CONSTRUCTION OF FINITE ELEMENT WAVELETS ON GENERAL DOMAINS OR MANIFOLDS

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1. Introduction

We present a construction of wavelets that

- is applicable on general d-dimensional domains or manifolds
- yields continuous wavelets that generate Riesz bases for a range of Sobolev spaces
- which moreover have in principal any number of vanishing moments $\tilde{m}$
- as well as any approximation order m
- are uniformly local
- and possibly even have uniformly local dual wavelets

Having such wavelets available gives the following advantages when it concerns solving *operator equations:*

- (uniformly) well-conditioned stiffness matrices
- fast convergence
- sparse representations of (boundary) integral equations ([Sch98])
- adaptive wavelet methods that convergence with an optimal rate ([CDD98, CDD00])

When also the dual wavelets are uniformly local, one obtains

- an efficient decomposition scheme

which is essential for 'classical' wavelet applications like signal analysis or image compression.

Two approaches are known to realize wavelets that satisfy aforementioned requirements:

- Domain decomposition approach. Starting with wavelets on the interval, construct wavelets on a unit cube by taking tensor products; write the domain or manifold as a disjoint union of parametric images of this cube; lift the wavelets living on the cube to the manifold using these parametrizations. Apply a kind of gluing to connect the wavelets over the interfaces. There are a number of approaches that differ in the way this gluing is realized ([DS99a, CTU99, DS99b]).

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- Finite element wavelets. Construct wavelets bases for basically standard finite element spaces ([Ste98, DSt99, FQ99, CES00, Ste00, NS01]).

We focus on the second approach, although the material in sections 2 and 3 is general applicable. Missing proofs can usually be found in [Ste98], [Ste00] or [NS01].

The following notations will be used in this paper. In order to avoid the repeated use of generic but unspecified constants, by $C \lesssim D$ we mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

For some countable collection Φ of functions in a separable Hilbert space H with scalar product $\langle\ ,\ \rangle$ and norm $\|\ \|$, and for $\mathbf{c} = (c_\phi)_{\phi\in\Phi}$ a vector of scalars, with $\mathbf{c}^T\Phi$ we will mean the expansion $\sum_{\phi\in\Phi} c_\phi \phi$. We always consider spaces of scalar vectors as being equipped with scalar product $\langle \mathbf{c}, \mathbf{d}\rangle_{\ell^2} = \sum_{\phi\in\Phi} c_\phi \overline{d_\phi}$ and norm $\|\mathbf{c}\|_{\ell^2} = \langle \mathbf{c}, \mathbf{c}\rangle_{\ell^2}^{\frac{1}{2}}$, and consequently, the spaces of possibly infinite matrices as being equipped with the corresponding operator norm. For $x \in H$, with $\langle \Phi, x\rangle$ and $\langle x, \Phi\rangle$ we will mean the column- and row-vectors with coefficients $\langle \phi, x\rangle$ and $\langle x, \phi\rangle$, $\phi \in \Phi$. More generally, when $\tilde{\Phi}$ is another countable collection in H, with $\langle \Phi, \tilde{\Phi}\rangle$ is meant the matrix $(\langle \phi, \tilde{\phi}\rangle)_{\phi\in\Phi,\tilde{\phi}\in\tilde{\Phi}}$. A collection Φ is called a *Riesz system* when $\|\mathbf{c}^T\Phi\| \eqsim \|\mathbf{c}\|_{\ell^2}$, and Φ is called a *Riesz basis* when it is in addition a basis for H.

2. Biorthogonal space decompositions

The clue to construct 'stable' wavelets is given by the following theorem:

Theorem 2.1 ([Dah96, DSt99]). *Let*

$$V_0 \subset V_1 \subset \cdots$$
$$\tilde{V}_0 \subset \tilde{V}_1 \subset \cdots$$

be sequences of nested closed subspaces of L^2. *Suppose*

(a) $\exists$ *uniformly bounded projectors* $Q_j : L^2 \to L^2$ *with*

$$\mathrm{Im} Q_j = V_j \quad \mathrm{Im}(I - Q_j) = \tilde{V}_j^{\perp_{L^2}}$$

and so for the L^2*-adjoints*

$$\mathrm{Im} Q_j^* = \tilde{V}_j \quad \mathrm{Im}(I - Q_j^*) = V_j^{\perp_{L^2}}$$

(b) *both sequences satisfy* Jackson estimates *with parameters* $m, \tilde{m} > 0$, *i.e.,*

$$\inf_{V_j} \|v - v_j\|_{L^2} \lesssim 2^{-jm}\|v\|_{H^m} \qquad (v \in H^m)$$
$$\inf_{\tilde{V}_j} \|v - \tilde{v}_j\|_{L^2} \lesssim 2^{-j\tilde{m}}\|v\|_{H^{\tilde{m}}} \qquad (v \in H^{\tilde{m}})$$

(c) *as well as* Bernstein estimates *with parameters* $0 < \gamma < m$, $0 < \tilde{\gamma} < \tilde{m}$, *i.e.*

$$\|v_j\|_{H^s} \lesssim 2^{js}\|v_j\|_{L^2} \qquad (0 \leq s < \gamma,\, v_j \in V_j)$$
$$\|\tilde{v}_j\|_{H^s} \lesssim 2^{js}\|\tilde{v}_j\|_{L^2} \qquad (0 \leq s < \tilde{\gamma},\, \tilde{v}_j \in \tilde{V}_j)$$

Then, with $Q_{-1} := 0$,

$$\|u\|^2_{H^s} \eqsim \sum_{j=0}^{\infty} 4^{js}\|(Q_j - Q_{j-1})u\|^2_{L^2} \quad (s \in (-\tilde{\gamma}, \gamma),\, u \in H^s)$$
$$\|u\|^2_{H^s} \eqsim \sum_{j=0}^{\infty} 4^{js}\|(Q^*_j - Q^*_{j-1})u\|^2_{L^2} \quad (s \in (-\gamma, \tilde{\gamma}),\, u \in H^s)$$

