf. ($\mathcal{J}$) and Proposition 3.1), with which it will be shown that the induced approximation spaces on Γ are nested, and have the same rate of approximation. We will make some assumptions (($\mathcal{J}1$) and ($\mathcal{J}2$)) connecting primal and dual multiresolution analyses to ensure the existence and uniform boundedness of the biorthogonal projector (cf. Proposition 5.2). Finally, we will assume the existence of a suitable "initial stable completion".

For $j \in \mathbb{N}_0$, let $I_j^\square \subset \overline{\square}$ be some index set with

$$\pi(I_j^\square) = I_j^\square \quad (\pi \in \Pi), \quad \sup_{y\in\square} \#(I_j^\square \cap B(y; 2^{-j})) \lesssim 1,$$

(see Figure 2.1). For completeness, for $A \subset \mathbb{R}^n$ and $\delta \geq 0$, with $B(A;\delta)$ we mean $\{y \in \mathbb{R}^n : \mathrm{dist}(y, A) \leq \delta\}$, and $B(\emptyset;\delta) := \emptyset$. For $j \in \mathbb{N}_0$, we assume a collection $\Phi_j^\square = (\phi_{j,x}^\square)_{x\in I_j^\square} \subset C(\overline{\square})$, usually referred to as being the set of *scaling functions*,

such that

($\mathcal{L}$) $\exists$ constant $\varepsilon > 0$, $\operatorname{supp} \phi^{\square}_{j,x} \subset B(x; \varepsilon 2^{-j})$.

($\mathcal{V}$) $\phi^{\square}_{j,x}$ vanishes on any face of $\square$ that does not contain x.

($\mathcal{S}$) $\phi^{\square}_{j,x} = \phi^{\square}_{j,\pi(x)} \circ \pi, \quad (\pi \in \Pi)$.

($\mathcal{R}$) $\Phi^{\square}_j$ is a uniform $L_2(\square)$-Riesz system.

($\mathcal{J}$) There exists a collection of functionals $\Lambda^{\square}_j = (\lambda^{\square}_{j,x})_{x \in I^{\square}_j} \subset C(\overline{\square})'$ such that

(i) $\exists$ constant $\vartheta > 0, \operatorname{supp} \lambda^{\square}_{j,x} \subset B(x; \vartheta 2^{-j})$.

(ii) If $x \in \partial\square$, then $\operatorname{supp} \lambda^{\square}_{j,x}$ is contained in the lowest dimensional face of $\square$ that contains x.

(iii) $\langle u, \lambda^{\square}_{j,\pi(x)} \rangle_{L_2(\square)} = \langle u \circ \pi, \lambda^{\square}_{j,x} \rangle_{L_2(\square)}, \quad (\pi \in \Pi)$.

(iv) $|\langle u, \lambda^{\square}_{j,x} \rangle_{L_2(\square)}| \lesssim 2^{-jn/2} \|u\|_{L_\infty(\operatorname{supp} \lambda^{\square}_{j,x})}$.

(v) $\langle \Phi^{\square}_j, \Lambda^{\square}_j \rangle_{L_2(\square)} = Id$.

(vi) For some $\frac{n}{2} < d \in \mathbb{N}$, $P_{d-1}(\square) \subset \mathcal{S}(\Phi^{\square}_j)$.

($\mathcal{N}$) $\mathcal{S}(\Phi^{\square}_j) \subset \mathcal{S}(\Phi^{\square}_{j+1})$.

($\mathcal{B}$) For some $\gamma > 0$, and any $s \in [0, \gamma)$, it holds that

$$\|u_j\|_{H^s(\square)} \lesssim 2^{sj} \|u_j\|_{L_2(\square)}, \quad (u_j \in \mathcal{S}(\Phi^{\square}_j)). \qquad .$$

Note that, in particular, ($\mathcal{J}$)(ii) implies that for x being a vertex of $\square$, $\langle u, \lambda^{\square}_{j,x} \rangle_{L_2(\square)}$ is a multiple of $u(x)$. Examples of such collections will be given at the end of this section.

PROPOSITION 3.1. *For the projector $P^{\square}_j : u \mapsto \langle u, \Lambda^{\square}_j \rangle_{L_2(\square)} \Phi^{\square}_j$ onto $\mathcal{S}(\Phi^{\square}_j)$, we have*

$$\|u - P^{\square}_j u\|_{L_2(\Diamond)} \lesssim 2^{-dj} |u|_{H^d(B(\Diamond; (\vartheta + 3\varepsilon) 2^{-j}) \cap \square)}, \quad (\Diamond \subset \square, u \in H^d(\square)).$$

The proof given in [22] follows standard lines.

Apart from the above collection $\Phi^{\square}_j$ of *primal* scaling functions, for $j \in \mathbb{N}_0$ we assume the existence of a collection $\tilde{\Phi}^{\square}_j = (\tilde{\phi}^{\square}_{j,x})_{x \in I^{\square}_j} \subset C(\overline{\square})$ of *dual* scaling functions. This collection should also satisfy all of ($\mathcal{L}$)-($\mathcal{B}$) with the same index set $I^{\square}_j$, but with generally different parameters and functionals in ($\mathcal{B}$) and ($\mathcal{J}$) that we will denote as $\tilde{\gamma} > 0$, $\tilde{d} > \frac{n}{2}$, $\tilde{\Lambda}^{\square}_j$, and $\tilde{\varepsilon}, \tilde{\vartheta} > 0$. The resulting projector $\tilde{P}^{\square}_j : u \mapsto \langle u, \tilde{\Lambda}^{\square}_j \rangle_{L_2(\square)} \tilde{\Phi}^{\square}_j$ satisfies the analogue of Proposition 3.1 with $(d, \vartheta, \varepsilon)$ replaced by $(\tilde{d}, \tilde{\vartheta}, \tilde{\varepsilon})$.

Since $\Phi^{\square}_j$ and $\tilde{\Phi}^{\square}_j$ are uniform $L_2(\square)$-Riesz systems, the matrix $\langle \Phi^{\square}_j, \tilde{\Phi}^{\square}_j \rangle_{L_2(\square)}$ defines a uniformly bounded linear operator on $\ell_2(I^{\square}_j)$. A relation between $\mathcal{S}(\Phi^{\square}_j)$ and $\mathcal{S}(\tilde{\Phi}^{\square}_j)$ is established by assuming that that its real part satisfies

($\mathcal{J}1$) $$\Re \langle \Phi^{\square}_j, \tilde{\Phi}^{\square}_j \rangle_{L_2(\square)} \gtrsim Id.$$

Finally, for $j \in \mathbb{N}_0$, let $J^{\square}_j \subset \overline{\square}$ be some index set with $\pi(J^{\square}_j) = J^{\square}_j$ $(\pi \in \Pi)$, $\sup_{y \in \square} \#(J^{\square}_j \cap B(y; 2^{-j})) \lesssim 1$, and for e being either $\overline{\square}$ or any face of $\square$, $\#((I^{\square}_j \cup$

$J_j^\square) \cap e) = \#(I_{j+1}^\square \cap e)$. In case $I_j^\square \subset I_{j+1}^\square$, a natural candidate is $J_j^\square = I_{j+1}^\square \backslash I_j^\square$. We assume the existence of collections $\Theta_j^\square = (\theta_{j,x}^\square)_{x\in I_j^\square}$ with

$$\langle \Theta_j^\square, \tilde{\Phi}_j^\square\rangle_{L_2(\square)} = Id, \tag{$\mathcal{J}$2}$$

and $\Xi_j^\square = (\xi_{j,x}^\square)_{x\in J_j^\square}$ such that the union $\Upsilon_{j+1}^\square := \begin{bmatrix}(\Theta_j^\square)^T & (\Xi_j^\square)^T\end{bmatrix}^T$ satisfies $(\mathcal{L})$-$(\mathcal{R})$, and $\mathcal{S}(\Upsilon_{j+1}^\square) = \mathcal{S}(\Phi_{j+1}^\square)$.

Remark 3.2. "Classical" wavelet constructions start with assuming biorthogonal scaling functions, i.e., $\langle \Phi_j^\square, \tilde{\Phi}_j^\square\rangle_{L_2(\square)} = Id$, in which case ($\mathcal{J}$1) and ($\mathcal{J}$2) are satisfied with $\Theta_j^\square = \Phi_j^\square$. When there is no need for locally supported dual wavelets, biorthogonality of the scaling functions can be relaxed to the conditions given here, with generally $\Theta_j^\square$ different from $\Phi_j^\square$, and in particular not contained in $\mathcal{S}(\Phi_j)$. For the case that $\Theta_j^\square = \Phi_j^\square$, in the literature the set $\Xi_j^\square$ is sometimes called an initial "stable" completion of $\Phi_j^\square$, that is, a completion of $\Phi_j^\square$ to a uniform $L_2(\square)$-Riesz basis for $\mathcal{S}(\Phi_{j+1}^\square)$. The wavelets to be constructed are then thought of being the target stable completion.

Remark 3.3. The condition ($\mathcal{J}$2) can be further relaxed, which turned out to be useful in [20]. Instead of assuming that $\langle \Theta_j^\square, \tilde{\Phi}_j^\square\rangle_{L_2(\square)}$ is diagonal, more generally it is also sufficient when, for some fixed p, $I_j^\square$ is the union of disjoint sets $I_{j,1}^\square, \dots, I_{j,p}^\square$, with $\pi(I_{j,i}^\square) = I_{j,i}^\square$ ($\pi \in \Pi$, $1 \le i \le p$), such that, with respect to this partitioning, $\langle \Theta_j^\square, \tilde{\Phi}_j^\square\rangle_{L_2(\square)}$ is a block triangular matrix, with diagonal blocks that are identity matrices. Although all results from this paper are also valid under this relaxed assumption, for ease of presentation we will stick to assumption ($\mathcal{J}$2).

Because of $\mathcal{S}(\Phi_j^\square) \subset \mathcal{S}(\Phi_{j+1}^\square)$, $\mathcal{S}(\tilde{\Phi}_j^\square) \subset \mathcal{S}(\tilde{\Phi}_{j+1}^\square)$ and $\mathcal{S}(\Upsilon_{j+1}^\square) = \mathcal{S}(\Phi_{j+1}^\square)$, it holds that $\Phi_j^\square = \langle \Phi_j^\square, \Lambda_{j+1}^\square\rangle_{L_2(\square)}\Phi_{j+1}^\square$, $\tilde{\Phi}_j^\square = \langle \tilde{\Phi}_j^\square, \tilde{\Lambda}_{j+1}^\square\rangle_{L_2(\square)}\tilde{\Phi}_{j+1}^\square$, and $\Upsilon_{j+1}^\square = \langle \Upsilon_{j+1}^\square, \Lambda_{j+1}^\square\rangle_{L_2(\square)}\Phi_{j+1}^\square$, where $\langle \Upsilon_{j+1}^\square, \Lambda_{j+1}^\square\rangle_{L_2(\square)}$ is uniformly boundedly invertible.

LEMMA 3.4. *For the matrix* $\mathbf{R}_j^\square$ *being either* $\langle \Phi_j^\square, \Lambda_{j+1}^\square\rangle_{L_2(\square)}$, $\langle \tilde{\Phi}_j^\square, \tilde{\Lambda}_{j+1}^\square\rangle_{L_2(\square)}$, $\langle \Upsilon_{j+1}^\square, \Lambda_{j+1}^\square\rangle_{L_2(\square)}$ *or* $\langle \Upsilon_{j+1}^\square, \Lambda_{j+1}^\square\rangle_{L_2(\square)}^{-1}$, *it holds that*

(a) $(\mathbf{R}_j^\square)_{\pi(x),\pi(y)} = (\mathbf{R}_j^\square)_{x,y}$ *($\pi \in \Pi$).*

(b) $(\mathbf{R}_j^\square)_{x,y} = 0$ *when y is on a face of $\square$ that does not contain x.*

Proof. Part (a) follows from the assumptions ($\mathcal{S}$) or ($\mathcal{J}$)(iii) for the involved collections of functions and functionals, respectively. Similarly, for the first three matrices, Part (b) follows from the assumptions ($\mathcal{V}$) or ($\mathcal{J}$)(ii). Now let e be a face of $\square$. With respect to the partitioning of the index sets for $\Upsilon_{j+1}^\square$ and $\Lambda_{j+1}^\square$ into indices on e and indices not on e, $\langle \Upsilon_{j+1}^\square, \Lambda_{j+1}^\square\rangle_{L_2(\square)}$ is a 2×2 upper block triangular matrix with square diagonal blocks, and thus so is its inverse, which shows (b) also for $\langle \Upsilon_{j+1}^\square, \Lambda_{j+1}^\square\rangle_{L_2(\square)}^{-1}$. ∎

By our assumptions, the matrices $\langle \Phi_j^\square, \Lambda_{j+1}^\square\rangle_{L_2(\square)}$ and $\langle \Upsilon_{j+1}^\square, \Lambda_{j+1}^\square\rangle_{L_2(\square)}$ are *uniformly local*, by which we mean that only entries with indices (x,y) with $|x-y| \lesssim 2^{-j}$ might be nonzero. As a consequence, for the wavelets we are going to construct, the basis transformation from wavelet- to single scale basis will be of optimal computational complexity.

For some applications, it is also essential to have a basis transformation from single scale to wavelet basis that is of optimal computational complexity. In that case, one has to assume *both* that

$$\Theta_j^\square = \Phi_j^\square,$$

with which (J1) can be dropped since it is implied by (J2), *and* also that

$$\langle \Upsilon_{j+1}^{\square}, \Lambda_{j+1}^{\square} \rangle_{L_2(\square)}^{-1} \text{ is } \textit{uniformly local}. \tag{3.1}$$

Note that, for $\Theta_j^{\square} = \Phi_j^{\square}$, $\langle \Upsilon_{j+1}^{\square}, \Lambda_{j+1}^{\square} \rangle_{L_2(\square)}^{-1}$ is the basis transformation from $\Phi_{j+1}^{\square}$ to the two-level basis $\Phi_j^{\square} \cup \Xi_j^{\square}$.

All conditions imposed in this section are satisfied by the collections $\Phi_j^{\square}$, $\tilde{\Phi}_j^{\square}$, $\Theta_j^{\square}$, $\Xi_j^{\square}$ underlying the *finite element wavelets* introduced in [14]. With this construction, $\mathcal{S}(\Phi_j^{\square})$, $\mathcal{S}(\tilde{\Phi}_j^{\square})$ are standard Lagrange finite element spaces, so that $\gamma = \tilde{\gamma} = \frac{3}{2}$, of orders d and $\tilde{d}$, respectively, with respect to a j times repeated uniform dyadic refinement of an initial simplicial partition of a polytope. For the present setting, we take this polytope to be a reference *n-simplex*. Thinking of $\tilde{d} \geq d$, these orders are chosen such that there is an $m \in \mathbb{N}$ with $2^m(d-1) = \tilde{d} - 1$, so that with the initial partition at the dual side being the reference simplex itself, and at the primal side being created by applying m dyadic recursive refinements to this simplex, we have $\dim \mathcal{S}(\Phi_j^{\square}) = \dim \mathcal{S}(\tilde{\Phi}_j^{\square})$. So each "element" at the dual side is equal to a macro-element at the primal side consisting of 2^m "elements". The collections $\tilde{\Phi}_j^{\square}$ at the dual side, and $\Phi_j^{\square}$, $\Theta_j^{\square}$, $\Xi_j^{\square}$ at the primal side are now assembled in the standard finite element fashion from local collections, of a small, fixed dimension, of functions defined on the individual elements or macro-elements, respectively. Each of these collections is a copy, or more precisely, a push-forward using an affine bijection of such a collection created once and for all on a reference (macro-) element. The functionals from $\tilde{\Lambda}_j^{\square}$ and $\Lambda_j^{\square}$ are assembled in the same manner from local collections, and are either simply scaled function evaluations in the "nodal points" $I_j^{\square}$, or local linear combinations of these. Actually, in the present paper, we will repeat the idea of assembling functions and functionals from collections defined on (macro-) elements, which in turn are push-forwards of collections defined on a reference (macro-) element. Now the role of the (macro-) elements will be played by the patches Γ_q, and that of the reference (macro-) element by $\kappa_q^{-1}(\Gamma_q)$. A difference is that the number of patches is fixed, and that, as a consequence, the dimension of the local collections grows with the level. The major difficulty we have to deal with is that generally the κ_q are not affine, so that the Jacobian determinants are not constants.

