Report No. 9823, Department of Mathematics, University of Nijmegen, November 1998. Revised version, January 2001, University of Utrecht. Accepted for publication in *Math. Comp.*

AN ANALYSIS OF NONCONFORMING MULTI-GRID METHODS, LEADING TO AN IMPROVED METHOD FOR THE MORLEY ELEMENT

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Abstract.

We recall and slightly refine the convergence theory for nonconforming multi-grid methods for symmetric positive definite problems developed by Bramble, Pasciak and Xu. We derive new results to verify the regularity and approximation assumption, and the assumption on the smoother. From the analysis it will appear that most efficient multi-grid methods can be expected for fully regular problems, and for prolongations for which the energy-norm of the iterated prolongations is uniformly bounded.

Guided by these observations, we develop a new multi-grid method for the biharmonic equation discretized with Morley finite elements, or equivalently, for the Stokes equations discretized with the P_0 - nonconforming P_1 pair. Numerical results show that the new method is superior to standard ones.

1. Introduction

We reconsider the convergence theory for nonconforming multi-grid methods for symmetric positive definite problems developed by Bramble, Pasciak and Xu in [BPX91]. With nonconforming methods, the coarse-grid correction is not a projection, and in many applications it defines an iteration that is even divergent. As a consequence, a W-cycle multi-grid method is not a safe choice, since with a fixed number of smoothing steps it may result in a preconditioned system that is indefinite. On the other hand, a V-cycle type method yields preconditioned systems that are always positive definite. Moreover, for $m(k)$ denoting the number of pre- and post smoothing steps on level $k = 1, \ldots, j$, it was proved that the resulting preconditioner is optimal when for some $\beta > 1$, $m(k) \geq \beta^{j-k}$ (*variable* V-cycle).

In this paper, we investigate to what extent this increase of the number of smoothing steps when going to coarser levels can be reduced, meanwhile preserving optimality. Apart from scientific interest, for a parallel implementation a reduction of the work on lower levels is important. A slight adaptation of the theory from [BPX91] will show that for a 'fully regular' problem, already $\sum_{k=0}^{j-1} \frac{1}{m(j-k)} \lesssim 1$ ensures optimality (*mildly* variable V-cycle) (cf. note at the end of this introduction).

Date: March 19, 2001.

1991 *Mathematics Subject Classification.* 65N55, 65N30, 65F10.

Key words and phrases. Multi-grid method, nonconforming finite elements, biharmonic equation, Morley finite element space, Stokes equations.

Aiming at minimizing the work on lower levels, the best method is clearly the standard "non-variable" V-cycle. Unfortunately, in the framework of nonconforming methods, we are not able to prove optimality of this cycle. Yet, we present an estimate that demonstrates that the condition number of the preconditioned system corresponding to the standard V-cycle may depend critically on the energy norm of the iterated prolongations alternated with smoothers. That is, if this norm grows exponentially with the numbers of levels, then the condition number may do so as well, whereas if this norm is uniformly bounded, then the standard V-cycle is at least suboptimal. These observations will be confirmed by numerical experiments. As a further support of these findings, we will recall some results obtained by Oswald in [Osw97], which show that in order to get a suitable *additive* multi-grid method, it is essential to use prolongations for which the energy-norm of the iterated prolongations is uniformly bounded.

The main application that we will discuss is the biharmonic equation on some convex polygon, discretized with Morley elements. The biharmonic operator is not fully regular, which means that we can only rely on the variable V-cycle. Yet, as is well-known, the Stokes equations discretized with the P_0 - nonconforming P_1 pair give rise to the same algebraic system. This equivalence has been exploited more often, in the sense that the biharmonic formulation was used to analyze multi-grid methods applied to the Stokes problem. Here we will follow the opposite approach.

The advantage of the Stokes formulation is that it defines a fully regular problem. Yet, since the usual basis for the finite element space is not uniformly L^2-stable, we have to pay for switching to this framework by the fact that standard smoothers do not satisfy the necessary assumptions. We develop a new type of smoothers that involve a call of a conforming multi-grid method to solve a discretized Laplacian. Using these smoothers, we show that the *mildly* variable V-cycle yields an optimal preconditioner.

It turns out that with the prolongation usually applied to above biharmonic, or equivalently Stokes problem, the energy norm of the iterated prolongations increases exponentially with the number of levels. We introduce a new prolongation, for which, at least in a model case, this energy-norm is uniformly bounded.

Using the standard V-cycle, we compare numerically the new smoother and prolongation with common choices. Both the new smoother and the new prolongation turn out to strongly reduce the condition numbers. With both improvements implemented, the condition numbers are "small", and they appear to be even uniformly bounded. Moreover, the new method can be implemented at the same costs as a standard method.

The remainder of this paper is organized as follows: We start with giving a description of the class of multi-grid methods that will be considered. This description is not only basis independent, but it also avoids making use of some scalar products, for which usually the L^2-scalar product is taken. As a consequence, the abstract formulation can be translated more easily in terms of an actual implementation.

We recall and slightly refine the multi-grid convergence theory from [BPX91]. We give new, general applicable criteria to verify whether a smoother satisfies the assumption necessary for this convergence theory. The proofs are based on some simple algebraic arguments only. We give a short proof of a new theorem, requiring more or less minimal assumptions, to obtain the full regularity and approximation assumption.

Mainly to underline the role of the iterated prolongations in the behaviour of multi-grid methods, we recall the convergence theory for additive multi-grid methods developed in [Osw97]. We relate the assumptions on the smoother in the multiplicative and the additive case.

Finally, we discuss some applications. Apart from the aforementioned application to the Morley element, we briefly discuss applications to the nonconforming P_1 and rotated Q_1 elements.

In order to avoid the repeated use of generic but unspecified constants, in this paper by $C \lesssim D$ we mean that C can be bounded by a multiple of D, independently of parameters which C and D may depend on. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

Note: The referee pointed out that a publication of J.H. Bramble and X. Zhang is going to appear in which it is proved that, with α being the regularity parameter, optimality of a V-cycle type method is guaranteed when $\sum_{k=0}^{j-1} \frac{1}{m(j-k)^\alpha} \lesssim 1$, which generalizes our finding in the $\alpha = 1$ case. Although this generalization means a quantitative improvement for $\alpha < 1$, it is not in conflict with the approach followed in this paper to reformulate the less-regular biharmonic problem as a fully regular Stokes problem with the aim to reduce the number of necessary smoothing steps on lower levels.

2. Multi-grid methods

2.1. **Algorithm.** We describe the symmetric (multiplicative) multi-grid method in a general setting. Let

$$V_0, V_1, \dots, V_j, \dots,$$

be a sequence of finite dimensional linear spaces over $\mathbb{R}$ or $\mathbb{C}$. With V_j' we denote the linear space of (anti-) linear functionals g on V_j, i.e., $\overline{g}$ is linear. Assuming that a_j is some scalar product on V_j, for given $g \in V_j'$ we are interested in finding $u \in V_j$ such that

$$a_j(u, v) = g(v) \qquad (v \in V_j). \tag{2.1}$$

Defining $A_j : V_j \to V_j'$ by

$$(A_j u)(v) = a_j(u, v) \qquad (u, v \in V_j), \tag{2.2}$$

we see that (2.1) is equivalent to

$$A_j u = g. \tag{2.3}$$

To define a multi-grid method for solving (2.3) iteratively, for $1 \le k \le j$ we need suitable linear mappings $I_k : V_{k-1} \to V_k$, which are called *prolongations*. The dual mappings $I_k' : V_k' \to V_{k-1}'$, defined by $(I_k' g)(w) = g(I_k w)$, are then called *restrictions*.

Furthermore, to define the *smoothers*, for $1 \leq k \leq j$ we need possibly non-Hermitian auxiliary sesqui-linear forms c_k on V_k, that give rise to operators $C_k, C_k^\dagger : V_k \to V_k'$ defined by

$$(C_k u)(v) = c_k(u,v), \quad (C_k^\dagger u)(v) = \overline{c_k(v,u)} \qquad (u, v \in V_k). \tag{2.4}$$

We set

$$C_{k,\ell} = \begin{cases} C_k & \text{if } \ell \text{ is odd,} \\ C_k^\dagger & \text{if } \ell \text{ is even.} \end{cases}$$

We assume that $\{u \in V_k : c_k(u, V_k) = 0\} = \{0\}$, which means that C_k and $C_k^\dagger$ are invertible.

The multi-grid operator $B_k : V_k' \to V_k$ is now defined by induction as follows:

Let $B_0 = A_0^{-1}$. Assume that B_{k-1} has been defined and define $B_k g$ for $g \in V_k'$ as follows:

1. Set $x^{(0)} = 0$ and $q^{(0)} = 0$
2. Define $x^{(\ell)}$ for $\ell = 1, \dots, m(k)$ by
$$x^{(\ell)} = x^{(\ell-1)} + C_{k,\ell+m(k)}^{-1}(g - A_k x^{(\ell-1)})$$
3. Define $y^{(m(k))} = x^{(m(k))} + I_k q^{(p)}$, where $q^{(i)}$ for $i = 1, \dots, p$ is defined by
$$q^{(i)} = q^{(i-1)} + B_{k-1}(I_k'(g - A_k x^{(m(k))}) - A_{k-1} q^{(i-1)})$$
4. Define $y^{(\ell)}$ for $\ell = m(k)+1, \dots, 2m(k)$ by
$$y^{(\ell)} = y^{(\ell-1)} + C_{k,\ell+m(k)}^{-1}(g - A_k y^{(\ell-1)})$$
5. Set $B_k g = y^{(2m(k))}$

Remark 2.1. Above description of the multi-grid method follows [BPX91] quite closely. A difference is that instead of using dual spaces V_k', in [BPX91] and in many other papers, after equipping the spaces V_k with some additional scalar products $(\, , \,)_{0,k}$, all multi-grid components are defined between primal spaces by (implicitly) applying Riesz' representation theorem. Then, as a consequence, all these components depend on the particular choice of $(\, , \,)_{0,k}$, whereas the preconditioned system $B_j A_j$ does not. In other words, the influence of $(\, , \,)_{0,k}$ on the multi-grid *algorithm* is artificial. On the other hand, the introduction of suitable scalar products $(\, , \,)_{0,k}$ has turned out to be essential for the *convergence theory.*

A definition of the multi-grid algorithm directly in terms of its implementation, i.e., in terms of matrices and vectors, can for example be found in [Hac85]. Actually, when $(\, , \,)_{0,k}$ is the Euclidean scalar product corresponding to the basis one wants to apply, the definitions from [BPX91] and [Hac85] are similar.

Our definition is basis independent. Because it also does not depend on scalar products $(\, , \,)_{0,k}$, the description of the implementation is straightforward, since it does not involve mass matrices and inverses of these matrices.

2.2. **Implementation.** For all k, let V_k be equipped with some basis $\{\phi_{k,m} : m \in I_k\}$, and think of V_k' as being equipped with the corresponding dual basis $\{\phi'_{k,m} : m \in I_k\}$ defined by $\phi'_{k,m}(\phi_{k,n}) = \delta_{n,m}$. From $g = \sum_{m\in I_k} g(\phi_{k,m})\phi'_{k,m}$, we see that $\mathbf{g} = (g(\phi_{k,m}))_{m\in I_k}$ is the vector representation of $g \in V_k'$. It is easily verified that the operator $A_k : V_k \to V_k'$ is represented by the *stiffness matrix* $\mathbf{A}_k = (a(\phi_{k,n}, \phi_{k,m}))_{m,n\in I_k}$, and that if $\mathbf{p}_k$ denotes the matrix representation of $I_k : V_{k-1} \to V_k$, then the matrix transpose $\mathbf{p}_k^T$ is the representation of I_k'.

Noting that $a_k(u, v) = \langle \mathbf{A}_k\mathbf{u}, \mathbf{v}\rangle$, where the vectors $\mathbf{u}$ and $\mathbf{v}$ denote the representations of $u \in V_k$ and $v \in V_k$ respectively and $\langle\ ,\ \rangle$ is the Euclidean scalar product, a natural way to construct the sesqui-linear forms c_k is the following: For $\mathbf{C}_k$ being some (easily) invertible approximation of $\mathbf{A}_k$, define

$$c_k(u, v) = \langle \mathbf{C}_k\mathbf{u}, \mathbf{v}\rangle. \tag{2.5}$$

Then the representations of the operators $C_k, C_k^\dagger : V_k \to V_k'$ are given by $\mathbf{C}_k$ and its matrix adjoint $\mathbf{C}_k^H$ respectively.

2.3. **Convergence theory.** We recall convergence results obtained in [BPX91], and give some additional estimates.

We will denote with $(\ )^*$ an adjoint with respect to the 'energy'-scalar product a_k. Then setting $K_k = I - C_k^{-1}A_k$, we have $K_k^* = I - C_k^{-\dagger}A_k$. Defining

$$\tilde{K}_k^{(m)} = \begin{cases} (K_k^*K_k)^{m/2} & \text{if } m \text{ is even,} \\ (K_k^*K_k)^{(m-1)/2}K_k^* & \text{if } m \text{ is odd,} \end{cases}$$

and by noting that

$$I_k^* = A_{k-1}^{-1}I_k'A_k,$$

for $k > 0$ we have

$$I - B_kA_k = (\tilde{K}_k^{(m(k))})^* \left[(I - I_kI_k^*) + I_k(I - B_{k-1}A_{k-1})^pI_k^*\right] \tilde{K}_k^{(m(k))}. \tag{2.6}$$

Note that $(B_kA_k)^* = B_kA_k$, and that $(\tilde{K}_k^{(m(k))})^*(I - I_kI_k^*)\tilde{K}_k^{(m(k))}$ is the error amplification operator of the corresponding *two-grid* method.