Corollary 2.2 (wavelets). *Let* $\Psi_j = \{\psi_{j,x} : x \in J_j\}$, $\tilde{\Psi}_j = \{\tilde{\psi}_{j,x} : x \in J_j\}$ *be uniform* L^2*-Riesz bases for*

$$V_{j+1} \cap \tilde{V}_j^{\perp_{L^2}} = \mathrm{Im}(Q_{j+1} - Q_j),$$
$$\tilde{V}_{j+1} \cap V_j^{\perp_{L^2}} = \mathrm{Im}(Q^*_{j+1} - Q^*_j)$$

($V_{-1} = \tilde{V}_{-1} := \{0\}$*). Then*

$$\cup_{j=0}^{\infty} 2^{-js}\Psi_{j-1}, \qquad \cup_{j=0}^{\infty} 2^{-js}\tilde{\Psi}_{j-1},$$

are Riesz bases for H^s *when*

$$s \in (-\tilde{\gamma}, \gamma) \quad \textit{or} \quad s \in (-\gamma, \tilde{\gamma})$$

respectively.

Remark 2.3. From the biorthogonality of the space decompositions, we find that

$$\langle \Psi_j, \tilde{\Psi}_\ell \rangle_{L^2} = 0 \quad \text{when } j \neq \ell.$$

In addition it can be shown that given Ψ_j, $\exists!\tilde{\Psi}_j$ s.t.

$$\langle \Psi_j, \tilde{\Psi}_j \rangle_{L^2} = I. \quad \textbf{(biorthogonal wavelets)} \tag{2.1}$$

We will always assume that Ψ_j, $\tilde{\Psi}_j$ are related according to (2.1).

We give two applications:

Application 2.4 (*Decomposition scheme)***.**
For $s \in (-\tilde{\gamma}, \gamma)$, let $u \in H^s$, then $u = \sum_j \mathbf{c}^T_{j-1} 2^{-js}\Psi_{j-1}$, where

$$\mathbf{c}_{j-1} = 2^{js}\langle \tilde{\Psi}_{j-1}, u \rangle_{L^2}.$$

Application 2.5 (*Transformation of a well-posed operator equation into a well-posed* ℓ^2*-problem)***.** For $t \in (-\tilde{\gamma}, \gamma)$ and $f \in H^{-t}$, consider

$$Lu = f, \tag{2.2}$$

where

$$L : H^t \to H^{-t} \quad \text{is bounded invertible.} \tag{2.3}$$

Writing $u = \sum_j \mathbf{c}_{j-1}^T 2^{-jt}\Psi_{j-1}$, (2.2) is equivalent to

$$\sum_j \mathbf{c}_{j-1}^T 2^{-jt} L\Psi_{j-1} = f,$$

which equality in $(H^t)'$ in turn is equivalent to

$$\sum_j 2^{-(j+\ell)t}\langle \Psi_{\ell-1}, L\Psi_{j-1}\rangle_{L^2}\mathbf{c}_{j-1} = 2^{-\ell t}\langle \Psi_{\ell-1}, f\rangle_{L^2} \qquad (\ell \in I\!N).$$

The infinite block matrix $\left[2^{-(j+\ell)}\langle \Psi_{\ell-1}, L\Psi_{j-1}\rangle_{L^2}\right]_{\ell j}$ (*stiffness matrix*) is nothing else than the representation of $L : H^t \to H^{-t}$ with H^t and H^{-t} equipped with $\cup_j 2^{-jt}\Psi_{j-1}$ and $\cup_j 2^{jt}\tilde{\Psi}_{j-1}$ respectively. Since L is bounded invertible and these bases are Riesz bases, it follows that this matrix defines a bounded invertible operator from $\ell^2 \to \ell^2$.

In any case when L is elliptic, it follows that principal submatrices of this matrix (discretizations) are bounded invertible uniformly in their size.

Remark 2.6. In view of practical computations, for Application 2.5 it is necessary that the Ψ_j are *uniformly local* meaning that

$$\operatorname{diam}(\operatorname{supp}\psi_{j,x}) \eqsim 2^{-j}$$

For Application 2.4, even *both* the Ψ_j and $\tilde{\Psi}_j$ should be uniformly local.

It is not hard to find sequences (V_j), $(\tilde{V}_j)$ that satisfy Jackson and Bernstein estimates. For example, finite element spaces with respect to nested sequences of underlying triangulations will do. The remaining condition (a) from Theorem 2.1 can be verified using the following result:

Proposition 2.7 ([Ste00]). *Let V, $\tilde{V}$ be closed subspaces of a separable Hilbert space H. Equivalent are:*

- $$\inf_{0\neq v\in V}\sup_{0\neq \tilde{v}\in\tilde{V}} \frac{|\langle v,\tilde{v}\rangle|}{\|v\|\|\tilde{v}\|} = \inf_{0\neq \tilde{v}\in\tilde{V}}\sup_{0\neq v\in V} \frac{|\langle \tilde{v},v\rangle|}{\|\tilde{v}\|\|v\|} \geq \sigma > 0,$$
- $\exists!$ *projector $Q : H \to H$ with*
  $$\operatorname{Im} Q = V \quad \operatorname{Im}(I-Q) = \tilde{V}^{\perp}$$
  and
  $$\|Q\| \leq \frac{1}{\sigma}$$
- *There exist Riesz bases Φ and $\tilde{\Phi}$ for V and $\tilde{V}$ such that $\langle \Phi, \tilde{\Phi}\rangle : \ell^2 \to \ell^2$ is bounded invertible.*
- *For any Riesz bases Φ and $\tilde{\Phi}$ for V and $\tilde{V}$, the matrix $\langle \Phi, \tilde{\Phi}\rangle$ is bounded invertible. The Riesz basis $\langle \Phi, \tilde{\Phi}\rangle^{-T}\tilde{\Phi}$ is the unique dual collection for Φ in $\tilde{V}$.*

Obviously, when $V = \tilde{V}$, above Q is just the orthogonal projector onto V giving $\sigma = 1$. A reason why it can be useful to consider $\tilde{V}_j \neq V_j$ is given in the following remark.

