ELEMENT-BY-ELEMENT CONSTRUCTION OF WAVELETS SATISFYING STABILITY AND MOMENT CONDITIONS

WOLFGANG DAHMEN* AND ROB STEVENSON†

Abstract. In this paper, we construct a class of locally supported wavelet bases for C^0 Lagrange finite element spaces on possibly non-uniform meshes on n-dimensional domains or manifolds. The wavelet bases are stable in the Sobolev spaces $\mathcal{H}^s$ for $|s| < \frac{3}{2}$ ($|s| \leq 1$ on Lipschitz' manifolds), and the wavelets can, in principal, be arranged to have any desired order of vanishing moments. As a consequence, these bases can be used e.g. for constructing an optimal solver of discretized $\mathcal{H}^s$-elliptic problems for s in above ranges.

The construction of the wavelets consists of two parts: An implicit part involves some computations on a reference element which, for each type of finite element space, have to be performed only once. In addition there is an explicit part which takes care of the necessary adaptations of the wavelets to the actual mesh. The only condition we need for this construction to work is that the refinements of initial elements are uniform.

We will show that the wavelet bases can be implemented efficiently.

Key words. Wavelets, stable biorthogonal space decompositions, operator equations, vanishing moments, matrix compression, optimal multi-level preconditioners, non-uniform meshes.

AMS subject classifications. 65N30, 65N55, 65F35, 65R20.

1. Introduction.

This paper is concerned with the construction of *finite element* based wavelet bases with respect to arbitrary initial triangulations. This introductory section is devoted to a summary of relevant background information which, in particular, motivates specific requirements on the wavelet bases concerning stability in Sobolev spaces and moment conditions.

1.1. Motivation and background.

Throughout this paper, $\mathcal{H}^s$ for $s \in \mathbb{R}$ or $|s| \leq t$, will denote a scale of Sobolev spaces, possibly incorporating essential boundary conditions, on an n-dimensional domain or sufficiently smooth manifold. We will denote $\mathcal{H}^0$ also as L_2, and when $s < 0$ the space $\mathcal{H}^s$ is understood to be the *dual* of $\mathcal{H}^{-s}$. Adjoints with respect to the L_2-scalar product will always be denoted by $(\,)^*$.

Consider the variational problem: Given $f \in \mathcal{H}^{-r}$, find $u \in \mathcal{H}^r$ such that

$$a(u, v) = f(v) \qquad (v \in \mathcal{H}^r), \tag{1.1}$$

where a is a scalar product satisfying

$$a(v, v) \eqsim \|v\|^2_{\mathcal{H}^r}, \tag{1.2}$$

i.e., the problem (1.1) is symmetric and *elliptic of order* $2r$. In order to avoid the repeated use of generic but unspecified constants, by $C \lesssim D$ we mean that C can be bounded by a multiple of D, independently of parameters on which C and D may depend. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

As typical model examples for (1.1) we have in mind (variational formulations of)

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[*] Institut für Geometrie und Praktische Mathematik, RWTH Aachen, 52056 Aachen, Germany. (dahmen@igpm.rwth-aachen.de).

[†] Department of Mathematics, Nijmegen University, P.O. Box 9010, NL-6500 GL Nijmegen, The Netherlands. (stevenso@sci.kun.nl).

– the *differential equation* $-\nabla \cdot B\nabla u + cu = f$ on a domain Ω, where $B(x) \eqsim I$ and $0 \leq c(x) \lesssim 1$, supplemented with suitable boundary conditions ($r = 1$),

– reformulations of Laplace's equation on Ω or $I\!R^n \backslash \Omega$ as an *integral equation* on $\partial\Omega$, like the single layer potential equation ($r = -\frac{1}{2}$) or the hypersingular equation ($r = \frac{1}{2}$).

Although for simplicity the advantages of suitable wavelet bases are demonstrated here for the symmetric case, another important application is the (non-symmetric) double layer potential equation that has order zero, and which also arises from a reformulation of Laplace's equation as a boundary integral equation.

Suppose we are given a sequence of nested closed subspaces, also called a *multiresolution analysis*,

$$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \mathcal{S}_j \subset \cdots \subset \mathcal{H}^r.$$

Then the Galerkin discretization of (1.1) reads as follows: Find $u_j \in \mathcal{S}_j$ such that

$$a(u_j, v_j) = f(v_j) \qquad (v_j \in \mathcal{S}_j). \tag{1.3}$$

By fixing a basis $\{\varphi_{j,x} : x \in I_j\}$ of $\mathcal{S}_j$, (1.3) leads to a linear system of equations

$$\mathbf{A}_j \mathbf{u}_j = \mathbf{f}_j. \tag{1.4}$$

The size of the *stiffness matrix* $\mathbf{A}_j = (a(\varphi_{j,y}, \varphi_{j,x}))_{x,y \in I_j}$ in realistic applications often excludes the use of *direct solvers*. In order to solve (1.4) iteratively in an efficient or rather optimal way, the following two questions are relevant:

– Can we select $\{\varphi_{j,x} : x \in I_j\}$ such that $\mathbf{A}_j$ is spectrally equivalent to its diagonal $\mathbf{D}_j$ uniformly in j. It is easily verified that this property is equivalent to the uniform $\mathcal{H}^r$*-stability* of the bases, defined as

$$\| \sum_{x \in I_j} c_{j,x} \varphi_{j,x} \|_{\mathcal{H}^r}^2 \eqsim \sum_{x \in I_j} |c_{j,x}|^2 \|\varphi_{j,x}\|_{\mathcal{H}^r}^2. \tag{1.5}$$

(For convenience, in the sequel we often refer to the property (1.5) by saying that $\{\varphi_{j,x} : x \in I_j\}$ is an $\mathcal{H}^r$-stable basis of $\mathcal{S}_j$, where we thus mean the uniform $\mathcal{H}^r$-stability of the *sequence* of bases $\{\varphi_{j,x} : x \in I_j\}$ for $j = 0, 1, \cdots$).

The following obvious but important observation concerns operator equations in which the operator is a sum of elliptic operators of different orders: If for $i \in \{1, 2\}$, $\mathbf{A}_j^{(i)}$ is uniformly spectrally equivalent to its diagonal $\mathbf{D}_j^{(i)}$, then $\mathbf{A}_j = \mathbf{A}_j^{(1)} + \mathbf{A}_j^{(2)}$ is uniformly spectrally equivalent to its diagonal $\mathbf{D}_j = \mathbf{D}_j^{(1)} + \mathbf{D}_j^{(2)}$. This is relevant for combinations of integral and differential operators but also for the first example above. There it ensures robustness of preconditioners with respect to dominating diffusion or reaction terms.

Secondly, considering integral equations which generally lead to dense matrices:

– Can we select $\{\varphi_{j,x} : x \in I_j\}$ so that it is possible to find a well-conditioned and sparse approximation of $\mathbf{A}_j$, meaning that this approximation has only $\mathcal{O}(\dim \mathcal{S}_j)$ non-zero entries, such that the resulting approximate solution has, as function of j, the same order of convergence as the exact Galerkin solution. This process of finding a sparse, accurate approximation is usually called *matrix compression.*

In a sequence of papers (see [9] and the references cited there), it was shown that both questions concerning stability and compression can be answered affirmatively when suitable wavelet bases are employed. Let us briefly point out what is meant by *suitable wavelet bases* in this context. Suppose that $\mathcal{W}_{k+1} \subset \mathcal{S}_{k+1}$ is chosen such that

$$\mathcal{S}_{k+1} = \mathcal{W}_{k+1} \oplus \mathcal{S}_k$$

and let $\Psi_{k+1} = \{\psi_{k+1,x} : x \in J_{k+1}\}$ be a basis of $\mathcal{W}_{k+1}$. For convenience let us also set $\mathcal{W}_0 = \mathcal{S}_0$ equipped with some basis Ψ_0. This gives rise to the direct sum decomposition

$$\mathcal{S}_j = \oplus_{k=0}^{j} \mathcal{W}_k,$$

which is sometimes called a *multiscale decomposition.* Consequently, $\Psi^j := \cup_{k=0}^{j} \Psi_k$ is a basis of $\mathcal{S}_j$, called *multiscale*-, or *wavelet* basis.