In [20], we reconsidered the finite element wavelets, and constructed collections with $\Theta_j^{\square} = \Phi_j^{\square}$, so that also the resulting dual wavelets are locally supported. In this case, the dual spaces, although consisting of continuous piecewise polynomials, are not standard finite element spaces.

Other examples of collections $\Phi_j^{\square}$, $\tilde{\Phi}_j^{\square}$, $\Theta_j^{\square}$, $\Xi_j^{\square}$ satisfying our assumptions, with $\Theta_j^{\square} = \Phi_j^{\square}$, and now with $\square$ being the *n-cube*, are given in [12], which underly the construction of *biorthogonal spline wavelets*. These collections are slight modifications of those developed in [11], and, for $n > 1$, they are simply generated using tensor products from univariate collections $\Phi_j^{[0,1]}$, $\tilde{\Phi}_j^{[0,1]}$, $\Xi_j^{[0,1]}$ defined on $[0,1]$. For given $\tilde{d} \geq d \geq 2$ with $d + \tilde{d}$ even, $\mathcal{S}(\Phi_j^{[0,1]})$ is the spline space of order d, so that $\gamma = d - \frac{1}{2}$, with respect to the knot sequence

$$(\underbrace{0, \ldots, 0}_{d \text{ times}}, r2^{-j}, r2^{-j} + 2^{-j}, \ldots, 1 - r2^{-j}, \underbrace{1, \ldots, 1}_{d \text{ times}}).$$

where $\mathbb{N} \ni r \geq d-1$ is some parameter that one can choose. The collection $\tilde{\Phi}_j^{[0,1]}$ is such that $P_{\tilde{d}-1}[0,1] \subset \mathcal{S}(\tilde{\Phi}_j^{[0,1]})$ and $\langle \Phi_j^{[0,1]}, \tilde{\Phi}_j^{[0,1]} \rangle_{L_2([0,1])} = Id$, where $\tilde{\gamma}$ grows linearly with $\tilde{d}$. For x not near the endpoints 0 or 1, $\phi_{j,x}^{[0,1]} = 2^{j/2}\phi(2^j \cdot -x)$ and $\tilde{\phi}_{j,x}^{[0,1]} = 2^{j/2}\tilde{\phi}(2^j \cdot -x)$, where $(\phi, \tilde{\phi})$ is a biorthogonal pair constructed in [6]. Also the functionals from $\Lambda_j^{\square}$ and $\tilde{\Lambda}_j^{\square}$ are constructed from the collections of univariate functionals $\Lambda_j^{[0,1]}$ and $\tilde{\Lambda}_j^{[0,1]}$ using tensor products, where $\lambda_{j,x}^{[0,1]} = \tilde{\phi}_{j,x}^{[0,1]}$, $\tilde{\lambda}_{j,x}^{[0,1]} = \phi_{j,x}^{[0,1]}$ for $x \notin \{0,1\}$, and where they are simply scaled function evaluations in 0 or 1, respectively, otherwise.

4. Induced, continuous multiresolution analyses on Γ. By lifting the collections of functions on $\square$ to the patches of Γ, and by connecting those that do not vanish at the interfaces continuously with ones from other patches, we will construct nested sequences of primal and dual spaces that satisfy Jackson estimates and Bernstein inequalities.

We define the index sets $I_j \subset \overline{\Gamma} \backslash \partial\Gamma_D$, and analogously J_j, by

$$I_j = (\cup_{q=1}^M \kappa_q(I_j^{\square})) \cap (\overline{\Gamma} \backslash \partial\Gamma_D)$$

(see Figure 2.1). By $(\mathcal{M})$ and $\pi(I_j^{\square}) = I_j^{\square}$ $(\pi \in \Pi)$, for any $1 \leq q, q' \leq M$ with $\overline{\Gamma_q} \cap \overline{\Gamma_{q'}} \neq \emptyset$, the sets $\kappa_q(I_j^{\square})$ and $\kappa_{q'}(I_j^{\square})$ restricted to this interface coincide. For $x \in \overline{\Gamma}$, we set $k(x) = \#\{q : x \in \overline{\Gamma_q}\}$.

For $j \in \mathbb{N}_0$, we define the collection $\Phi_j = (\phi_{j,x})_{x \in I_j} \subset C(\overline{\Gamma})$ by

$$\phi_{j,x}(y) = k(x)^{-\frac{1}{2}} \begin{cases} \phi^{\square}_{j,\kappa_q^{-1}(x)}(\kappa_q^{-1}(y)) & \text{when } x, y \in \overline{\Gamma_q} \text{ for some } 1 \leq q \leq M, \\ 0 & \text{elsewhere.} \end{cases} \tag{4.1}$$

Note that by $(\mathcal{S})$, $(\mathcal{V})$ and (2.1), $\phi_{j,x}$ is well-defined, indeed continuous and it vanishes on $\partial\Gamma_D$. By assumption $(\mathcal{L})$, the collection Φ_j is *uniformly local*, by which we mean that $x \in \operatorname{supp} \phi_{j,x}$, and that $d_\Gamma(x,y) \lesssim 2^{-j}$ for any $y \in \operatorname{supp} \phi_{j,x}$, where $d_\Gamma(x,y)$ denotes the geodesic distance of x and y over Γ, i.e., the length of the shortest curve on Γ connecting x and y.

With $\mathbf{E}_{j,q} : \ell_2(I_j^{\square}) \to \ell_2(I_j)$ defined by

$$(\mathbf{E}_{j,q}\mathbf{c}_j^{\square})_x = k(x)^{-\frac{1}{2}} \begin{cases} \mathbf{c}^{\square}_{j,\kappa_q^{-1}(x)} & x \in \overline{\Gamma_q}, \\ 0 & \text{otherwise,} \end{cases} \tag{4.2}$$

and similarly $\mathbf{F}_{j,q} : \ell_2(J_j^{\square}) \to \ell_2(J_j)$, we have $\sum_{q=1}^M \mathbf{E}_{j,q}\mathbf{E}_{j,q}^T = Id$ and $\sum_{q=1}^M \mathbf{F}_{j,q}\mathbf{F}_{j,q}^T = Id$. By construction of Φ_j from $\Phi_j^{\square}$, we have

$$\langle \Phi_j, \Phi_j \rangle_{L_2(\Gamma)} = \sum_{q=1}^M \mathbf{E}_{j,q} \langle \Phi_j^{\square}, \Phi_j^{\square} \rangle_{L_2(\square), |\partial\kappa_q|} \mathbf{E}_{j,q}^T, \tag{4.3}$$

so that, because of $\langle \Phi_j^{\square}, \Phi_j^{\square} \rangle_{L_2(\square), |\partial\kappa_q|} \eqsim \langle \Phi_j^{\square}, \Phi_j^{\square} \rangle_{L_2(\square)} \eqsim Id$ by $|\partial\kappa_q| \eqsim 1$ and $(\mathcal{R})$, Φ_j *is a uniform $L_2(\Gamma)$-Riesz system.*

Proposition 4.1. *Setting $\Lambda_j = (\lambda_{j,x})_{x \in I_j} \subset C(\overline{\Gamma})'$ by*

$$\lambda_{j,x}(u) = k(x)^{\frac{1}{2}} \lambda^{\square}_{j,\kappa_q^{-1}(x)}(u \circ \kappa_q) \quad \textit{when } x \in \overline{\Gamma}_q,$$

we have $\langle \Phi_j, \Lambda_j\rangle_{L_2(\Gamma)} = Id$. *The projector* $P_j : u \mapsto \langle u, \Lambda_j\rangle_{L_2(\Gamma)}\Phi_j$ *onto* $\mathcal{S}(\Phi_j)$ *satisfies*

$$\|(Id-P_j)u\|_{L_2(\Omega)} \lesssim 2^{-dj}\sum_{q=1}^{M} |u\circ\kappa_q|_{H^d(B(\kappa_q^{-1}(\Omega\cap\Gamma_q);(\vartheta+3\varepsilon)2^{-j})\cap\square)} \quad (\Omega\subset\Gamma,\, u\in\mathcal{H}_d(\Gamma)).$$

Proof. Assumption $(\mathcal{J})$(ii) shows that $\lambda_{j,x}(u)$ is well-defined for $u \in C(\overline{\Gamma})$, also when x is on an interface between patches, and, because of (4.1), that $(P_ju)\circ\kappa_q = P_j^{\square}(u\circ\kappa_q)$ when u vanishes on $\partial\Gamma_D$. Condition $(\mathcal{J})$(v) shows that $\langle\Phi_j,\Lambda_j\rangle_{L_2(\Gamma)} = Id$. By Proposition 3.1, we have

$$\|(Id-P_j)u\|_{L_2(\Omega)} \eqsim \sum_{q=1}^{M}\|((Id-P_j)u)\circ\kappa_q\|_{L_2(\kappa_q^{-1}(\Omega\cap\Gamma_q))} =$$

$$\sum_{q=1}^{M}\|(Id-P_j^{\square})(u\circ\kappa_q)\|_{L_2(\kappa_q^{-1}(\Omega\cap\Gamma_q))} \lesssim 2^{-dj}\sum_{q=1}^{M}|u\circ\kappa_q|_{H^d(B(\kappa_q^{-1}(\Omega\cap\Gamma_q);(\vartheta+3\varepsilon)2^{-j})\cap\square)}.$$

□

By substituting $\Omega = \Gamma$ in Proposition 4.1, we have the following *Jackson estimate*:

$$\inf_{u_j\in\mathcal{S}(\Phi_j)}\|u-u_j\|_{L_2(\Gamma)} \lesssim 2^{-dj}\|u\|_d, \qquad (u\in\mathcal{H}_d(\Gamma)). \tag{4.4}$$

A direct consequence of $(\mathcal{B})$ is the following *Bernstein inequality*: For $s\in[0,\gamma)$,

$$\|u_j\|_s \lesssim 2^{sj}\|u_j\|_{L_2(\Gamma)}, \quad (u_j\in\mathcal{S}(\Phi_j)). \tag{4.5}$$

Thanks to properties of a Sobolev scale, (4.5) gives rise to the following extended version that will be used in the appendix:

LEMMA 4.2. *For any* $t\le s<\gamma$ *with* $t\le 0$,

$$\|u_j\|_s \lesssim 2^{(s-t)j}\|u_j\|_t, \quad (u_j\in\mathcal{S}(\Phi_j)).$$

The short proof of this lemma can be found in [22].

As $\Phi_j^{\square}$, via (4.1), gave rise to a uniformly local, uniform $L_2(\Gamma)$-Riesz system Φ_j, analogously the collections $\Upsilon_{j+1}^{\square} = \left[(\Theta_j^{\square})^T \quad (\Xi_j^{\square})^T\right]^T$ and $\tilde{\Phi}_j^{\square}$ yield *uniformly local, uniform* $L_2(\Gamma)$*-Riesz systems* $\Upsilon_{j+1} = \left[\Theta_j^T \quad \Xi_j^T\right]^T$ and $\tilde{\Phi}_j$, respectively.

We have the analogue of Proposition 4.1 at the dual side, with functionals and a projector denoted as $\tilde{\Lambda}_j = (\tilde{\lambda}_{j,x})_{x\in I_j}$ and $\tilde{P}_j$, respectively, and with Φ_j, d, ϑ, ε replaced by $\tilde{\Phi}_j$, $\tilde{d}$, $\tilde{\vartheta}$, $\tilde{\varepsilon}$. In particular, we have the *Jackson estimate*

$$\inf_{u_j\in\mathcal{S}(\tilde{\Phi}_j)}\|u-u_j\|_{L_2(\Gamma)} \lesssim 2^{-\tilde{d}j}\|u\|_{\tilde{d}}, \qquad (u\in\mathcal{H}_{\tilde{d}}(\Gamma)), \tag{4.6}$$

and furthermore also the *Bernstein inequality*: For any $s\in[0,\tilde{\gamma})$,

$$\|u_j\|_s \lesssim 2^{sj}\|u_j\|_{L_2(\Gamma)}, \quad (u_j\in\mathcal{S}(\tilde{\Phi}_j)), \tag{4.7}$$

which can be extended analogously to Lemma 4.2.

Analogously to [12, Prop. 4.3.1], using Lemma 3.4, one may verify the following easily implementable formulas for the representations of the global embeddings in terms of corresponding representations of local embeddings.

PROPOSITION 4.3. *It holds that*

$$\Phi_j = \sum_{q=1}^{M} \mathbf{E}_{j,q} \langle \Phi_j^{\square}, \Lambda_{j+1}^{\square} \rangle_{L_2(\square)} \mathbf{E}_{j+1,q}^T \Phi_{j+1},$$

$$\tilde{\Phi}_j = \sum_{q=1}^{M} \mathbf{E}_{j,q} \langle \tilde{\Phi}_j^{\square}, \tilde{\Lambda}_{j+1}^{\square} \rangle_{L_2(\square)} \mathbf{E}_{j+1,q}^T \tilde{\Phi}_{j+1},$$

$$\Upsilon_{j+1} = \sum_{q=1}^{M} \left[\begin{array}{c|c} \mathbf{E}_{j,q} & 0 \\ \hline 0 & \mathbf{F}_{j,q} \end{array}\right] \langle \Upsilon_{j+1}^{\square}, \Lambda_{j+1}^{\square} \rangle_{L_2(\square)} \mathbf{E}_{j+1,q}^T \Phi_{j+1},$$

$$\Phi_{j+1} = \sum_{q=1}^{M} \mathbf{E}_{j+1,q} \langle \Upsilon_{j+1}^{\square}, \Lambda_{j+1}^{\square} \rangle_{L_2(\square)}^{-1} \left[\begin{array}{c|c} \mathbf{E}_{j,q}^T & 0 \\ \hline 0 & \mathbf{F}_{j,q}^T \end{array}\right] \Upsilon_{j+1}.$$

So, in particular, $\mathcal{S}(\Phi_j) \subset \mathcal{S}(\Phi_{j+1})$, $\mathcal{S}(\tilde{\Phi}_j) \subset \mathcal{S}(\tilde{\Phi}_{j+1})$ *and* $\mathcal{S}(\Upsilon_{j+1}) = \mathcal{S}(\Phi_{j+1})$.

5. Biorthogonal space decompositions and wavelets. We have constructed primal and dual sequences of nested spaces that satisfy Jackson and Bernstein estimates. To conclude existence and stability, with respect to a range of Sobolev norms, of the corresponding biorthogonal space decompositions, the only thing left to show is the existence and uniform $L_2(\Gamma)$-boundedness of the biorthogonal projector.

Results similar to the next lemma are often used in the context of saddle point problems. A proof of (the non-trivial part of) this lemma can be found in, e.g., [14, Theorem 2.1(a)].