For $(\ ,\ )_{0,k}$ some additional scalar product on V_k, and $\|\cdot\|_{0,k} := (\cdot,\cdot)_{0,k}^{\frac{1}{2}}$, we define

$$\|u\|_{2,k} = \sup_{0\neq v\in V_k} \frac{|a_k(u, v)|}{\|v\|_{0,k}} \qquad (u \in V_k) \tag{2.7}$$

and

$$\rho_k = \sup_{0\neq u\in V_k} \frac{a_k(u, u)}{\|u\|_{0,k}^2}. \tag{2.8}$$

We make the following assumptions:

(A) (*regularity and approximation assumption*) There exists an $\alpha \in (0, 1]$ such that

$$|a_k((I - I_kI_k^*)u, u)| \lesssim (\rho_k^{-1}\|u\|_{2,k}^2)^\alpha a_k(u, u)^{1-\alpha} \qquad (u \in V_k),$$

and furthermore

$$\text{(B)} \qquad a_k(u,u) - a_k(K_k u, K_k u) \gtrsim \rho_k^{-1}\|u\|_{2,k}^2 \qquad (u \in V_k).$$

If in addition

$$\text{(C1)} \qquad \rho(I_k^* I_k) \leq 1,$$

then the

– W-cycle, i.e., $p = 2$, and $m(1) = \cdots = m(j) \geq 1$,

– variable V-cycle, i.e., $p = 1$, and for some $\beta > 1$, $m(j-k) \geq \beta^k m(j)$ and $m(j) \geq 1$,

and when $\alpha = 1$, the

– standard V-cycle, i.e., $p = 1$, and $m(1) = \cdots = m(j) \geq 1$,

all have been shown to yield B_j that satisfy

$$\sigma(I - B_j A_j) \subset [0, \tfrac{\delta_j}{1+\delta_j}], \text{ where } \delta_j \lesssim m(j)^{-\alpha}. \tag{2.9}$$

A particular case for which (C1) is valid is $I_k^* I_k = I$, i.e., $a_{k-1}(u,v) = a_k(I_k u, I_k v)$ ('Galerkin approach'). For this case a lot of additional multi-grid convergence theory is available, even for which a regularity assumption is not necessary. In [Che99] an application is described of the Galerkin approach to nonconforming finite element discretizations, which involves a redefinition of the energy-scalar products on lower levels.

If only

$$\text{(C2)} \qquad \rho(I_k^* I_k) \leq 2,$$

then the W-cycle has been shown to yield B_j that satisfy

$$\sigma(I - B_j A_j) \subset [\tfrac{-\delta_j}{1+\delta_j}, \tfrac{\delta_j}{1+\delta_j}], \text{ where } \delta_j \lesssim m(j)^{-\alpha}. \tag{2.10}$$

Unfortunately, for the usual nonconforming multi-grid methods, (C1) does not hold, whereas (C2) has been shown only for an I_k used for the rotated Q_1 element ([CO98]). The common I_k used for the nonconforming P_1 element or the Morley element generally do not satisfy (C2), and neither does the alternative I_k for the Morley element that will be introduced in this paper.

Without (C2), the W-cycle satisfies (2.10) for $m(k) = m$ sufficiently large. In fact, it can be shown that it is sufficient that

$$\text{(C3)} \qquad \rho(I_k^* \tilde{K}_k^{(m-1)} (\tilde{K}_k^{(m-1)})^* I_k) \leq 2,$$

which generalizes (C2). By (A) and (B), the forthcoming Lemma 2.2 and (2.13) show that indeed (C3) is valid for m sufficiently large.

If (C3) is not valid, then the W-cycle may result in a preconditioned system that is indefinite. On the other hand, when $p = 1$, from (B) it can be deduced that $B_j A_j$ is positive definite anyway, although not necessarily $\rho(I - B_j A_j) < 1$. In the remainder of this subsection, we will study some *V-cycle variants*, i.e., $p = 1$, to be used as *preconditioners*, only assuming (A) and (B).

For the variable V-cycle, i.e., $m(j-k)$ grows exponentially with k, it has been shown that

$$\lambda_{\max}(I-B_jA_j) \leq \tfrac{\delta_j}{1+\delta_j}, \text{ where } \delta_j \lesssim m(j)^{-\alpha}. \tag{2.11}$$

Yet, from the analysis presented in [BPX91], it can be deduced that when $\alpha=1$, (2.11) is even valid for any $(m(k))_{1\leq k\leq j}$ with $m(k)\geq m(j)$.

We now consider $\lambda_{\min}(I-B_jA_j) = -\lambda_{\max}(B_jA_j-I)$. A repeated use of (2.6) shows that

$$a_j((B_jA_j-I)u,u) = \sum_{k=1}^{j} a_k((\tilde{K}_k^{(m(k))})^*(I_kI_k^*-I)\tilde{K}_k^{(m(k))}\tilde{I}^*_{j\leftarrow k}u, \tilde{I}^*_{j\leftarrow k}u),$$

where

$$\tilde{I}_{j\leftarrow k} := (\tilde{K}_j^{(m(j))})^* I_j \tilde{I}_{j-1\leftarrow k} \quad (j>k), \quad \text{and} \quad \tilde{I}_{k\leftarrow k} := I.$$

With

$$q_k := \max\{0, \lambda_{\max}((\tilde{K}_k^{(m(k))})^*(I_kI_k^*-I)\tilde{K}_k^{(m(k))})\},$$

we find that

$$\lambda_{\max}(B_jA_j-I) \leq \sum_{k=1}^{j} q_k \rho(\tilde{I}^*_{j\leftarrow k}\tilde{I}_{j\leftarrow k}) \tag{2.12}$$

For convenience, from [BPX91] we recall the following result that concerns the two-grid method:

Lemma 2.2. *Assume that* (A) *and* (B) *hold. Then*

$$q_k \leq \rho((\tilde{K}_k^{(m(k))})^*(I-I_kI_k^*)\tilde{K}_k^{(m(k))}) \lesssim m(k)^{-\alpha}.$$

Proof. With $\overline{K}_k = \begin{cases} K_k^*K_k & \text{if } m(k) \text{ is even} \\ K_kK_k^* & \text{if } m(k) \text{ is odd} \end{cases}$, (A) and (B) show that

$$\begin{aligned} |a_k((I-I_kI_k^*)\tilde{K}_k^{(m(k))}u, \tilde{K}_k^{(m(k))}u)| &\lesssim \left(\rho_k^{-1}\|\tilde{K}_k^{(m(k))}u\|^2_{2,k}\right)^{\alpha} a_k(\overline{K}_k^{m(k)}u,u)^{1-\alpha} \\ &\lesssim a_k((I-\overline{K}_k)\overline{K}_k^{m(k)}u,u)^{\alpha} a_k(\overline{K}_k^{m(k)}u,u)^{1-\alpha} \end{aligned}$$

Since $\sigma(\overline{K}_k)\subset[0,1]$, which follows from (B), we infer that

$$a_k((I-\overline{K}_k)\overline{K}_k^{m(k)}u,u) \leq a_k(u,u)\max_{\lambda\in[0,1]}(1-\lambda)\lambda^{m(k)} \lesssim m(k)^{-1}a_k(u,u)$$

and $a_k(\overline{K}_k^{m(k)}u,u)\leq a_k(u,u)$, which completes the proof. ☐

To estimate $\rho(\tilde{I}^*_{j\leftarrow k}\tilde{I}_{j\leftarrow k})$, we use that

$$\begin{aligned} &a_k(I^*_{k+1}\tilde{K}_{k+1}^{(m(k+1))}u, I^*_{k+1}\tilde{K}_{k+1}^{(m(k+1))}u) \\ &\quad = a_{k+1}((\tilde{K}_{k+1}^{(m(k+1))})^*(I_{k+1}I^*_{k+1}-I)\tilde{K}_{k+1}^{(m(k+1))}u,u) + a_{k+1}(\overline{K}_{k+1}^{m(k+1)}u,u) \\ &\quad \leq (q_{k+1}+1)a_{k+1}(u,u), \end{aligned} \tag{2.13}$$

and so $\rho(\tilde{I}^*_{j\leftarrow k}\tilde{I}_{j\leftarrow k}) \leq \prod_{i=k+1}^{j}(q_i+1)$. From (2.12) and Lemma 2.2, we conclude that

$$\begin{aligned}\lambda_{\max}(B_jA_j - I) &\leq \sum_{k=1}^{j} q_k \prod_{i=k+1}^{j}(q_i+1) = -1 + \prod_{k=1}^{j}(q_k+1) \\ &\leq -1 + e^{\sum_{k=1}^{j} q_k} \lesssim -1 + e^{\sum_{k=0}^{j-1} m(j-k)^{-\alpha}}.\end{aligned}$$

For the variable V-cycle, we infer that $\lambda_{\max}(B_jA_j - I) \lesssim m(j)^{-\alpha}$, and so by (2.11), $\kappa(B_jA_j) - 1 \lesssim m(j)^{-\alpha}$ as was also noted in [BPX91]. However, a milder increase of $m(j-k)$ as function of k is already sufficient. For example, $m(j-k) \gtrsim m(j) + k^{\beta}$ for some $\beta > \frac{1}{\alpha}$ yields $\lambda_{\max}(B_jA_j - I) \lesssim m(j)^{-\alpha+\frac{1}{\beta}}$. What is more, to get $\lambda_{\max}(B_jA_j - I)$ that is just uniformly bounded, it is obviously already sufficient when $\sum_{k=0}^{j-1} m(j-k)^{-\alpha} \lesssim 1$. Combining this with the bound (2.11) on $\lambda_{\max}(I - B_jA_j)$, which for $\alpha = 1$ is valid for any $(m(k))_{1\leq k\leq j}$ with $m(k) \geq m(j)$, yields the following result:

Theorem 2.3. (‘*mildly*’ variable V-cycle) *Assume (A) with $\alpha = 1$ and (B). Let $p = 1$, $m(k) \geq m(j)$, and $\sum_{k=0}^{j-1} \frac{1}{m(j-k)} \lesssim 1$, then $\kappa(B_jA_j) \lesssim 1$.*

So besides that upper bounds on the condition number corresponding to various cycles decrease faster as function of the number of smoothing steps when α is larger, Theorem 2.3 is another indication that for more regular problems one may expect more efficient multi-grid methods, in particular when $\alpha = 1$.

Aiming at minimizing the number of operations on lower levels, obviously the best algorithm is the *standard V-cycle*. Unfortunately, a proof of optimality of this cycle applied to general nonconforming discretizations cannot be deduced from above estimates. However, some useful observations can be made. As stated before, for $\alpha = 1$ the upper bound (2.11) on $\lambda_{\max}(I - B_jA_j)$ is also valid for the standard V-cycle. Furthermore, again for the standard V-cycle, (2.12) shows that the behaviour of $\lambda_{\max}(B_jA_j - I)$ might depend critically on the factors $\rho(\tilde{I}^*_{j\leftarrow k}\tilde{I}_{j\leftarrow k})$, that is, on the squared energy-norm of the ‘iterated prolongations alternated with smoothers’. Indeed, in case these factors are uniformly bounded, then at least $\lambda_{\max}(B_jA_j - I) \lesssim j\, m(j)^{-\alpha}$, and so $\kappa(B_jA_j) \lesssim j\, m(j)^{-1}$ if $\alpha = 1$. However, if $\rho(\tilde{I}^*_{j\leftarrow k}\tilde{I}_{j\leftarrow k})$ increases exponentially with $j-k$, note which is not excluded by the preceeding analysis, then $\kappa(B_jA_j)$ might increase exponentially as function of j.

Numerical results for an $\alpha = 1$ case presented in Section 3.3.3 show that this exponential increase of the condition number indeed occurs with a prolongation that is commonly used. With a new prolongation, that is developed with the aim of getting bounded factors $\rho(\tilde{I}^*_{j\leftarrow k}\tilde{I}_{j\leftarrow k})$, the condition number of the standard V-cycle turns out to be even uniformly bounded.

2.4. **Assumption (B) for inexact Gauss-Seidel and damped Jacobi smoothers.** By substituting $u = A_k^{-1} C_k w$, we see that (B) can be rewritten as

$$\rho_k(2\Re c_k(w,w) - a_k(w,w)) \gtrsim \sup_{0 \neq v \in V_k} \frac{|c_k(w,v)|^2}{\|v\|_{0,k}^2} \qquad (w \in V_k). \tag{2.14}$$

Having fixed some bases of the spaces V_k, and with c_k and $\mathbf{C}_k$ related according to (2.5), and *mass matrices* $\mathbf{M}_k$ defined by $\langle \mathbf{M}_k \mathbf{u}, \mathbf{v} \rangle = (u,v)_{0,k}$ $(u, v \in V_k)$, by noting that

$$\rho_k = \rho(\mathbf{M}_k^{-1} \mathbf{A}_k),$$

(2.14) in turn can be written as

$$\rho(\mathbf{M}_k^{-1}\mathbf{A}_k)(\mathbf{C}_k + \mathbf{C}_k^H - \mathbf{A}_k) \gtrsim \mathbf{C}_k^H \mathbf{M}_k^{-1} \mathbf{C}_k. \tag{2.15}$$

In two propositions, we give sufficient conditions for (B). Dealing with 'inexact smoothers', i.e., smoothers that involve an inexact, possibly non-symmetric inner solver, these propositions generalize results from the literature, e.g. from [BP92]. These generalizations turn out to be particularly useful in cases where the mass matrices are not uniformly well-conditioned, a situation that we will encounter in practical applications.