Aiming at computationally optimal implementations, we will always assume the availability of *single-scale* bases $\Phi_k = \{\phi_{k,x} : x \in I_k\}$ of the spaces $\mathcal{S}_k$, which are *locally finite*, i.e., $\sup_{x \in I_k} \#\{y \in I_k : \operatorname{supp}\phi_{k,x} \cap \operatorname{supp}\phi_{k,y} \neq \emptyset\} \lesssim 1$. Viewing bases as (column) vectors whose components are the basis functions, the nesting $\mathcal{S}_k \subset \mathcal{S}_{k+1}$ implies the existence of a refinement relation

$$\Phi_k^T = \Phi_{k+1}^T \mathbf{P}_{k+1,0}. \tag{1.6}$$

Here the xth column of the $\#I_{k+1} \times \#I_k$ matrix $\mathbf{P}_{k+1,0}$ contains the coefficients for the linear combination of $\phi_{k,x}$ in terms of the $\phi_{k+1,y}$. We will assume that Φ_k is *local with respect to* Φ_{k+1}, with which we mean that the rows and columns of $\mathbf{P}_{k+1,0}$ have a uniformly bounded number of non-vanishing entries. This situation is encountered for any standard finite element discretization based on nested partitions. Clearly the matrix $\mathbf{P}_{k+1,0}$ represents a *prolongation operator* in multigrid terminology.

Given the single-scale bases Φ_k, we put $\Psi_0 = \Phi_0$ and search for complement bases Ψ_{k+1} , such that Ψ_{k+1} is local with respect to Φ_{k+1}. In terms of two-scale relations this means that we have to find $\#I_{k+1} \times \#J_{k+1}$ matrices $\mathbf{P}_{k+1,1}$ which are *sparse* in the above sense so that

$$\Psi_{k+1}^T = \Phi_{k+1}^T \mathbf{P}_{k+1,1}. \tag{1.7}$$

In that case $\Psi_{k+1} \cup \Phi_k$ is also local with respect to Φ_{k+1}. The fact that $\Phi_k \cup \Psi_{k+1}$ is also a basis for $\mathcal{S}_k$ is equivalent to saying that the matrix $\mathbf{P}_{k+1} := [\mathbf{P}_{k+1,0}, \mathbf{P}_{k+1,1}]$ is invertible. The matrices $\mathbf{P}_{k+1}, k = 0, \ldots, j-1$ are the core ingredients of the basis transformation $\mathbf{T}_j$ in $\mathcal{S}_j$ that takes the multiscale coefficients of an element in $\mathcal{S}_j$ into its single-scale coefficients. It can be implemented recursively from bottom-to-top as a pyramid scheme which, on account of the sparseness of the $\mathbf{P}_{k+1}$, requires $\mathcal{O}(\dim \mathcal{S}_j)$ operations.

Even when the order $2r$ of the problem (1.1) is negative, we will always assume that $\mathcal{S}_k \subset L_2$. Then the question of $\mathcal{H}^r$ -stability of wavelet bases can be separated into two issues:

Remark 1.1. If

$$\Psi_k \text{ is an } L_2\text{-stable basis of } \mathcal{W}_k. \tag{1.8}$$

and

$$\|\sum_{k=0}^{\infty} v_k\|_{\mathcal{H}^r}^2 \eqsim \sum_{k=0}^{\infty} \lambda_{k,r} \|v_k\|_{L_2}^2 \qquad (v_k \in \mathcal{W}_k) \tag{1.9}$$

for some arbitrary constants $\lambda_{k,r}$, then $\Psi^j = \cup_{k=0}^j \Psi_k$ is an $\mathcal{H}^r$-stable basis of $\mathcal{S}_j$.

Indeed, from (1.9) and (1.8), we infer

$$\begin{aligned}\|\sum_{k,x} c_{k,x}\psi_{k,x}\|_{\mathcal{H}^r}^2 &\eqsim \sum_k \lambda_{k,r}\|\sum_x c_{k,x}\psi_{k,x}\|_{L_2}^2 \eqsim \sum_{k,x}|c_{k,x}|^2\lambda_{k,r}\|\psi_{k,x}\|_{L_2}^2\\ &\eqsim \sum_{k,x}|c_{k,x}|^2\|\psi_{k,x}\|_{\mathcal{H}^r}^2.\end{aligned}$$

To interpret the above norm equivalences for $r < 0$ our assumption $\mathcal{S}_k \subset L_2$ requires, in particular, to decide how to identify an L_2-function with a functional in $\mathcal{H}^r$. It is important to note that this has to be related to the variational formulation (1.1) in order to draw conclusions on preconditioning. In fact, recall that the variational problem (1.1) is derived from some operator equation $Au = f$ by setting $a(u, v) := (Au, v)$ where $(\cdot,\cdot)$ is an L_2-scalar product. In this setting, for $r < 0$ the relation $a(v, v) \eqsim \|v\|_{\mathcal{H}^r}^2$ is established first for smooth functions, which now *determines* the identification of an L_2-function u with the functional $u(v) := (u, v)$ through the L_2-scalar product used in the definition of $a(\cdot,\cdot)$ or through an equivalent one. Here 'equivalent' means that the Riesz map relating two such scalar products not only has to be an isomorphism in L_2 but *also* in $\mathcal{H}^r$. In fact, a subtle point here is that *different* L_2-scalar products give rise to different functionals, which generally have *non-equivalent* $\mathcal{H}^r$-norms, even when these scalar products yield equivalent L_2-norms, see also the comments below in §1.2. In practical applications such as those involving the above examples, the analysis of the variational form is based on the standard L_2-scalar product $(\cdot,\cdot)_{L_2}$, which leads us to the following remark:

Remark 1.2. When $r < 0$, the $\mathcal{H}^r$-norm of an L_2-function will always refer to the canonical embedding of L_2 in $\mathcal{H}^r$ induced by $u(v) = (u, v)_{L_2}$.

The stability (1.8) on each refinement level is, in principal, relatively easy to check. For example, assuming a normalization such that $\|\phi_{k,y}\|_{L_2}$, $\|\psi_{k,x}\|_{L_2} \eqsim 1$ (uniformly in k, $x \in I_k$, $y \in J_k$), the L_2-stability of the basis $\Phi_{k-1} \cup \Psi_k$ is equivalent to the fact that

$$\|\mathbf{P}_k\|,\ \|\mathbf{P}_k^{-1}\| \eqsim 1, \tag{1.10}$$

where $\|\cdot\|$ denotes the spectral norm.

By contrast, it is much less apparent how to assert the stability (1.9) across all levels. In [8], it was proved that (1.9) is valid when $\mathcal{W}_{k+1} = (\mathcal{S}_k^*)^{\perp_{L_2}} \cap \mathcal{S}_{k+1}$, where

$$\mathcal{S}_0^* \subset \mathcal{S}_1^* \subset \cdots \mathcal{S}_j^* \subset \cdots \subset L_2,$$

is a "dual" sequence of nested closed subspaces, such that both $(\mathcal{S}_k)_k$ as $(\mathcal{S}_k^*)_k$ satisfy certain *direct* (or Jackson) and *inverse* (or Bernstein) estimates with respect to suitable Sobolev norms. Here (1.9) for $r < 0$ should be understood with respect to an identification of L_2-functions as functionals using the same L_2-scalar product that is applied in the orthogonality construction, which in this paper will therefore be the canonical L_2-scalar product. A precise formulation of above result together with a new proof can be found in §2 and the appendix respectively. In this setting, we also have $\mathcal{S}_{k+1}^* = \mathcal{W}_{k+1}^* \oplus \mathcal{S}_k^*$, where $\mathcal{W}_{k+1}^* = (\mathcal{S}_k)^{\perp_{L_2}} \cap \mathcal{S}_{k+1}^*$. Since therefore

$$\mathcal{S}_k \perp_{L_2} \mathcal{W}_{k+1}^*, \quad \mathcal{S}_k^* \perp_{L_2} \mathcal{W}_{k+1}, \tag{1.11}$$

and so $\mathcal{W}_\ell^* \perp_{L_2} \mathcal{W}_k$ ($\ell \neq k$), the space decompositions $\oplus_k \mathcal{W}_k$ and $\oplus_k \mathcal{W}_k^*$ are called *biorthogonal space decompositions.*

Now we turn to the question of matrix compression. Using the Bramble-Hilbert lemma, aforementioned direct estimates are usually enforced by demanding that, relative to their underlying meshes, $\mathcal{S}_k$ and $\mathcal{S}_k^*$ contain all piecewise polynomials, possibly satisfying some global smoothness conditions, of sufficiently high degree, say degree $d-1$, and d^*-1, respectively. As a consequence, then the wavelets are L_2-orthogonal to all polynomials of degree less than d^*, or the wavelets are said to have *vanishing moments* of order d^*. As first observed in [1], in case of an integral operator it is this property that ensures that the stiffness matrix with respect to the wavelet basis is close to a sparse one.