LEMMA 5.1. *Let* V, U *be closed subspaces of a Hilbert space* H*. Then the following statements are equivalent:*

(a) $\gamma := \inf_{0 \neq u \in U} \sup_{0 \neq v \in V} \frac{|\langle u,v \rangle|}{\|u\|\|v\|} > 0$*, and for any* $v \in V$*, there exists a* $u \in U$ *with* $\langle u, v \rangle \neq 0$.

(b) *There exists a bounded projector* $Q : H \to H$ *with* $\Im(Q) = V$ *and* $\Im(I-Q) = U^{\perp}$*, which is therefore appropriately called a* biorthogonal projector.

In either case it holds that $\gamma = \|Q\|^{-1}$*, and the adjoint* Q^* *satisfies* $\Im(Q^*) = U$ *and* $\Im(I - Q^*) = V^{\perp}$.

When Σ *and* Δ *are Riesz bases for* U *and* V*, respectively, then* (a) *or* (b) *are equivalent to the existence of a bounded inverse of* $\langle \Sigma, \Delta \rangle : \ell_2(\Delta) \to \ell_2(\Sigma)$*. In that case it holds that*

$$\|\langle \Sigma, \Sigma \rangle^{-1}\|^{-\frac{1}{2}} \|\langle \Delta, \Delta \rangle^{-1}\|^{-\frac{1}{2}} \leq \frac{\|Q\|}{\|\langle \Sigma, \Delta \rangle^{-1}\|} \leq \|\Sigma\|\|\Delta\|.$$

To be able to transfer results valid on the reference parameter domain to the manifold, in particular those concerning $L_2(\square)$- or $L_2(\Gamma)$-angles between spaces, we will have to assume that the coarsest "mesh" is sufficiently fine in order to control the influence of the generally non-constant Jacobian determinants.

PROPOSITION 5.2. *For* $j \geq j_0$ *being large enough, there exists a uniformly bounded projector* $Q_j : L_2(\Gamma) \to L_2(\Gamma)$ *with* $\Im(Q_j) = \mathcal{S}(\Phi_j)$ *and* $\Im(I - Q_j) = \mathcal{S}(\tilde{\Phi}_j)^{\perp_{L_2(\Gamma)}}$.

Proof. Setting

$$\Delta_{j,q}^{\square} = \operatorname{diag}(|\partial\kappa_q(x)|)_{x\in I_j^{\square}}, \tag{5.1}$$

by $(\mathcal{L})$ for both $\Phi_j^{\square}, \tilde{\Phi}_j^{\square}$, the smoothness of $z \mapsto |\partial\kappa_q(z)|$, and the uniform boundedness of $\|\phi_{j,x}^{\square}\|_{L_2(\square)}$, $\|\tilde{\phi}_{j,x}^{\square}\|_{L_2(\square)}$, we have

$$\|\langle\Phi_j^{\square}, \tilde{\Phi}_j^{\square}\rangle_{L_2(\square),\,|\partial\kappa_q|} - (\Delta_{j,q}^{\square})^{\frac{1}{2}}\langle\Phi_j^{\square}, \tilde{\Phi}_j^{\square}\rangle_{L_2(\square)}(\Delta_{j,q}^{\square})^{\frac{1}{2}}\| \lesssim 2^{-j}. \tag{5.2}$$

By assumption $(\mathcal{J}1)$ and $|\partial\kappa_q| \gtrsim 1$, we have

$$\Re((\Delta_{j,q}^{\square})^{\frac{1}{2}}\langle\Phi_j^{\square}, \tilde{\Phi}_j^{\square}\rangle_{L_2(\square)}(\Delta_{j,q}^{\square})^{\frac{1}{2}}) = (\Delta_{j,q}^{\square})^{\frac{1}{2}}\Re\langle\Phi_j^{\square}, \tilde{\Phi}_j^{\square}\rangle_{L_2(\square)}(\Delta_{j,q}^{\square})^{\frac{1}{2}} \gtrsim Id,$$

so that for $j \geq j_0$ large enough, $\Re\langle\Phi_j^{\square}, \tilde{\Phi}_j^{\square}\rangle_{L_2(\square),\,|\partial\kappa_q|} \gtrsim Id$. Similarly to (4.3), we find that

$$\Re\langle\Phi_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)} = \sum_{q=1}^{M} \mathbf{E}_{j,q}\Re\langle\Phi_j^{\square}, \tilde{\Phi}_j^{\square}\rangle_{L_2(\square),\,|\partial\kappa_q|}\mathbf{E}_{j,q}^T \gtrsim \sum_{q=1}^{M} \mathbf{E}_{j,q}\mathbf{E}_{j,q}^T = Id.$$

Since apparently, for $j \geq j_0$, $\langle\Phi_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)}$ is uniformly boundedly invertible, and Φ_j and $\tilde{\Phi}_j$ are uniform $L_2(\Gamma)$-Riesz systems, an application of Lemma 5.1 completes the the proof. ∎

For $j \geq j_0$, the nesting $\mathcal{S}(\tilde{\Phi}_j) \subset \mathcal{S}(\tilde{\Phi}_{j+1})$ gives $Q_j^* = Q_{j+1}^* Q_j^*$ or $Q_j = Q_j Q_{j+1}$, from which it follows that

$$\Im(Q_{j+1} - Q_j) = \mathcal{S}(\Phi_{j+1}) \cap \mathcal{S}(\tilde{\Phi}_j)^{\perp_{L_2(\Gamma)}}.$$

Analogously, $\mathcal{S}(\Phi_j) \subset \mathcal{S}(\Phi_{j+1})$ implies that

$$\Im(Q_{j+1}^* - Q_j^*) = \mathcal{S}(\tilde{\Phi}_{j+1}) \cap \mathcal{S}(\Phi_j)^{\perp_{L_2(\Gamma)}}.$$

From the Jackson estimates and Bernstein inequalities at primal and dual side (4.4), (4.5), (4.6) and (4.7), and the existence and uniform $L_2(\Gamma)$-boundedness of the biorthogonal projectors Q_j from Proposition 5.2, we have

THEOREM 5.3 (cf. e.g. [8], [14, Theorem 2.1]). *With $Q_{j_0-1} := 0$, it holds that*

$$\|\!|\sum_{j=j_0}^{\infty} w_j|\!\|_s^2 \lesssim \sum_{j=j_0}^{\infty} 4^{sj}\|w_j\|_{L_2(\Gamma)}^2, \qquad (w_j \in \Im(Q_j - Q_{j-1}),\ s \in (-\tilde{d}, \gamma)), \tag{5.3}$$

and

$$\sum_{j=j_0}^{\infty} 4^{sj}\|(Q_j - Q_{j-1})u\|_{L_2(\Gamma)}^2 \lesssim \|\!|u|\!\|_s^2, \qquad (u \in \mathcal{H}_s(\Gamma),\ s \in (-\tilde{\gamma}, d)). \tag{5.4}$$

For $s \in (-\min\{\tilde{\gamma}, \tilde{d}\}, \min\{\gamma, d\})$, $(w_j)_{j\geq j_0} \mapsto \sum_{j=j_0}^{\infty} w_j$ and $u \mapsto ((Q_j - Q_{j-1})u)_{j\geq j_0}$, which mappings are bounded in the sense of (5.3) and (5.4), are each others inverse.

Analogous results are valid with (Q_j) replaced by (Q_j^*) and with interchanged roles of (γ, d) and $(\tilde{\gamma}, \tilde{d})$.

Next, we construct a uniform $L_2(\Gamma)$-Riesz basis for $\Im(Q_{j+1} - Q_j)$, which elements are called *wavelets*.

PROPOSITION 5.4.

(a) *For $j \geq j_0$ being large enough, there exists a uniformly bounded projector $\bar{Q}_j : L_2(\Gamma) \to L_2(\Gamma)$ with $\Im(\bar{Q}_j) = \mathcal{S}(\Theta_j)$ and $\Im(Id - \bar{Q}_j) = \mathcal{S}(\tilde{\Phi}_j)^{\perp_{L_2(\Gamma)}}$.*

(b) *This projector can be computed as $\bar{Q}_j u = \langle u, \tilde{\Phi}_j\rangle_{L_2(\Gamma)}\langle \Theta_j, \tilde{\Phi}_j\rangle^{-1}_{L_2(\Gamma)}\Theta_j$.*

(c) *The collection of wavelets*

$$\boxed{\Psi_j := \Xi_j - \langle \Xi_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)}\langle \Theta_j, \tilde{\Phi}_j\rangle^{-1}_{L_2(\Gamma)}\Theta_j,} \tag{5.5}$$

is a uniform $L_2(\Gamma)$-Riesz basis for $\mathcal{S}(\Phi_{j+1}) \cap \mathcal{S}(\tilde{\Phi}_j)^{\perp_{L_2(\Gamma)}}$.

So by taking j_0 to be the maximum of the values from (a) *and that of Proposition 5.2, for $s \in (-\min\{\tilde{\gamma}, \tilde{d}\}, \min\{\gamma, d\})$,*

$$\Phi_{j_0} \cup \cup_{j\geq j_0} 2^{-sj}\Psi_j \text{ is a Riesz basis for } \mathcal{H}_s(\Gamma),$$

and thus, in view of (2.4)*, when in addition $|s| < \frac{3}{2}$, $|s| < s_\Gamma \notin \mathbb{N}$ or $|s| \leq s_\Gamma \in \mathbb{N}$, it is a Riesz basis for $\mathcal{H}^s(\Gamma)$.*

Proof. (a) Since Θ_j and $\tilde{\Phi}_j$ are uniform $L_2(\Gamma)$-Riesz bases, by Lemma 5.1 we have to show that, for $j \geq j_0$ being large enough, $\langle \Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)}$ is uniformly boundedly invertible, which follows from (J2) similarly to the proof of Proposition 5.2.

(b) Using that $\langle \Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)}$ is uniformly boundedly invertible, one easily verifies that $\bar{Q}_j$, as given in (b), indeed has the properties listed in (a).

(c) Let $u \in \mathcal{S}(\Phi_{j+1})$, then $u = \mathbf{c}_j^T\Theta_j + \mathbf{d}_j^T\Xi_j$ with $\|u\|_{L_2(\Gamma)} \eqsim (\|\mathbf{c}_j\|^2 + \|\mathbf{d}_j\|^2)^{\frac{1}{2}}$. If, in addition, $u \in \mathcal{S}(\tilde{\Phi}_j)^{\perp_{L_2(\Gamma)}}$, then $u = (Id - \bar{Q}_j)u = (Id - \bar{Q}_j)\mathbf{d}_j^T\Xi_j = \mathbf{d}_j^T\Psi_j$, and so with $\|u\|_{L_2(\Gamma)} \lesssim (1 + \|\bar{Q}_j\|_{L_2(\Gamma)\to L_2(\Gamma)})\|\mathbf{d}_j^T\Xi_j\|_{L_2(\Gamma)} \lesssim \|\mathbf{d}_j\| \lesssim \|u\|_{L_2(\Gamma)}$. Noting that $\Psi_j \subset \mathcal{S}(\Phi_{j+1}) \cap \mathcal{S}(\tilde{\Phi}_j)^{\perp_{L_2(\Gamma)}}$, we conclude that it is a uniform $L_2(\Gamma)$-Riesz basis for this space. The last statements are now consequences of Theorem 5.3. ☐

Note that Proposition 4.1 at the dual side implies that $\Psi_j = (\psi_{j,x})_{x\in J_j}$, yielded by (5.5), satisfies

$$\begin{aligned}|\langle \psi_{j,x}, u\rangle_{L_2(\Gamma)}| &= |\langle \psi_{j,x}, (Id - \tilde{P}_j)u\rangle_{L_2(\Gamma)}| \\ &\lesssim 2^{-\tilde{d}j}\sum_{q=1}^{M} |u\circ\kappa_q|_{H^{\tilde{d}}(B(\kappa_q^{-1}(\operatorname{supp}\psi_{j,x}\cap\Gamma_q);(\tilde{\vartheta}+3\tilde{\varepsilon})2^{-j})\cap\square)} \quad (u \in \mathcal{H}^{\tilde{d}}_{\tilde{d}}(\Gamma)).\end{aligned} \tag{5.6}$$

This property of the collections Ψ_j, with $\tilde{\vartheta} + 3\tilde{\varepsilon}$ replaced by an arbitrary but fixed $\tilde{\eta} \geq 0$ and the semi-norms $|\cdot|_{H^{\tilde{d}}(\cdots)}$ replaced by the norms $\|\cdot\|_{H^{\tilde{d}}(\cdots)}$, will be referred to as the *uniform cancellation property of order $\tilde{d}$*.

6. Stability of *approximate* wavelet bases. Similarly to (4.3), the definition of the collections Θ_j and $\tilde{\Phi}_j$ via (4.1) shows that

$$\langle \Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)} = \sum_{q=1}^{M} \mathbf{E}_{j,q}\langle \Theta_j^{\square}, \tilde{\Phi}_j^{\square}\rangle_{L_2(\square),|\partial\kappa_q|}\mathbf{E}_{j,q}^T. \tag{6.1}$$

So if, for each q, $z \mapsto |\partial\kappa_q(z)|$ is a *constant function*, then (J2) shows that $\langle \Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)}$ is diagonal, and the collection of wavelets Ψ_j given in (5.5) is *uniformly local*. Unfortunately, only a restricted class of manifolds can be described as the union of patches that are the images of $\square$ under parametrizations that have constant Jacobians. In case *not all Jacobians are constants*, then, generally, $\langle \Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)}$ is not diagonal

and its inverse is densely populated, so that (5.5) yields wavelets Ψ_j that have *global supports*.

A possibility to circumvent this problem, pursued in [12], is to carry out the whole wavelet construction outlined so far using the modified scalar product $\langle\!\langle\, ,\, \rangle\!\rangle_0$ instead of $\langle\, ,\, \rangle_{L_2(\Gamma)}$. Indeed, $\langle\!\langle \Theta_j, \tilde{\Phi}_j \rangle\!\rangle_0 = \sum_{q=1}^M \mathbf{E}_{j,q} \langle \Theta_j^\square, \tilde{\Phi}_j^\square \rangle_{L_2(\square)} \mathbf{E}_{j,q}^T = \sum_{q=1}^M \mathbf{E}_{j,q}\ \mathbf{E}_{j,q}^T = Id$, and so uniformly local wavelets are obtained. What is more, by employing this scalar product, it is always possible to take the coarsest level $j_0 = 0$.

As was already recognized in [12], this approach, however, has two limitations: Firstly, the obtained wavelets will be orthogonal to the constant function with respect to $\langle\!\langle\, ,\, \rangle\!\rangle_0$. As a consequence, if the function

$$J : \cup_{q=1}^M \Gamma_q \to \mathbb{R} : x \mapsto |\partial \kappa_{q'}(\kappa_{q'}^{-1}(x))| \quad \text{when } x \in \Gamma_q'$$

has discontinuities, or more precisely, cannot be extended to a continuous function on Γ, then wavelets with supports that are not contained in one patch will generally *not* have a zero mean value with respect to the canonical Lebesgue measure on Γ, meaning that they have *no cancellation property* with respect to $\langle \cdot, \cdot \rangle_{L_2(\Gamma)}$. The application of wavelets we focus on is that for the solution of differential- or integral equations in variational form using the duality pairing with respect to $\langle \cdot, \cdot \rangle_{L_2(\Gamma)}$ (taking a different scalar product here yields other disadvantages, cf. [14, §1.2]). For obtaining nearly sparse representations of these operators in wavelet coordinates, and with that algorithms of optimal computational complexity, the wavelets should have a cancellation property of sufficiently high order (cf. [10, 21] or the surveys [9, 3]), with respect thus to $\langle \cdot, \cdot \rangle_{L_2(\Gamma)}$. In a non-adaptive setting, under circumstances it might be possible that the fact that only wavelets along the lower dimensional patch interfaces do not have cancellation properties does not spoil optimal complexity. In an adaptive setting, however, such an argument cannot be applied.