Proposition 2.4. *Let $\tilde{\mathbf{D}}_k$, $\mathbf{D}_k$ and $\mathbf{L}_k$ be some matrices of the same size as $\mathbf{A}_k$, and let $\mathbf{C}_k := \tilde{\mathbf{D}}_k + \mathbf{L}_k$. If*
(a). $0 < \mathbf{D}_k = \mathbf{D}_k^H \lesssim \rho(\mathbf{M}_k^{-1}\mathbf{A}_k)\mathbf{M}_k$,
(b). $\mathbf{A}_k \leq \mathbf{D}_k + \mathbf{L}_k + \mathbf{L}_k^H$,
(c). $\|\mathbf{D}_k^{-\frac{1}{2}} \mathbf{L}_k \mathbf{D}_k^{-\frac{1}{2}}\| \lesssim 1$,
(d). $1 - \|\mathbf{I} - \mathbf{D}_k^{\frac{1}{2}} \tilde{\mathbf{D}}_k^{-1} \mathbf{D}_k^{\frac{1}{2}}\| \gtrsim 1$,
with $\|\cdot\| = \langle \cdot, \cdot \rangle^{\frac{1}{2}}$ being the Euclidean-norm, then (B) is valid.

Proof. From the definition of $\mathbf{C}_k$, (b) and (2.15), it is sufficient to prove that

$$\rho(\mathbf{M}_k^{-1}\mathbf{A}_k)(\tilde{\mathbf{D}}_k + \tilde{\mathbf{D}}_k^H - \mathbf{D}_k) \gtrsim (\tilde{\mathbf{D}}_k^H + \mathbf{L}_k)\mathbf{M}_k^{-1}(\tilde{\mathbf{D}}_k + \mathbf{L}_k).$$

From

$$\tilde{\mathbf{D}}_k + \tilde{\mathbf{D}}_k^H - \mathbf{D}_k = \tilde{\mathbf{D}}_k^H \mathbf{D}_k^{-\frac{1}{2}} \left(\mathbf{I} - (\mathbf{I} - \mathbf{D}_k^{\frac{1}{2}} \tilde{\mathbf{D}}_k^{-H} \mathbf{D}_k^{\frac{1}{2}})(\mathbf{I} - \mathbf{D}_k^{\frac{1}{2}} \tilde{\mathbf{D}}_k^{-1} \mathbf{D}_k^{\frac{1}{2}}) \right) \mathbf{D}_k^{-\frac{1}{2}} \tilde{\mathbf{D}}_k,$$

which is valid for any invertible $\tilde{\mathbf{D}}_k$ and $\mathbf{D}_k = \mathbf{D}_k^H > 0$, we infer that

$$\tilde{\mathbf{D}}_k + \tilde{\mathbf{D}}_k^H - \mathbf{D}_k \geq (1 - \|\mathbf{I} - \mathbf{D}_k^{\frac{1}{2}} \tilde{\mathbf{D}}_k^{-1} \mathbf{D}_k^{\frac{1}{2}}\|^2) \tilde{\mathbf{D}}_k^H \mathbf{D}_k^{-1} \tilde{\mathbf{D}}_k. \tag{2.16}$$

Because of this result together with (d), and $\mathbf{M}_k^{-1} \lesssim \rho(\mathbf{M}_k^{-1}\mathbf{A}_k)\mathbf{D}_k^{-1}$ by (a), it is sufficient to show that

$$\mathbf{D}_k^{-1} \gtrsim (\mathbf{I} + \tilde{\mathbf{D}}_k^{-H} \mathbf{L}_k) \mathbf{D}_k^{-1} (\mathbf{I} + \mathbf{L}_k \tilde{\mathbf{D}}_k^{-1}).$$

The latter relation follows from

$$\|\mathbf{D}_k^{-\frac{1}{2}} (\mathbf{I} + \mathbf{L}_k \tilde{\mathbf{D}}_k^{-1}) \mathbf{D}_k^{\frac{1}{2}}\| \leq 1 + \|\mathbf{D}_k^{-\frac{1}{2}} \mathbf{L}_k \mathbf{D}_k^{-\frac{1}{2}}\| \|\mathbf{D}_k^{\frac{1}{2}} \tilde{\mathbf{D}}_k^{-1} \mathbf{D}_k^{\frac{1}{2}}\| \lesssim 1,$$

by (c) and (d). $\square$

Remark 2.5. Condition (d) of Proposition 2.4 means that $\tilde{\mathbf{D}}_k^{-1}$ is an approximate inverse of $\mathbf{D}_k$ which defines a uniformly convergent iteration in the 'energy'-norm $\langle \mathbf{D}_k \cdot, \cdot \rangle^{\frac{1}{2}}$.

Remark 2.6. (('inexact') point or block Gauss-Seidel). In the setting of Proposition 2.4, let $\mathbf{A}_k = \mathbf{D}_k + \mathbf{L}_k + \mathbf{L}_k^H$ be a partitioning of $\mathbf{A}_k$ into its (block) diagonal, its (block) lower triangular and its (block) upper triangular part, so that (b) is trivially valid.

Now $\mathbf{D}_k > 0$ follows from $\mathbf{A}_k > 0$. With $\Delta_k := \operatorname{diag}(\mathbf{M}_k)$, $\hat{\mathbf{A}}_k := \Delta_k^{-\frac{1}{2}} \mathbf{A}_k \Delta_k^{-\frac{1}{2}}$, $\hat{\mathbf{D}}_k := \Delta_k^{-\frac{1}{2}} \mathbf{D}_k \Delta_k^{-\frac{1}{2}}$ and $\hat{\mathbf{M}}_k := \Delta_k^{-\frac{1}{2}} \mathbf{M}_k \Delta_k^{-\frac{1}{2}}$, the relation $\mathbf{D}_k \lesssim \rho(\mathbf{M}_k^{-1}\mathbf{A}_k)\mathbf{M}_k$ can be rewritten as $\hat{\mathbf{D}}_k \lesssim \rho(\hat{\mathbf{M}}_k^{-1}\hat{\mathbf{A}}_k)\hat{\mathbf{M}}_k$. From $\hat{\mathbf{D}}_k \leq \rho(\hat{\mathbf{A}}_k)\mathbf{I} \leq \kappa(\hat{\mathbf{M}}_k)\rho(\hat{\mathbf{M}}_k^{-1}\hat{\mathbf{A}}_k)\hat{\mathbf{M}}_k$, a sufficient condition for (a) is $\kappa(\hat{\mathbf{M}}_k) \lesssim 1$. From

$$\frac{\langle \mathbf{M}_k \mathbf{u}, \mathbf{u} \rangle}{\langle \Delta_k \mathbf{u}, \mathbf{u} \rangle} = \frac{\| \sum_{m \in I_k} \mathbf{u}_{k,m} \phi_{k,m} \|_{0,k}^2}{\sum_{m \in I_k} |\mathbf{u}_{k,m}|^2 \|\phi_{k,m}\|_{0,k}^2},$$

we see that $\kappa(\hat{\mathbf{M}}_k) \lesssim 1$ means uniform stability of the normalized bases of V_k with respect to $\| \, \|_{0,k}$.

The Cauchy-Schwarz inequality $|\langle \mathbf{A}_k \mathbf{u}, \mathbf{v} \rangle| \leq \langle \mathbf{A}_k \mathbf{u}, \mathbf{u} \rangle^{\frac{1}{2}} \langle \mathbf{A}_k \mathbf{v}, \mathbf{v} \rangle^{\frac{1}{2}}$ implies that all elements or blocks of $\mathbf{D}_k^{-\frac{1}{2}} \mathbf{L}_k \mathbf{D}_k^{-\frac{1}{2}}$ have absolute values or Euclidean-norms less or equal to one. So in case the number of elements or blocks, or more generally, the number of non-zero elements or blocks in each row and column is uniformly bounded, then (c) is valid. Applications are given by (inexact, i.e., $\tilde{\mathbf{D}}_k \neq \mathbf{D}_k$) point or block Gauss-Seidel iterations based on lexicographical or multicolor orderings of the unknowns.

Proposition 2.7. *Let $\tilde{\mathbf{D}}_k$, $\mathbf{D}_k$ and $\mathbf{L}_k$ be some matrices of the same size as $\mathbf{A}_k$, and let $\mathbf{C}_k := \rho(\mathbf{D}_k^{-1}\mathbf{A}_k)\tilde{\mathbf{D}}_k$. If*

(a). $0 < \mathbf{D}_k = \mathbf{D}_k^H \lesssim \frac{\rho(\mathbf{M}_k^{-1}\mathbf{A}_k)}{\rho(\mathbf{D}_k^{-1}\mathbf{A}_k)} \mathbf{M}_k$,

(b). $1 - \|\mathbf{I} - \mathbf{D}_k^{\frac{1}{2}} \tilde{\mathbf{D}}_k^{-1} \mathbf{D}_k^{\frac{1}{2}}\| \gtrsim 1$,

then (B) is valid.

Proof. In this case, (2.15) reads as

$$\rho(\mathbf{M}_k^{-1}\mathbf{A}_k)(\tilde{\mathbf{D}}_k + \tilde{\mathbf{D}}_k^H - \rho(\mathbf{D}_k^{-1}\mathbf{A}_k)^{-1}\mathbf{A}_k) \gtrsim \rho(\mathbf{D}_k^{-1}\mathbf{A}_k)\tilde{\mathbf{D}}_k^H \mathbf{M}_k^{-1} \tilde{\mathbf{D}}_k. \tag{2.17}$$

The proof follows from $\mathbf{A}_k \leq \rho(\mathbf{D}_k^{-1}\mathbf{A}_k)\mathbf{D}_k$, an application of (2.16), (b) and (a). $\square$

Remark 2.8. (preconditioned Richardson). If in Proposition 2.7, $\tilde{\mathbf{D}}_k = \omega^{-1}\mathbf{D}_k$ for some fixed $\omega \in \mathbb{R}$, then (b) means $\omega \in (0,2)$. Moreover, since after substituting $\tilde{\mathbf{D}}_k = \omega^{-1}\mathbf{D}_k$, (2.17) is equivalent to

$$2\omega \mathbf{I} - \omega^2 \rho(\mathbf{D}_k^{-1}\mathbf{A}_k)^{-1} \mathbf{D}_k^{-\frac{1}{2}} \mathbf{A}_k \mathbf{D}_k^{-\frac{1}{2}} \gtrsim \frac{\rho(\mathbf{D}_k^{-1}\mathbf{A}_k)}{\rho(\mathbf{M}_k^{-1}\mathbf{A}_k)} \mathbf{D}_k^{\frac{1}{2}} \mathbf{M}_k^{-1} \mathbf{D}_k^{\frac{1}{2}},$$

in this case both $\omega \in (0,2)$ and (a) are also necessary conditions for (B).

Remark 2.9. Writing $\mathbf{A}_k - \mathbf{D}_k$ in the form $\mathbf{L}_k + \mathbf{L}_k^H$, condition (a) of Proposition 2.7 follows from $0 < \mathbf{D}_k \lesssim \rho(\mathbf{M}_k^{-1}\mathbf{A}_k)\mathbf{M}_k$ and $\rho(\mathbf{D}_k^{-1}\mathbf{A}_k) \leq 1 + 2\|\mathbf{D}_k^{-\frac{1}{2}}\mathbf{L}_k\mathbf{D}_k^{-\frac{1}{2}}\| \lesssim 1$, that is, from assumptions (a) and (c) of Proposition 2.4.

In particular, when $\mathbf{D}_k$ is a (block) diagonal part of $\mathbf{A}_k$, sufficient conditions for these assumptions are discussed in Remark 2.6. In this case, and with $\tilde{\mathbf{D}}_k = \omega^{-1}\mathbf{D}_k$, the iteration from Proposition 2.7 is known as the *damped (block) Jacobi iteration* with damping parameter $\rho(\mathbf{D}_k^{-1}\mathbf{A}_k)^{-1}\omega$. Iterations with $\tilde{\mathbf{D}}_k$ unequal to some multiple of $\mathbf{D}_k$ will be called inexact damped (block) Jacobi iterations.

2.5. **The regularity and approximation assumption (A) with $\alpha = 1$ in a nonconforming framework.** We consider the following usual nonconforming finite element setting: Let

$$\mathcal{H}^2 \hookrightarrow \mathcal{H}^1 \hookrightarrow \mathcal{H}^0$$

be continuously embedded Hilbert spaces. We assume that for all k,

$$V_k \subset \mathcal{H}^0,$$

and put

$$\|\cdot\|_{0,k} := \|\cdot\|_{\mathcal{H}^0}.$$

Furthermore, we assume that there exists a scalar product a on $\mathcal{H}^1$ satisfying

$$a(\cdot,\cdot) \eqsim \|\cdot\|^2_{\mathcal{H}^1},$$

such that for all k, a_k can be extended to a scalar product on $\mathcal{H}^1 + V_k$, which reduces to a on $\mathcal{H}^1$. Finally, we assume that the sequence $(\rho_k)_k$ defined in (2.8) satisfies $\rho_{k+1} \lesssim \rho_k$.

Theorem 2.10. *For $f \in \mathcal{H}^0$, let $u \in \mathcal{H}^1$ denote the solution of*

$$a(u,v) = (f,v)_{\mathcal{H}^0} \qquad (v \in \mathcal{H}^1).$$

Then, if

(a). $u \in \mathcal{H}^2$ *with* $\|u\|_{\mathcal{H}^2} \lesssim \|f\|_{\mathcal{H}^0}$ *(*'full' regularity*),*

(b). $|a_k(u,v_k) - (f,v_k)_{\mathcal{H}^0}| \lesssim \rho_k^{-\frac{1}{2}} a_k(v_k,v_k)^{\frac{1}{2}}\|f\|_{\mathcal{H}^0}$ $\quad (v_k \in \mathcal{H}^1 + V_k)$ *(*consistency*)*

(c). $\inf_{v_k \in V_k} a_k(v - v_k, v - v_k)^{\frac{1}{2}} \lesssim \rho_k^{-\frac{1}{2}}\|v\|_{\mathcal{H}^2}$ $\quad (v \in \mathcal{H}^2)$ *(*approximation*)*

and there exist mappings $\Pi_k : \mathcal{H}^2 \to V_k$ such that

(d). $\|I - \Pi_k\|_{\mathcal{H}^0 \leftarrow \mathcal{H}^2} \lesssim \rho_k^{-1}$,

(e). $\|\Pi_k - I_k\Pi_{k-1}\|_{\mathcal{H}^0 \leftarrow \mathcal{H}^2} \lesssim \rho_k^{-1}$,

and finally,

(f). $\|I_k\|_{\mathcal{H}^0 \leftarrow \mathcal{H}^0} \lesssim 1$,

then

$$|a_k((I - I_kI_k^*)v_k, v_k)| \lesssim \rho_k^{-1}\|v_k\|^2_{2,k} \qquad (v_k \in V_k)$$

(Assumption (A) with $\alpha = 1$).