For this biorthogonal setting, at the end of a sequence of papers ([10, 12, 25]), it was proved in [12, 20] that if

$$d^* > d - 2r, \tag{1.12}$$

and if the Schwarz kernel of the integral operator is smooth off the diagonal and exhibits a certain asymptotic behavior under differentiation, then the stiffness matrix can be compressed to a sparse one by dropping small elements in an à priori way, i.e., without computing these elements, in such a way that the order of convergence is not reduced. Note that for $r \leq 0$, condition (1.12) rules out the orthogonal space decompositions, i.e., $\mathcal{S}_k^* = \mathcal{S}_k$ or $\mathcal{W}_{k+1} = \mathcal{S}_{k+1} \ominus^{\perp_{L_2}} \mathcal{S}_k$.

1.2. Construction of suitable wavelet bases. So far, for $d' > d$, which is needed for integral operators of non-positive order, wavelet bases satisfying all desirable conditions have been constructed primarily only for spaces $\mathcal{S}_j$ that are spanned by cardinal B-splines on uniform partitions of $I\!R$ in [7], and of $[0,1]$ in [11]. Clearly, by taking tensor products, these constructions can be extended to $I\!R^n$ or $[0,1]^n$.

Using these wavelet bases as building blocks, in [13] and [4], wavelet bases were constructed on manifolds that can be represented as disjoint unions of smooth parametric images of the unit cube. To put the present alternative approach into a proper perspective with regard to these constructions a few comments are in order. In [13, 4], the underlying space decompositions are *biorthogonal* with respect to some *modified* L_2-scalar product $(v,u)_{L_2,w} = (wv,u)_{L_2}$, where both w and $1/w$ are bounded and *piecewise smooth* functions on the underlying domain or manifold, but which generally have *jumps* across the interfaces of adjacent patches. These weight functions and their discontinuous character result from the underlying isoparametric concept. The norm equivalences derived in [13, 4] therefore refer for $r < 0$ to the non-canonical embedding of L_2 in $\mathcal{H}^r$ based on the identification of $u \in L_2$ as the functional $u(v) = (u,v)_{L_2,\omega}$. In fact, the $\mathcal{H}^r$-norm of such a functional is equal to $\|wu\|_{\mathcal{H}^r}$ in the present understanding of $u(v) = (u,v)_{L_2}$ required by the variational formulation (1.1) as explained above. However, for $r \leq -\frac{1}{2}$, the norms $\|w \cdot \|_{\mathcal{H}^r}$ and $\| \cdot \|_{\mathcal{H}^r}$ are *not* equivalent, since that would imply that $\|wv\|_{\mathcal{H}^{-r}} \eqsim \|v\|_{\mathcal{H}^{-r}}$ $(v \in \mathcal{H}^{-r})$, which is not true for $r \leq -\frac{1}{2}$ when w has jump discontinuities. We conclude that the bases from [13, 4] are not optimal for preconditioning purposes when $r \leq -\frac{1}{2}$.

One may argue that the problems for $r \leq -\frac{1}{2}$ can be circumvented by switching from (1.1), for instance, to the equivalent variational problem $\tilde{a}(\tilde{u},\tilde{v}) = \tilde{f}(\tilde{v})$ $(\tilde{v} \in \mathcal{H}^r)$, where $\tilde{a}(\tilde{u},\tilde{v}) := a(w\tilde{u}, w\tilde{v})$ and $\tilde{f}(\tilde{v}) := f(w\tilde{v})$. Indeed, based on the identification of L_2-functions as functionals using $(\cdot,\cdot)_{L_2,\omega}$, one has $\tilde{a}(\tilde{v},\tilde{v}) \eqsim \|\tilde{v}\|^2_{\mathcal{H}^r}$, i.e., the new formulation is elliptic of order $2r$ using the alternative identification. However, the solution $\tilde{u} = u/w$ of the reformulated problem can be expected to be less smooth than u. Thus a Galerkin discretization based on spaces $\mathcal{S}_j$ containing only continuous functions are expected to produce reduced convergence orders.

Recently, in [14], see also [9], the difficulties for $r \leq -\frac{1}{2}$ were also overcome yielding in principle optimal results for any r. Yet, this construction involves certain extension operators which have to be carefully chosen depending on the problem at hand. Moreover, common to all three mentioned constructions is that they are based solely on smooth parametrizations of the unit cube, which may have several disadvantages from a practical point of view.

As an alternative approach, for C^0 Lagrange finite element spaces of order $d \geq 2$ based on subdivisions into n-simplices ("triangulations") of domains in n dimensional Euclidean space, in this paper we introduce a construction of wavelet bases that meets all aforementioned requirements in the following sense: The wavelet bases are $\mathcal{H}^r$-stable for $|r| < \frac{3}{2}$, the wavelets are local with respect to the nodal bases, and they have, in principle, any order $d' \geq 2$ of vanishing moments. The underlying space decompositions are biorthogonal with respect to the canonical L_2-scalar product, which means that we are in the scope of Remark 1.2. The construction is applicable to arbitrary initial meshes and for arbitrary boundary conditions. The only condition we need is that of *uniform* dyadic *refinements.* Apart from this condition, the appreciated flexibility of finite elements is fully retained.

The whole construction carries directly over to finite element type spaces on certain Lipschitz manifolds. More precisely, those manifolds are covered that consist of patches, each of them the parametric image of a domain with triangulations as above, such that the images of the triangulations match at the interfaces, and on each domain the Jacobian determinant is piecewise constant with respect to the initial triangulation. This means that we can handle manifolds consisting of patches that are for example parts of hyperplanes, spheres or cylinders. The construction yields $\mathcal{H}^r$-stable wavelet bases for the in this case *maximal* range $r \in [-1, 1]$. Since we again realize biorthogonality with respect to the canonical L_2-scalar product, the stability results have the favourable interpretation in the sense of Remark 1.2.

When dealing with boundary integral equations on such manifolds the cancellation properties are important. Since some of the resulting wavelets may have support which intersects more than a single patch one can no longer resort to standard polynomial moment conditions. Nevertheless, employing a different strategy, we are still able to confirm certain cancellation properties needed to establish optimal decay estimates for matrix compression.

Our construction requires solving once and for all some system on a reference element, which depends on d, d' and n. Having solved this system, the necessary element-by-element adaptations of the wavelets to the mesh on the domain or manifold, and possibly to boundary conditions are given *explicitly.* We will perform the computations on the reference element for $d = 2$ (piecewise linears) in combination with $d' = 2$ ($n = 1$, 2 or 3) or $d' = 3$ and $d' = 5$ ($n = 1$ or 2). Note that $d' = 2$, 3 and 5 suffices for the hypersingular equation, the double layer potential equation and the single layer potential equation, respectively. The results for $d = d' = 2$ were published earlier in [22].

Compared with existing approaches, our construction seems easy and, as we will show, it can be implemented efficiently. The clue of our approach is that we drop one condition that is usually imposed, viz. the existence of a well localized wavelet basis associated with the spaces $\mathcal{S}_k^*$ which is biorthogonal to the wavelet basis associated with the spaces $\mathcal{S}_k$. We will refer to these bases as dual respectively primal bases. Whereas for other applications of wavelets, such as signal analysis, the availability of both the primal and the dual basis is essential and enters the computations, for our present goals viz. stability and matrix compression, there is no need for explicitly knowing in addition to the primal basis also the dual one. It appears that without this requirement the possibilities of constructing efficient "flexible" wavelet bases are increased dramatically.