Remark 2.8. Since $\psi_{j,x} \perp_{L^2} \tilde{V}_j$, the Jackson estimate at the dual side implies that for $v \in H^{\tilde{m}}$,

$$|\langle v, \psi_{j,x}\rangle_{L^2}| = \inf_{\tilde{v}_j \in \tilde{V}_j} |\langle v - \tilde{v}_j, \psi_{j,x}\rangle_{L^2}| \lesssim 2^{-j\tilde{m}} \|v\|_{H^{\tilde{m}}}$$

For all practical constructions of uniformly local wavelets actually it can be proved that

$$|\langle v, \psi_{j,x}\rangle_{L^2}| \lesssim 2^{-j(\tilde{m}+\frac{d}{2})} \|v\|_{W^{\infty,\tilde{m}}(\mathrm{supp}\psi_{j,x})}$$

which property is referred to as the **cancellation property of order** $\tilde{m}$.

As shown in [Sch98], if L is a singular integral operator of order $2t$ and the wavelets have the cancellation property of order $\tilde{m} > m - 2t$, then the stiffness matrices $\left[2^{-(j+\ell)}\langle \Psi_{\ell-1}, L\Psi_{j-1}\rangle_{L^2}\right]_{0\leq \ell,j\leq J}$ can be replaced by sparse matrices (more precisely, matrices for which the application to a vector can be computed in $\mathcal{O}(2^{Jd})$ operations), without lowering the order of convergence as function of J.

Note that $t \leq 0$ requires $\tilde{m} > m$.

Given sequences (V_j), $(\tilde{V}_j)$ that satisfy the conditions of Theorem 2.1, in the next section we will show how to construct (primal) wavelets, i.e., uniform L^2-Riesz bases for the spaces $V_j \cap \tilde{V}_{j-1}^{\perp_{L^2}}$ for $j \geq 1$.

3. Stable completions

Theorem 3.1. *Let $V_{j+1} = W_j \oplus Z_j$ such that*

(a) $\exists$ *uniformly bounded projectors $P_j : L^2 \to L^2$ with*

$$\mathrm{Im}P_j = W_j \quad \mathrm{Im}(I - P_j) = \tilde{V}_j^{\perp_{L^2}}$$

(cf. Prop. 2.7)

(b)

$$\sup_j \sup_{0\neq w_j\in W_j,\, 0\neq w_j \in Z_j} \frac{|\langle w_j, z_j\rangle_{L^2}|}{\|w_j\|_{L^2}\|z_j\|_{L^2}} < 1$$

then

$$(I - P_j)|_{Z_j} : Z_j \to V_{j+1} \cap \tilde{V}_j^{\perp_{L^2}} \quad \textit{is uniformly } L^2\textit{-bounded invertible.}$$

See Figure 1 for an illustration.

A consequence of this theorem is that the P_j's map uniform L^2-Riesz bases ('initial stable completions' in the terminology of [CDP96]) for any of such Z_j onto uniform L^2-Riesz bases for $V_{j+1} \cap \tilde{V}_j^{\perp_{L^2}}$ ('target stable completions').

To implement the action of P_j, we need bases for W_j and $\tilde{V}_j$.

Corollary 3.2. *Let Ξ_j, Θ_j, $\tilde{\Phi}_j$ be uniform L^2-Riesz bases for Z_j, W_j, $\tilde{V}_j$, then*

$$\Psi_j = (I - P_j)\Xi_j = \Xi_j - \langle \Xi_j, \tilde{\Phi}_j\rangle_{L^2}\langle \Theta_j, \tilde{\Phi}_j\rangle_{L^2}^{-1}\Theta_j \tag{3.1}$$

are uniform L^2-Riesz bases for $V_{j+1} \cap \tilde{V}_j^{\perp_{L^2}}$, i.e., (primal) wavelets.

Corollary 3.3. *If*

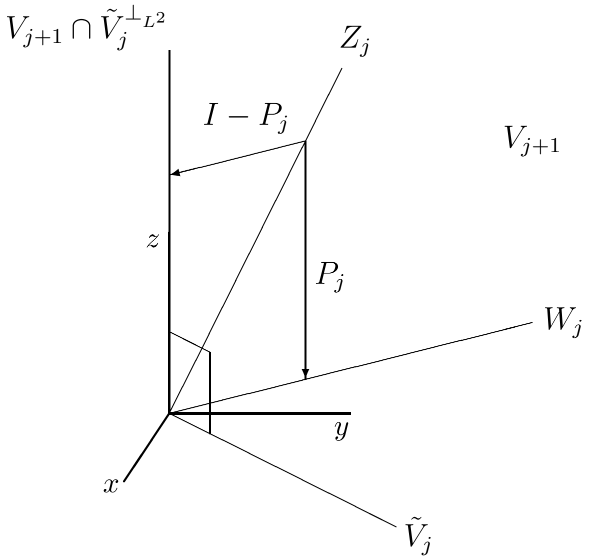

FIGURE 1. V_{j+1} is represented by the vertical plane $x = 0$. $\tilde{V}_j$ is contained in the horizontal plane $z = 0$.

(a) *the* Ξ_j, Θ_j, $\tilde{\Phi}_j$ *are uniformly local,*
(b) $\langle \Theta_j, \tilde{\Phi}_j \rangle_{L^2} = I$ *(or more generally,* $\langle \Theta_j, \tilde{\Phi}_j \rangle_{L^2}^{-1}$ *are uniformly sparse),*

then the Ψ_j *are uniformly local.*

Remark 3.4 (*Wavelets and dual wavelets that are* both *uniformly local)***.** A special choice for W_j usually made in the literature is $W_j = V_j$. As already imposed by Theorem 2.1, this choice satisfies (a) of Theorem 3.1. For this case, we will denote the basis for $W_j = V_j$ as Φ_j instead of Θ_j.