Proof. By the definition of $\| \, \|_{2,k}$ in (2.7), it follows that

$$|a_k(w_k, v_k)| \leq \|w_k\|_{\mathcal{H}^0} \|v_k\|_{2,k} \qquad (w_k, v_k \in V_k), \tag{2.18}$$

which means that it is sufficient to show that

$$\|(I - I_k I_k^*) v_k\|_{\mathcal{H}^0} = \sup_{0 \neq f \in \mathcal{H}^0} \frac{|(f, (I - I_k I_k^*) v_k)_{\mathcal{H}^0}|}{\|f\|_{\mathcal{H}^0}} \lesssim \rho_k^{-1} \|v_k\|_{2,k} \qquad (u_k \in V_k). \tag{2.19}$$

Given $f \in \mathcal{H}^0$, we define $u \in \mathcal{H}^1$, and for each k, $u_k \in V_k$ by

$$\begin{aligned} a(u, w) &= (f, w)_{\mathcal{H}^0} \qquad && (w \in \mathcal{H}^1), \\ a_k(u_k, w_k) &= (f, w_k)_{\mathcal{H}^0} \qquad && (w_k \in V_k). \end{aligned}$$

From

$$\begin{aligned} (f, (I - I_k I_k^*) v_k)_{\mathcal{H}^0} &= a_k(u_k, (I - I_k I_k^*) v_k) \\ &= a_k(u_k - I_k u_{k-1}, v_k) + a_k(I_k(u_{k-1} - I_k^* u_k), v_k), \end{aligned}$$

together with (2.18) and (f), we see that (2.19) will follow from

$$\|u_k - I_k u_{k-1}\|_{\mathcal{H}^0} \lesssim \rho_k^{-1} \|f\|_{\mathcal{H}^0} \quad \text{and} \tag{2.20}$$

$$\|u_{k-1} - I_k^* u_k\|_{\mathcal{H}^0} \lesssim \rho_k^{-1} \|f\|_{\mathcal{H}^0}. \tag{2.21}$$

By (a), (b) and (c), the well-known Aubin-Nitsche lemma (cf. e.g. [Cia78, Ex. 4.2.3]) shows that

$$\|u - u_k\|_{\mathcal{H}^0} \lesssim \rho_k^{-1} \|f\|_{\mathcal{H}^0} \qquad \textit{(convergence)}. \tag{2.22}$$

By writing

$$u_k - I_k u_{k-1} = (u_k - \Pi_k u) + (\Pi_k u - I_k \Pi_{k-1} u) + I_k(\Pi_{k-1} u - u_{k-1}),$$

(2.20) follows from (2.22), (d), (e), (f) and (a).

To establish (2.21), we write

$$\|u_{k-1} - I_k^* u_k\|_{\mathcal{H}^0} = \sup_{0 \neq g \in \mathcal{H}^0} \frac{|(g, u_{k-1} - I_k^* u_k)_{\mathcal{H}^0}|}{\|g\|_{\mathcal{H}^0}},$$

and, for given $g \in \mathcal{H}^0$, we define $z \in \mathcal{H}^1$, and for each k, $z_l \in V_l$ by

$$\begin{aligned} a(z, w) &= (g, w)_{\mathcal{H}^0} \qquad && (w \in \mathcal{H}^1), \\ a_k(z_k, w_k) &= (g, w_k)_{\mathcal{H}^0} \qquad && (w_k \in V_k). \end{aligned}$$

By writing

$$\begin{aligned} (g, u_{k-1} - I_k^* u_k)_{\mathcal{H}^0} &= (g, u_{k-1})_{\mathcal{H}^0} - a_{k-1}(z_{k-1}, I_k^* u_k) \\ &= (g, u_{k-1})_{\mathcal{H}^0} - a_k(I_k z_{k-1}, u_k) \\ &= (g, u_{k-1} - u_k)_{\mathcal{H}^0} + a_k(z_k - I_k z_{k-1}, u_k) \\ &= (g, u_{k-1} - u_k)_{\mathcal{H}^0} + (z_k - I_k z_{k-1}, f)_{\mathcal{H}^0}, \end{aligned}$$

(2.21) follows from (2.22) and (2.20). ∎

Remark 2.11. Under the assumptions of Theorem 2.10, most proofs from the literature yield assumption (A) only for $\alpha = \frac{1}{2}$. An exception is [Bre99], which our proof is partly based upon. Yet, compared to that paper, our proof is shorter and it needs less assumptions. On the other hand, the arguments from [Bre99] are not restricted to the 'full' regularity case. In view of the multi-grid convergence theory from Section 2.3, it is desirable to have (A) with α as high as possible, and in particular to know whether it is valid for $\alpha = 1$.

In [BDH99, §4] a result similar to Theorem 2.10 was proved. Instead of (b) and (c), there (2.22) was assumed, which is clearly also a sufficient condition for the present proof. Moreover, instead of (d), (e) and (f), it was assumed that $I_k : V_{k-1} \to V_k$ can be extended to an $\mathcal{H}^0$-bounded *projector* $\hat{I}_k$ from $V_{k-1} + V_k$ onto V_k. Obviously, (2.20) can also be deduced from this property, which means that our proof applies as well. Although in applications often the condition involving $\hat{I}_k$ is more easily verified, in connection with the Morley element we will encounter an I_k of practical interest for which (d), (e) and (f) are valid, but the condition involving $\hat{I}_k$ is not.

2.6. **Additive multi-grid method.** In particular to underline the role of the iterated prolongations in the behaviour of multi-grid methods, we briefly discuss the additive variant. Given some scalar products e_k on V_k, we define $E_k : V_k \to V_k'$, determining a *Hermitian* smoother, by

$$(E_k u)(v) = e_k(u, v) \qquad (u, v \in V_k).$$

The additive multi-grid operator $B_k^{(\mathrm{add})}$ is now defined by

$$B_k^{(\mathrm{add})} = \begin{cases} E_k^{-1} + I_k B_{k-1}^{(\mathrm{add})} I_k' & \text{if } k > 0, \\ A_0^{-1} & \text{if } k = 0. \end{cases}$$

The following result can be deduced from [Osw97]:

Theorem 2.12. *For $k \geq 0$, let $P_k : V_{k+1} \to V_k$ be some mappings. Put*

$$I_{j\leftarrow k} := \begin{cases} I_j I_{j-1\leftarrow k} & \text{if } j > k \\ I & \text{if } j = k \end{cases} \quad \text{and } P_{k\leftarrow j} := \begin{cases} P_{k\leftarrow j-1} P_{j-1} & \text{if } j > k \\ I & \text{if } j = k. \end{cases}$$

Then, with $I_0 P_{-1} := 0$,

$$\inf_{0 \neq u \in V_j} \frac{e_j(u,u)}{a_j(u,u)} \cdot \max_{0 \leq k \leq j} \sup_{0 \neq u \in V_k} \frac{a_j(I_{j\leftarrow k} u, I_{j\leftarrow k} u)}{e_k(u,u)} \leq \kappa(B_j^{(\mathrm{add})} A_j) \leq$$

$$\sup_{0 \neq u \in V_j} \frac{\sum_{k=0}^{j} e_k((I - I_k P_{k-1}) P_{k\leftarrow j} u, (I - I_k P_{k-1}) P_{k\leftarrow j} u)}{a_j(u,u)} \cdot \sum_{k=0}^{j} \sup_{0 \neq u \in V_k} \frac{a_j(I_{j\leftarrow k} u, I_{j\leftarrow k} u)}{e_k(u,u)}.$$

In particular, if

$$a_k(u,u) \lesssim e_k(u,u) \lesssim \rho_k \|u\|_{0,k}^2 \qquad (u \in V_k), \tag{2.23}$$

then with

$$t_j := \sup_{0\neq u\in V_j} \frac{\sum_{k=0}^{j} \rho_k \|(I - I_k P_{k-1}) P_{k\leftarrow j} u\|_{0,k}^2}{a_j(u,u)},$$

it follows that

$$\max_{0\leq k\leq j} \sup_{0\neq u\in V_k} \frac{a_j(I_{j\leftarrow k}u, I_{j\leftarrow k}u)}{e_k(u,u)} \lesssim \kappa(B_j^{(\mathrm{add})}A_j) \lesssim t_j \sum_{k=0}^{j} \rho(I_{j\leftarrow k}^* I_{j\leftarrow k}). \tag{2.24}$$

Remark 2.13. Let us assume (2.23). In our applications, it will appear that the P_k can be selected such that either $t_j \lesssim 1$ or $t_j \lesssim j$, and so $\kappa(B_j^{(\mathrm{add})}A_j) \lesssim j \max_{0\leq k\leq j} \rho(I_{j\leftarrow k}^* I_{j\leftarrow k})$ or $\kappa(B_j^{(\mathrm{add})}A_j) \lesssim j^2 \max_{0\leq k\leq j} \rho(I_{j\leftarrow k}^* I_{j\leftarrow k})$. On the other hand, for any *fixed* k, based on $a_k(\,,\,) \eqsim e_k(\,,\,)$, we have $\kappa(B_j^{(\mathrm{add})}A_j) \gtrsim \rho(I_{j\leftarrow k}^* I_{j\leftarrow k})$ $(j \geq k)$. Together these bounds show that the quality of the additive multi-grid preconditioner depends critically on the energy-norm of the iterated prolongations.

Recall that at the end of Subsection 2.3, for B_j defined by the (multiplicative) standard V-cycle, we observed that the upper bound on $\kappa(B_jA_j)$ depends critically on the energy-norm of the 'iterated prolongations alternated with smoothers', i.e., on the factors $\rho(\tilde{I}_{j\leftarrow k}^* \tilde{I}_{j\leftarrow k})$. Since by assumption (B), $\rho(K_k^* K_k) \leq 1$, it is likely that $\rho(I_{j\leftarrow k}^* I_{j\leftarrow k}) \lesssim 1$ would imply that $\rho(\tilde{I}_{j\leftarrow k}^* \tilde{I}_{j\leftarrow k}) \lesssim 1$. On the other hand, if for example $\rho(I_{j\leftarrow k}^* I_{j\leftarrow k})$ is an exponentially increasing function of $j-k$, then for general smoothers it cannot be expected that $\rho(\tilde{I}_{j\leftarrow k}^* \tilde{I}_{j\leftarrow k}) \lesssim 1$.

Remark 2.14. Even when $t_j \lesssim 1$ and $\rho(I_{j\leftarrow k}^* I_{j\leftarrow k}) \lesssim 1$, Theorem 2.12 only shows that the additive multi-grid preconditioner is *sub*optimal, i.e., $\kappa(B_j^{(\mathrm{add})}A_j) \lesssim j$. Yet, under the conditions that were imposed, this is the best result one can expect. Indeed, consider the case that $0 \neq V_0 \subset \cdots \subset V_{j-1} \subsetneq V_j$, for all $1 \leq k \leq j$, $e_k = a_k = a_j$, and I_k is the trivial injection. It is not difficult to show that then $\kappa(B_j^{(\mathrm{add})}A_j) = j+1$.

Below we comment on the construction of Hermitian smoothers that satisfy (2.23). If the forms c_k introduced in Section 2.1 are Hermitian and (B) is valid, then the equivalence of (B) and (2.14) shows that $2c_k(u,u) > a_k(u,u)$ and $c_k(u,u) \lesssim \rho_k \|u\|_{0,k}^2$, or $e_k = c_k$ satisfies (2.23).

The reverse is not valid; taking $c_k = e_k$, where e_k satisfies (2.23), does not imply (B). Indeed, note for example that (2.23) does not guarantee that $2e_k > a_k$, that is, convergence of the corresponding iteration.

In case c_k is non-Hermitian, an obvious candidate for a suitable Hermitian smoother is the *symmetrized smoother* defined as follows: Let C_k, $C_k^\dagger$ be defined as in (2.4). For $g \in V_k'$, put $G_k g = x^{(2)}$, where $x^{(0)} = 0$ and

$$\begin{cases} x^{(1)} = x^{(0)} + C_k^{-1}(g - A_k x^{(0)}), \\ x^{(2)} = x^{(1)} + C_k^{-\dagger}(g - A_k x^{(1)}), \end{cases}$$

or $G_k = C_k^{-1} + C_k^{-\dagger} - C_k^{-\dagger} A_k C_k^{-1}$.

Proposition 2.15. *$E_k := G_k^{-1}$ exists and $e_k(u,v) = (E_k u)(v)$ are scalar products that satisfy (2.23) if and only if the c_k satisfy (B).*

Proof. Assumption (B) can be rewritten as

$$a_k(G_k A_k u, u) \gtrsim \rho_k^{-1} \|u\|_{2,k}^2 \qquad (u \in V_k), \tag{2.25}$$

which shows that E_k exists.

From $a_k((I - G_k A_k)u, u) = a_k(K_k^* K_k u, u) \geq 0$, we have $a_k(G_k A_k u, u) \leq a_k(u,u)$, or, by substituting $u = (G_k A_k)^{-\frac{1}{2}} v$,

$$a_k(v,v) \leq a_k(A_k^{-1} E_k v, v) = e_k(v,v) \qquad (v \in V_k). \tag{2.26}$$

Substituting $u = (G_k A_k)^{-1} w$ in (2.25) yields

$$e_k(w,w) \gtrsim \rho_k^{-1} \sup_{0 \neq v \in V_k} \frac{|e_k(w,v)|^2}{\|v\|_{0,k}^2}, \tag{2.27}$$

and so by taking $v = w$, in particular

$$e_k(w,w) \lesssim \rho_k \|w\|_{0,k}^2 \qquad (w \in V_k). \tag{2.28}$$

Together, formulas (2.26) and (2.28) show that (2.23) is valid.