Although our wavelet construction can be implemented efficiently, simple second order problems of Poisson type should not be viewed as typical target problems. In fact, earlier comparisons from [18] indicate that (at least for uniform discretizations) schemes like the BPX-preconditioner or multiplicative multigrid schemes are more efficient if perhaps somewhat less robust with respect to zero order terms. We see a much more promising potential in connection with boundary integral operators or combinations of integral- and differential operators, in particular, when norms of negative order are involved. For these problems, BPX-type preconditioners are not useful, even when the essential compression question is not taken into account. An impression of the huge benefits in practical computations of the wavelet approach for such problems can be obtained from [16]. Compared with the *discontinuous* wavelets based on *orthogonal* decompositions ($d = d'$) applied in [16], and that were introduced in [24], the wavelet bases constructed here have a larger range of stability ($|s| < \frac{3}{2}$ vs. $|s| < \frac{1}{2}$) in the above sense which now covers also the stability range required by the single layer potential operator. Moreover, due to the more flexible concept of *biorthogonality*, d' can be chosen to be larger than d, which is essential for asymptotically optimal matrix compression in connection with integral equations of negative order. The second very promising application context is the design of highly adaptive methods for partial differential as well as integral equations suggested by the results in [6]. The present construction meets all the requirements needed there to ensure *optimal convergence rates.* In this sense we view the present contribution as one essential ingredient (among possibly others) in a more complex application context rather than a competitive selfcontained method when only single aspects like preconditioning are focussed upon.

The remainder of this paper is organized as follows: In §2 we formulate a modified version of a crucial theorem concerning stability of biorthogonal space decompositions, that was first proved in a somewhat more general Hilbert space setting in [8]. Its proof given in the appendix for the slight specialization is shorter and, as we think, better accessible.

The construction of stable bases of the subspaces generating a biorthogonal space decomposition, i.e., the construction of the wavelets, is treated in §3.

In §4, the approach from §3 is specialized to C^0 Lagrange finite element spaces. We will perform the computations on the reference element for the cases mentioned above. Special attention will be paid to showing that the wavelet construction carries over to compact manifolds, and that resulting wavelets have the right *cancellation properties* essential for matrix compression. We give a quantitative analysis of the costs of performing the wavelet transform.

2. Biorthogonal space decompositions.

In this section, we formulate sufficient conditions for existence and stability of biorthogonal space decompositions. The essential distinction of the following Theorem 2.1 from related results in [8] is the practicality of the hypotheses in the present context. While the assumptions in [8] refer to special projectors which in practice already require some a-priori knowledge about suitable biorthogonal bases the present assumptions concern only multiresolution spaces that to some extent can be chosen.

Theorem 2.1 (cf. [8]). *Let $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots$ and $\mathcal{S}_0^* \subset \mathcal{S}_1^* \subset \mathcal{S}_2^* \subset \cdots$ be sequences of nested closed subspaces of L_2, and let $\rho > 1$ be some constant.*

(a). Suppose that

$$\text{(C1)} \qquad \inf_{0 \neq u_k \in \mathcal{S}_k} \sup_{0 \neq u_k^* \in \mathcal{S}_k^*} \frac{|(u_k, u_k^*)_{L_2}|}{\|u_k\|_{L_2} \|u_k^*\|_{L_2}} \gtrsim 1,$$

as well as the analogous condition $(\text{C1})^*$ *with interchanged roles of $(\mathcal{S}_k)_k$ and $(\mathcal{S}_k^*)_k$ hold.*
Then, there exists a sequence $(\mathcal{Q}_k)_k$ of uniformly bounded projectors $\mathcal{Q}_k : L_2 \to L_2$, with $\Im(\mathcal{Q}_k) = \mathcal{S}_k$, $\Im(I - \mathcal{Q}_k) = (\mathcal{S}_k^*)^{\perp_{L_2}}$, and likewise for the adjoint projectors, $\Im(\mathcal{Q}_k^*) = \mathcal{S}_k^*$, $\Im(I - \mathcal{Q}_k^*) = (\mathcal{S}_k)^{\perp_{L_2}}$. One has $\mathcal{Q}_k \mathcal{Q}_{k+1} = \mathcal{Q}_k$ and $\mathcal{Q}_k^* \mathcal{Q}_{k+1}^* = \mathcal{Q}_k^*$.

(b). In addition, assume that there exist $0 < \gamma < d$ such that

$$\text{(C2)} \qquad \inf_{u_k \in \mathcal{S}_k} \|u - u_k\|_{L_2} \lesssim \rho^{-sk} \|u\|_{\mathcal{H}^s} \quad (u \in \mathcal{H}^s,\ 0 \leq s \leq d)$$

(direct or Jackson estimate), and

$$\text{(C3)} \quad \|u_k\|_{\mathcal{H}^s} \lesssim \rho^{sk} \|u_k\|_{L_2} \quad (u_k \in \mathcal{S}_k,\ 0 \leq s < \gamma) \quad (\textit{inverse or Bernstein estimate}),$$

and that analogous assumptions $(\text{C2})^*$ *and* $(\text{C3})^*$ *with constants $0 < \gamma^* < d^*$ hold for $(\mathcal{S}_k^*)_k$.*

Then, with $\mathcal{Q}_{-1} := 0$, one has

$$\text{(2.1)} \qquad \Big\| \sum_{k=0}^{\infty} v_k \Big\|_{\mathcal{H}^s}^2 \lesssim \sum_{k=0}^{\infty} \rho^{2sk} \|v_k\|_{L_2}^2 \qquad (v_k \in \Im(\mathcal{Q}_k - \mathcal{Q}_{k-1}),\ s \in (-d^*, \gamma))$$

and

$$\text{(2.2)} \qquad \sum_{k=0}^{\infty} \rho^{2sk} \|(\mathcal{Q}_k - \mathcal{Q}_{k-1}) u\|_{L_2}^2 \lesssim \|u\|_{\mathcal{H}^s}^2 \qquad (u \in \mathcal{H}^s,\ s \in (-\gamma^*, d)).$$

For $s \in (-\gamma^*, \gamma)$, $u \mapsto ((\mathcal{Q}_k - \mathcal{Q}_{k-1})u)_k$ is a bounded mapping from $\mathcal{H}^s$ onto

$$\ell_{2,s}(\mathcal{Q}) := \{(v_k)_k : v_k \in \Im(\mathcal{Q}_k - \mathcal{Q}_{k-1}),\ \|(v_k)_k\|_{\ell_{2,s}(\mathcal{Q})} := (\sum_{k=0}^{\infty} \rho^{2sk} \|v_k\|_{L_2}^2)^{\frac{1}{2}} < \infty\},$$

with bounded inverse $(v_k)_k \mapsto \sum_{k=0}^{\infty} v_k$. That is, for $s \in (-\gamma^*, \gamma)$, the symbols "$\lesssim$" in (2.1) *and* (2.2) *can be replaced by "$\eqsim$" symbols.*

Analogous results $(2.1)^*$, $(2.2)^*$ *are valid with $(\mathcal{Q}_k)_k$ replaced by $(\mathcal{Q}_k^*)_k$ and with interchanged roles of (γ, d) and (γ^*, d^*).*

Remark 2.2. To relate the results of this theorem to the concepts discussed in §1, let us denote by $\mathcal{W}_k := \Im(\mathcal{Q}_k - \mathcal{Q}_{k-1})$ the range of the difference operators $\mathcal{Q}_k - \mathcal{Q}_{k-1}$. Then $\mathcal{W}_0 = \mathcal{S}_0$, and by $\mathcal{Q}_k \mathcal{Q}_{k+1} = \mathcal{Q}_k$, one easily confirms that $\mathcal{S}_{k+1} = \mathcal{W}_{k+1} \oplus \mathcal{S}_k$ and $\mathcal{W}_{k+1} = (\mathcal{S}_k^*)^{\perp_{L_2}} \cap \mathcal{S}_{k+1}$. Under its conditions, Theorem 2.1 shows that (1.9) holds for $r \in (-\gamma^*, \gamma)$, where $\lambda_{k,r} = \rho^{2kr}$. Analogous results hold for $\mathcal{W}_k^* := \Im(\mathcal{Q}_k^* - \mathcal{Q}_{k-1}^*)$, i.e., (1.11) is valid here.

Partly based on arguments from [20], in the appendix we give a new, relatively short proof of Theorem 2.1. Here we briefly comment only on the hypotheses.

As spaces $\mathcal{S}_k$ and $\mathcal{S}_k^*$, we have finite element spaces in mind. In that context the validity of direct and inverse estimates are standard facts (cf. e.g. [19]).

When the functions from the spaces $\mathcal{S}_k$ and $\mathcal{S}_k^*$ are piecewise smooth and globally t respectively t^* times continuously differentiable, under the usual assumptions for quasi-uniform meshes with mesh sizes $\sim \rho^{-k}$ on level k, the inverse estimates (C2) and (C2)* are valid with $\gamma = t + \frac{3}{2}$ and $\gamma^* = t^* + \frac{3}{2}$, respectively ($t\,(t^*) := -1$ in case of no continuity between elements).