The second limitation has to do with the interpretation for $s < 0$ of the statement that $\Phi_{j_0} \cup \cup_{j \geq j_0} 2^{-sj} \Psi_j$ is a Riesz basis for $\mathcal{H}_s(\Gamma)$, which is a consequence of Theorem 5.3. In case biorthogonality is realized with respect to $\langle \cdot, \cdot \rangle_{L_2(\Gamma)}$, then an expansion in terms of the basis $\Phi_{j_0} \cup \cup_{j \geq j_0} 2^{-sj} \Psi_j$ should be interpreted as an element of $\mathcal{H}_s(\Gamma)$, i.e., as a functional, using the embedding $L_2(\Gamma) \to \mathcal{H}_s(\Gamma) : u \mapsto (v \mapsto \langle v, u \rangle_{L_2(\Gamma)})$. Replacing $\langle \cdot, \cdot \rangle_{L_2(\Gamma)}$ by $\langle\!\langle \cdot, \cdot \rangle\!\rangle_0$ means that also the embedding should be changed into $u \mapsto (v \mapsto \langle\!\langle v, u \rangle\!\rangle_0)$. One can show (cf. [17, §4]) that if J has discontinuities, then for $s \leq -\frac{1}{2}$ a set $\Phi_{j_0} \cup \cup_{j \geq j_0} 2^{-sj} \Psi_j$ which is a Riesz basis for $\mathcal{H}_s(\Gamma)$ using the embedding $u \mapsto (v \mapsto \langle\!\langle v, u \rangle\!\rangle_0)$ *cannot* be a Riesz basis for $\mathcal{H}_s(\Gamma)$ using the embedding $u \mapsto (v \mapsto \langle v, u \rangle_{L_2(\Gamma)})$. For $s > -\frac{1}{2}$, the property of being a Riesz basis for $\mathcal{H}_s(\Gamma)$ is the same for both embeddings. Again, since in applications duality pairing with respect to $\langle \cdot, \cdot \rangle_{L_2(\Gamma)}$ is used, in this paper the property of a collection of functions to be a Riesz basis for $\mathcal{H}_s(\Gamma)$ for $s < 0$ will always be interpreted with respect to the canonical embedding $u \mapsto (v \mapsto \langle v, u \rangle_{L_2(\Gamma)})$.

In this paper, we propose another approach to solve the problem that generally $\langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)}^{-1}$ is densely populated, so that the wavelets yielded by (5.5) have global supports. As we will see, $\langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)}^{-1}$ can be well approximated by uniformly local matrices, so that close to the collections of the wavelets Ψ_j, there are collections of uniformly local functions of which suitable ones might be applied instead. In the following main theorem of this paper we derive general criteria under which such *approximate wavelets* satisfy the *same* conditions as Ψ_j concerning *both* stability with

respect to a range of Sobolev norms *and* the order of the cancellation property, where moreover, in contrast to Ψ_j, they are uniformly local.

THEOREM 6.1. *If, for $j \geq j_0$,*

(i) $\breve{\Psi}_j = (\breve{\psi}_{j,x})_{x\in J_j} \subset \mathcal{S}(\Phi_{j+1})$ *is uniformly local,*

(ii) $\breve{\Psi}_j$ *has the uniform cancellation property of order $\tilde{d}$ (w.r.t. $\langle\cdot,\cdot\rangle_{L_2(\Gamma)}$),*

(iii) *for some $\omega \in (0,1)$, $\|\Psi_j - \breve{\Psi}_j\|_{L_2(\Gamma)} \lesssim \omega^j$,*

then, possibly for a larger value of j_0, for $s \in (-\min\{\tilde{\gamma},\tilde{d}\}, \min\{\gamma, d\})$,

$$\Phi_{j_0} \cup \cup_{j\geq j_0} 2^{-sj}\breve{\Psi}_j \text{ is a Riesz basis for } \mathcal{H}_s(\Gamma),$$

and thus, in view of (2.4), when in addition $|s| < \frac{3}{2}$, $|s| < s_\Gamma \notin \mathbb{N}$ or $|s| \leq s_\Gamma \in \mathbb{N}$, it is a Riesz basis for $\mathcal{H}^s(\Gamma)$.

The rather lengthy proof of this theorem is postponed to Appendix A. The new aspect of this theorem is that instead of assuming (iii) with $\omega \leq 2^{-\min\{\tilde{\gamma},\tilde{d}\}}$, which would yield the statement by "brute force" arguments, it is allowed that ω is arbitrarily close to 1 when, in addition, (ii) is valid, which property we like to have anyway. So although in two of our three constructions of approximate wavelets in the next section, ω will be equal to $\frac{1}{2}$, we nevertheless thus end up with Riesz bases for $\mathcal{H}_s(\Gamma)$ for the full range of s allowed by $\tilde{\gamma}$, $\tilde{d}$, γ, and d. It is easily seen that (ii) alone is not sufficient to guarantee that the approximate wavelets generate a Riesz basis for any $\mathcal{H}_s(\Gamma)$.

The proof of Theorem 6.1 is based on perturbation arguments making use of the fact that we know that the true $L_2(\Gamma)$-biorthogonal wavelets generate Riesz bases for the full range of Sobolev spaces. We derived this fact in Section 5 by generalizing upon the well-known concept of stable completions developed in [2]. Note that in our setting we did not assume to have explicitly available $L_2(\Gamma)$-biorthogonal collections of scaling functions. Theorem 6.1 and the applications in the following sections show the value of this generalization.

Remark 6.2. The approximate wavelets $\breve{\Psi}_j$ we are going to construct will be of type $\breve{\Psi}_j = \Xi_j - \mathbf{Z}_j\Theta_j$, where $\mathbf{Z}_j$ is a uniformly local $\#J_j \times \#I_j$ matrix. Since the basis transformation $\langle \Upsilon_{j+1}, \Lambda_{j+1}\rangle^T_{L_2(\Gamma)}$ from $\Upsilon_{j+1} = \left[\Theta_j^T \quad \Xi_j^T\right]^T$ to Φ_{j+1} is uniformly local, so is the basis transformation from $\left[\Theta_j^T \quad \breve{\Psi}_j^T\right]^T$ to Φ_{j+1}, and the transformation from multi-scale basis $\Phi_{j_0} \cup \cup_{k=j_0}^{j-1}\breve{\Psi}_k$ to single-scale basis Φ_j has linear complexity.

In the special case that $\Theta_j = \Phi_j$ and $\langle \Upsilon_{j+1}, \Lambda_{j+1}\rangle^{-T}_{L_2(\Gamma)}$ is uniformly local, which by Proposition 4.3 holds assuming (3.1), the basis transformation from Φ_{j+1} to $\left[\Phi_j^T \quad \breve{\Psi}_j^T\right]^T$ will also be uniformly local, and so the inverse transformation from single-scale basis Φ_j to multi-scale basis $\Phi_{j_0} \cup \cup_{k=j_0}^{j-1}\breve{\Psi}_k$ has also linear complexity. However, since $\mathcal{S}(\breve{\Psi}_j)$ is only approximately $L_2(\Gamma)$-orthogonal to $\mathcal{S}(\Phi_j)$, the corresponding dual wavelets will not be explicitly given.

7. Construction of uniformly local approximate wavelet bases.

7.1. Approximating $\langle\Theta_j, \tilde{\Phi}_j\rangle^{-1}_{L_2(\Gamma)}$ using Jacobi iteration. As we will see, for $j \to \infty$, the matrix $\langle\Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)}$ is increasingly close to its diagonal, and so it makes sense to approximate its inverse by a few Jacobi iteration steps ($\tilde{d}$ steps will be sufficient). We will denote the resulting collection of approximate wavelets as Ψ_j^{Jc}, where "Jc" refers to Jacobi iteration.

THEOREM 7.1. *With* $\mathbf{D}_j := \operatorname{diag}\langle\Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)}$*, and for* $j \geq j_0$ *large enough,*

$$\Psi_j^{\mathrm{Jc}} := \Xi_j - \langle\Xi_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)}\Big[\sum_{k=0}^{\tilde{d}-1}(Id - \mathbf{D}_j^{-1}\langle\Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)})^k\mathbf{D}_j^{-1}\Big]\Theta_j$$

is uniformly local, it has the uniform cancellation property of order $\tilde{d}$*, and finally, for* $s \in (-\min\{\tilde{\gamma}, \tilde{d}\}, \min\{\gamma, d\})$*,*

$$\Phi_{j_0} \cup \cup_{j\geq j_0} 2^{-sj}\Psi_j^{\mathrm{Jc}} \text{ is a Riesz basis for } \mathcal{H}_s(\Gamma).$$

In view of (2.4)*, when in addition* $|s| < \frac{3}{2}$*,* $|s| < s_\Gamma \notin \mathbb{N}$ *or* $|s| \leq s_\Gamma \in \mathbb{N}$*, the collection is thus a Riesz basis for* $\mathcal{H}^s(\Gamma)$*.*

Proof. With $\Delta_{j,q}^{\square}$ as defined in (5.1), and by using (J2), similar as in (5.2) we have $\|\langle\Theta_j^{\square}, \tilde{\Phi}_j^{\square}\rangle_{L_2(\square),|\partial\kappa_q|} - \Delta_{j,q}^{\square}\| \lesssim 2^{-j}$. Since

$$\bar{\mathbf{D}}_j := \sum_{q=1}^{M}\mathbf{E}_{j,q}\Delta_{j,q}^{\square}\mathbf{E}_{j,q}^T$$

is diagonal, by (6.1) we have

$$\begin{aligned}\|\mathbf{D}_j - \bar{\mathbf{D}}_j\| &\leq \max_{x,y\in I_j}|(\mathbf{D}_j - \bar{\mathbf{D}}_j)_{x,y}| \leq \max_{x,y\in I_j}|(\langle\Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)} - \bar{\mathbf{D}}_j)_{x,y}| \\ &\leq \|\langle\Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)} - \bar{\mathbf{D}}_j\| \lesssim 2^{-j},\end{aligned}$$

and so $\|\mathbf{D}_j - \langle\Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)}\| \leq \|\mathbf{D}_j - \bar{\mathbf{D}}_j\| + \|\bar{\mathbf{D}}_j - \langle\Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)}\| \lesssim 2^{-j}$.

As we have seen in the proof of Proposition 5.4, for $j \geq j_0$ large enough, the matrix $\langle\Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)}$ is uniformly boundedly invertible, and thus, possibly for a larger j_0, so is $\mathbf{D}_j$. We infer that

$$\begin{aligned}&\|\langle\Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)}^{-1} - \sum_{k=0}^{\tilde{d}-1}(Id - \mathbf{D}_j^{-1}\langle\Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)})^k\mathbf{D}_j^{-1}\| \\ &\qquad = \|\langle\Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)}^{-1}((\mathbf{D}_j - \langle\Theta_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)})\mathbf{D}_j^{-1})^{\tilde{d}}\| \lesssim 2^{-\tilde{d}j},\end{aligned} \tag{7.1}$$

and thus by $\|\langle\Xi_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)}\| \lesssim 1$, that

$$\|\Psi_j - \Psi_j^{\mathrm{Jc}}\|_{L_2(\Gamma)} \lesssim 2^{-\tilde{d}j}\|\Theta_j\|_{L_2(\Gamma)} \lesssim 2^{-\tilde{d}j}. \tag{7.2}$$

Since furthermore Ψ_j^{Jc} is uniformly local, in view of Theorem 6.1 the only thing left to show is that Ψ_j^{Jc} has the uniform cancellation property of order $\tilde{d}$.

Although Ψ_j has the uniform cancellation property of order $\tilde{d}$, we cannot immediately conclude this from (7.2) for Ψ_j^{Jc}. Indeed, since the wavelets from Ψ_j generally have global supports, invoking (7.2) and the cancellation property of $\psi_{j,x}$ would yield a bound for $|\langle\psi_{j,x}^{\mathrm{Jc}}, u\rangle_{L_2(\Gamma)}|$ in terms of the global H^d-norms of $u \circ \kappa_q$, whereas the definition of the cancellation property requires a bound in terms of the H^d-norms of $u \circ \kappa_q$ in a neighbourhood of $\operatorname{supp}(\psi_{j,x}^{\mathrm{Jc}} \circ \kappa_q)$ with diameter of order 2^{-j}. To arrive at this result, we split u into $(Id - \tilde{P}_j)u$ and $\tilde{P}_j u$, and then replace $\tilde{P}_j u$ by a function, equal to $\tilde{P}_j u$ on $\operatorname{supp}\psi_{j,x}^{\mathrm{Jc}}$ and still in $\mathcal{S}(\tilde{\Phi}_j)$, that has a support with diameter of order 2^{-j}. The details can be found in [22]. ∎

Remark 7.2. The construction of the approximate wavelets Ψ_j^{Jc} in Theorem 7.1 has some similarities to the construction of approximate "prewavelets" in [23]. There the inverse of a mass matrix with respect to a standard finite element basis is approximated by a number of steps of an iterative method, as the Jacobi or symmetric Gauss-Seidel method. A difference is that in our case the matrix $\mathbf{D}_j$ *converges to* $\langle\Theta_j,\tilde{\Phi}_j\rangle_{L_2(\Gamma)}$ as $j\to\infty$, allowing us to derive much stronger results concerning the generation of Riesz bases by the resulting approximate wavelets.

Compared to the approximate wavelets one gets by simply replacing $\langle\cdot,\cdot\rangle_{L_2(\Gamma)}$ by $\langle\!\langle\cdot,\cdot\rangle\!\rangle_0$ in (5.5), for $\tilde{d}\geq 2$ the approximate wavelets $\psi_{j,x}^{\mathrm{Jc}}$ have relatively large supports. Although this has not so much an effect on the multi-scale to single scale transform, that can be implemented much more efficient than is suggested by the sizes of the supports, it is a disadvantage for example when it concerns the compression of the stiffness matrix of an integral operator with respect to these approximate wavelets. In the following two subsections, we construct approximate wavelets with smaller supports.

As a preparation, the next proposition facilitates the verification of the third condition from Theorem 6.1, in case different constructions of approximate wavelets are used on different parts of Γ. In the proof, the problem of the generally global supports of the true biorthogonal wavelets is circumvented by approximating them by sufficiently accurate, uniformly local approximate wavelets generated by the Jacobi iteration approach.