On the other hand, if $e_k(u,v) := (E_k u)(v)$ are scalar products that satisfy (2.23) and thus (2.28), then by an application of the Cauchy-Schwarz inequality we infer (2.27) and so (2.25), that is, (B) is valid.

□

3. Applications

3.1. **Nonconforming P_1 element.** Let $\tau_0, \tau_1, \ldots$ be a sequence of conforming triangulations of some bounded, convex polygon $\Omega \subset \mathbb{R}^2$, such that τ_{k+1} is generated from τ_k by refinement, $\sup_{T \in \tau_k} \operatorname{diam}(T) \eqsim 2^{-k}$, and the triangles satisfy a shape regularity condition uniformly over the levels. We define $\overline{E}_k$ and E_k as the sets of all or internal edges of τ_k respectively. For $e \in \overline{E}_k$, m_e will denote the midpoint of e. We take $V_k = V_k^{(P_1)}$, where

$$V_k^{(P_1)} = \{v \in \textstyle\prod_{T \in \tau_k} P_1(T)\colon v \text{ is continuous at } m_e \text{ for } e \in E_k, \\ \text{and it vanishes at } m_e \text{ for } e \in \overline{E}_k \backslash E_k\},$$

and define

$$a_k(u,v) = \sum_{T \in \tau_k} \int_T \nabla u \cdot \nabla v.$$

With $\| \, \|_{0,k} := \| \, \|_{L^2(\Omega)}$, we find that ρ_k defined in (2.8) satisfies $\rho_k \eqsim 4^k$.

We define the prolongation in the usual way, that is,

$$(I_k u)(m_e) = \text{ average}_i \text{ of } u|_{T_i}(m_e) \qquad (e \in E_k),$$

where $\tau_{k-1} \ni T_i \supset e$.

In the setting of Section 2.5, let

$$\mathcal{H}^0 = L^2(\Omega), \quad \mathcal{H}^1 = H_0^1(\Omega), \quad \mathcal{H}^2 = H^2(\Omega) \cap H_0^1(\Omega),$$

and

$$a(u,v)=\int_\Omega \nabla u\cdot\nabla v.$$

With these definitions, it is well-known that conditions (a)-(c) of Theorem 2.10 are satisfied (for (b) and (c), see e.g. [BS94, §8.3]). We define $\Pi_k : H^2(\Omega)\cap H_0^1(\Omega)\to V_k^{(P_1)}$ by $(\Pi_k u)(m_e)=u(m_e)$. Then using the local reproduction by I_k of first degree polynomials, standard arguments as the Sobolev embedding theorem and the Bramble-Hilbert lemma show the remaining conditions (c), (d) and (f). From Theorem 2.10 we conclude that assumption (A) with $\alpha=1$ is valid.

We now equip the spaces $V_k^{(P_1)}$ with nodal bases $\{\eta_{k,e}: e\in E_k\}$, defined by

$$\eta_{k,e}(m_{\tilde e})=\delta_{e,\tilde e}\qquad (e,\tilde e\in E_k). \tag{3.1}$$

These bases are $L^2(\Omega)$-orthogonal, and so, as demonstrated in Section 2.4, (inexact) standard Gauss-Seidel and damped Jacobi smoothers satisfy assumption (B).

Since (A) and (B) are valid, with a number of smoothing steps m that is sufficiently large, the W-cycle yields uniformly convergent iterations. Yet, since generally $\rho(I_k^* I_k)>2$, which has been demonstrated in [Che97, Ex.1], for any m that happens not to be large enough the W-cycle might result in a preconditioned system that is indefinite. On the other hand, the variable V-cycle yields preconditioned systems that have uniformly bounded condition numbers. Since (A) is valid with $\alpha=1$, Theorem 2.3 even shows that this also holds for the *mildly* variable V-cycle.

We now consider the additive multi-grid method. By taking P_k to be the restriction to $V_{k+1}^{(P_1)}$ of the $a_{k+1}(\ ,\ )$-orthogonal projector from $V_{k+1}^{(P_1)}+V_k^{(P_1)}$ to $V_k^{(P_1)}$, in [Osw97] it has been proved that the scalars t_j from Theorem 2.12 are uniformly bounded. Furthermore, assuming uniform dyadic refinements and under some technical assumptions concerning the degree of the nodes in the mesh, in [Osw92] it was shown that $\rho(I_{j\leftarrow k}^* I_{j\leftarrow k})\lesssim 1$. Assuming a Hermitian smoother that satisfies (2.23), from Theorem 2.12 we conclude that the additive multi-grid preconditioner $B_j^{(\mathrm{add})}$ satisfies $\kappa(B_j^{(\mathrm{add})}A_j)\lesssim j$. Since regularity plays no role in the analysis of the additive method, this result is also valid for non-convex Ω.

Because of $\rho(I_{j\leftarrow k}^* I_{j\leftarrow k})\lesssim 1$, it is likely that also $\rho(\tilde I_{j\leftarrow k}^* \tilde I_{j\leftarrow k})\lesssim 1$. According to the observations made at the end of Section 2.3, this would mean that the standard (multiplicative) V-cycle yields a preconditioner B_j for which at least $\kappa(B_jA_j)\lesssim j$.

3.2. **Rotated Q_1 element.** Now let $\tau_0,\tau_1,\ldots$ be a sequence of conforming subdivisions of some bounded, convex polygon $\Omega\subset\mathbb{R}^2$ into *parallelograms*, such that τ_{k+1} is generated from τ_k by refinement, $\sup_{T\in\tau_k}\mathrm{diam}(T)\eqsim 2^{-k}$, and the parallelograms satisfy a shape regularity condition uniformly over the levels. We define $\overline{E}_k$ and E_k as the sets of all or internal edges of τ_k respectively. For $e\in\overline{E}_k$, m_e will denote the midpoint of e. For each $T\in\tau_k$, we consider the space $P_T=\{v\in L^2(T): v\circ F_T\in\mathrm{span}\{1,x,y,x^2-y^2\}\}$, where F_T is an affine bijection between $[-1,1]^2$ and T. There are two usual options to identify

$v \in P_T$ uniquely, namely either by

$$\{v(m_e) : e \in E_k\}, \tag{3.2}$$

or by

$$\{\tfrac{1}{|e|}\textstyle\int_e v : e \in E_k\}. \tag{3.3}$$

Both choices give rise to different finite element spaces $V_k = V_k^{(Q_1)}$ defined by

$$V_k^{(Q_1)} = \{v \in \textstyle\prod_{T\in\tau_k} P_T\text{: the degrees of freedom match at } e \in E_k, \\ \text{and they vanish at } e \in \overline{E}_k \backslash E_k\},$$

Similar to Section 3.1, we take

$$a_k(u,v) = \sum_{T\in\tau_k} \int_T \nabla u \cdot \nabla v,$$

and with $\| \, \|_{0,k} := \| \, \|_{L^2(\Omega)}$, we find that ρ_k defined in (2.8) satisfies $\rho_k \eqsim 4^k$.

As in Section 3.1, the prolongations I_k can be defined by averaging the degrees of freedom at $e \in E_k$, and by setting them equal to zero at $e \in \overline{E}_k \backslash E_k$. For both (3.2) and (3.3), the resulting I_k reproduces first degree polynomials.

With $\mathcal{H}^0$, $\mathcal{H}^1$, $\mathcal{H}^2$ and $a(\, , \,)$ as in Section 3.1, all conditions of Theorem 2.10 can be verified analogously. We conclude that assumption (A) with $\alpha = 1$ is valid.

Since for both (3.2) and (3.3), the normalized bases of $V_k^{(Q_1)}$ with respect to the degrees of freedom are uniformly $L^2(\Omega)$-stable, from Section 2.4 we learn that (inexact) standard Gauss-Seidel and damped Jacobi smoothers satisfy (B). We conclude that the W-cycle with a sufficiently large number of smoothing iterations, the variable V-cycle and even the *mildly* variable V-cycle all yield uniformly convergent iterations or preconditioned systems with uniformly bounded condition numbers.

Assuming that τ_k corresponds to a uniform partition of Ω into squares, for the choice (3.3), in [CO98] it was shown that $\rho(I_k^* I_k) \leq 2$ (but generally $\rho(I_k^* I_k) > 1$, see [Che97]). This means that even for *any* positive number of smoothing iterations the W-cycle yields uniformly convergent iterations.

Again for (3.3), and under the same assumption on the mesh, in [CO98] it was shown that $\rho(I_{j\leftarrow k}^* I_{j\leftarrow k}) \lesssim 1$. Since for P_k being the restriction to $V_{k+1}^{(Q_1)}$ of the $a_{k+1}(\, , \,)$-orthogonal projector from $V_{k+1}^{(Q_1)} + V_k^{(Q_1)}$ to $V_k^{(Q_1)}$, it was proved that $t_j \lesssim 1$, Theorem 2.12 now shows that with a suitable Hermitian smoother the additive multi-grid preconditioner is suboptimal, i.e., $\kappa(B_j^{(\mathrm{add})} A_j) \lesssim j$. Since it is likely that as a consequence also $\rho(\tilde{I}_{j\leftarrow k}^* \tilde{I}_{j\leftarrow k}) \lesssim 1$, this would mean that the standard (multiplicative) V-cycle yields a preconditioner that is at least suboptimal.

On the other hand, for the choice (3.2), it holds that $\rho(I_k^* I_k) > 2$ ([Che97]). This means that with a number a smoothing steps that is not large enough, the W-cycle might result in a preconditioned system that is indefinite. Moreover, again for (3.2), numerical results from [Osw97] indicate that for fixed k, $\rho(I_{j\leftarrow k}^* I_{j\leftarrow k})$ increases exponentially as a function of $j - k$. By Remark 2.13, this means that also $\kappa(B_j^{(\mathrm{add})} A_j)$ is an exponentially growing function of $j \geq k$. Moreover, since in this case for general smoothers it cannot be expected that $\rho(\tilde{I}^*_{j\leftarrow k}\tilde{I}_{j\leftarrow k}) \lesssim 1$, the discussion at the end of Section 2.3 shows that the standard (multiplicative) V-cycle might give unsatisfactory results as well.

3.3. Morley element.

3.3.1. *The discretized biharmonic.* Let $\tau_0, \tau_1, \dots$ be a sequence of conforming triangulations of some bounded, convex polygon $\Omega \subset I\!R^2$, such that τ_{k+1} is generated from τ_k by refinement, $\sup_{T\in\tau_k} \operatorname{diam}(T) \eqsim 2^{-k}$, and the triangles satisfy a shape regularity condition uniformly over the levels. We define $\overline{E}_k$, $\overline{N}_k$ as the set of all edges and vertices of τ_k, and E_k, N_k as the set of internal edges and vertices of τ_k. For $e \in \overline{E}_k$, m_e will denote the midpoint of e, and $\mathbf{n}_e$ a unit vector normal to e. We take $V_k = M_k$, where M_k is the Morley finite element space corresponding to τ_k, i.e.,

$$\begin{aligned} M_k = \{v \in \textstyle\prod_{T\in\tau_k} P_2(T)\colon\ & v \text{ is continuous at } p \in N_k, \text{ and it vanishes at } p \in \overline{N}_k\backslash N_k;\\ & \partial_{\mathbf{n}_e} v \text{ is continuous at } m_e \text{ for } e \in E_k, \text{ and it vanishes at } m_e \text{ for } e \in \overline{E}_k\backslash E_k\}. \end{aligned}$$

Since $v \in M_k$ is piecewise quadratic, its derivative tangential to $e \in \overline{E}_k$ in m_e from either side of e (if $e \in E_k$) can be expressed as a divided difference in terms of the values of v at the endpoints of e. The continuity of v at the vertices therefore shows that also these tangential derivatives are continuous for $e \in E_k$ and that they vanish for $e \in \overline{E}_k\backslash E_k$.

We define $a_k = a_k^{(\mathrm{Bih})}$ by

$$a_k^{(\mathrm{Bih})}(u,v) := \sum_{T\in\tau_k} \int_T \sum_{i,j=1}^{2} \tfrac{\partial^2 u}{\partial x_i \partial x_j} \overline{\tfrac{\partial^2 v}{\partial x_i \partial x_j}}.$$

The prolongation $I_k^{(\mathrm{Bih},1)} : M_{k-1} \to M_k$ commonly used in connection with the Morley finite element spaces was introduced in [Bre89], and is defined by

$$\begin{aligned} (I_k^{(\mathrm{Bih},1)}u)(p) &= \text{average}_i \text{ of } u|_{T_i}(p) \qquad (p \in N_k),\\ \partial_{\mathbf{n}_e}(I_k^{(\mathrm{Bih},1)}u)(m_e) &= \text{average}_i \text{ of } \partial_{\mathbf{n}_e}(u|_{T_i})(m_e) \qquad (e \in E_k), \end{aligned} \tag{3.4}$$

where $\tau_{k-1} \ni T_i \ni p$ or $\tau_{k-1} \ni T_i \supset e$.

As appears from numerical results reported in [Osw97, Che97], a disadvantage of $I_k^{(\mathrm{Bih},1)}$ is that, for fixed k, $\rho((I_{j\leftarrow k}^{(\mathrm{Bih},1)})^* I_{j\leftarrow k}^{(\mathrm{Bih},1)})$ generally grows exponentially with $j-k$. As exposed before, this has an adverse effect on the additive and (V-cycle type) multiplicative multigrid methods.