The direct estimates (C3) and (C3)* are usually enforced by demanding that, relative to their meshes, $\mathcal{S}_k$ and $\mathcal{S}_k^*$ contain all t or t^* times continuously differentiable piecewise polynomials of degree $d-1$ and d^*-1 respectively. In that case, since by (1.11) $\mathcal{W}_{k+1} \subset (\mathcal{S}_k^*)^{\perp_{L_2}}$, basis functions of $\mathcal{W}_{k+1}$, i.e., the wavelets, will have d^* vanishing moments. As pointed out in §1, the property of having sufficiently many vanishing moments is essential for the use of the wavelet basis for optimal compression (see [12, 20]).

Finally, (C1) or (C1)* are conditions concerning invertibility of certain linear operators that are usually encountered in connection with saddle point problems. Their validity will be checked using the following criterion.

LEMMA 2.3. *Let $\Xi_k = \{\xi_{k,x} : x \in I_k\}$ and $\Xi_k^* = \{\xi_{k,x}^* : x \in I_k\}$ be L_2-stable bases (cf.* (1.5)*) of $\mathcal{S}_k$ and $\mathcal{S}_k^*$ respectively. (Note that* (C1) *and* (C1)* *can be valid simultaneously only if the bases for $\mathcal{S}_k$ and $\mathcal{S}_k^*$ have the same cardinality, so it is no restriction to use the same index set I_k for both bases.) Define the possibly infinite matrix*

$$\mathbf{M}_k = \left(\frac{(\xi_{k,y}, \xi_{k,x}^*)_{L_2}}{\|\xi_{k,y}\|_{L_2} \|\xi_{k,x}^*\|_{L_2}} \right)_{x \in I_k, y \in I_k},$$

Then (C1) *is equivalent to $\|\mathbf{M}_k \mathbf{c}_k\|_{\ell_2(I_k)} \gtrsim \|\mathbf{c}_k\|_{\ell_2(I_k)}$, where $\mathbf{c}_k = (c_{k,x})_{x \in I_k}$ and $\|\mathbf{c}_k\|_{\ell_2(I_k)} = (\sum_{x \in I_k} |c_{k,x}|^2)^{\frac{1}{2}}$, and (C1)* is equivalent to $\|\mathbf{M}_k^T \mathbf{c}_k\|_{\ell_2(I_k)} \gtrsim \|\mathbf{c}_k\|_{\ell_2(I_k)}$.*

Proof. Write $u_k = \sum_{x \in I_k} c_{k,x} \frac{\xi_{k,x}}{\|\xi_{k,x}\|_{L_2}} \in \mathcal{S}_k$ and $u_k^* = \sum_{x \in I_k} d_{k,x} \frac{\xi_{k,x}^*}{\|\xi_{k,x}^*\|_{L_2}} \in \mathcal{S}_k^*$. The stability of Ξ_k^* and Ξ_k shows that

$$\sup_{0 \neq u_k^* \in \mathcal{S}_k^*} \frac{|(u_k, u_k^*)_{L_2}|}{\|u_k^*\|_{L_2}} \eqsim \sup_{0 \neq \mathbf{d}_k \in \ell_2(I_k)} \frac{|(\mathbf{M}_k \mathbf{c}_k, \mathbf{d}_k)_{\ell_2(I_k)}|}{\|\mathbf{d}_k\|_{\ell_2(I_k)}} = \|\mathbf{M}_k \mathbf{c}_k\|_{\ell_2(I_k)}$$

and $\|u_k\|_{L_2} \eqsim \|\mathbf{c}_k\|_{\ell_2(I_k)}$ respectively. From these results the proof follows immediately. □

3. A general construction of wavelets. Let $(\mathcal{S}_k)_k$, $(\mathcal{S}_k^*)_k$ be nested sequences of closed subspaces as in Theorem 2.1. In this section, we present a way of constructing L_2-stable bases of the spaces $\mathcal{W}_{k+1} = (\mathcal{S}_k^*)^{\perp_{L_2}} \cap \mathcal{S}_{k+1}$. Having these stable bases at hand, and assuming an L_2-stable basis of $\mathcal{W}_0 = \mathcal{S}_0$ as well, Theorem 2.1 implies that for $s \in (-\gamma^*, \gamma)$, the union of all basis functions, called *wavelets*, of all spaces $\mathcal{W}_k$ is a stable basis of $\mathcal{H}^s$. In particular, in this case the basis of the space $\mathcal{S}_j$ defined as the union of the bases of $\mathcal{W}_0, \ldots, \mathcal{W}_j$ is (uniformly) $\mathcal{H}^s$-stable (cf. Remark 1.1).

THEOREM 3.1. *Let $\Phi_k^* = \{\phi_{k,y}^* : y \in I_k\} \subset \mathcal{S}_k^*$, $\Theta_{k+1} = \{\theta_{k+1,y} : y \in I_k\} \subset \mathcal{S}_{k+1}$ and $\Upsilon_{k+1} = \{\upsilon_{k+1,x} : x \in J_{k+1}\} \subset \mathcal{S}_{k+1}$ have the following properties:*

(a). Φ_k^* is an L_2-stable basis of $\mathcal{S}_k^*$,

(b). $\Upsilon_{k+1} \cup \Theta_{k+1}$ is an L_2-stable basis of $\mathcal{S}_{k+1}$,

(c). Θ_{k+1} is a biorthogonal basis *of Φ_k^*, i.e.*

$$(\theta_{k+1,y}, \phi_{k,z}^*)_{L_2} \eqsim \delta_{y,z} \|\theta_{k+1,y}\|_{L_2} \|\phi_{k,z}^*\|_{L_2} \qquad (y,\, z \in I_k).$$

Then, defining

$$\psi_{k+1,x} := \upsilon_{k+1,x} - \sum_{y\in I_k} \frac{(\upsilon_{k+1,x}, \phi^*_{k,y})_{L_2}}{(\theta_{k+1,y}, \phi^*_{k,y})_{L_2}} \theta_{k+1,y}, \tag{3.1}$$

the set $\Psi_{k+1} := \{\psi_{k+1,x} : x \in J_{k+1}\}$ *is an* L_2*-stable basis of* $\mathcal{W}_{k+1} = (\mathcal{S}^*_k)^{\perp_{L_2}} \cap \mathcal{S}_{k+1}$ *and* $\Psi_{k+1} \cup \Theta_{k+1}$ *is an* L_2*-stable basis of* $\mathcal{S}_{k+1}$*.*

Proof. Let us first show that $\Psi_{k+1} \cup \Theta_{k+1}$ is an L_2-stable basis of $\mathcal{S}_{k+1}$. To this end, consider the (possibly infinite) matrix

$$\mathbf{B}_k = \left(\frac{(\upsilon_{k+1,x}, \phi^*_{k,y})_{L_2} \|\theta_{k+1,y}\|_{L_2}}{(\theta_{k+1,y}, \phi^*_{k,y})_{L_2} \|\upsilon_{k+1,x}\|_{L_2}} \right)_{y\in I_k, x\in J_{k+1}}.$$

For any $\mathbf{d}_{k+1} = (d_{k+1,x})_{x\in J_{k+1}}$ and $\mathbf{c}_k = (c_{k,y})_{y\in I_k}$ the L_2-stability of Υ_{k+1} and Φ^*_k and condition (c) provide

$$\begin{aligned}
(\mathbf{B}_k \mathbf{d}_{k+1}, \mathbf{c}_k)_{\ell_2(I_k)} &= \Big(\sum_{x\in J_{k+1}} d_{k+1,x} \tfrac{\upsilon_{k+1,x}}{\|\upsilon_{k+1,x}\|_{L_2}}, \sum_{y\in I_k} c_{k,y} \tfrac{\|\theta_{k+1,y}\|_{L_2} \|\phi^*_{k,y}\|_{L_2}}{(\theta_{k+1,y}, \phi^*_{k,y})_{L_2}} \tfrac{\phi^*_{k,y}}{\|\phi^*_{k,y}\|_{L_2}} \Big)_{L_2} \\
&\lesssim \Big(\sum_{x\in J_{k+1}} |d_{k+1,x}|^2 \Big)^{\frac12} \Big(\sum_{y\in I_k} |c_{k,y}|^2 \tfrac{\|\theta_{k+1,y}\|^2_{L_2} \|\phi^*_{k,y}\|^2_{L_2}}{|(\theta_{k+1,y}, \phi^*_{k,y})_{L_2}|^2} \Big)^{\frac12} \\
&\lesssim \|\mathbf{d}_{k+1}\|_{\ell_2(J_{k+1})} \|\mathbf{c}_k\|_{\ell_2(I_k)},
\end{aligned}$$

which means that

$$\|\mathbf{B}_k\|_{\ell_2(J_{k+1}) \to \ell_2(I_k)} \lesssim 1. \tag{3.2}$$

The definition of $\psi_{k+1,x}$ and the L_2-stability of $\Upsilon_{k+1} \cup \Theta_{k+1}$ imply that

$$\frac{\|\psi_{k+1,x}\|^2_{L_2}}{\|\upsilon_{k+1,x}\|^2_{L_2}} \eqsim 1 + \sum_{y\in I_k} |(\mathbf{B}_k)_{y,x}|^2,$$

and so, by the boundedness of $\mathbf{B}_k$, we have $\|\psi_{k+1,x}\|_{L_2} \eqsim \|\upsilon_{k+1,x}\|_{L_2}$.