PROPOSITION 7.3. *Let $\omega\in(0,1)$ and let $\breve{\Psi}_j=(\breve{\psi}_{j,x})_{x\in J_j}$ be uniformly local. Then $\|\Psi_j-\breve{\Psi}_j\|_{L_2(\Gamma)}\lesssim\omega^j$ if and only if $\sup_{x\in J_j}\|\psi_{j,x}-\breve{\psi}_{j,x}\|_{L_2(\Gamma)}\lesssim\omega^j$.*

Proof. Let $\sup_{x\in J_j}\|\psi_{j,x}-\breve{\psi}_{j,x}\|_{L_2(\Gamma)}\lesssim\omega^j$. Selecting $m\in\mathbb{N}$ such that $2^{-m}\leq\omega$, from the proof of Theorem 7.1 we learn that there exists a uniformly local Ψ_j^{Jc} with $\|\Psi_j-\Psi_j^{\mathrm{Jc}}\|_{L_2(\Gamma)}\lesssim 2^{-mj}\leq\omega^j$, and so $\sup_{x\in J_j}\|\psi_{j,x}-\psi_{j,x}^{\mathrm{Jc}}\|_{L_2(\Gamma)}\lesssim\omega^j$ and thus $\sup_{x\in J_j}\|\psi_{j,x}^{\mathrm{Jc}}-\breve{\psi}_{j,x}\|_{L_2(\Gamma)}\lesssim\omega^j$. Since both Ψ_j^{Jc} and $\breve{\Psi}_j$ are uniformly local, this implies $\|\Psi_j^{\mathrm{Jc}}-\breve{\Psi}_j\|_{L_2(\Gamma)}\lesssim\omega^j$, and thus that $\|\Psi_j-\breve{\Psi}_j\|_{L_2(\Gamma)}\leq\|\Psi_j-\Psi_j^{\mathrm{Jc}}\|_{L_2(\Gamma)}+\|\Psi_j^{\mathrm{Jc}}-\breve{\Psi}_j\|_{L_2(\Gamma)}\lesssim\omega^j$. The proof of the other implication is trivial. □

7.2. Ignoring the Jacobian determinants away from the interfaces. In this subsection, we show that *away from the patch interfaces*, we may replace the wavelets from Ψ_j^{Jc} by the corresponding ones from

$$\Psi_j^{(0)}:=\Xi_j-\langle\!\langle\Xi_j,\tilde{\Phi}_j\rangle\!\rangle_0\Theta_j.$$

This is the collection of biorthogonal wavelets one obtains when biorthogonality is realized with respect to $\langle\!\langle\,,\,\rangle\!\rangle_0$ instead of $\langle\,,\,\rangle_{L_2(\Gamma)}$, i.e., when, in the wavelet formula (5.5), all Jacobian determinants are replaced by the constant 1.

Recalling that for $x\in\overline{\Gamma}$, $k(x)=\#\{q:x\in\overline{\Gamma_q}\}$, we set $I_j^\circ=\{x\in I_j:k(x)=1\}$ and

$$J_j^\circ=\{x\in J_j:k(x)=1\text{ and }\langle\xi_{j,x},\tilde{\phi}_{j,y}\rangle_{L_2(\Gamma)}=0\text{ for all }y\in I_j\backslash I_j^\circ\},$$

which set is designed such that for $x\in J_j^\circ$, $\psi_{j,x}^{(0)}$ is fully supported inside one patch $\overline{\Gamma_q}$.

THEOREM 7.4. *The set $\breve{\Psi}_j=\{\breve{\psi}_{j,x}:x\in J_j\}$, defined by $\breve{\psi}_{j,x}=\psi_{j,x}^{(0)}$ when $x\in J_j^\circ$, and $\breve{\psi}_{j,x}=\psi_{j,x}^{\mathrm{Jc}}$ when $x\in J_j\backslash J_j^\circ$, is uniformly local, it has the uniform cancellation*

property of order $\tilde{d}$, and for any $s \in (-\min\{\tilde{\gamma}, \tilde{d}\}, \min\{\gamma, d\})$ and j_0 large enough,

$$\Phi_{j_0} \cup \cup_{j \geq j_0} 2^{-sj} \breve{\Psi}_j \text{ is a Riesz basis for } \mathcal{H}_s(\Gamma).$$

In view of (2.4)*, when in addition $|s| < \frac{3}{2}$, $|s| < s_\Gamma \notin \mathbb{N}$ or $|s| \leq s_\Gamma \in \mathbb{N}$, this collection is thus a Riesz basis for $\mathcal{H}^s(\Gamma)$.*

Proof. Using that, for $j \geq j_0$ large enough, $\|\langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)}^{-1} - \mathbf{D}_j^{-1}\| \lesssim 2^{-j}$, which follows as a special case from (7.1), and that the mappings $z \mapsto |\partial \kappa_q(z)|$ are smooth, one easily verifies that for $x \in J_j^\circ$,

$$\|\psi_{j,x} - \psi_{j,x}^{(0)}\| \lesssim 2^{-j}.$$

Since for $x \in J_j \backslash J_j^\circ$, $\|\psi_{j,x} - \psi_{j,x}^{\mathrm{Jc}}\| \lesssim 2^{-\tilde{d}j}$, and $\breve{\Psi}_j$ is uniformly local, in view of Theorem 6.1 and Proposition 7.3 the only thing left to show is that $\breve{\Psi}_j$ has the uniform cancellation property of order $\tilde{d}$. Knowing this for Ψ_j^{Jc}, we only have to consider $\psi_{j,x}^{(0)}$ for $x \in J_j^\circ$.

Let $x \in J_j^\circ$, say $x \in \Gamma_q$, and let u be a globally continuous, patchwise smooth function on Γ that is zero on $\partial\Gamma_D$. Let v be some arbitrary extension of the mapping $x \mapsto u(x)|\partial\kappa_q(\kappa_q^{-1}(x))|$ on Γ_q to a globally continuous, patchwise smooth function on Γ that is zero on $\partial\Gamma_D$. Since $\operatorname{supp} \psi_{j,x}^{(0)} \subset \overline{\Gamma_q}$, from Proposition 4.1 at the dual side we have

$$\begin{aligned}
|\langle \psi_{j,x}^{(0)}, u \rangle_{L_2(\Gamma)}| &= |\langle\!\langle \psi_{j,x}^{(0)}, v \rangle\!\rangle_0| = |\langle\!\langle \psi_{j,x}^{(0)}, ((Id - \tilde{P}_j)v)|_{\operatorname{supp} \psi_{j,x}^{(0)}} \rangle\!\rangle_0| \\
&\lesssim 2^{-\tilde{d}j} |v \circ \kappa_q|_{H^{\tilde{d}}(B(\kappa_q^{-1}(\operatorname{supp} \psi_{j,x}^{(0)}); (\tilde{\vartheta} + 3\tilde{\varepsilon})2^{-j}) \cap \square)} \\
&\lesssim 2^{-\tilde{d}j} \|u \circ \kappa_q\|_{H^{\tilde{d}}(B(\kappa_q^{-1}(\operatorname{supp} \psi_{j,x}^{(0)}); (\tilde{\vartheta} + 3\tilde{\varepsilon})2^{-j}) \cap \square)},
\end{aligned}$$

which completes the proof. □

Note that for $x \in J_j^\circ$, generally it only holds that $\|\psi_{j,x} - \psi_{j,x}^{(0)}\|_{L_2(\Gamma)} \eqsim 2^{-j}$. So in contrast to Ψ_j^{Jc}, for the approximate wavelets $\breve{\Psi}_j$ from this subsection, generally $\|\Psi_j - \breve{\Psi}_j\|_{L_2(\Gamma)} \not\lesssim 2^{-\tilde{d}j}$ when $\tilde{d} > 1$. The same will hold true for the collections $\breve{\Psi}_j$ that will be constructed in the next subsection. As follows from Theorem 6.1, however, this fact does not limit the range of s for which the approximate wavelets generate a Riesz basis for $\mathcal{H}^s(\Gamma)$.

7.3. Approximate wavelets with small supports near the interfaces. As we saw in the previous subsection, for $x \in J_j^\circ$ we can replace $\psi_{j,x}^{\mathrm{Jc}}$ by $\psi_{j,x}^{(0)}$, that, for $\tilde{d} \geq 2$, has a much smaller support. In this subsection, we investigate whether also near the interfaces we can find appropriate approximate wavelets $\breve{\psi}_{j,x}$ with smaller supports.

We set

$$\mathbb{P}_{\tilde{d}-2}(\Gamma) := C(\Gamma) \cap \prod_{q=1}^{M} \kappa_q(\mathbb{P}_{\tilde{d}-2}(\square)),$$

where when $\square$ is the interior of an n-simplex, $\mathbb{P}_{\tilde{d}-2}(\square) := P_{\tilde{d}-2}(\square)$, and when $\square = (0,1)^n$, $\mathbb{P}_{\tilde{d}-2}(\square) := Q_{\tilde{d}-2}(\square)$, being the tensor product space of the univariate polynomial spaces $P_{\tilde{d}-2}(0,1)$ in the n coordinate directions. In the latter case, in

addition to the assumption that $P_{\tilde{d}-1}(\square) \subset \mathcal{S}(\tilde{\Phi}_j^{\square})$ ($(\mathcal{J})$(ii) at the dual side), in this subsection we assume that

$$(\tilde{\mathcal{J}}\text{e}) \qquad Q_{\tilde{d}-2}(\square) \subset \mathcal{S}(\tilde{\Phi}_j^{\square}).$$

For $z \in \Gamma$ and $\varepsilon \geq 0$, let $B_\Gamma(z;\varepsilon) = \{y \in \overline{\Gamma} : d_\Gamma(z,y) \leq \varepsilon\}$. With, for some constant $\rho \geq 0$, setting

$$\tilde{V}_{j,x,\rho} = \Big\{v \in \mathbb{P}_{\tilde{d}-2}(\Gamma) : v|_{\partial\Gamma_D \cap B_\Gamma(x;\rho 2^{-j})} = 0\Big\},$$

for $x \in J_j \backslash J_j^\circ$ we will search

$$\breve{\psi}_{j,x} \perp_{L_2(\Gamma)} \tilde{V}_{j,x,\rho} \text{ with } \|\breve{\psi}_{j,x} - \hat{\psi}_{j,x}\|_{L_2(\Gamma)} \lesssim 2^{-j}. \tag{7.3}$$

We note that by taking $\tilde{V}_{j,x,\rho}$ to be the smaller set $\{v \in \mathbb{P}_{\tilde{d}-2}(\Gamma) : v|_{\partial\Gamma_D} = 0\}$, $\breve{\psi}_{j,x}$ would not necessarily have the cancellation property of order $\tilde{d}$, and on the other hand, as we will see later, without incorporating boundary conditions in the definition of $\tilde{V}_{j,x,\rho}$, generally we cannot expect that $\|\breve{\psi}_{j,x} - \psi_{j,x}\|_{L_2(\Gamma)} \lesssim 2^{-j}$. For the moment assuming that such $\breve{\psi}_{j,x}$ can be found, which topic will be treated later in this subsection, the following Theorem 7.5 shows that they have the uniform cancellation property of order $\tilde{d}$, which, by Theorem 6.1, additionally yields the Riesz basis property for the full range of s. This may look surprising since, ignoring boundary conditions, the condition $\breve{\psi}_{j,x} \perp_{L_2(\Gamma)} \tilde{V}_{j,x,\rho}$ seems only to imply the uniform cancellation property of order $\tilde{d}-1$. Yet, given a $u \in \mathcal{H}_{\tilde{d}}(\Gamma)$, an interpolant $v \in \tilde{V}_{j,x,\rho}$ can be constructed such that $\|u-v\|_{L_2(\operatorname{supp}\breve{\psi}_{j,x})} \lesssim 2^{-(\tilde{d}-1)j}$, and, using $(\tilde{\mathcal{J}}\text{e})$, such that $(Id - \tilde{P}_j)v$ vanishes on $\operatorname{supp}\breve{\psi}_{j,x}$. Now by writing

$$\begin{aligned}\langle \breve{\psi}_{j,x}, u\rangle_{L_2(\Gamma)} &= \langle \breve{\psi}_{j,x}, u-v\rangle_{L_2(\Gamma)} = \langle \breve{\psi}_{j,x}, (Id-\tilde{P}_j)u\rangle_{L_2(\Gamma)} + \langle \breve{\psi}_{j,x}, \tilde{P}_j(u-v)\rangle_{L_2(\Gamma)} \\ &= \langle \breve{\psi}_{j,x}, (Id-\tilde{P}_j)u\rangle_{L_2(\Gamma)} + \langle \breve{\psi}_{j,x} - \psi_{j,x}, \tilde{P}_j(u-v)\rangle_{L_2(\Gamma)},\end{aligned}$$

and, for the second term, using the additional factor 2^{-j} from $\|\breve{\psi}_{j,x} - \psi_{j,x}\|_{L_2(\Gamma)} \lesssim 2^{-j}$, the cancellation property of order $\tilde{d}$ can be shown. For a detailed proof of Theorem 7.5, we refer to [22].

THEOREM 7.5. *Let $\breve{\Psi}_j = \{\breve{\psi}_{j,x} : x \in J_j\} \subset \mathcal{S}(\Phi_{j+1})$ be a uniformly local set, with $\breve{\psi}_{j,x} = \psi_{j,x}^{(0)}$ when $x \in J_j^\circ$, and such for $x \in J_j \backslash J_j^\circ$, (7.3) is valid. Then, for $j \geq j_0$ large enough, $\breve{\Psi}_j$ has the uniform cancellation property of order $\tilde{d}$, and for $s \in (-\min\{\tilde{\gamma}, \tilde{d}\}, \min\{\gamma, d\})$,*

$$\Phi_{j_0} \cup \cup_{j \geq j_0} 2^{-sj}\breve{\Psi}_j \text{ is a Riesz basis for } \mathcal{H}_s(\Gamma).$$

In view of (2.4)*, when in addition $|s| < \frac{3}{2}$, $|s| < s_\Gamma \notin \mathbb{N}$ or $|s| \leq s_\Gamma \in \mathbb{N}$, this collection is thus a Riesz basis for $\mathcal{H}^s(\Gamma)$.*

Next, we discuss a *construction* of $\breve{\Psi}_j$ as in Theorem 7.5. Consider for $x \in J_j \backslash J_j^\circ$, the first order approximation $\hat{\psi}_{j,x}$ for $\psi_{j,x}$ from the collection

$$\hat{\Psi}_j := \Xi_j - \langle \Xi_j, \tilde{\Phi}_j\rangle_{L_2(\Gamma)} \mathbf{D}_j^{-1} \Theta_j. \tag{7.4}$$

As a special case of (7.1), we have $\|\psi_{j,x} - \hat{\psi}_{j,x}\|_{L_2(\Gamma)} \lesssim 2^{-j}$, where generally $\hat{\psi}_{j,x}$ only has the cancellation property of order 1. We will construct $\breve{\psi}_{j,x}$ from $\hat{\psi}_{j,x}$ by adding

correction terms. In view of our requirement that $\|\psi_{j,x} - \breve{\psi}_{j,x}\|_{L_2(\Gamma)} \lesssim 2^{-j}$, we first show that $\hat{\psi}_{j,x}$ is already nearly orthogonal to $\tilde{V}_{j,x,\rho}$, so that the correction can be small. For this to be true, the incorporation of boundary conditions in the definition of $\tilde{V}_{j,x,\rho}$ is essential.

We set $\bar{I}_j = \cup_{q=1}^M \kappa_q(I_j^\square)$, i.e., without the exclusion of possible points on $\partial\Gamma_D$, and define $\tilde{\phi}_{j,y}$ and $\tilde{\lambda}_{j,y}$ similarly as in (4.1) and in Proposition 4.1, respectively, but now also for $y \in \bar{I}_j \backslash I_j$.