Assuming biorthogonality of the collections of "scaling functions" Φ_j and $\tilde{\Phi}_j$ for all j (cf. (b) of Corollary 3.3), and thus of $\Phi_j \cup \Psi_j$ and $\tilde{\Phi}_j \cup \tilde{\Psi}_j$ (cf. (2.1)), the uniqueness of such biorthogonal collections shows that

$$\begin{bmatrix} \tilde{\Phi}_j^T & \tilde{\Psi}_j^T \end{bmatrix} = \tilde{\Phi}_{j+1}^T \mathbf{M}_j^{-T},$$

with matrix $\mathbf{M}_j$ being defined by

$$\begin{bmatrix} \Phi_j^T & \Psi_j^T \end{bmatrix} = \Phi_{j+1}^T \mathbf{M}_j.$$

Since there is a lot of freedom to select Z_j and thus Ξ_j, it is usually not a problem to choose it such that both $\hat{\mathbf{M}}_j$ and $\hat{\mathbf{M}}_j^{-1}$, defined by $\begin{bmatrix} \Phi_j^T & \Xi_j^T \end{bmatrix} = \Phi_{j+1}^T \hat{\mathbf{M}}_j$, are uniformly sparse. From (3.1), it follows that

$$\mathbf{M}_j = \hat{\mathbf{M}}_j \begin{bmatrix} \mathbf{I} & -\langle \tilde{\Phi}_j, \Xi_j \rangle_{L^2} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}.$$

We conclude that the $\mathbf{M}_j^{-T}$ are uniformly sparse, showing that in this case also the dual wavelets $\tilde{\Psi}_j$ are uniformly local.

We stress however that the condition that V_j and $\tilde{V}_j$ can be equipped with biorthogonal bases Φ_j, $\tilde{\Phi}_j$ that are both uniformly local, in addition to the conditions already imposed in Theorem 2.1 put severe restrictions to the choice of the sequences $(V_j)_j$, $(\tilde{V}_j)_j$. Recall that for Application 2.5 access to dual wavelets is not needed.

To end this section, we summarize all conditions we imposed so far. As a preparation for the following section, we recast them all in terms of bases only.

The wavelets we constructed are determined by a sequence of quadruples $(\Phi_j, \tilde{\Phi}_j, \Theta_j, \Xi_j)_j$ of uniformly local, uniform L^2-Riesz systems such that

- $V_j := \mathrm{cl}_{L^2}\mathrm{span}\,\Phi_j$, $\tilde{V}_j := \mathrm{cl}_{L^2}\mathrm{span}\,\tilde{\Phi}_j$ are nested and satisfy Jackson and Bernstein estimates.
- $\langle \Phi_j, \tilde{\Phi}_j\rangle_{L^2}$ are uniformly bounded invertible (equivalent to condition (a) of Theorem 2.1)
- $\Theta_j \cup \Xi_j$ are uniform L^2-Riesz bases for V_{j+1} (implies (b) of Theorem 3.1 with $W_j := \mathrm{cl}_{L^2}\mathrm{span}\,\Theta_j$ and $Z_j := \mathrm{cl}_{L^2}\mathrm{span}\,\Xi_j$).
- $\langle \Theta_j, \tilde{\Phi}_j\rangle_{L^2}$ are uniformly bounded invertible (implies (a) of Theorem 3.1) *and* $\langle \Theta_j, \tilde{\Phi}_j\rangle_{L^2}^{-1}$ uniformly sparse.

The special case $\Phi_j = \Theta_j$ yields dual wavelets that are also uniformly local (at least under the mild condition that the basis transformations between $\Theta_j \cup \Xi_j$ and Φ_{j+1} and vice versa are both uniformly sparse.)

4. Finite element wavelets

We will select both $(V_j)_j$ and $(\tilde{V}_j)_j$ as being sequences of finite element type spaces w.r.t. uniformly dyadically refined nested sequences of underlying triangulations, starting with an *arbitrary* initial triangulation of some domain.

4.1. **Global spaces.**

Let $\Omega \subset I\!R^d$ be a polygonal domain. To avoid some technicalities we assume $d \leq 3$.

Let $\boldsymbol{T}$ be some reference d-simplex (interval, triangle, tetrahedron) written in barycentric coordinates. For any d-simplex T, let $\lambda_T : T \to \boldsymbol{T}$ be an affine bijection. Fix some dyadic refinement of $\boldsymbol{T}$ into $\boldsymbol{T}_1, \ldots, \boldsymbol{T}_{2^d}$ (one possibility when $d \leq 2$).

Let τ_0 be an arbitrary conforming subdivision of Ω into a set of d-simplices (elements). For $j \in I\!N$, we define τ_{j+1} as the collection of d-simplices obtained by splitting each $T \in \tau_j$ into $\cup_{k=1}^{2^d} \lambda_T^{-1}(\boldsymbol{T}_k)$ (uniform dyadic refinements).

Assuming that spaces $\boldsymbol{V}, \tilde{\boldsymbol{V}}$ of continuous functions on $\boldsymbol{T}$ are available, we define

$$V_j = \{u \in C(\Omega) : u \circ \lambda_T^{-1} \in \boldsymbol{V}, T \in \tau_j\}$$

(possibly intersected with $H_0^1(\Omega)$), and analogously $\tilde{V}_j$.

By the continuity over the element boundaries, the Bernstein estimates are valid with $\gamma = \tilde{\gamma} = \frac{3}{2}$ (if not limited by the smoothness of functions in $\boldsymbol{V}$ or $\tilde{\boldsymbol{V}}$).

4.2. **Local spaces and bases.** We assume that $\boldsymbol{V}, \tilde{\boldsymbol{V}} \subset C(\boldsymbol{T})$ are *refinable*, i.e.,

$$\boldsymbol{V} \subset \boldsymbol{V}^{(r)} := \{u \in C(\boldsymbol{T}) : u \circ \lambda_{\boldsymbol{T}_k}^{-1} \in \boldsymbol{V},\, 1 \leq k \leq 2^d\}.$$

meaning that $V_j \subset V_{j+1}$ and $\tilde{V}_j \subset \tilde{V}_{j+1}$.

We assume that $P_{m-1}(\boldsymbol{T}) \subset \boldsymbol{V}$, $P_{\tilde{m}-1}(\boldsymbol{T}) \subset \tilde{\boldsymbol{V}}$, which implies the Jackson estimates with parameters m and $\tilde{m}$.

Let $\boldsymbol{I} \subset \boldsymbol{T}$ be some index set that is refinable, i.e., $\boldsymbol{I} \subset \boldsymbol{I}^{(r)} := \cup_k \lambda_{\boldsymbol{T}_k}^{-1}(\boldsymbol{I})$. An example is given by the set of vertices.