Given any $\mathbf{c}_{k+1} = (c_{k+1,x})_{x\in J_{k+1}}$, $\mathbf{d}_k = (d_{k,y})_{y\in I_k}$, the expansion

$$\sum_{x\in J_{k+1}} c_{k+1,x} \tfrac{\psi_{k+1,x}}{\|\psi_{k+1,x}\|_{L_2}} + \sum_{y\in I_k} d_{k,y} \tfrac{\theta_{k+1,y}}{\|\theta_{k+1,y}\|_{L_2}}$$

can be rewritten as

$$\sum_{x\in J_{k+1}} e_{k+1,x} \tfrac{\upsilon_{k+1,x}}{\|\upsilon_{k+1,x}\|_{L_2}} + \sum_{y\in I_k} f_{k,y} \tfrac{\theta_{k+1,y}}{\|\theta_{k+1,y}\|_{L_2}},$$

where with $\mathbf{\Delta}_{k+1} = \operatorname{diag}\left(\tfrac{\|\upsilon_{k+1,x}\|_{L_2}}{\|\psi_{k+1,x}\|_{L_2}}\right)_{x\in J_{k+1}}$ and $\mathbf{L}_{k+1} = \begin{bmatrix} \mathbf{\Delta}_{k+1} & \mathbf{0} \\ -\mathbf{B}_k \mathbf{\Delta}_{k+1} & \mathbf{I} \end{bmatrix}$, the arrays $\mathbf{e}_{k+1} = (e_{k+1,x})_{x\in J_{k+1}}$ and $\mathbf{f}_k = (f_{k,y})_{y\in I_k}$ are given by $[\mathbf{e}_{k+1}\ \ \mathbf{f}_k]^T = \mathbf{L}_{k+1}[\mathbf{c}_{k+1}\ \ \mathbf{d}_k]^T$. From $\|\psi_{k+1,x}\|_{L_2} \eqsim \|\upsilon_{k+1,x}\|_{L_2}$ and (3.2), we infer that both $\mathbf{L}_{k+1}$ and $\mathbf{L}^{-1}_{k+1}$ are uniformly bounded mappings from $\ell_2(I_k \cup J_{k+1}) \to \ell_2(I_k \cup J_{k+1})$. Since, by assumption, $\Upsilon_{k+1} \cup \Theta_{k+1}$ is an L_2-stable basis of $\mathcal{S}_{k+1}$, the boundedness of both $\mathbf{L}_{k+1}$ and $\mathbf{L}^{-1}_{k+1}$ is equivalent to the property that $\Psi_{k+1} \cup \Theta_{k+1}$ is an L_2-stable basis of $\mathcal{S}_{k+1}$.

Finally, suppose that $\upsilon_{k+1} = \sum_{x \in J_{k+1}} c_{k+1,x}\psi_{k+1,x} + \sum_{y \in I_k} d_{k,y}\theta_{k+1,y}$ belongs to $\mathcal{W}_{k+1} = (\mathcal{S}_k^*)^{\perp_{L_2}} \cap \mathcal{S}_{k+1}$. One easily verifies that $\psi_{k+1,x} \in (\mathcal{S}_k^*)^{\perp_{L_2}}$ $(x \in J_{k+1})$. Now by taking L_2-scalar products with basis functions $\phi_{k,z}^*$, we see that $d_{k,z} = 0$ for all $z \in I_k$. Therefore Ψ_{k+1} spans $\mathcal{W}_{k+1}$ and hence is an L_2-stable basis of $\mathcal{W}_{k+1}$, which shows the remaining part of the claim. $\square$

Remark 3.2. As was already noted in §1, in order to get an optimal implementation, given some family of (locally finite, L_2-stable) "single-scale" bases Φ_k, so that Φ_k is local with respect to Φ_{k+1}, we need Ψ_{k+1} that is local with respect to Φ_{k+1}. Generally in the context of Theorem 3.1, this necessarily means that both Υ_{k+1} and Θ_{k+1} are local with respect to Φ_{k+1}, and that the matrix

$$\big((\phi_{k+1,x}, \phi_{k,y}^*)_{L_2}\big)_{x \in I_{k+1}, y \in I_k} \text{ is sparse.} \tag{3.3}$$

Remark 3.3. The construction presented in Theorem 3.1 is closely related to the concept of so-called *stable completions* proposed in [5]. Differences are that in [5], it is assumed that

- $\Theta_{k+1} = \Phi_k$, i.e. (Φ_k, Φ_k^*) is assumed to be a so-called *biorthogonal pair* of L_2-stable bases of $\mathcal{S}_k$ and $\mathcal{S}_k^*$ respectively, and
- $(\phi_{k,y}, \phi_{k,y}^*)_{L_2} = 1$ $(y \in I_k)$.

Under these two assumptions, the wavelets defined in (3.1) are exactly those yielded by [5, Theorem 3.3]. In fact, since $\mathcal{W}_{k+1} = (\mathcal{S}_k^*)^{\perp_{L_2}} \cap \mathcal{S}_{k+1}$ does not depend on $\mathcal{S}_k$, we may even replace in [5, Theorem 3.3] $\mathcal{S}_k$ by $\operatorname{span}\Theta_{k+1}$, and in particular, Φ_k by Θ_{k+1}, so that, upon assuming $(\phi_{k,y}, \phi_{k,y}^*)_{L_2} = 1$, this theorem also yields the wavelets from our Theorem 3.1.

Furthermore, in [5] it is assumed that the basis functions are scaled such that $\|\phi_{k,y}\|_{L_2}$, $\|\phi_{k,y}^*\|_{L_2}$, $\|\upsilon_{k+1,x}\| \eqsim 1$. The fact that we avoid assuming some L_2 normalization of basis functions, and correspondingly, that our definition of (1.5) of stability of a basis is independent of the scaling of the basis functions involved, is of minor importance from a mathematical point of view. However, as was noticed in [21], in practical computations, in particular cases of non-uniform meshes, the absence of scalings permits a more efficient implementation.

The generalization that we allow Θ_{k+1} to be *some* biorthogonal basis of Φ_k^* in $\mathcal{S}_{k+1}$, instead of sticking to the initially given coarse generator basis Φ_k, is a crucial point of this paper. Remark 3.2 says that we need Φ_k to be local with respect to Φ_{k+1}, whereas in case $\Theta_{k+1} = \Phi_k$, and thus (Φ_k, Φ_k^*) a biorthogonal pair of bases, (3.3) is equivalent to the property that Φ_k^* is local with respect to Φ_{k+1}^*. So far, such biorthogonal bases have been constructed for pairs $((\mathcal{S}_k), (\mathcal{S}_k^*))$ satisfying the conditions of Theorem 2.1 only for spaces $\mathcal{S}_k$ which are spanned by B-splines on uniform dyadic partitions of the real line (in [7]), or of the interval (in [11]). In [7], and as a consequence partly also in [11], the construction of the space $\mathcal{S}_k^*$ relies through the construction of biorthogonal generators still on Fourier techniques, which restricts the field of applications essentially to uniform meshes and adaptive refinements of such.

In contrast, as we will see in the next sections, our generalization concerning the choice of Θ_{k+1} permits us to take for both $\mathcal{S}_k$ and $\mathcal{S}_k^*$ standard finite element spaces, also in more space dimensions and for non-uniform meshes.