LEMMA 7.6. *Let the constant ρ in the definition of $\tilde{V}_{j,x,\rho}$ be sufficiently large such that for all $x \in J_j$ and $y \in \bar{I}_j$ with $\operatorname{supp}\tilde{\phi}_{j,y} \cap \operatorname{supp}\hat{\psi}_{j,x} \neq \emptyset$, $\operatorname{supp}\tilde{\lambda}_{j,y} \subset B_\Gamma(x;\rho 2^{-j})$. Then*

$$|\langle \hat{\psi}_{j,x}, p\rangle_{L_2(\Gamma)}| \lesssim 2^{-j}\|p\|_{L_2(\operatorname{supp}\hat{\psi}_{j,x})} \quad (p \in \tilde{V}_{j,x,\rho}).$$

Proof. Since $p \in \mathbb{P}_{\tilde{d}-2}(\Gamma)$, by the inclusion of possible points on $\partial\Gamma_D$ and $(\tilde{\mathcal{J}}\mathrm{e})$, we have $p = \sum_{y\in\bar{I}_j}\langle p, \tilde{\lambda}_{j,y}\rangle_{L_2(\Gamma)}\tilde{\phi}_{j,y}$. Terms in this sum for $y \in \bar{I}_j\backslash I_j$ vanish on $\operatorname{supp}\hat{\psi}_{j,x}$ by $(\mathcal{J})$(ii) and because p vanishes on $\partial\Gamma_D \cap B_\Gamma(x;\rho 2^{-j})$. Setting $p_{j,x} = \sum_{\{y\in I_j:\operatorname{supp}\tilde{\phi}_{j,y}\cap\operatorname{supp}\hat{\psi}_{j,x}\neq\emptyset\}}\langle p, \tilde{\lambda}_{j,y}\rangle_{L_2(\Gamma)}\tilde{\phi}_{j,y}$, which is a function in $\mathcal{S}(\tilde{\Phi}_j)$, we find that

$$\begin{aligned}|\langle \hat{\psi}_{j,x}, p\rangle_{L_2(\Gamma)}| = |\langle \hat{\psi}_{j,x}, p_{j,x}\rangle_{L_2(\Gamma)}| &= |\langle \hat{\psi}_{j,x} - \psi_{j,x}, p_{j,x}\rangle_{L_2(\Gamma)}| \\ &\lesssim 2^{-j}\|p_{j,x}\|_{L_2(\Gamma)} \lesssim 2^{-j}\|p\|_{L_2(\operatorname{supp}\hat{\psi}_{j,x})},\end{aligned}$$

where in the last step we used $(\mathcal{J})$(iv). ∎

Let us first consider the *special case* $\tilde{d} = 2$ making the natural assumption that $|\int \xi_{j,x}d\mu| \gtrsim 2^{-jn/2}$. Let ρ be as in Lemma 7.6. If $x \in J_j\backslash J_j^\circ$ is such that $\partial\Gamma_D \cap B_\Gamma(x;\rho 2^{-j}) \neq \emptyset$, then $\tilde{V}_{j,x,\rho} = \{0\}$, and we can take $\breve{\psi}_{j,x} = \hat{\psi}_{j,x}$. Otherwise, we take $\breve{\psi}_{j,x} := \hat{\psi}_{j,x} - [\int_\Gamma \hat{\psi}_{j,x}d\mu / \int_\Gamma \xi_{j,x}d\mu]\xi_{j,x}$. Obviously $\breve{\psi}_{j,x} \perp_{L_2(\Gamma)} 1$, i.e., $\breve{\psi}_{j,x} \perp \tilde{V}_{j,x,\rho}$, and Lemma 7.6 shows that $|\int_\Gamma \hat{\psi}_{j,x}d\mu| \lesssim 2^{-j(1+n/2)}$, so that indeed $\|\hat{\psi}_{j,x} - \breve{\psi}_{j,x}\|_{L_2(\Gamma)} \lesssim 2^{-j}$. In view of our aim to replace $\psi_{j,x}^{\mathrm{Jc}}$ for $x \in J_j\backslash J_j^\circ$ by an approximate wavelet with smaller support, note that the support of $\breve{\psi}_{j,x}$ is *equal* to that of $\hat{\psi}_{j,x}$ (which is equal to that of $\psi_{j,x}^{(0)}$).

For $\tilde{d} > 2$, generally we have to add more than one degree of freedom to find a correction of $\hat{\psi}_{j,x}$ that is orthogonal to $\tilde{V}_{j,x,\rho}$. We will search the correction from the span of $\theta_{j,y}$ with $d_\Gamma(x,y) \lesssim 2^{-j}$. Instead of adding as many degrees of freedom as $\dim(\tilde{V}_{j,x,\rho})$, generally we add more degrees of freedom, but then solve the resulting underdetermined problem in a minimal norm sense to end up with a correction term that is as small as possible. The resulting approximate wavelets will be denoted as $\psi_{j,x}^{\mathrm{ls}}$, where 'ls' refers to least-squares. In Theorem 7.7 it is stated that if, for sufficiently large δ, we use all $\theta_{j,y}$ for $y \in I_j$ with $d_\Gamma(x,y) \leq \delta 2^{-j}$, then the constrained minimization problem has a unique solution $\psi_{j,x}^{\mathrm{ls}}$, with $\|\psi_{j,x}^{\mathrm{ls}} - \psi_{j,x}\|_{L_2(\Gamma)} \lesssim 2^{-j}$. Note that although in our numerical example we end up with $\psi_{j,x}^{\mathrm{ls}}$ that has the *same* support as $\hat{\psi}_{j,x}$, which thus in particular is much smaller than the support of $\psi_{j,x}^{\mathrm{Jc}}$, we cannot prove this in general.

THEOREM 7.7. *Let ρ be as in Lemma 7.6. For a sufficiently large constant $\delta > 0$, and with $\Theta_{j,x}^\delta := \{\theta_{j,y} : y \in I_j \cap B_\Gamma(x;\delta 2^{-j})\}$, for any $x \in J_j\backslash J_j^\circ$ the problem of*

determining

$$\operatorname*{argmin}_{\psi^{\mathrm{ls}}_{j,x}\in\hat{\psi}_{j,x}+\mathcal{S}(\Theta^{\delta}_{j,x})}\{\|\psi^{\mathrm{ls}}_{j,x}-\hat{\psi}_{j,x}\|_{L_2(\Gamma)}:\psi^{\mathrm{ls}}_{j,x}\perp_{L_2(\Gamma)}\tilde{V}_{j,x,\rho}\}$$

has a unique solution with $\|\psi^{\mathrm{ls}}_{j,x}-\psi_{j,x}\|_{L_2(\Gamma)}\lesssim 2^{-j}$*, so that Theorem 7.5 applies.*

For a proof of this theorem we refer to [22]. From the theory of saddle point problems we know that the constrained minimization problem has a solution if and only if for each $p\in\tilde{V}_{j,x,\rho}$, there exists a $v_j\in\mathcal{S}(\Theta^{\delta}_{j,x})$ with $\langle p,v_j\rangle_{L_2(\Gamma)}>0$. It can be observed that on a sufficiently large neighbourhood of $\operatorname{supp}\hat{\psi}_{j,x}$ with diameter $\eqsim 2^{-j}$, any $p\in\mathbb{P}_{\tilde{d}-2}$ can be well approximated by a linear combination of those $\tilde{\phi}_{j,y}$ $(y\in I_j)$ that are fully supported in this neighbourhood. Since $\langle\Theta_j,\tilde{\Phi}_j\rangle_{L_2(\Gamma)}$ is nearly diagonal, for any function w_j in the span of these $\tilde{\phi}_{j,y}$, a v_j can be found in the span of the $\theta_{j,y}$, for the same set of y, with $\langle w_j,v_j\rangle_{L_2(\Gamma)}>0$. By combining both properties, the existence and uniqueness of the constrained minimization problem can be inferred. The estimate $\|\psi^{\mathrm{ls}}_{j,x}-\psi_{j,x}\|_{L_2(\Gamma)}\lesssim 2^{-j}$ can be derived from the fact that $\hat{\psi}_{j,x}$ is already nearly orthogonal to $\tilde{V}_{j,x,\rho}$, as was shown in Lemma 7.6.

Remark 7.8. In case $\Theta_j\neq\Phi_j$, it is not of any interest that $\breve{\Psi}_j-\Xi_j\in\mathcal{S}(\Theta_j)$ (cf. Remark 6.2). In that case, in Theorem 7.7 we may search $\psi^{\mathrm{ls}}_{j,x}$ in the larger space $\hat{\psi}_{j,x}+\mathcal{S}(\Phi^{\delta}_{j+1,x})$, with $\Phi^{\delta}_{j+1,x}:=\{\phi_{j+1,y}:x\in I_{j+1}\cap B_\Gamma(x;\delta 2^{-j})\}$, which opens the possibility that we may take a smaller δ, and so reduce the support of the resulting $\psi^{\mathrm{ls}}_{j,x}$.

Remark 7.9. Both the construction of $\psi^{\mathrm{Jc}}_{j,x}$ from Theorem 7.1, and that of $\psi^{\mathrm{ls}}_{j,x}$ from Theorem 7.7 requires the evaluation of $L_2(\Gamma)$-scalar products. For general parametrizations κ_q, these scalar products cannot be evaluated exactly, and therefore have to be approximated using numerical quadrature. Theorem 6.1 shows that if the quadrature is organized such that it causes an $L_2(\Gamma)$-error $\lesssim 2^{-\tilde{d}j}$ in the resulting approximate wavelet, then all results concerning cancellation properties and the generation of Riesz bases remain valid.

8. Numerical example. We consider a 2-dimensional Lipschitz manifold $\Gamma=\cup_{i=1}^{4}\overline{\Gamma_q}\subset\mathbb{R}^3$ as illustrated in Figure 8.1, which, together with its parametrization that satisfies $(\mathcal{M})$ is defined as follows. Let P be a tetrahedron in $\mathbb{R}^3$, with vertices on the unit sphere, geometric centroid in $(0,0,0)$, and one of its four facets $F_1,\ldots,F_4$, say F_4, parallel to and below the $x_3=0$ plane. Let $\square$ be the interior of a reference 2-simplex

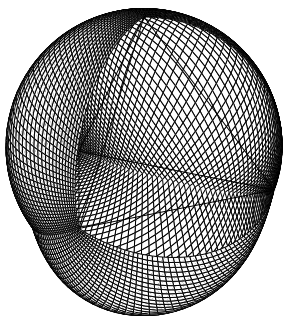

FIG. 8.1. *The manifold* Γ*, excluding one of the patches* F_1*,* F_2*, or* F_3*, and the tetrahedron* P*.*

in $\mathbb{R}^2$, with $\operatorname{vol}(\square)=1$, and for $1\leq q\leq 4$, let $B_q:\square\to F_q$ some affine bijection. The

parametrizations $\kappa_q : \square \to \Gamma_q$ are defined by $\kappa_q(z) = B_q(z)/\|B_q(z)\|$ for $1 \leq q \leq 3$, and by $\kappa_4(z) = B_q(z)/\|B_q(z)\| - \frac{27}{4}(0, 0, \lambda_1(z)\lambda_2(z)\lambda_3(z))$, where $(\lambda_1(z), \lambda_2(z), \lambda_3(z))$ are the barycentric coordinates of z with respect to $\square$. So without the perturbation by this cubic bubble, Γ would be the unit sphere. We added this perturbation term so that $J : x \mapsto |\partial\kappa_q(\kappa_q^{-1}(x))|$ when $x \in \Gamma_q$, cannot be extended to a continuous function on Γ. This means that constructions based on ignoring the Jacobian determinants will yield wavelets of which those that have supports that intersect an interface between Γ_4 and one of the three other patches have no cancellation properties. Furthermore, in view of our discussion at the beginning of §6, note that $z \mapsto |\partial\kappa_q(z)|$ are not constant functions.

We consider two examples of collections $\Phi_j^\square$, $\tilde{\Phi}_j^\square$, $\Theta_j^\square$, $\Xi_j^\square$, $\Lambda_j^\square$, $\tilde{\Lambda}_j^\square$, both based on the construction of finite element wavelets from [14]. These collections satisfy all conditions formulated in §3, where $\square$ is a reference 2-simplex, $\gamma = \tilde{\gamma} = \frac{3}{2}$, $d = 2$, and $\tilde{d} = 2$ or $\tilde{d} = 3$. With S_j being the standard Lagrange finite element space of order 2 with respect to a j times repeated uniform dyadic refinement of $\square$, for $\tilde{d} = 2$, $\mathcal{S}(\Phi_j^\square) = S_j$, and for $\tilde{d} = 3$, $\mathcal{S}(\Phi_j^\square) = S_{j+1}$. The index set $I_j^\square$ is the set of all vertices in the triangulation underlying the finite element space, and $J_j^\square = I_{j+1}^\square \backslash I_j^\square$. For more details, we refer to [22].

We implemented the approximate wavelet constructions from §7.2 and §7.3, that away from the patch interfaces both yield the approximate wavelets from the collection $\Psi_j^{(0)} = \Xi_j - \langle\!\langle \Xi_j, \tilde{\Phi}_j \rangle\!\rangle_0 \langle\!\langle \Theta_j, \tilde{\Phi}_j \rangle\!\rangle_0^{-1} \Theta_j$ obtained by ignoring the Jacobian determinants. The pull-backs of these wavelets to the parameter domain are illustrated in Figure 8.2, which functions are continuous piecewise linear with respect to the indicated triangulation.

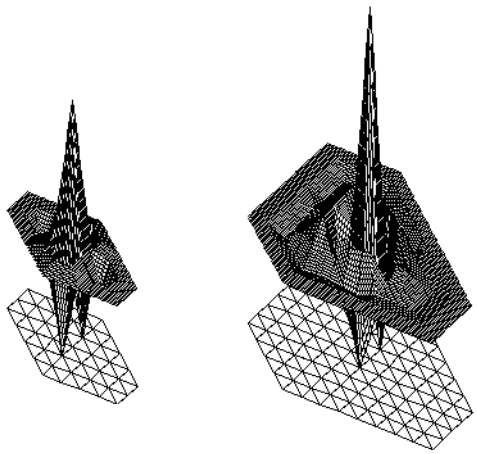

FIG. 8.2. *Wavelets $\psi_{j,x}^{(0)}$ away from the patch interfaces for $\tilde{d} = 2$ and $\tilde{d} = 3$ (one of the two different types), and their supports in terms of the underlying triangulation.*

With the approach from §7.2, wavelets along the patch interfaces are taken from the collection $\Psi_j^{\mathrm{Jc}} := \Xi_j - \langle \Xi_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)} \left[\sum_{k=0}^{\tilde{d}-1} (Id - \mathbf{D}_j^{-1} \langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)})^k \mathbf{D}_j^{-1} \right] \Theta_j$, where $\mathbf{D}_j := \mathrm{diag} \langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)}$. Illustrations of the, naturally joined, patchwise pull-backs of these wavelets can be found in Figure 8.3. For $j = 0$, the Neumann series does not converge, and as a consequence $\sum_{k=0}^{\tilde{d}-1} (Id - \mathbf{D}_j^{-1} \langle \Theta_j, \tilde{\Phi}_j \rangle_{L_2(\Gamma)})^k \mathbf{D}_j^{-1}$

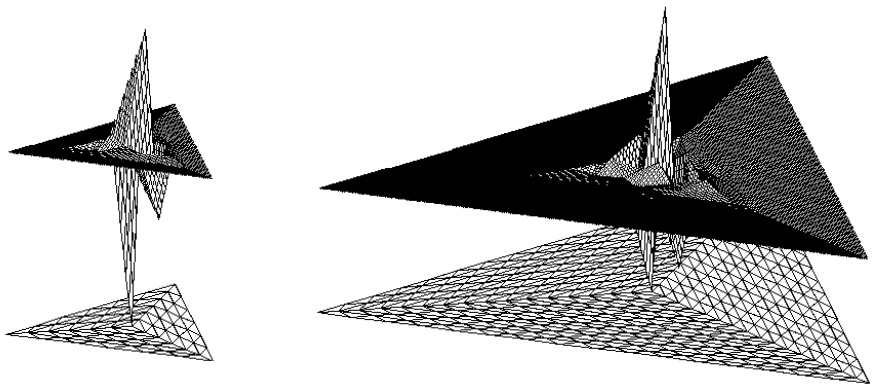

FIG. 8.3. *Wavelets $\psi^{\mathrm{Jc}}_{4,x}$ for $\tilde{d}=2$ and $\psi^{\mathrm{Jc}}_{3,x}$ for $\tilde{d}=3$ near the "north pole", and their supports in terms of the underlying triangulation.*

provides a very poor approximation for $\langle\Theta_j,\tilde{\Phi}_j\rangle^{-1}_{L_2(\Gamma)}$. We redefined $\breve{\Psi}_0 := \Xi_0 - \langle\Xi_j,\tilde{\Phi}_j\rangle_{L_2(\Gamma)}\langle\Theta_j,\tilde{\Phi}_j\rangle^{-1}_{L_2(\Gamma)}\Theta_0$.