In this paper, we therefore introduce an alternative prolongation $I_k^{(\mathrm{Bih},2)}$. For ease of presentation, we restrict ourselves to the case of uniform dyadic refinements. Let $E_k^{(\mathrm{new})}$ denote the set of new edges, i.e., $E_k^{(\mathrm{new})} = \{e \in E_k : e \not\subset \tilde{e} \text{ for any } \tilde{e} \in E_{k-1}\}$. Then $I_k^{(\mathrm{Bih},2)}$ is defined by

$$(I_k^{(\mathrm{Bih},2)}u)(p) = \text{average}_i \text{ of } u|_{T_i}(p) \qquad (p \in N_k), \tag{3.5}$$

where $\tau_{k-1} \ni T_i \ni p$,

$$\partial_{\mathbf{n}_e}(I_k^{(\mathrm{Bih},2)}u)(m_e) = \partial_{\mathbf{n}_e}u(m_e) \qquad (e \in E_k^{(\mathrm{new})}) \tag{3.6}$$

and

$$a_k^{(\mathrm{Bih})}(I_k^{(\mathrm{Bih},2)}u, \check{M}_k) = 0, \tag{3.7}$$

where $\check{M}_k$ is defined as the span of the remaining degrees of freedom, i.e.,

$$\check{M}_k = \{u \in M_k : u(N_k) = 0,\ \partial_{\mathbf{n}_e}u(m_e) = 0\ (e \in E_k^{(\mathrm{new})})\}.$$

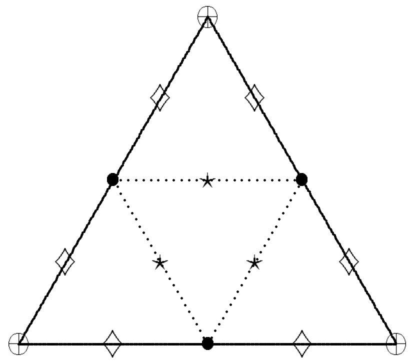

FIGURE 1. Degrees of freedom of M_k: Vertices in N_{k-1} ($\oplus$), or in $N_k \backslash N_{k-1}$ ($\bullet$); midpoints of $e \in E_k^{(\mathrm{new})}$ ($\star$), or of $e \in E_k \backslash E_k^{(\mathrm{new})}$ ($\Diamond$)

Note that (3.5)-(3.6) coincide with the corresponding definitions of $I_k^{(\mathrm{Bih},1)}$, and that for each $\tilde{p} \in N_{k-1}$, equation (3.7) involves solving a small system with unknowns the values $\partial_{\mathbf{n}_e}(I_k^{(\mathrm{Bih},2)}u)(m_e)$ for all edges $e \in E_k \backslash E_k^{(\mathrm{new})}$ that contain $\tilde{p}$.

Since among all prolongations $I_k^{(\mathrm{Bih})} : M_{k-1} \to M_k$ satisfying (3.5)-(3.6), $I_k^{(\mathrm{Bih},2)}$ is the one for which $a_k(I_k^{(\mathrm{Bih})}u, I_k^{(\mathrm{Bih})}u)$ is minimal, we have $\rho((I_k^{(\mathrm{Bih},2)})^* I_k^{(\mathrm{Bih},2)}) \leq \rho((I_k^{(\mathrm{Bih},1)})^* I_k^{(\mathrm{Bih},1)})$.

Remark 3.1. In view of a practical implementation, we note that, in case of a *multiplicative* multi-grid method, if a prolongation $I_k^{(\mathrm{Bih})}$ is followed by a block Gauss-Seidel smoother for which the first block, that is assumed to be inverted exactly, corresponds to the degrees of freedom at the midpoints m_e of $e \in E_k \backslash E_k^{(\mathrm{new})}$, then the values $\partial_{\mathbf{n}_e}(I_k^{(\mathrm{Bih})}u)(m_e)$ for these m_e are irrelevant. That is, when applying such a smoother, it does not matter whether $I_k^{(\mathrm{Bih},1)}$ or $I_k^{(\mathrm{Bih},2)}$ is used, and what is more, the values $\partial_{\mathbf{n}_e}(I_k^{(\mathrm{Bih})}u)(m_e)$ for these m_e do not have to be computed. Obviously, an analogous comment applies to the restriction that is preceded by the adjoint smoother. The 'trick' described here is well-known in the multi-grid literature in connection with multicolor Gauss-Seidel smoothing.

To be able to prove assumption (A), later on we will need that, as $I_k^{(\mathrm{Bih},1)}$, the prolongation $I_k^{(\mathrm{Bih},2)}$ locally preserves second order polynomials: Let $\tilde{p} \in N_{k-1}$, and assume that $u \in M_{k-1}$ is equal to some second degree polynomial on the union V of all $T \in \tau_{k-1}$ that contain $\tilde{p}$. It holds that $\partial_{\mathbf{n}_e}(I_k^{(\mathrm{Bih},2)}u)(m_e) = \partial_{\mathbf{n}_e}u(m_e)$ for all $e \in E_k^{(\mathrm{new})}$, and, since u is continuous on V, $(I_k^{(\mathrm{Bih},2)}u)(p) = u(p)$ for all $p \in N_k$ in the interior of V. Now let U be the union of all triangles $T_1, \ldots, T_q \in \tau_k$ that contain $\tilde{p}$. Since the first order partial derivatives of $v \in \check{M}_k$ are continuous at m_e for $e \in E_k \backslash E_k^{(\mathrm{new})}$, and vanish at m_e for $e \in E_k^{(\mathrm{new})}$, integration by parts and an application of the midpoint quadrature rule shows that

$$\sum_{\ell=1}^{q} \int_{T_\ell} \sum_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} \overline{\frac{\partial^2 v}{\partial x_i \partial x_j}} = \sum_{\ell=1}^{q} \sum_{j} \int_{\partial T_\ell} (\nabla \tfrac{\partial u}{\partial x_j} \cdot \mathbf{n}) \overline{\tfrac{\partial v}{\partial x_j}} = 0.$$

We conclude that $(u - I_k^{(\mathrm{Bih},2)}u)|_U = 0$, so that indeed $I_k^{(\mathrm{Bih},2)}$ locally preserves second order polynomials.

Since we were not able to derive useful theoretical upper bounds for $\rho((I_{j\leftarrow k}^{(\mathrm{Bih},2)})^* I_{j\leftarrow k}^{(\mathrm{Bih},2)})$, we give some numerical results for a model case.

Example 3.2. Let τ_0 be the triangulation of $\Omega = [0,1]^2$ into two triangles, and for $k > 0$, let τ_k be generated from τ_{k-1} by uniform dyadic refinement. Numerically computed values of $\rho((I_{10\leftarrow k}^{(\mathrm{Bih},2)})^* I_{10\leftarrow k}^{(\mathrm{Bih},2)})$ and $\rho((I_{10\leftarrow k}^{(\mathrm{Bih},1)})^* I_{10\leftarrow k}^{(\mathrm{Bih},1)})$ are given in Table 1. The results indicate that,

TABLE 1. $\rho((I_{10\leftarrow k}^{(\mathrm{Bih,i})})^* I_{10\leftarrow k}^{(\mathrm{Bih,i})})$

k	9	8	7	6	5	4	3	2	1	0
$i=2$	2.97	4.66	6.36	7.65	8.66	9.29	9.35	8.43	7.45	0.620
$i=1$	4.19	1.18e1	3.05e1	7.45e1	1.76e2	4.02e2	8.64e2	1.57e3	1.93e3	7.39e2

in contrast to $\rho((I_{j\leftarrow k}^{(\mathrm{Bih},1)})^* I_{j\leftarrow k}^{(\mathrm{Bih},1)})$, the values $\rho((I_{j\leftarrow k}^{(\mathrm{Bih},2)})^* I_{j\leftarrow k}^{(\mathrm{Bih},2)})$ are uniformly bounded. The column $k = 9$ however shows that generally $\rho((I_k^{(\mathrm{Bih},2)})^* I_k^{(\mathrm{Bih},2)}) > 2$, which means that also for $I_k^{(\mathrm{Bih},2)}$, when the number of smoothing steps is not large enough the W-cycle might result in a preconditioned system that is indefinite.

Remark 3.3. Instead of minimizing the energy-norm over the degrees of freedom at the midpoints of $e \in E_k \backslash E_k^{(\mathrm{new})}$, equally well we could have modified the standard prolongation $I_k^{(\mathrm{Bih},1)}$ by minimizing the energy with respect to the degrees of freedom at midpoints of $e \in E_k^{(\mathrm{new})}$. Although obviously this also yields a prolongation with a smaller energy-norm, in the model case of Example 3.2, the energy-norm of the sufficiently many iterated prolongations turned out to be even larger than with the standard prolongation $I_k^{(\mathrm{Bih},1)}$.

For the following convergence analysis, we take $\|\ \|_{0,k} = \|\ \|_{L^2(\Omega)}$, which implies that ρ_k, defined in (2.8), satisfies $\rho_k \eqsim 16^k$.

First we consider the *additive* multi-grid method. For the prolongation $I_k^{(\mathrm{Bih},1)}$, and with P_{k-1} being the restriction to M_k of the $a_k^{(\mathrm{Bih})}(\ ,\ )$-orthogonal projection from $M_k + M_{k-1}$ to M_{k-1}, in [Osw97] it was proved that the scalars t_j from Theorem 2.12 are uniformly bounded. However, because of the generally exponential growth of $\rho((I_{j\leftarrow k}^{(\mathrm{Bih},1)})^* I_{j\leftarrow k}^{(\mathrm{Bih},1)})$ as function of $j \geq k$, from Remark 2.13 we learn that with this prolongation the condition numbers of the preconditioned system are exponentially growing as well.

If we replace $I_k^{(\mathrm{Bih},1)}$ by $I_k^{(\mathrm{Bih},2)}$ and use the same mappings P_{k-1}, then unfortunately the proof of $t_j \lesssim 1$ does not carry over. In view of [Osw97, Lemma 7], the problem is that, in contrast to $I_k^{(\mathrm{Bih},1)}$, the prolongation $I_k^{(\mathrm{Bih},2)} : M_{k-1} \to M_k$ cannot be extended to an $L^2(\Omega)$-bounded projector from $M_k + M_{k-1}$ onto M_k (cf. also Remark 2.11). Yet, by definition of P_k, it follows that

$$a_k^{(\mathrm{Bih})}(P_k u, P_k u) = a_{k+1}^{(\mathrm{Bih})}(P_k u, P_k u) \leq a_{k+1}^{(\mathrm{Bih})}(u, u) \qquad (u \in M_{k+1}). \tag{3.8}$$

So if we can prove that

$$\|(I - I_k^{(\mathrm{Bih},2)} P_{k-1})u\|_{L^2(\Omega)}^2 \lesssim 16^{-k} \sum_{T \in \tau_k} |u|_{H^2(T)}^2 \qquad (u \in M_k), \tag{3.9}$$

then the suboptimal result $t_j \lesssim j$ is valid.

To show (3.9), we write

$$I - I_k^{(\mathrm{Bih},2)} P_{k-1} = (I - I_k^{(\mathrm{Bih},2)}) P_{k-1} + (I - P_{k-1}).$$

In [Osw97], it was proved that

$$\|(I - P_{k-1})u\|_{L^2(\Omega)}^2 \lesssim 16^{-k} \sum_{T \in \tau_k} |u|_{H^2(T)}^2 \qquad (u \in M_k),$$

which together with (3.8) and the following lemma shows (3.9) and thus $t_j \lesssim j$.

Lemma 3.4. *With $I_k^{(\mathrm{Bih})}$ being either $I_k^{(\mathrm{Bih},1)}$ or $I_k^{(\mathrm{Bih},2)}$, we have*

$$\|(I - I_k^{(\mathrm{Bih})})u\|_{L^2(\Omega)}^2 \lesssim 16^{-k} \sum_{T \in \tau_{k-1}} |u|_{H^2(T)}^2 \qquad (u \in M_{k-1}).$$

Proof. By the exactness of the midpoint quadrature rule on first degree polynomials, in (3.6) we may replace $\partial_{\mathbf{n}_e} u(m_e)$ by $\frac{1}{|e|} \int_e \partial_{\mathbf{n}_e} u$, and in (3.4) we can make analogous replacements for $\partial_{\mathbf{n}_e}(u|_{T_i})(m_e)$. Although these modification thus do not change the definitions of the prolongations on M_{k-1}, in contract to the original ones, the new definitions allow for canonical extensions of $I_k^{(\mathrm{Bih})}$ to mappings $\tilde{I}_k^{(\mathrm{Bih})} : M_{k-1} + H_0^2(\Omega) \to M_k$. Since $\tilde{I}_k^{(\mathrm{Bih})}$ locally preserves first (even second) degree polynomials, the Bramble-Hilbert lemma and a homogeneity argument show that

$$\|(I - \tilde{I}_k^{\mathrm{Bih})})u\|_{L^2(\Omega)} \lesssim 4^{-k} \|u\|_{H^2(\Omega)} \qquad (u \in H_0^2(\Omega)). \tag{3.10}$$

Let $\tilde{M}_{k-1} \subset H_0^2(\Omega)$ be the Hsieh-Clough-Tocher macro element space corresponding to τ_{k-1} (see e.g [Cia78]). In [Bre99], a mapping $E_{k-1} : M_{k-1} \to \tilde{M}_{k-1}$ was constructed satisfying

$$\|(I - E_{k-1})u\|_{L^2(\Omega)}^2 \lesssim 16^{-k} \sum_{T \in \tau_{k-1}} |u|_{H^2(T)}^2 \qquad (u \in M_{k-1}), \tag{3.11}$$

and so in particular,

$$\|E_{k-1}u\|^2_{H^2(\Omega)} \lesssim \sum_{T\in\tau_{k-1}} |u|^2_{H^2(T)} \qquad (u \in M_{k-1}). \tag{3.12}$$

A simple scaling argument shows that

$$\|\tilde{I}_k^{(\mathrm{Bih})}u\|_{L^2(\Omega)} \lesssim \|u\|_{L^2(\Omega)} \qquad (u \in M_{k-1} + \tilde{M}_{k-1}). \tag{3.13}$$

By writing $I - I_k^{(\mathrm{Bih})} = (I - \tilde{I}_k^{(\mathrm{Bih})})E_{k-1} + (I - \tilde{I}_k^{(\mathrm{Bih})})(I - E_{k-1})$, the proof of the lemma follows from (3.10), (3.12), (3.13) and (3.11). ☐

Assuming that indeed $\rho((I_{j\leftarrow k}^{(\mathrm{Bih},2)})^* I_{j\leftarrow k}^{(\mathrm{Bih},2)}) \lesssim 1$, from $t_j \lesssim j$ and Theorem 2.12 we conclude that with a Hermitian smoother that satisfies (2.23) the additive multi-grid preconditioner $B_j^{(\mathrm{add})}$ satisfies $\kappa(B_j^{(\mathrm{add})}A_j) \lesssim j^2$.