On the other hand, as is known from the literature, for applications different from the generation of well-conditioned stiffness matrices or matrix compression, the wavelet construction starting with suitable biorthogonal pairs of bases offers some potential advantages as will be described in the next remark.

Remark 3.4 (cf. [5]). Let (Φ_k, Φ_k^*) be a biorthogonal pair of bases, i.e., suppose $\Theta_{k+1} = \Phi_k$. Suppose Φ_k and Φ_k^* are local with respect to Φ_{k+1} and Φ_{k+1}^* respectively, i.e., both $\mathbf{P}_{k+1,0}$ and $\tilde{\mathbf{P}}_{k+1,0}$ from the refinement relations $\Phi_k^T = \Phi_{k+1}^T \mathbf{P}_{k+1,0}$ and ${\Phi_k^*}^T = {\Phi_{k+1}^*}^T \tilde{\mathbf{P}}_{k+1,0}$ are sparse matrices. Furthermore, assume that Υ_{k+1} can be selected such that both $\hat{\mathbf{P}}_{k+1} = [\mathbf{P}_{k+1,0} \quad \hat{\mathbf{P}}_{k+1,1}]$, defined by $[\Phi_k^T \quad \Upsilon_{k+1}^T] = \Phi_{k+1}^T \hat{\mathbf{P}}_{k+1}$, and $\hat{\mathbf{P}}_{k+1}^{-1}$ are sparse. Examples can be found in [11]. Since $\Theta_{k+1} = \Phi_k$, the definition of Ψ_{k+1} from Theorem 3.1 shows that $[\Phi_k^T \quad \Psi_{k+1}^T] = \Phi_{k+1}^T \mathbf{P}_{k+1}$, where $\mathbf{P}_{k+1} = \hat{\mathbf{P}}_{k+1} \begin{bmatrix} \mathbf{I} & -\mathbf{Z}_k \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$, $\mathbf{Z}_k = \left(\frac{(\upsilon_{k+1,x}, \phi_{k,y}^*)_{L_2}}{(\phi_{k,y}, \phi_{k,y}^*)_{L_2}} \right)_{y \in I_k, x \in J_{k+1}} = \hat{\mathbf{\Delta}}_k^{-1} \tilde{\mathbf{P}}_{k+1,0}^* \hat{\mathbf{\Delta}}_{k+1} \hat{\mathbf{P}}_{k+1,1}$ and $\hat{\mathbf{\Delta}}_k = \operatorname{diag}((\phi_{k,y}, \phi_{k,y}^*)_{L_2})_{y \in I_k}$. From the assumptions on $\hat{\mathbf{P}}_{k+1}$, $\hat{\mathbf{P}}_{k+1}^{-1}$ and $\tilde{\mathbf{P}}_{k+1,0}$, we conclude that in this case not only $\mathbf{P}_{k+1}$, but also $\mathbf{P}_{k+1}^{-1}$ is sparse.

As a first consequence, we note that now besides $\mathbf{T}_j$, also the basis transformation $\mathbf{T}_j^{-1}$ from single scale basis Φ_j to multi-scale basis $\Psi^j = \cup_{k=0}^j \Psi_k$ can be implemented in $\mathcal{O}(\dim \mathcal{S}_j)$ operations, by a recursive top-to-bottom application of $\mathbf{P}_{k+1}^{-1}$.

For the second application, it is convenient to assume $(\phi_{k,y}, \phi_{k,y}^*)_{L_2} = 1$ or $\hat{\mathbf{\Delta}}_k = \mathbf{I}$ and $\tilde{\mathbf{P}}_{k+1,0}^* \mathbf{P}_{k+1,0} = \mathbf{I}$. Then, writing $(\hat{\mathbf{P}}_{k+1}^{-1})^* = [\mathbf{H}_{k+1,0} \quad \mathbf{H}_{k+1,1}]$, there holds $(\mathbf{P}_{k+1}^{-1})^* = [\tilde{\mathbf{P}}_{k+1,0} \quad \mathbf{H}_{k+1,1}]$. Defining $(\Psi_{k+1}^*)^T$ by $[{\Phi_k^*}^T \quad {\Psi_{k+1}^*}^T] = {\Phi_{k+1}^*}^T (\mathbf{P}_{k+1}^{-1})^*$, it follows that Ψ_{k+1}^* is an L_2-stable basis of $\mathcal{W}_{k+1}^* = \mathcal{S}_k^{\perp_{L_2}} \cap \mathcal{S}_{k+1}^*$, which is local with respect to Φ_{k+1}^* and biorthogonal to Ψ_{k+1} (with scalar products equal to one when the indices match). So in this case we obtain biorthogonal bases $\bigcup_{j \geq 0} \Psi_j$ and $\bigcup_{j \geq 0} \Psi_j^*$, where the dual wavelets for the spaces $\mathcal{S}_k^*$ have similar favorable properties as the primal ones.

As a consequence, in the context of Theorem 2.1, for $s \in (-\gamma^*, \gamma)$ and $f \in \mathcal{H}^s$, the coefficients in the expansion $f = \sum_{k,x} c_{k,x} \psi_{k,x}$ are now given by $c_{k,x} = (f, \psi_{k,x}^*)_{L_2}$. Assuming that Φ_j^* is locally finite, the coefficients $(f, \psi_{k,x}^*)_{L_2}$ for $k \leq j$ can be computed recursively from top-to-bottom in an optimal way.

4. Wavelet bases of C^0 Lagrange finite element spaces.

In this section, we will choose for both $\mathcal{S}_k$ and $\mathcal{S}_k^*$ C^0 Lagrange finite element spaces, on possibly non-uniform meshes. We confirm the validity of conditions (C2), (C2)*, (C3) and (C3)*. Furthermore, we reduce the problems of verifying (C1) and (C1)*, and the construction of the single-scale bases Φ_k, as well as the bases Φ_k^*, Θ_{k+1} and Υ_{k+1}, to corresponding problems on the *reference element*. These questions on the reference element will be treated for a number of concrete cases.

4.1. Finite element spaces.

For some fixed $m \in \mathbb{N}$, let $\mathcal{T}_{-m}$ be a collection of closed n-simplices such that $\cup_{\tau \in \mathcal{T}_{-m}} \tau$ is a partition, also called triangulation, of the closure of some open domain $\Omega \subset \mathbb{R}^n$. The construction of wavelets on manifolds will be addressed in §4.3. We assume that the triangulation is conforming, i.e., the intersection of any two simplices $\tau, \tilde{\tau}$ in $\mathcal{T}_{-m}$ is either empty or a common lower-dimensional face.

We include the possibility of imposing essential boundary conditions on part of $\partial\Omega$. That is, we consider functions on $\overline{\Omega}$ that are zero on $\Gamma_D \subset \partial\Omega$, where Γ_D is the union of a number (possibly zero) of faces of simplices from $\mathcal{T}_{-m}$. We define $\mathcal{H}^1$ as the completion in $H^1(\Omega)$ of $\{u \in C^\infty(\Omega) \cap H^1(\Omega) : \operatorname{supp} u \cap \Gamma_D = \emptyset\}$, and for $0 < s < 1$, we define $\mathcal{H}^s$ by interpolation between $L_2 = L_2(\Omega)$ and $\mathcal{H}^1$.

Let us now describe the hierarchy of meshes and associated finite element spaces. For $-m < k \in \mathbb{Z}$, let $\mathcal{T}_k$ be the collection of n-simplices generated from $\mathcal{T}_{k-1}$ by uniform, regular, dyadic refinement, i.e., each $\tau \in \mathcal{T}_{k-1}$ is subdivided into 2^n congruent n-simplices. Note that for $n > 2$, there are several possibilities to divide each simplex in this way. For example, for $n = 3$ there exist three possibilities to subdivide a tetrahedron into 8 congruent tetrahedra, see [**?**] for more details. However, again for $n = 3$ this ambiguity concerns only subsimplices whose 2-faces are not contained in any 2-face of the mother simplex. Thus the refinement of adjacent simplices in $\mathcal{T}_{-m}$ does not affect compatibility on common faces. Therefore, to ease presentation, we will assume that $n \leq 3$, which means that refinements can be made on an element-by-element basis with automatic matching of the triangulation at interfaces of adjacent macro elements. The set of vertices of all $\tau \in \mathcal{T}_k$ is denoted by V_k.