For the construction from §7.3, for each $x\in J_j\backslash J_j^\circ$ we have to specify a subspace $A_{j,x}\subset\mathcal{S}(\Phi_{j+1})$ that defines $\psi^{\mathrm{ls}}_{j,x}$ via

$$\operatorname*{argmin}_{\psi^{\mathrm{ls}}_{j,x}\in\hat{\psi}_{j,x}+A_{j,x}}\{\|\psi^{\mathrm{ls}}_{j,x}-\hat{\psi}_{j,x}\|_{L_2(\Gamma)}:\psi^{\mathrm{ls}}_{j,x}\perp_{L_2(\Gamma)}\mathbb{P}_{\tilde{d}-2}(\Gamma)\}.$$

For $\tilde{d}=2$, we took $A_{j,x}=\mathcal{S}(\{\xi_{j,x}\})$, so that $\psi^{\mathrm{ls}}_{j,x}=\hat{\psi}_{j,x}+\alpha\xi_{j,x}$ with α such that $\int_\Gamma\psi^{\mathrm{ls}}_{j,x}=0$. For $\tilde{d}=3$, we took $A_{j,x}=\mathcal{S}(\{\phi_{j+1,y}:\operatorname{supp}\phi_{j+1,y}\subset\operatorname{supp}\hat{\psi}_{j,x}\})$, which space turns out to be sufficiently large so that the constrained minimization problem has a solution $\psi^{\mathrm{ls}}_{j,x}$, with $\|\psi^{\mathrm{ls}}_{j,x}-\hat{\psi}_{j,x}\|_{L_2(\Gamma)}\lesssim 2^{-j}$. The, naturally joined, patchwise pull-backs of the resulting $\psi^{\mathrm{ls}}_{j,x}$ are illustrated in Figure 8.4. By definition they have the *same supports* as the corresponding $\psi^{(0)}_{j,x}$ that one obtains by ignoring the Jacobian determinants also along the interfaces. As with the Jacobi iteration

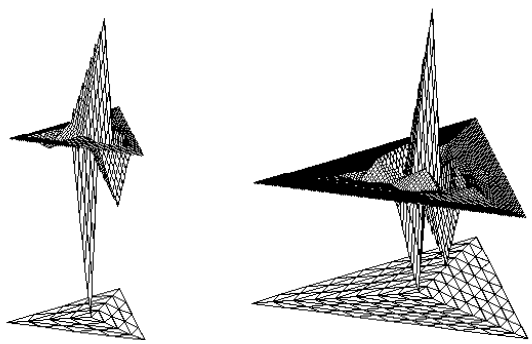

FIG. 8.4. *Wavelets $\psi^{\mathrm{ls}}_{4,x}$ for $\tilde{d}=2$ and $\psi^{\mathrm{ls}}_{3,x}$ for $\tilde{d}=3$ near the "north pole", and their supports in terms of the underlying triangulation.*

approach, for $j=0$, we redefined $\breve{\Psi}_0:=\Xi_0-\langle\Xi_j,\tilde{\Phi}_j\rangle_{L_2(\Gamma)}\langle\Theta_j,\tilde{\Phi}_j\rangle^{-1}_{L_2(\Gamma)}\Theta_0$.

Let

$$\breve{\Psi}^{(j)} := \begin{cases} \Phi_0 \cup \cup_{k=0}^{j-1} \breve{\Psi}_k & \text{when } \tilde{d}=2, \\ \Phi_0 \cup \cup_{k=0}^{j-2} \breve{\Psi}_k & \text{when } \tilde{d}=3, \end{cases}$$

so in both cases being a basis for $C(\Gamma) \cap \prod_{q=1}^4 \kappa_q(S_j)$. With

$$\kappa_{\Sigma,\|\cdot\|} := \sup_{0\neq \mathbf{c}=(c_\sigma)_{\sigma\in\Sigma}} \frac{\left\|\sum_{\sigma\in\Sigma} c_\sigma \frac{\sigma}{\|\sigma\|}\right\|^2}{\|\mathbf{c}\|^2} \Bigg/ \inf_{0\neq \mathbf{c}=(c_\sigma)_{\sigma\in\Sigma}} \frac{\left\|\sum_{\sigma\in\Sigma} c_\sigma \frac{\sigma}{\|\sigma\|}\right\|^2}{\|\mathbf{c}\|^2},$$

that, in case $\|\cdot\|$ corresponds to a scalar product, is the spectral norm of the corresponding mass matrix, for $j \leq 6$ we computed $\kappa_{\breve{\Psi}^{(j)},\|\cdot\|_{L_2(\Gamma)}}$, $\kappa_{\breve{\Psi}^{(j)},|||\cdot|||_1}$, and $\kappa_{\breve{\Psi}^{(j)},|||\cdot|||_{-1,6}}$, where on $C(\Gamma)\cap\prod_{q=1}^4 \kappa_q(S_m)$, $|||u|||_{-1,m} := \sup_{v\in C(\Gamma)\cap\prod_{q=1}^4 \kappa_q(S_m)} \frac{|\langle u,v\rangle_{L_2(\Gamma)}|}{|||v|||_1}$. The uniform boundedness in $|||\cdot|||_1$ of the $L_2(\Gamma)$-orthogonal projector onto $C(\Gamma)\cap \prod_{q=1}^4 \kappa_q(S_m)$, which is a consequence of Theorem 5.3, shows that $|||\cdot|||_{-1} \eqsim |||\cdot|||_{-1,m}$ on $C(\Gamma)\cap\prod_{q=1}^4 \kappa_q(S_j)$ uniformly in m and $j \leq m$.

Since the functions from Φ_0, and for $\tilde{d}=2$, from $\breve{\Psi}_0$ and $\breve{\Psi}_1$, and for $\tilde{d}=3$, from $\breve{\Psi}_0$ have global supports anyway, for computing the condition numbers we replaced each of these collections by a orthonormalized version with respect to either $\langle\cdot,\cdot\rangle_{L_2(\Gamma)}$, $\langle\!\langle\cdot,\cdot\rangle\!\rangle_1$ or $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{-1,6}$, the latter being the scalar product corresponding to the norm $\|\cdot\|_{-1,6}$. The results for the Jacobi approximation or least squares approximation along the interfaces are presented in Tables 8.1 and 8.2, respectively. Although

TABLE 8.1
Condition numbers for $\tilde{d}=2$.

	Jacobi approximation			least squares approximation		
j	$\kappa_{\Psi^{(j)},\|\cdot\|_{L_2(\Gamma)}}$	$\kappa_{\breve{\Psi}^{(j)},|||\cdot|||_1}$	$\kappa_{\breve{\Psi}^{(j)},|||\cdot|||_{-1,6}}$	$\kappa_{\breve{\Psi}^{(j)},\|\cdot\|_{L_2(\Gamma)}}$	$\kappa_{\breve{\Psi}^{(j)},|||\cdot|||_1}$	$\kappa_{\breve{\Psi}^{(j)},|||\cdot|||_{-1,6}}$
1	1.35e0	2.30e0	3.06e0	1.35e0	2.30e0	3.09e0
2	1.45e0	7.86e0	8.65e0	1.82e0	7.64e0	1.09e1
3	1.75e1	4.82e1	2.79e1	2.42e1	6.84e1	3.41e1
4	1.79e1	6.51e1	4.53e1	2.46e1	9.06e1	5.21e1
5	1.79e1	7.66e1	6.20e1	2.46e1	1.06e2	6.69e1
6	1.79e1	8.25e1	7.34e1	2.46e1	1.14e2	7.84e1

TABLE 8.2
Condition numbers for $\tilde{d}=3$.

	Jacobi approximation			least squares approximation		
j	$\kappa_{\Psi^{(j)},\|\cdot\|_{L_2(\Gamma)}}$	$\kappa_{\breve{\Psi}^{(j)},|||\cdot|||_1}$	$\kappa_{\breve{\Psi}^{(j)},|||\cdot|||_{-1,6}}$	$\kappa_{\breve{\Psi}^{(j)},\|\cdot\|_{L_2(\Gamma)}}$	$\kappa_{\breve{\Psi}^{(j)},|||\cdot|||_1}$	$\kappa_{\breve{\Psi}^{(j)},|||\cdot|||_{-1,6}}$
2	2.80e0	1.26e1	4.18e1	2.80e0	1.26e1	4.22e0
3	1.36e1	5.47e1	3.36e1	1.85e1	8.51e1	3.82e1
4	1.81e1	7.88e1	5.76e1	2.34e1	1.19e2	6.86e1
5	1.97e1	8.83e1	8.00e1	2.44e1	1.32e2	9.65e1
6	2.03e1	9.20e1	9.64e1	2.47e1	1.36e2	1.34e2

it turns out that unfortunately, in particular in the $\|\cdot\|_{-1,6}$-norm, the condition numbers were not completely stabilized yet, we stopped our computations at $j=6$

since mainly due to the normalization of the wavelets, in particular with respect to the $\|\cdot\|_{-1,6}$-norm, on higher levels they became too time consuming.

Appendix A. Proof of Theorem 6.1. The proof consists of steps **(I)**-**(VI)**. Although basically **(V)** and **(VI)** can be found in [19, Theorem 3.1], which in turn was based on [24, Appendix], for convenience we include a complete proof.

(I) Since, as shown in Proposition 5.4, for $j \geq j_0$ large enough, Ψ_j is a uniform $L_2(\Gamma)$-Riesz system, and by Condition (iii),

$$\begin{aligned}&\|\langle\breve{\Psi}_j,\breve{\Psi}_j\rangle_{L_2(\Gamma)} - \langle\Psi_j,\Psi_j\rangle_{L_2(\Gamma)}\|\\ &= \|\langle\breve{\Psi}_j-\Psi_j,\Psi_j\rangle_{L_2(\Gamma)} + \langle\Psi_j,\breve{\Psi}_j-\Psi_j\rangle_{L_2(\Gamma)} + \langle\breve{\Psi}_j-\Psi_j,\breve{\Psi}_j-\Psi_j\rangle_{L_2(\Gamma)}\| \lesssim \omega^j,\end{aligned}$$

we infer that, possibly for a larger value of j_0, for $j \geq j_0$, $\breve{\Psi}_j$ is a uniform $L_2(\Gamma)$-Riesz system.

(II) Next we investigate whether $\mathcal{S}(\Phi_j)+\mathcal{S}(\breve{\Psi}_j)$ is a uniformly $L_2(\Gamma)$-stable *two-level* decomposition of $\mathcal{S}(\Phi_{j+1})$:

PROPOSITION A.1. *Let V, W be subspaces of a Hilbert space H. Then equivalent are*

(a) *$H = V + W$ and $\alpha := \sup_{0\neq v\in V,\, 0\neq w\in W} \frac{|\langle v,w\rangle|}{\|v\|\|w\|} < 1$,*

(b) *There exists a bounded projector $Q : H \to H$ with $\Im(Q) = V$ and $\Im(Id-Q) = W$. Moreover, $\|Q\| = (1-\alpha^2)^{-\frac{1}{2}}$.*

Now let (a)*, or equivalently* (b) *be satisfied, and let $\breve{W}$ be another subspace of H for which there exists a linear mapping $R : \breve{W} \to W$ with $\|Id - R\| < \frac{1-\alpha}{1+\alpha}$. Then*

$$\breve{\alpha} := \sup_{0\neq v\in V,\, 0\neq\breve{w}\in\breve{W}} \frac{|\langle v,\breve{w}\rangle|}{\|v\|\|\breve{w}\|} \leq \alpha + (1+\alpha)\|Id-R\| < 1.$$

With $\breve{Q} : H \to H$ being the bounded projector with $\Im(\breve{Q}) = V$ and $\Im(Id-\breve{Q}) = \breve{W}$, it holds that

$$\|Q - \breve{Q}\| \leq \frac{\|Id-R\|}{(1-\alpha^2)^{\frac{1}{2}}(1-\breve{\alpha}^2)^{\frac{1}{2}}}.$$

Proof. If (a) is valid, then $H = V \oplus W$, and so there exists a projector Q with $\Im(Q) = V$ and $\Im(Id-Q) = W$. For any $u \in H$, we have

$$\begin{aligned}\|u\|^2 = \|Qu + (Id-Q)u\|^2 &\geq \|Qu\|^2 - 2|\langle Qu,(Id-Q)u\rangle| + \|(Id-Q)u\|^2\\ &\geq \|Qu\|^2 - 2\alpha\|Qu\|\|(Id-Q)u\| + \|(Id-Q)u\|^2 \geq (1-\alpha^2)\|Qu\|^2,\end{aligned}$$

or $\|Q\| \leq (1-\alpha^2)^{-\frac{1}{2}}$.

Now let (b) be valid. Suppose there exist nonzero $v \in V$, $w \in W$ such that $\mu := \frac{|\langle v,w\rangle|}{\|v\|\|w\|} > (1-\|Q\|^{-2})^{\frac{1}{2}}$. Then there exist nonzero $v \in V$, $w \in W$ with $\langle v,w\rangle = -\mu\|v\|\|w\|$, moreover which can be chosen such that $\|w\| = -\mu\|v\|$. From

$$\|Q\|^{-2}\|v\|^2 \leq \|v+w\|^2 = \|v\|^2 + 2\langle v,w\rangle + \|w\|^2 = (1-\mu^2)\|v\|^2$$

we conclude a contradiction, so that $\sup_{0\neq v\in V,\, 0\neq w\in W} \frac{|\langle v,w\rangle|}{\|v\|\|w\|} \leq (1-\|Q\|^{-2})^{\frac{1}{2}}$.