We now turn to the verification of assumptions (A) and (B) for the *multiplicative* multi-grid method. It is well-known that the normalized bases corresponding to the degrees of freedom defining the Morley finite element space are uniformly $L^2(\Omega)$-stable. So Section 2.4 shows that (inexact) standard Gauss-Seidel and damped Jacobi smoothers satisfy assumption (B).

Using the local reproduction by both $I_k^{(\mathrm{Bih},1)}$ and $I_k^{(\mathrm{Bih},2)}$ of second order polynomials, a proof of assumption (A) with $\alpha = \frac{1}{2}$ can be deduced from [Bre99]. Since with

$$a^{(\mathrm{Bih})}(u,v) := \int_\Omega \sum_{i,j=1}^{2} \frac{\partial^2 u}{\partial x_i \partial x_j} \overline{\frac{\partial^2 v}{\partial x_i \partial x_j}},$$

for $f \in H^{-2}(\Omega)$ the problem of finding $u \in H_0^2(\Omega)$ satisfying $a^{(\mathrm{Bih})}(u,v) = f(v)$ $(v \in H_0^2(\Omega))$ is *not* fully regular, i.e., $\|u\|_{H^4(\Omega)} \lesssim \|f\|_{L^2(\Omega)}$ is generally not valid, but instead only

$$\|u\|_{H^3(\Omega)} \lesssim \|f\|_{H^{-1}(\Omega)} \tag{3.14}$$

can be shown, we stress that assumption (A) with $\alpha > \frac{1}{2}$ cannot be expected. As a consequence, we may only conclude that the W-cycle with a number of smoothing steps that is sufficiently large is a uniformly convergent iteration, and that the variable V-cycle yields preconditioned systems having uniformly bounded condition numbers. On the other hand, the mildly variable V-cycle does not necessarily yield uniformly bounded condition numbers, and, compared to an $\alpha = 1$ case, less favourable results can be expected for the standard V-cycle. We will develop better V-cycle type methods in the next subsection.

3.3.2. *An equivalent discretized Stokes problem.* Let $\tau_0, \tau_1, \ldots$ a the sequence of triangulations as in Section 3.3.1. Let

$$\mathbf{Z}_k = \{\mathbf{u} \in (V_k^{(P_1)})^2 : \mathrm{div}_k \mathbf{u} = 0\},$$

where $V_k^{(P_1)}$ is the nonconforming P_1 space from Section 3.1, and $(\mathrm{div}_k\mathbf{u})|_T := \mathrm{div}\,\mathbf{u}|_T$ $(T \in \tau_k)$.

With $(\mathbf{curl}_k v)|_T := \mathbf{curl}\, v|_T$ $(T \in \tau_k)$, in [FM90] it was proved that $\mathbf{curl}_k : M_k \to \mathbf{Z}_k$ is a bijection, and moreover that

$$a_k^{(\mathrm{Bih})}(u, v) = a_k^{(\mathrm{St})}(\mathbf{curl}_k u, \mathbf{curl}_k v), \tag{3.15}$$

where

$$a_k^{(\mathrm{St})}(\mathbf{u}, \mathbf{v}) := \sum_{T \in \tau_k} \int_T \sum_{i=1}^{2} \nabla \mathbf{u}_i \cdot \nabla \mathbf{v}_i.$$

So when $g = \mathbf{f} \circ \mathbf{curl}_j$, the problems of solving

$$a_j^{(\mathrm{Bih})}(u, v) = g(v) \qquad (v \in M_j) \tag{3.16}$$

and

$$a_j^{(\mathrm{St})}(\mathbf{u}, \mathbf{v}) = \mathbf{f}(\mathbf{v}) \qquad (\mathbf{v} \in \mathbf{Z}_j) \tag{3.17}$$

are equivalent, in the sense that $\mathbf{u} = \mathbf{curl}_j u$.

Remark 3.5. The equation (3.17) can be identified as characterizing the velocity components of a discretized Stokes problem. In [Ste00], an efficient and stable post-processing procedure is presented for finding an approximation for the pressure component assuming that some approximations of the velocity components are available.

A consequence of the equivalence of (3.16) and (3.17) is that if on all levels we relate smoothers and prolongations for both problems according to

$$c_k^{(\mathrm{Bih})}(u, v) = c_k^{(\mathrm{St})}(\mathbf{curl}_k u, \mathbf{curl}_k v),$$

and

$$\mathbf{curl}_k I_k^{(\mathrm{Bih})} = I_k^{(\mathrm{St})} \mathbf{curl}_{k-1}, \tag{3.18}$$

then the resulting multi-grid methods are equivalent. Moreover, if we equip $\mathbf{Z}_k$ with a basis generated by applying $\mathbf{curl}_k$ to all basis functions of M_k, then from Section 2.2 it appears that for both multi-grid methods the matrix representations $\mathbf{A}_k$, $\mathbf{p}_k$, $\mathbf{p}_k^T$, $\mathbf{C}_k^{-1}$ and $\mathbf{C}_k^{-H}$ of all individual components are equal, and also that the vector representations of the right-hand sides and the solutions are equal.

This equivalence of both multi-grid methods was used earlier in the literature, e.g. in [Bre90], in the sense that the discretized biharmonic was used to analyze the behaviour of the multi-grid method applied to the discretized Stokes problem. Here we will follow the opposite approach.

In our general nonconforming multi-grid framework, let $V_k = \mathbf{Z}_k$ and $a_k = a_k^{(\mathrm{St})}$. With $\| \, \|_{0,k} := \| \, \|_{L^2(\Omega)^2}$, we find that $\rho_k \eqsim 4^k$. Since an analysis using the Stokes formulation of the additive method does not yield new insights, we concentrate on the *multiplicative* multi-grid method. In the setting of Section 2.5, we take

$$\mathcal{H}^0 = L^2(\Omega)^2, \quad \mathcal{H}^1 = \{\mathbf{u} \in H_0^1(\Omega)^2 : \operatorname{div} \mathbf{u} = 0\}, \quad \mathcal{H}^2 = H^2(\Omega)^2 \cap \mathcal{H}^1$$

and $a = a^{(\mathrm{St})}$ defined by

$$a^{(\mathrm{St})}(\mathbf{u},\mathbf{v}) = \int_\Omega \sum_{i=1}^{2} \nabla \mathbf{u}_i \cdot \nabla \mathbf{v}_i.$$

With these definitions, conditions (a)-(c) of Theorem 2.10 are satisfied. In fact, using the continuous equivalent of (3.15), condition (a) ('full' regularity) can be shown to be equivalent to (3.14). For the verification of (b) and (c) one may consult [Ste99, §6.3], where they correspond to conditions (I) and (G) respectively.

Because $\mathbf{curl} : H_0^2(\Omega) \cap H^3(\Omega) \to \mathcal{H}^2$ is a homeomorphism (see e.g. [GR86]), the conditions (d)-(f) of Theorem 2.10 involving $I_k^{(\mathrm{St})} : \mathbf{Z}_{k-1} \to \mathbf{Z}_k$ and some suitable $\Pi_k^{(\mathrm{St})} : \mathcal{H}^2 \to \mathbf{Z}_k$ can be rewritten as

$$\sum_{T \in \tau_k} |(I - \Pi_k^{(\mathrm{Bih})})u|^2_{H^1(T)} \lesssim 16^{-k} \|u\|^2_{H^3(\Omega)} \qquad (u \in H_0^2(\Omega) \cap H^3(\Omega)), \tag{3.19}$$

$$\sum_{T \in \tau_k} |(\Pi_k^{(\mathrm{Bih})} - I_k^{(\mathrm{Bih})}\Pi_{k-1}^{(\mathrm{Bih})})u|^2_{H^1(T)} \lesssim 16^{-k} \|u\|^2_{H^3(\Omega)} \qquad (u \in H_0^2(\Omega) \cap H^3(\Omega)), \tag{3.20}$$

and

$$\sum_{T \in \tau_k} |I_k^{(\mathrm{Bih})}u|^2_{H^1(T)} \lesssim \sum_{T \in \tau_{k-1}} |u|^2_{H^1(T)} \qquad (u \in M_{k-1}), \tag{3.21}$$

where $I_k^{(\mathrm{Bih})}$, $I_k^{(\mathrm{St})}$ and $\Pi_k^{(\mathrm{Bih})}$, $\Pi_k^{(\mathrm{St})}$ are related according to (3.18) and $\mathbf{curl}_k \Pi_k^{(\mathrm{Bih})} = \Pi_k^{(\mathrm{St})}\mathbf{curl}$ respectively.

Let $I_k^{(\mathrm{Bih})}$ be either $I_k^{(\mathrm{Bih},1)}$ or $I_k^{(\mathrm{Bih},2)}$, and $\Pi_k^{(\mathrm{Bih})}$ be defined by $(\Pi_k^{(\mathrm{Bih})}u)(p) = u(p)$ $(p \in N_k)$ and $\partial_{\mathbf{n}_e}(\Pi_k^{(\mathrm{Bih})}u)(m_e) = \partial_{\mathbf{n}_e}u(m_e)$ $(e \in E_k)$. Then using the local reproduction by $\Pi_k^{(\mathrm{Bih})}$ and $I_k^{(\mathrm{Bih})}$ of second order polynomials, (3.20) and (3.21) follow from the Bramble-Hilbert lemma and a homogeneity argument. The inverse inequalities $|\cdot|_{H^2(T)} \lesssim 2^k |\cdot|_{H^1(T)}$ and $|\cdot|_{H^1(T)} \lesssim 2^k \|\cdot\|_{L^2(T)}$ on $P_2(T)$, $T \in \tau_k$, together with Lemma 3.4 show (3.21). Theorem 2.10 shows that within this Stokes framework assumption (A) with $\alpha = 1$ is valid.

Now we turn to the verification of assumption (B). Applying $\mathbf{curl}_k$ to the canonical basis of M_k that corresponds to the degrees of freedom defining this space, we obtain a basis for $\mathbf{Z}_k$ that we denote by

$$\{\mathbf{v}_{k,e} : e \in E_k\} \cup \{\mathbf{w}_{k,p} : p \in N_k\}. \tag{3.22}$$

One may verify that

$$\mathbf{v}_{k,e} = \eta_{k,e}\mathbf{t}_e,$$

where $\mathbf{t}_e = [(\mathbf{n}_e)_2 \;\; -(\mathbf{n}_e)_1]^T$ is a unit vector tangential to e and $\eta_{k,e}$ is the canonical basis function of $V_k^{(P_1)}$ corresponding to e defined in (3.1), and furthermore that

$$\mathbf{w}_{k,p} = \sum_{i=1}^{\ell} |e_i|^{-1} \eta_{k,e_i} \mathbf{n}_{e_i,p},$$

where $e_1, \dots, e_\ell \in E_k$ are all edges that contain p, and $\mathbf{n}_{e_i,p}$ is the unit vector normal to e_i pointing in the counterclockwise direction with respect to p, see Figure 2.

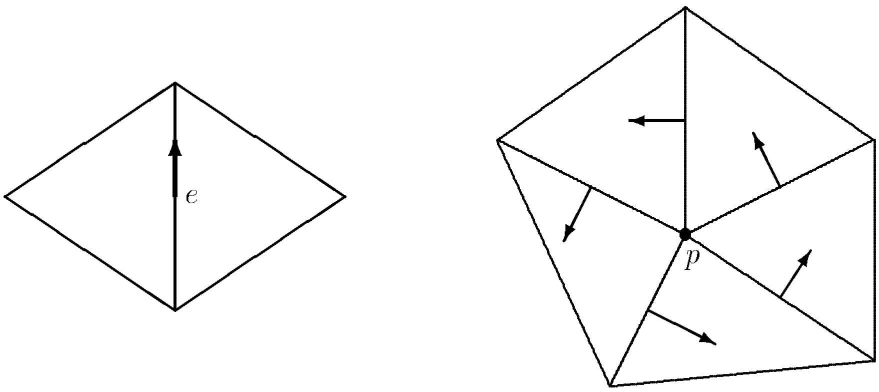

FIGURE 2. Basis functions $\mathbf{v}_{k,e}$ and $\mathbf{w}_{k,p}$ of the space $\mathbf{Z}_k$

Using that $\{\eta_{k,e} : e \in E_k\}$ is an $L^2(\Omega)$-orthogonal basis for $V_k^{(P_1)}$, for vectors $\mathbf{c} = (\mathbf{c}_e)_{e \in E_k}$ and $\mathbf{d} = (\mathbf{d}_p)_{p \in N_k}$, we infer that

$$\|\sum_{e \in E_k} \mathbf{c}_e \mathbf{v}_{k,e} + \sum_{p \in N_k} \mathbf{d}_p \mathbf{w}_{k,p}\|^2_{L^2(\Omega)^2} = \sum_{e \in E_k} |\mathbf{c}_e|^2 \|\eta_{k,e}\|^2_{L^2(\Omega)} + \|\sum_{p \in N_k} \mathbf{d}_p \mathbf{w}_{k,p}\|^2_{L^2(\Omega)^2} \tag{3.23}$$

and

$$\|\sum_{p \in N_k} \mathbf{d}_p \mathbf{w}_{k,p}\|^2_{L^2(\Omega)^2} = \sum_{e \in E_k} |\mathbf{d}_{p_e} - \mathbf{d}_{\tilde{p}_e}|^2 |e|^{-2} \|\eta_{k,e}\|^2_{L^2(\Omega)}, \tag{3.24}$$

where $p_e, \tilde{p}_e \in \overline{N}_k$ denote both vertices on $e \in E_k$, and $\mathbf{d}_p := 0$ when $p \in \overline{N}_k \backslash N_k$. From (3.24) we conclude that the normalized bases (3.22) are *not* uniformly $L^2(\Omega)^2$-stable, and so that general Gauss-Seidel or damped Jacobi smoothers do not necessarily satisfy assumption (B).