Let

$$\begin{aligned}
&\boldsymbol{\Phi} = \{\boldsymbol{\phi}_\lambda : \lambda \in \boldsymbol{I}\} \text{ be a basis for } \boldsymbol{V},\\
&\tilde{\boldsymbol{\Phi}} = \{\tilde{\boldsymbol{\phi}}_\lambda : \lambda \in \boldsymbol{I}\} \text{ be a basis for } \tilde{\boldsymbol{V}},\\
&\boldsymbol{\Theta} = \{\boldsymbol{\theta}_\lambda : \lambda \in \boldsymbol{I}\} \subset \boldsymbol{V}^{(r)} \text{ s.t. } \langle \boldsymbol{\Theta}, \tilde{\boldsymbol{\Phi}} \rangle_{L^2(\boldsymbol{T})} = I
\end{aligned}$$

(in fact, a milder condition suffices, see [Ste00]), and finally

$$\boldsymbol{\Xi} = \{\boldsymbol{\xi}_\lambda : \lambda \in \boldsymbol{I}^{(r)} \backslash \boldsymbol{I}\} \subset \boldsymbol{V}^{(r)} \text{ s.t. } \boldsymbol{\Theta} \cup \boldsymbol{\Xi} \text{ is a basis for } \boldsymbol{V}^{(r)}.$$

We will connect these local bases over the element boundaries to get global bases. In order to be able to do so, we impose the following conditions on $\boldsymbol{\Phi}$:

($\mathcal{V}$) $\boldsymbol{\phi}_\lambda$ vanishes on any lower-dimensional face that does not include λ,

($\mathcal{S}$) $\boldsymbol{\phi}_\lambda = \boldsymbol{\phi}_{\pi(\lambda)} \circ \pi$ for any permutation $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$,

($\mathcal{I}$) For $\boldsymbol{e} = \boldsymbol{T}$, or for $\boldsymbol{e}$ being any lower-dimensional face of $\boldsymbol{T}$,

$$\{\boldsymbol{\phi}_\lambda|_{\boldsymbol{e}} : \lambda \in \boldsymbol{I} \cap \boldsymbol{e}\} \text{ is independent,}$$

and analogous conditions on $\tilde{\boldsymbol{\Phi}}$ and $\boldsymbol{\Theta} \cup \boldsymbol{\Xi}$ (with $\boldsymbol{I}^{(r)}$ replacing $\boldsymbol{I}$).

Finally, we assume that

$$\operatorname{Re}\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}} \rangle_{L^2(\boldsymbol{T})} > 0. \tag{4.1}$$

4.3. **Global bases and wavelets.** With $I_j := \cup_{T \in \tau_j} \lambda_T^{-1}(\boldsymbol{I})$ (possibly without points on $\partial\Omega$), we define

$$\Phi_j = \{\phi_{j,x} : x \in I_j\}$$

by

$$\phi_{j,x}(y) = \begin{cases} 2^{jd/2} \boldsymbol{\phi}_{\lambda_T(x)}(\lambda_T(y)) & \text{if } x, y \in T \in \tau_j \\ 0 & \text{elsewhere} \end{cases}$$

These functions are well-defined, continuous, uniformly local, and generate a basis for V_j. Analogously, we define

$$\begin{aligned}
\tilde{\Phi}_j &= \{\tilde{\phi}_{j,x} : x \in I_j\},\\
\Theta_j &= \{\theta_{j,x} : x \in I_j\},\\
\Xi_j &= \{\xi_{j,x} : x \in I_{j+1} \backslash I_j\}.
\end{aligned}$$

Both basis transformations between $\Theta_j \cup \Xi_j$ and Φ_{j+1} and vice versa are uniformly sparse.

Since $|\partial\lambda_T| = \frac{\mathrm{vol}(T)}{\mathrm{vol}(\boldsymbol{T})}$ is *constant*, for any functions $\omega_{j,x}$, $\nu_{j,y}$ from these collections, we have

$$\langle \omega_{j,x}, \nu_{j,y}\rangle_{L^2} = \sum_{\{T\in\tau_j : T\ni x,y\}} 2^{jd}\frac{\mathrm{vol}(T)}{\mathrm{vol}(\boldsymbol{T})}\langle \boldsymbol{\omega}_{\lambda_T(x)}, \boldsymbol{\nu}_{\lambda_T(y)}\rangle_{L^2(\boldsymbol{T})}.$$

From this relation, we deduce that

- Φ_j, $\tilde{\Phi}_j$, $\Theta_j \cup \Xi_j$ are uniform L^2-Riesz bases for V_j, $\tilde{V}_j$ and V_{j+1},
- $\mathrm{Re}\langle \Phi_j, \tilde{\Phi}_j\rangle_{L^2} \gtrsim I$ (from (4.1)) and thus $\langle \Phi_j, \tilde{\Phi}_j\rangle_{L^2}$ are uniformly bounded invertible,
- $\langle \Theta_j, \tilde{\Phi}_j\rangle_{L^2}$ are diagonal matrices, which are uniformly bounded invertible,

meaning that all conditions mentioned at the end of Sect. 3 are satisfied.

In particular, the wavelet formula (3.1) now reads as $\Psi_j = \{\psi_{j,y} : y \in I_{j+1}\backslash I_j\}$, where

$$\psi_{j,y} = \xi_{j,y} - \sum_{x\in I_j} \frac{\sum_{\{T\in\tau_j : T\ni x,y\}} \mathrm{vol}(T)\langle \boldsymbol{\xi}_{\lambda_T(y)}, \tilde{\boldsymbol{\phi}}_{\lambda_T(x)}\rangle_{L^2(\boldsymbol{T})}}{\sum_{\{T\in\tau_j : T\ni x\}} \mathrm{vol}(T)}\ \theta_{j,x}$$

Knowing the local scalar products $\langle \boldsymbol{\Xi}, \tilde{\boldsymbol{\Phi}}\rangle_{L^2(\boldsymbol{T})}$, we thus obtained an explicit formula for the wavelets in terms of the local topology of the mesh.