Now let (a) or, equivalently, (b) be valid, and let $\breve{W}$ be a subspace as in the proposition. For any $v \in V$, $\breve{w} \in \breve{W}$,

$$|\langle v, \breve{w}\rangle| = |\langle v, R\breve{w}\rangle + \langle v, (Id-R)\breve{w}\rangle| \le \alpha\|v\|\|R\breve{w}\| + \|v\|\|(Id-R)\breve{w}\|$$
$$\le \alpha\|v\|\|\breve{w}\| + (1+\alpha)\|v\|\|(Id-R)\breve{w}\| \le (\alpha + (1+\alpha)\|Id-R\|)\|v\|\|\breve{w}\|,$$

showing the statement about $\breve{\alpha}$. The last statement follows from $\|Q\| = (1-\alpha^2)^{-\frac{1}{2}}$, $\|Id - \breve{Q}\| = (1-\breve{\alpha}^2)^{-\frac{1}{2}}$, and $Q - \breve{Q} = Q(Id-R)(Id-\breve{Q})$ by $QR = 0$ and $Q\breve{Q} = \breve{Q}$, □

Returning to the proof of Theorem 6.1, with, for $j \ge j_0$, $Q_j^{(j+1)} := Q_j|_{\mathcal{S}(\Phi_{j+1})}$, it holds that $\Im(Q_j^{(j+1)}) = \mathcal{S}(\Phi_j)$ and $\Im(Id - Q_j^{(j+1)}) = \mathcal{S}(\Psi_j)$. Setting $R_j : \mathcal{S}(\breve{\Psi}_j) \to \mathcal{S}(\Psi_j) : \mathbf{c}_j^T\breve{\Psi}_j \mapsto \mathbf{c}_j^T\Psi_j$, by Condition (iii) and **(I)** we have

$$\|(Id-R_j)\mathbf{c}_j^T\breve{\Psi}_j\|_{L_2(\Gamma)} = \|\mathbf{c}_j^T(\breve{\Psi}_j - \Psi_j)\|_{L_2(\Gamma)} \le \|\mathbf{c}\|\|\breve{\Psi}_j - \Psi_j\|_{L_2(\Gamma)} \lesssim \|\mathbf{c}_j^T\breve{\Psi}_j\|_{L_2(\Gamma)}\omega^j,$$

or $\|Id - R_j\|_{L_2(\Gamma)\to L_2(\Gamma)} \lesssim \omega^j$. From Proposition A.1 we conclude that, possibly for a larger value of j_0, for $j \ge j_0$ there exists a uniformly $L_2(\Gamma)$-bounded projector $\breve{Q}_j^{(j+1)} : \mathcal{S}(\Phi_{j+1}) \to \mathcal{S}(\Phi_{j+1})$ with

$$\Im(\breve{Q}_j^{(j+1)}) = \mathcal{S}(\Phi_j), \quad \Im(Id - \breve{Q}_j^{(j+1)}) = \mathcal{S}(\breve{\Psi}_j),$$

and

$$\|Q_j^{(j+1)} - \breve{Q}_j^{(j+1)}\|_{L_2(\Gamma)\to L_2(\Gamma)} \lesssim \omega^j. \tag{A.1}$$

(III) By Condition (i), $\breve{\Psi}_j$ is uniformly local. Since furthermore, as shown in **(I)**, $\breve{\Psi}_j$ is a uniform $L_2(\Gamma)$-Riesz system, that, by Condition (ii), has the uniform cancellation property of order $\tilde{d}$, for some $\tilde{\eta} \ge 0$ we have

$$|\langle \mathbf{c}_j^T\breve{\Psi}_j, u\rangle_{L_2(\Gamma)}| \le \sum_{x\in J_j} |c_{j,x}||\langle \breve{\psi}_{j,x}, u\rangle_{L_2(\Gamma)}| \quad (\mathbf{c}_j \in \ell_2(J_j),\ u \in \mathcal{H}_{\tilde{d}}(\Gamma))$$
$$\lesssim 2^{-j\tilde{d}}\|\mathbf{c}_j\|\,[\sum_{x\in J_j}\sum_{q=1}^{M} \|u\circ\kappa_q\|^2_{H^{\tilde{d}}(B(\kappa_q^{-1}(\operatorname{supp}\breve{\psi}_{j,x}\cap\Gamma_q);\tilde{\eta}2^{-j})\cap\square)}]^{\frac{1}{2}} \lesssim 2^{-j\tilde{d}}\|\mathbf{c}_j^T\breve{\Psi}_j\|_{L_2(\Gamma)}|\!|\!|u|\!|\!|_{\tilde{d}},$$

or

$$|\!|\!|\cdot|\!|\!|_{-\tilde{d}} \lesssim 2^{-\tilde{d}j}\|\cdot\|_{L_2(\Gamma)} \quad \text{on } \mathcal{S}(\breve{\Psi}_j). \tag{A.2}$$

By the uniform $L_2(\Gamma)$-boundedness of $\breve{Q}_j^{(j+1)}$ for $j \ge j_0$, an direct consequence of the last result is that $|\!|\!|Id - \breve{Q}_j^{(j+1)}|\!|\!|_{0\to-\tilde{d}} \lesssim 2^{-\tilde{d}j}$. By the Jackson estimate at the dual side (4.6), and the uniform $L_2(\Gamma)$-boundedness of Q_j for $j \ge j_0$, we have $|\!|\!|Id - Q_j^{(j+1)}|\!|\!|_{0\to-\tilde{d}} \le |\!|\!|Id - Q_j|\!|\!|_{0\to-\tilde{d}} \lesssim 2^{-\tilde{d}j}$, and so $|\!|\!|Q_j^{(j+1)} - \breve{Q}_j^{(j+1)}|\!|\!|_{0\to-\tilde{d}} \lesssim 2^{-\tilde{d}j}$. By the extended Bernstein inequality Lemma 4.2, $|\!|\!|Q_j^{(j+1)} - \breve{Q}_j^{(j+1)}|\!|\!|_{-\tilde{d}\to-\tilde{d}} \lesssim 1$, so that by interpolation using (A.1) we infer that

$$|\!|\!|Q_j^{(j+1)} - \breve{Q}_j^{(j+1)}|\!|\!|_{s\to s} \lesssim \omega^{(1+\frac{s}{\tilde{d}})j}, \quad (s \in [-\tilde{d}, 0]). \tag{A.3}$$

(IV) Knowing (A.3), we are ready to investigate the stability of the *multi-level* decomposition defined by the collections $\breve{\Psi}_j$. We are going to construct a projector

$\breve{Q}_j$ defined on $\mathcal{H}_s(\Gamma)$ for some range of s, such that, for $j \geq j_0$, $\Im(\breve{Q}_j) = \mathcal{S}(\Phi_j)$ and $\Im(\breve{Q}_{j+1} - \breve{Q}_j) = \mathcal{S}(\breve{\Psi}_j)$.

By writing $Q_j = \sum_{k=j_0}^{j} Q_k - Q_{k-1}$, for any $s \in (-\min\{\tilde{\gamma}, \tilde{d}\}, \min\{\gamma, d\})$ and $u \in \mathcal{H}_s(\Gamma)$, Theorem 5.3 shows that $\|Q_j u\|_s^2 \lesssim \sum_{k=j_0}^{j} 4^{sk} \|(Q_k - Q_{k-1})u\|_{L_2(\Gamma)}^2 \leq \sum_{k=j_0}^{\infty} 4^{sk} \|(Q_k - Q_{k-1})u\|_{L_2(\Gamma)}^2 \lesssim \|u\|_s^2$ or $\|Q_j\|_{s\to s} \lesssim 1$.

For $\ell \geq j \geq j_0$, we define $\breve{Q}_j^{(\ell)} : \mathcal{S}(\Phi_\ell) \to \mathcal{S}(\Phi_j)$ by $\breve{Q}_\ell^{(\ell)} = Id$ and, for $j < \ell$, by $\breve{Q}_j^{(\ell)} = \breve{Q}_j^{(j+1)} \breve{Q}_{j+1}^{(\ell)}$. For some arbitrary, but fixed $t \in (-\min\{\tilde{\gamma}, \tilde{d}\}, 0]$, we set

$$\rho_j^{(\ell)} := \max_{j_0 \leq k \leq j} \|Q_k \breve{Q}_j^{(\ell)}\|_{t\to t}, \quad \varepsilon_j := \max_{j_0 \leq k \leq j} \|Q_k (\breve{Q}_j^{(j+1)} - Q_j^{(j+1)})\|_{t\to t}.$$

From $Q_k \breve{Q}_j^{(\ell)} = Q_k (\breve{Q}_j^{(j+1)} - Q_j^{(j+1)}) \breve{Q}_{j+1}^{(\ell)} + Q_k \breve{Q}_{j+1}^{(\ell)}$, we find $\rho_j^{(\ell)} \leq (\varepsilon_j + 1) \rho_{j+1}^{(\ell)}$, and so by $\rho_\ell^{(\ell)} = \max_{j_0 \leq k \leq j} \|Q_k\|_{t\to t} \lesssim 1$, and $\varepsilon_j \lesssim \omega^{(1+\frac{t}{d})j}$ by (A.3), we infer that

$$\|\breve{Q}_j^{(\ell)}\|_{t\to t} \leq \rho_j^{(\ell)} \lesssim \prod_{k=j}^{\ell-1} (\varepsilon_k + 1) \lesssim 1 + \sum_{k=j}^{\ell-1} \varepsilon_k \lesssim 1,$$

which thus hold uniformly in j and ℓ.

As a consequence of $Id = \sum_{j=j_0}^{\infty} (Q_j - Q_{j-1})$ on $\mathcal{H}_t(\Gamma)$ by Theorem 5.3, we have $\mathrm{clos}_{H_t(\Gamma)} \cup_{j \geq j_0} \mathcal{S}(\Phi_j) = \mathcal{H}_t(\Gamma)$. Since for any $u \in \mathcal{H}_t(\Gamma)$, $j \leq k \leq \ell$ and $u_k \in \mathcal{S}(\Phi_k)$, $\breve{Q}_j^{(\ell)} Q_\ell u = \breve{Q}_j^{(k)} u_k + \breve{Q}_j^{(\ell)} Q_\ell (u - u_k)$, from $\|\breve{Q}_j^{(\ell)} Q_\ell\|_{t\to t} \lesssim 1$ we infer that for any $j \geq j_0$, $(\breve{Q}_j^{(\ell)} Q_\ell u)_{\ell \geq j}$ is a Cauchy-sequence in $\mathcal{H}_t(\Gamma)$, and we set $\breve{Q}_j u = \lim_{\ell\to\infty} \breve{Q}_j^{(\ell)} Q_\ell u$. We conclude that $\breve{Q}_j : \mathcal{H}_t(\Gamma) \to \mathcal{H}_t(\Gamma)$ is uniformly bounded, with $\Im(\breve{Q}_j) = \mathcal{S}(\Phi_j)$ and $\Im(\breve{Q}_{j+1} - \breve{Q}_j) = \mathcal{S}(\breve{\Psi}_j)$.

(V) Let $s \in (t, \min\{\gamma, d\})$. With $\breve{Q}_{j_0 - 1} := 0$, we are going to prove that

$$\sum_{j=j_0}^{\infty} 4^{sj} \|(\breve{Q}_j - \breve{Q}_{j-1})u\|_{L_2(\Gamma)}^2 \lesssim \|u\|_s^2, \quad (u \in \mathcal{H}_s(\Gamma)). \tag{A.4}$$

For $u \in \mathcal{H}_s(\Gamma)$, $w_\ell := (Q_\ell - Q_{\ell-1})u$, Theorem 6.1 shows that $u = \sum_{\ell=j_0}^{\infty} w_\ell$, $\|u\|_s^2 \eqsim \sum_{\ell=j_0}^{\infty} 4^{s\ell} \|w_\ell\|_{L_2(\Gamma)}^2$, and $\|w_\ell\|_t \lesssim 2^{t\ell} \|w_\ell\|_{L_2(\Gamma)}$. Since $\|\breve{Q}_j\|_{t\to 0} \lesssim 2^{-tj}$, which follows from $\|\breve{Q}_j\|_{t\to t} \lesssim 1$ and the extended Bernstein inequality Lemma 4.2, and $\breve{Q}_j - \breve{Q}_{j-1}$ vanishes on $\mathcal{S}(\Phi_{j-1})$, we arrive at

$$\begin{aligned}
&\sum_{j=j_0}^{\infty} 4^{sj} \|(\breve{Q}_j - \breve{Q}_{j-1})u\|_{L_2(\Gamma)}^2 = \sum_{j=j_0}^{\infty} \sum_{\ell,\ell'=j_0}^{\infty} 4^{sj} \langle (\breve{Q}_j - \breve{Q}_{j-1}) w_\ell, (\breve{Q}_j - \breve{Q}_{j-1}) w_{\ell'} \rangle_{L_2(\Gamma)} \\
&= \sum_{\ell,\ell'=j_0}^{\infty} \sum_{j=j_0}^{\min\{\ell,\ell'\}} 4^{sj} \langle (\breve{Q}_j - \breve{Q}_{j-1}) w_\ell, (\breve{Q}_j - \breve{Q}_{j-1}) w_{\ell'} \rangle_{L_2(\Gamma)} \\
&\lesssim \sum_{\ell,\ell'=j_0}^{\infty} \sum_{j=j_0}^{\min\{\ell,\ell'\}} 4^{sj} 4^{-tj} \|w_\ell\|_t \|w_{\ell'}\|_t \lesssim \sum_{\ell,\ell'=j_0}^{\infty} 4^{(s-t)\min\{\ell,\ell'\}} \|w_\ell\|_t \|w_{\ell'}\|_t \\
&\lesssim \sum_{\ell,\ell'=j_0}^{\infty} 4^{(s-t)\min\{\ell,\ell'\}} 2^{(t-s)(\ell+\ell')} (2^{s\ell} \|w_\ell\|_{L_2(\Gamma)}) (2^{s\ell'} \|w_{\ell'}\|_{L_2(\Gamma)}) \\
&\lesssim \sum_{\ell=j_0}^{\infty} 4^{s\ell} \|w_\ell\|_{L_2(\Gamma)}^2 \eqsim \|u\|_s^2,
\end{aligned}$$

where we have used that the infinite matrix $[2^{(s-t)(2\min\{\ell,\ell'\}-\ell-\ell')}]_{\ell,\ell'\geq j_0}$ is bounded.

(VI) For $s\in[-\tilde{d},\gamma)$, it holds that

$$\|\!|\cdot|\!\|_s \lesssim 2^{sj}\|\cdot\|_{L_2(\Gamma)} \quad \text{on } \Im(\breve{Q}_j-\breve{Q}_{j-1}).$$

Indeed for $s>0$ this is just the Bernstein inequality (4.5), whereas for $s<0$ it is a consequence of the extended Bernstein inequality Lemma 4.2 and (A.2). Now let $s\in(-\tilde{d},\gamma)$, and let $\varepsilon>0$ be such that $s\pm\varepsilon\in[-\tilde{d},\gamma)$. Then for any $\breve{w}_j\in\Im(\breve{Q}_j-\breve{Q}_{j-1})$, with $\sum_{j=j_0}^\infty 4^{sj}\|\breve{w}_j\|^2_{L_2(\Gamma)}<\infty$, it holds that

$$\begin{aligned}\|\!|\sum_{j=j_0}^\infty \breve{w}_j|\!\|_s^2 &= \sum_{j,j'=j_0}^\infty \langle\!\langle \breve{w}_j,\breve{w}_{j'}\rangle\!\rangle_s \lesssim \sum_{j=j_0}^\infty\sum_{j'\geq j}^\infty \|\!|\breve{w}_j|\!\|_{s+\varepsilon}\|\!|\breve{w}_{j'}|\!\|_{s-\varepsilon}\\ &\lesssim \sum_{j=j_0}^\infty\sum_{j'\geq j}^\infty 2^{\varepsilon j}2^{-\varepsilon j'}(2^{sj}\|\breve{w}_j\|_{L_2(\Gamma)})(2^{sj'}\|\breve{w}_{j'}\|_{L_2(\Gamma)}) \lesssim \sum_{j=j_0}^\infty 4^{sj}\|\breve{w}_j\|^2_{L_2(\Gamma)}.\end{aligned}$$

Since, when $s\in(t,\min\{\gamma,d\})$, (A.4) shows that $\sum_{j=j_0}^\infty 4^{sj}\|\breve{w}_j\|^2_{L_2(\Gamma)}\lesssim\|\!|\sum_{j=j_0}^\infty\breve{w}_j|\!\|_s^2$, and Φ_{j_0} is an $L_2(\Gamma)$-Riesz basis for $\Im(\breve{Q}_{j_0})$, and for $j>j_0$, $\breve{\Psi}_{j-1}$ is a uniform $L_2(\Gamma)$-Riesz basis for $\Im(\breve{Q}_j-\breve{Q}_{j-1})$, we conclude that $\Phi_{j_0}\cup\cup_{j\geq j_0}2^{-sj}\breve{\Psi}_j$ is a Riesz system in $\mathcal{H}_s(\Gamma)$. Finally, since, as follows from Theorem 5.3, $\mathrm{clos}_{H_s(\Gamma)}\cup_{j\geq 0}\mathcal{S}(\Phi_j)=\mathcal{H}_s(\Gamma)$, we conclude it is even Riesz basis for this space, with which the proof of Theorem 6.1 is completed.

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