Now, let $\mathbf{Z}_k^{(0)} = \operatorname{span}\{\mathbf{w}_{k,p} : p \in N_k\}$, and for $1 \le i \le m$, $\mathbf{Z}_k^{(i)} = \operatorname{span}\{\mathbf{v}_{k,e} : e \in E_k^{(i)}\}$, where $\cup_{i=1}^m E_k^{(i)}$ is some partition of E_k into disjoint subsets. Then by definition of ρ_k and (3.23), we have

$$\sum_{i=0}^{m} a_k^{(\mathrm{St})}(\mathbf{u}^{(i)}, \mathbf{u}^{(i)}) \le \rho_k \sum_{i=0}^{m} \|\mathbf{u}^{(i)}\|^2_{L^2(\Omega)^2} = \rho_k \|\sum_{i=0}^{m} \mathbf{u}^{(i)}\|^2_{L^2(\Omega)^2} \qquad (\mathbf{u}^{(i)} \in \mathbf{Z}_k^{(i)}).$$

That is, if, with respect to the decomposition $\mathbf{Z}_k = \oplus_{i=0}^m \mathbf{Z}_k^{(i)}$, $\mathbf{D}_k$ is the *block* diagonal part of the stiffness matrix $\mathbf{A}_k$ corresponding to (3.22), then condition (a) of Proposition 2.4 is satisfied. So, if in addition for $i > 0$ the $\mathbf{Z}_k^{(i)}$ are selected such that, possibly after reordering, the decomposition $\mathbf{Z}_k = \oplus_{i=0}^m \mathbf{Z}_k^{(i)}$ satisfies (c) of Proposition 2.4 as well, then these resulting block Gauss-Seidel and damped block Jacobi smoothers do satisfy (B).

Since neither of the spaces $\mathbf{Z}_k^{(i)}$ contain smooth vector fields, we infer that

$$a^{(\mathrm{St})}(\mathbf{u}^{(i)}, \mathbf{u}^{(i)}) \eqsim \rho_k \|\mathbf{u}^{(i)}\|^2_{L^2(\Omega)^2} \qquad (\mathbf{u}^{(i)} \in \mathbf{Z}_k^{(i)},\ 0 \le i \le m). \tag{3.25}$$

So, in particular, (3.25) for $i = 0$ combined with (3.24) shows that a further decomposition of $\mathbf{Z}_k^{(0)}$ into subspaces each of them spanned by some uniformly bounded number of $\mathbf{w}_{k,p}$'s will generally *not* give rise to (block) Gauss-Seidel or damped (block) Jacobi smoothers that satisfy (B), because condition (a) of Proposition 2.4 will be violated.

At the same time, (3.25) for $i = 0$ combined with (3.24) shows that 'exact' block Gauss-Seidel or damped block Jacobi smoothers corresponding to $\mathbf{Z}_k = \oplus_{i=0}^m \mathbf{Z}_k^{(i)}$ are not feasible, since the diagonal block of $\mathbf{D}_k$ corresponding to $\mathbf{Z}_k^{(0)}$ cannot be inverted in $\mathcal{O}(\mathrm{dim}\mathbf{Z}_k)$ operations. However, from Propositions 2.4 and 2.7 we learn that in order to satisfy (B), it is sufficient to invert the diagonal blocks approximately, at least when the approximate inverses define iterations that converge uniformly in the corresponding 'energy'-norms.

Considering the diagonal block corresponding to $\mathbf{Z}_k^{(0)}$, one easily verifies that

$$\sum_{e \in E_k} |\mathbf{d}_{p_e} - \mathbf{d}_{\tilde{p}_e}|^2 |e|^{-2} \|\eta_{k,e}\|^2_{L^2(\Omega)} \eqsim \int_\Omega |\nabla d^I|^2 dx, \tag{3.26}$$

where d^I is the function in the *conforming* P_1 finite element space $C(\overline{\Omega}) \cap H_0^1(\Omega) \cap \prod_{T \in \tau_k} P_1(T)$ defined by $d^I(p) = \mathbf{d}_p$ $(p \in N_k)$. Optimal *conforming* multi-grid preconditioners that take only $\mathcal{O}(\# N_k)$ operations are available for the right-hand side of (3.26). So properly scaled, these preconditioners satisfy the assumptions to be used as inexact solvers for the diagonal block of $\mathbf{D}_k$ corresponding to $\mathbf{Z}_k^{(0)}$.

If not already invertible in $\mathcal{O}(\mathrm{dim}\mathbf{Z}_k)$ operations, (3.25) for $i > 0$ and (3.23) show that the other diagonal blocks of $\mathbf{D}_k$ are uniformly well-conditioned, so that also for these blocks suitable approximate inverses are available. From Proposition 2.4 or 2.7, we conclude that the above introduced 'inexact' block Gauss-Seidel or damped block Jacobi smoothers satisfy assumption (B), and that they can be performed in $\mathcal{O}(\mathrm{dim}\mathbf{Z}_k)$ operations.

Since in this Stokes framework (A) with $\alpha = 1$ and, with above smoothers, (B) are valid, compared to the discretized biharmonic formulation from Section 3.3.1, we obtain the following new results: The *mildly* variable V-cycle yields preconditioned systems that have uniformly bounded condition numbers. Furthermore, assuming that $\rho((\tilde{I}_{j\leftarrow k}^{(\mathrm{St},2)})^* \tilde{I}_{j\leftarrow k}^{(\mathrm{St},2)}) \lesssim 1$, the standard V-cycle using the prolongation $I_k^{(\mathrm{St},2)}$ yields a preconditioner that is at least suboptimal. This condition on the 'iterated prolongations alternated with smoothers' likely follows from $\rho((I_{j\leftarrow k}^{(\mathrm{St},2)})^* I_{j\leftarrow k}^{(\mathrm{St},2)}) = \rho((I_{j\leftarrow k}^{(\mathrm{Bih},2)})^* I_{j\leftarrow k}^{(\mathrm{Bih},2)}) \lesssim 1$ (for completeness, here $(\,)^*$ refers to energy-scalar products in the Stokes and biharmonic framework respectively). Numerical evidence for the latter result was found in the model case of Example 3.2.

3.3.3. *Practical algorithms and numerical results.* We apply the multiplicative *standard* V-cycle to the discretized biharmonic problem (3.16), or equivalently, the discretized Stokes problem (3.17), taking $m(k) \equiv m = 1$, i.e., one post-smoothing step and one pre-smoothing step with the 'adjoint' smoother.

We use either the *standard prolongation*

– $I_k^{(\mathrm{Bih},1)}$ ($I_k^{(\mathrm{St},1)}$),

or the *new prolongation*

– $I_k^{(\mathrm{Bih},2)}$ ($I_k^{(\mathrm{St},2)}$).

Equipping M_k or $\mathbf{Z}_k$ with standard basis (3.22), we use 'inexact' block Gauss-Seidel smoothing with respect to the following ordered subdivision of the degrees of freedom:

1. midpoints of $e \in E_k \backslash E_k^{(\mathrm{new})}$,
2. vertices,
3. midpoints of $e \in E_k^{(\mathrm{new})}$,

and with reversed ordering in the post-smoothing step.

Assuming uniform dyadic refinements, note that with respect to this partitioning the first and last diagonal block of the stiffness matrix $\mathbf{A}_k$ are block diagonal matrices with blocks of small, and uniformly bounded sizes. We invert the third diagonal block exactly, and the first one approximately by applying one damped point Jacobi iteration with $\omega = 1$ (cf. Remark 2.9).

We consider two options to approximate the inverse of the second diagonal block. We either apply

– one damped point Jacobi iteration with $\omega = 1$

or

– a properly scaled multi-grid iteration for a discretized Laplacian on the corresponding conforming P_1 finite element space.

As demonstrated in §3.3.1 and 3.3.2, the smoother corresponding to the first option satisfies (B) in the biharmonic framework but not in the Stokes framework, whereas the smoother corresponding to the second option satisfies (B) also in the Stokes framework. The first smoother can be expected to give qualitatively similar results as any arbitrary smoother, and we will refer to this smoother as the *'standard smoother'*. The second smoother will be referred to as the *'new smoother'*.

As noted in Remark 3.1, the combination of either smoother with $I_k^{(\mathrm{Bih},2)}$ can be implemented efficiently by *replacing* both the computation of the normal derivatives of the prolongated function in the midpoints of $e \in E_k \backslash E_k^{(\mathrm{new})}$ and the application of the approximate inverse of the first diagonal block, by the application of the exact inverse of this block.

We have performed numerical tests in the model case of τ_0 being the subdivision of $\Omega = [0,1]^2$ into two triangles, and for $k > 0$, τ_k being generated from τ_{k-1} by uniform dyadic refinement. In this case, the second diagonal block of $\mathbf{A}_k$ is just a multiple of the discretized Laplacian on the corresponding conforming P_1 finite element space. For the new smoother, as 'inner' multi-grid method we applied one standard V-cycle with one block Gauss-Seidel iteration with respect to a 'red-black' ordering of the unknowns as a pre-smoother, and one 'adjoint' iteration as the post-smoother. Note that Propositions 2.4 and 2.7 also allow for 'unsymmetric' inner multi-grid methods, but we did not test this possibility.

For all four combinations of $I_k^{(\mathrm{Bih},1)}$ or $I_k^{(\mathrm{Bih},2)}$, and standard or new smoother, we counted the number of arithmetic operations per degree of freedom necessary to perform one multi-grid iteration, where we let the number of levels tend to infinity. The results given in Table 2 show that implementing together $I_k^{(\mathrm{Bih},2)}$ and the new smoother does not increase the costs compared to $I_k^{(\mathrm{Bih},1)}$ and the standard smoother. Note that, on a regular mesh, the number of vertices is only $\frac{1}{4}$ of the total number of degrees of freedom of a Morley finite element space, which explains that the new smoother needs relatively few additional operations. Although for non-regular meshes the operation counts will be somewhat larger,

TABLE 2. Operation count per degree of freedom

	$I_k^{(\mathrm{Bih},1)}$	$I_k^{(\mathrm{Bih},2)}$
standard smoother	56	45
new smoother	70	57

the ratios will basically be the same.

Numerically computed condition numbers of the stiffness matrix $\mathbf{A}_j$ preconditioned with the standard V-cycle with each of the four smoother-prolongation combinations are presented in Figure 3.3.3. The results show that the condition numbers for the standard method increase almost exponentially with the number of levels. Both the new smoother and the new prolongation result in smaller condition numbers. With both improvements implemented, the condition numbers are 'small' and they even appear to be uniformly bounded.

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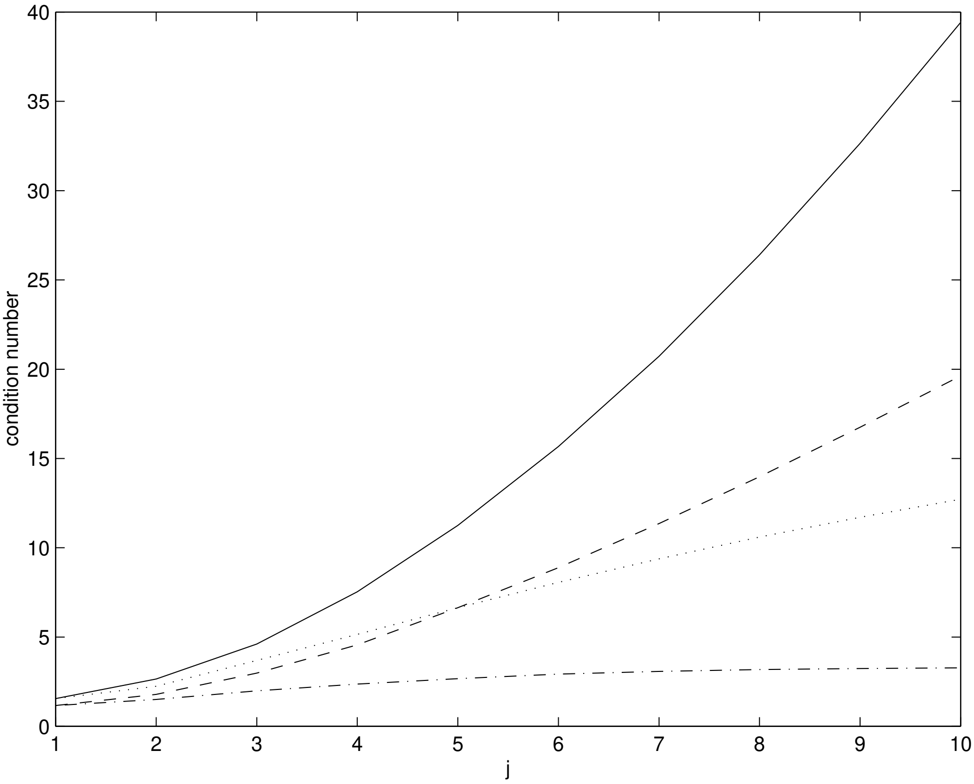

FIGURE 3. Standard smoother, standard prolongation (—); new smoother, standard prolongation (− −); standard smoother, new prolongation (···); new smoother, new prolongation (−·)

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