Making use of the way these wavelets are constructed, the inverse wavelet transform, i.e., the basis transformation from $\Phi_0 \cup \cup_{\ell=0}^{j-1}\Psi_\ell$ to Φ_j can be implemented much more efficiently than one might expect from the sizes of the supports of the wavelets ([DSt99]).

4.4. **Examples.** As we have seen, for global bases constructed as in Sect. 4.3, it is sufficient to check the conditions on $(\boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}}, \boldsymbol{\Theta}, \boldsymbol{\Xi})$, all mentioned in Sect. 4.2. Below we give a number of examples of such local quadruples satisfying these requirements.

Example 4.1 (*Orthogonal space decomposition)*. Let

- $\boldsymbol{I}$ be the set of vertices of some d-simplex $\boldsymbol{T}$
- $\boldsymbol{\Phi} = \tilde{\boldsymbol{\Phi}} = \{\boldsymbol{\phi}_\lambda : \lambda \in \boldsymbol{I}\}$ the nodal basis of $P_1(\boldsymbol{T})$ defined by $\boldsymbol{\phi}_\lambda(\mu) = \delta_{\lambda\mu}$ $(\lambda, \mu \in \boldsymbol{I})$

Then $\boldsymbol{V} = \tilde{\boldsymbol{V}}$ is refinable, and $m = \tilde{m} = 2$. The global spaces $V_j = \tilde{V}_j$ are the standard Lagrange P_1-finite element space with respect to τ_j.

With $\boldsymbol{\Phi}^{(r)} = \{\boldsymbol{\phi}_\lambda^{(r)} : \lambda \in \boldsymbol{I}^{(r)}\} \subset \boldsymbol{V}^{(r)}$ defined by $\boldsymbol{\phi}_\lambda^{(r)}(\mu) = \delta_{\lambda\mu}$ $(\lambda, \mu \in \boldsymbol{I}^{(r)})$, suitable sets $\boldsymbol{\Theta}$ and $\boldsymbol{\Xi}$ are given by

- $\boldsymbol{\Theta} = \left\{\dfrac{2^{d+1}(d+1)}{\mathrm{vol}(\boldsymbol{T})}(\boldsymbol{\phi}_\lambda^{(r)} - 2^{-(d+1)}\boldsymbol{\phi}_\lambda) : \lambda \in \boldsymbol{I}\right\}$
- $\boldsymbol{\Xi} = \{\boldsymbol{\phi}_\lambda^{(r)} : \lambda \in \boldsymbol{I}^{(r)}\backslash \boldsymbol{I}\}$

see Figures 2, 3 and 4.

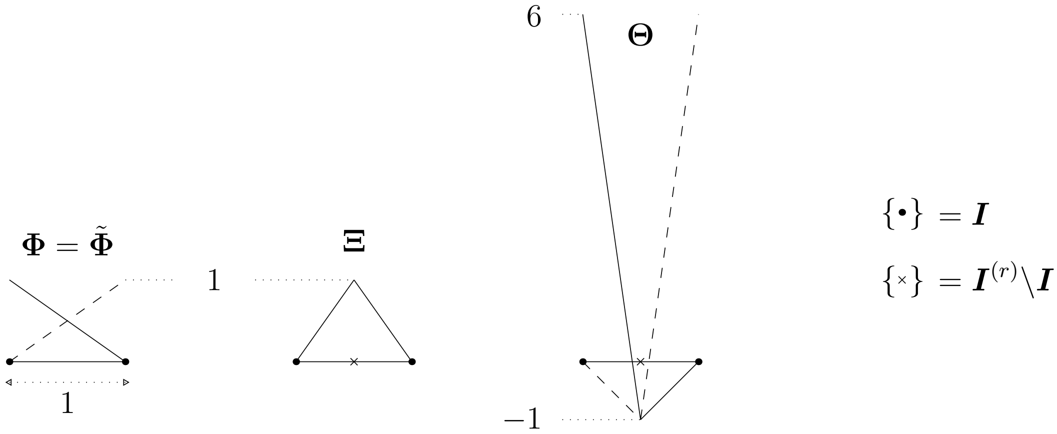

FIGURE 2. $(\boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}}, \boldsymbol{\Theta}, \boldsymbol{\Xi})$ from Example 4.1 for $d = 1$

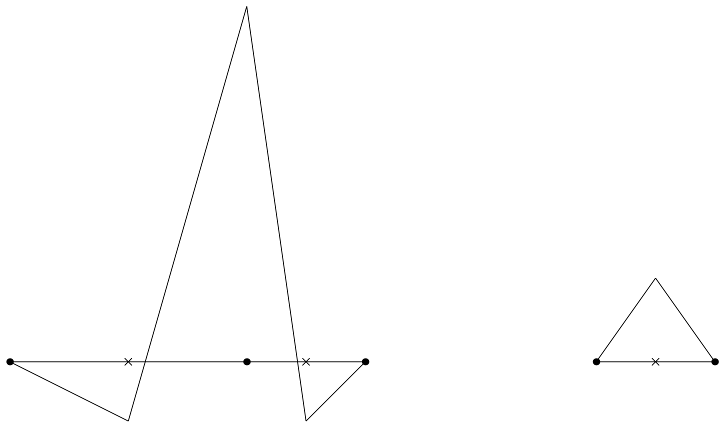

FIGURE 3. Resulting 'global' $\theta_{j,x}$ and $\xi_{j,y}$ corresponding to Figure 2

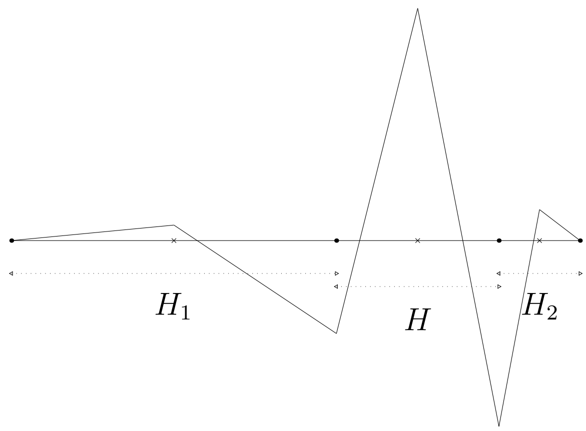

FIGURE 4. Resulting wavelet $\psi_{j,y} = \xi_{j,y} - \sum_{i\in\{1,2\}} \frac{1}{4}\frac{H}{H+H_i}\theta_{j,x_i}$ corresponding to Figure 2

The following examples all deal with the case $d = 1$. Corresponding constructions in two dimensions can be found [DSt99, Ste00].

Example 4.2 (*Cancellation property of higher order)***.** Let $\boldsymbol{T} = [-1, 1]$ and $\boldsymbol{I} = \{-1, 0, 1\}$. The quadruple is illustrated in Figure 5, where the functions from $\tilde{\boldsymbol{\Phi}}$ are quadratic polynomials. $\boldsymbol{V} :=$ span$\boldsymbol{\Phi}$ is the space of continuous piecewise linears with respect to the dyadic

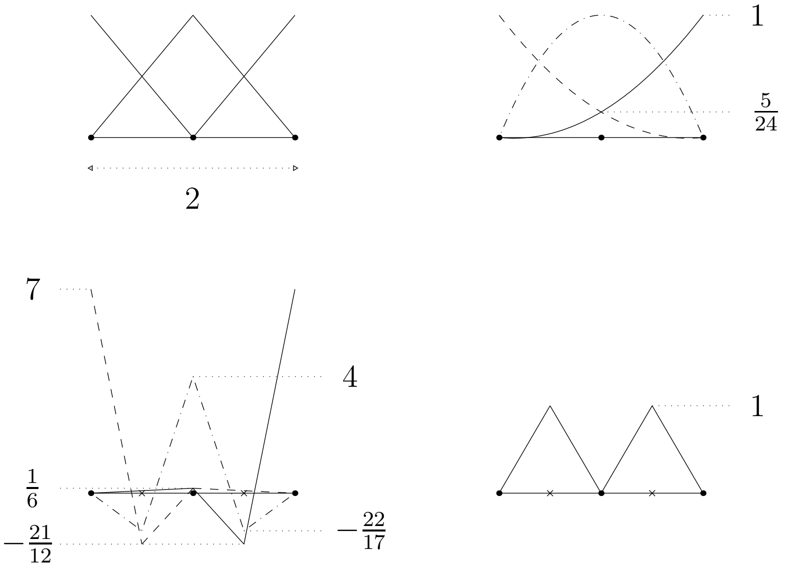

FIGURE 5. $(\boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}}, \boldsymbol{\Theta}, \boldsymbol{\Xi})$ from Example 4.2

refinement of $\boldsymbol{T}$, and $\tilde{\boldsymbol{V}} :=$ span$\boldsymbol{\Phi} = P_2(\boldsymbol{T})$. Both spaces are refinable. We have $m = 2$ and now $\tilde{m} = 3$. The spaces V_j, $\tilde{V}_j$ are the Lagrange P_1, P_2-finite element spaces with respect to τ_{j+1}, τ_j respectively.

In the same way as in Examples 4.1 and 4.2, in principal for any $m = 2^{\ell} + 1$, $\tilde{m} = 2^{\tilde{\ell}} + 1$ and d, uniformly local, primal wavelet bases can be constructed, giving rise to Riesz bases for $H^s(\Omega)$ ($|s| < \frac{3}{2}$), which correspond to the biorthogonal space decomposition of Lagrange P_{m-1}, $P_{\tilde{m}-1}$-finite element spaces with respect to $\tau_{j+\tilde{\ell}-\min\{\ell,\tilde{\ell}\}}$, $\tau_{j+\ell-\min\{\ell,\tilde{\ell}\}}$ respectively.

Example 4.3 (*Uniformly local dual wavelets)***.** Let $\boldsymbol{T} = [-1, 1]$, $\boldsymbol{I} = \{-1, 0, 1\}$,

- $\boldsymbol{N} = \{\boldsymbol{n}_{-1}, \boldsymbol{n}_0, \boldsymbol{n}_1\}$ the nodal basis of $C[-1, 1] \cap (P_1[-1, 0] \times P_1[0, 1])$ (first picture in Fig. 5),
- $\boldsymbol{M} = \{\boldsymbol{m}_{-1}, \boldsymbol{m}_1\}$ the nodal basis of $P_1[-1, 1]$,
- $\boldsymbol{m}_0$ the solution of

$$\boldsymbol{m}_0(x) = \boldsymbol{n}_0(x) - \tfrac{1}{2}(\boldsymbol{m}_0(2x - 1) + \boldsymbol{m}_0(2x + 1)).$$

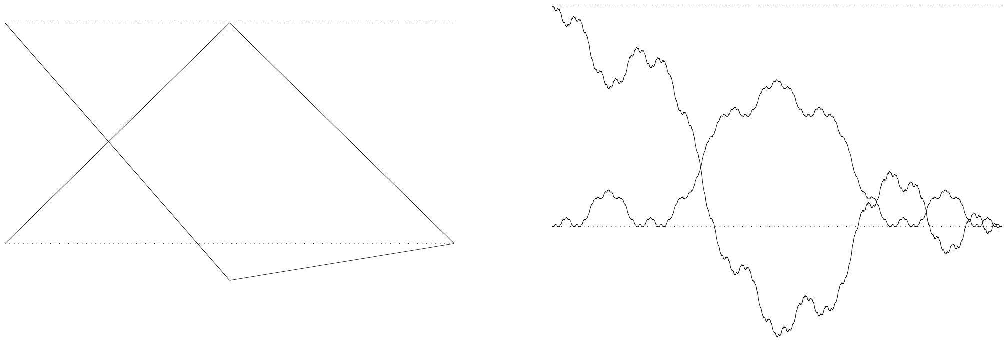

FIGURE 6. $\boldsymbol{\phi}_{-1}$, $\boldsymbol{\phi}_0$, and $\tilde{\boldsymbol{\phi}}_{-1}$, $\tilde{\boldsymbol{\phi}}_0$ ($\boldsymbol{\phi}_1$, $\tilde{\boldsymbol{\phi}}_1$ by symmetry)

It can be shown that $\boldsymbol{m}_0 \in H^s(-1,1)$ for all $s<1$, but $\boldsymbol{m}_0 \notin H^1(-1,1)$. We define $\boldsymbol{\Phi}$, $\tilde{\boldsymbol{\Phi}}$ by

$$\begin{bmatrix} \boldsymbol{\phi}_{-1} \\ \boldsymbol{\phi}_0 \\ \boldsymbol{\phi}_1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{6} & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{6} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{n}_{-1} \\ \boldsymbol{n}_0 \\ \boldsymbol{n}_1 \end{bmatrix}, \quad \begin{bmatrix} \tilde{\boldsymbol{\phi}}_{-1} \\ \tilde{\boldsymbol{\phi}}_0 \\ \tilde{\boldsymbol{\phi}}_1 \end{bmatrix} = \begin{bmatrix} 6 & -6 & 0 \\ 0 & 4 & 0 \\ 0 & -6 & 6 \end{bmatrix} \begin{bmatrix} \boldsymbol{m}_{-1} \\ \boldsymbol{m}_0 \\ \boldsymbol{m}_1 \end{bmatrix}$$

see Figure 6. $\boldsymbol{V} = \operatorname{span}\boldsymbol{\Phi}$, and thus V_j and m are as in Example 4.2. By construction $\tilde{\boldsymbol{V}}$ is refinable, it includes $P_1(\boldsymbol{T})$, meaning that $\tilde{m}=2$, and finally

$$\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}} \rangle_{L^2(\boldsymbol{T})} = I.$$

We conclude that with $\boldsymbol{\Theta} = \boldsymbol{\Phi}$, and for example again $\boldsymbol{\Xi}$ as in the last picture in Fig. 5, uniformly local dual wavelets become available.

In this example, the Bernstein estimate at the dual side is limited by the smoothness of $\boldsymbol{m}_0$, i.e., $\tilde{\gamma} = 1$ (and $\gamma = \frac{3}{2}$).

The following example shows that it is even possible to obtain explicitly given dual scaling functions and thus dual wavelets.

Example 4.4 (*Explicitly given duals)***.** We take $\boldsymbol{T} = [0,1]$, $\boldsymbol{I} = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ and $\boldsymbol{\Phi}$, $\tilde{\boldsymbol{\Phi}}$ as illustrated in Figure 7 (formulas in [Ste00]) with

$$\operatorname{span}\boldsymbol{\Phi} = C[0,1] \cap \prod_{i=0}^{4} P[\frac{i}{4}, \frac{i+1}{4}]$$

and

$$P_3([0,1] \subset \operatorname{span}\tilde{\boldsymbol{\Phi}} \subset C[0,1] \cap (P_3[0,\tfrac{1}{2}] \times P_3[\tfrac{1}{2},1]). \tag{4.2}$$

$\boldsymbol{V} = \operatorname{span}\boldsymbol{\Phi}$ is refinable, $m=2$, and V_j is the P_1 Lagrange finite element space w.r.t. τ_{j+2}. From (4.2) it follows that $\tilde{\boldsymbol{V}} = \operatorname{span}\tilde{\boldsymbol{\Phi}}$ is refinable and $\tilde{m}=4$. The collections $\boldsymbol{\Phi}$ and $\tilde{\boldsymbol{\Phi}}$ are chosen such that $\langle \boldsymbol{\Phi}, \tilde{\boldsymbol{\Phi}} \rangle_{L^2(\boldsymbol{T})} = I$. We conclude that with $\boldsymbol{\Theta} = \boldsymbol{\Phi}$, and for example $\boldsymbol{\Xi} = \{\boldsymbol{\xi}_\lambda : \lambda \in \boldsymbol{I}^{(r)} \backslash \boldsymbol{I}\} \subset \boldsymbol{V}^{(r)}$ defined by $\boldsymbol{\xi}_\lambda(\mu) = \delta_{\lambda\mu}$ ($\mu \in \boldsymbol{I}^{(r)}$), uniformly local, explicitly given dual wavelets become available.

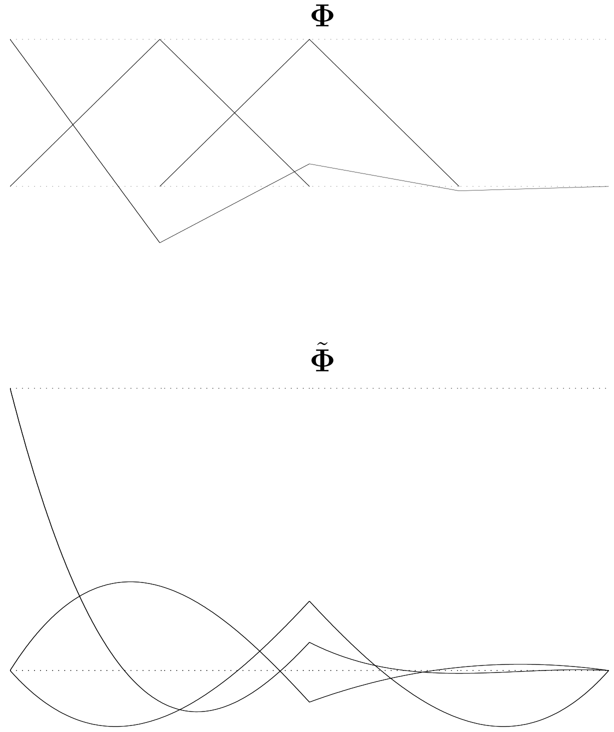

FIGURE 7. $\boldsymbol{\phi}_0$, $\boldsymbol{\phi}_{\frac{1}{4}}$, $\boldsymbol{\phi}_{\frac{1}{2}}$ and $\tilde{\boldsymbol{\phi}}_0$, $\tilde{\boldsymbol{\phi}}_{\frac{1}{4}}$, $\tilde{\boldsymbol{\phi}}_{\frac{1}{2}}$ (missing basis functions by symmetry)

5. Finite element wavelets on manifolds

Mainly in view of solving boundary integral equations, we will discuss a generalization of the construction of finite element wavelets to manifolds and non-polygonal domains ([NS01, Ste]).

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