For $\tilde{\ell} \in \mathbb{N}$, $k \in \mathbb{Z}$ with $\tilde{\ell} \leq m$, $k \geq \tilde{\ell} - m$, we define $S(\tilde{\ell}, k)$ as the C^0 Lagrange finite element space of degree $2^{\tilde{\ell}}$ corresponding to the set of "nodes" V_k and the $\tilde{\ell}$-times coarsened triangulation $\mathcal{T}_{k-\tilde{\ell}}$. Note that the dimension of $S(\tilde{\ell}, k)$ only depends on k, but not on $\tilde{\ell}$. Restricting the polynomial degrees to powers of two ensures that the sets of nodes are nested under refinement. Each $\tau \in \mathcal{T}_{k-\tilde{\ell}}$ contains $\binom{2^{\tilde{\ell}}+n}{n}$ nodes from V_k, which indeed equals the dimension of $\Pi_{2^{\tilde{\ell}}}$, the space of polynomials of total degree $2^{\tilde{\ell}}$ on $\mathbb{R}^n$. In particular, one may verify that $S(\tilde{\ell}, k)$ is well-defined (e.g. see [2, §3]).

Now fix $\ell, \ell^* \in \mathbb{N}$ and assume throughout the following that $m = \max\{\ell, \ell^*\}$. We define

$$\mathcal{S}_k = S(\ell, k) \cap \mathcal{H}^1, \;\; \mathcal{S}_k^* = S(\ell^*, k) \cap \mathcal{H}^1,$$

where the intersection with $\mathcal{H}^1$ is made to impose possibly essential boundary conditions.

Note that $k = 0$ is the lowest level on which both $\mathcal{S}_k$ and $\mathcal{S}_k^*$ are defined. In case $\ell \neq \ell^*$ then to define either $(\mathcal{S}_k)_{k \geq 0}$ or $(\mathcal{S}_k^*)_{k \geq 0}$, we could have started with a triangulation less coarse than $\mathcal{T}_{-m}$. Yet to relate both sequences of finite element spaces via biorthogonal constructions we will need this common 'ancestor' $\mathcal{T}_{-m}$.

Only for matters related to obtaining an efficient implementation, sometimes it will be convenient also to consider spaces $S(\tilde{\ell}, k)$ for some negative k.

Remark 4.1. The sequences $(\mathcal{S}_k)_{k \geq 0}$, $(\mathcal{S}_k^*)_{k \geq 0}$ satisfy the conditions (C2), (C2)*, (C3), (C3)* with $\rho = 2$ (dyadic refinement), and

$$\gamma = \gamma^* = \frac{3}{2} \qquad d = 2^{\ell} + 1, \;\; d^* = 2^{\ell^*} + 1.$$

We will consider space decompositions that are biorthogonal with respect to the canonical L_2-scalar product. Assuming that (C1) and (C1)* are satisfied as well, so that, on account of Remark 2.2, $\mathcal{W}_{k+1} = \mathcal{S}_{k+1} \cap (\mathcal{S}_k^*)^{\perp_{L_2}}$ is well-defined, elements $v_{k+1} \in \mathcal{W}_{k+1}$ will have d^* vanishing moments. In case of essential boundary conditions, this holds true for v_{k+1} for which $\operatorname{supp} v_{k+1} \subset \cup\{\tau \in \mathcal{T}_{k-\ell^*} : \tau \cap \Gamma_D = \emptyset\}$.

As index set for bases of $S(\tilde{\ell}, k) \cap \mathcal{H}^1$, we will use

$$I_k := V_k \backslash (\Gamma_D \cap V_k). \tag{4.1}$$

4.2. Reduction to a reference element. In this subsection, we will reduce the problems of constructing single-scale bases Φ_k, bases Φ_k^*, Θ_{k+1} and Υ_{k+1} needed in Theorem 3.1, as well as bases Ξ_k and Ξ_k^* needed in Lemma 2.3 for verifying (C1) and (C1)*, to corresponding problems on some reference (macro) element $\check\tau$, say with vertices $v_1,\ldots,v_{n+1}\in\mathbb{R}^n$ not all lying on one hyperplane.

Starting with the collection $\check{\mathcal{T}}_{-m}=\{\check\tau\}$ consisting of the reference simplex only, we define in analogy to the definitions in §4.1 sequences of triangulations $(\check{\mathcal{T}}_{\check k})_{\check k\geq -m}$, nodes $(\check V_{\check k})_{\check k\geq -m}$, and for $\tilde\ell\leq m$, finite element spaces $(\check S(\tilde\ell,\check k))_{\check k\geq\tilde\ell-m}$. In fact we will need these spaces only for $\check k\in\{0,1\}$, but for matters related to obtaining an efficient implementation on a few places we consider these spaces also for some $\check k<0$. In any case, for some arguments it is useful to keep in mind that the dimension of all spaces $\check S(\tilde\ell,\check k)$ that we consider is uniformly bounded, with a bound that only depends on $m=\max\{\ell,\ell^*\}$. In the following we will denote $\check V_{\check k}$ by $\check I_{\check k}$, which is in agreement with (4.1), since we do not impose boundary conditions on $\check\tau$.

To describe certain symmetry relations which are relevant for the subsequent construction consider the mapping

$$B:\{\alpha\in\mathbb{R}^{n+1}:\sum_{j=1}^{n+1}\alpha_j=1,\,\alpha_j\geq 0\}\to\check\tau:\alpha\mapsto\sum_{j=1}^{n+1}\alpha_j v_j$$

which is a bijection from the standard n-simplex to $\check\tau$ described in terms of barycentric coordinates. For any permutation of the coordinates $\pi:\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$, we define the bijection

$$S_\pi=B\circ\pi\circ B^{-1}:\check\tau\to\check\tau.$$

For some fixed $\check k\geq\tilde\ell-m$, let $\{\check\varphi_{\check k,\check x}:\check x\in\check I_{\check k}\}\subset\check S(\tilde\ell,\check k)$ be a collection of functions on the reference element satisfying

(P1). $(\check\varphi_{\check k,S_\pi(\check x)}\circ S_\pi)|_{\partial\check\tau}=\check\varphi_{\check k,\check x}|_{\partial\check\tau}$ for all permutations π and all $\check x\in\check I_{\check k}$,

(P2). $\check\varphi_{\check k,\check x}(\check y)=0$ if for some $i\in\{1,\ldots,n+1\}$, $(B^{-1}(\check y))_i=0$ and $(B^{-1}(\check x))_i\neq 0$.

Property (P1) means that $\{\check\varphi_{\check k,\check x}|_{\partial\check\tau}:\check x\in\check I_{\check k}\}$ is invariant under permutations of the barycentric coordinates, whereas (P2) says that $\check\varphi_{\check k,\check x}$ vanishes on all faces of $\check\tau$ that do not contain $\check x$.

For $k\geq\check k$ and $x\in I_k$, let us now define $\varphi_{k,x}\in S(\tilde\ell,k)\cap\mathcal{H}^1$ by

$$\varphi_{k,x}(y)=\begin{cases}\check\varphi_{\check k,T_\tau^{-1}(x)}(T_\tau^{-1}(y)) & \text{if } x,y\in\tau\in\mathcal{T}_{k-\check k-m}\\ 0 & \text{elsewhere}\end{cases}\quad, \tag{4.2}$$

where $T_\tau:\check\tau\to\tau$ is an affine bijection. Note that such a T_τ is a bijection between the nodes $\check V_{\check k}$ in the reference domain and $V_k\cap\tau$ in the physical domain. For $n\leq 2$, due to the uniqueness of uniform dyadic refinements, it is also a bijection between the triangulations $\check{\mathcal{T}}_{\check k-\tilde\ell}$ and $\{\tau_{k-\tilde\ell}\in\mathcal{T}_{k-\tilde\ell}:\tau_{k-\tilde\ell}\subset\tau\}$ in both domains. However, as mentioned before, for $n=3$ the latter property does not hold automatically, and for that case we therefore impose it as a condition on T_τ.

For any $\tau\neq\tilde\tau\in\mathcal{T}_{k-\check k-m}$ with $\tau\cap\tilde\tau\neq\emptyset$, and any T_τ, $T_{\tilde\tau}$ as above, there exists a permutation π such that $T_{\tilde\tau}\circ S_\pi\circ T_\tau^{-1}|_{\tilde\tau\cap\tau}=I$. Property (P1) now implies that $\varphi_{k,x}$ is well-defined, and that it is continuous on $\tau\cup\tilde\tau$ if $x\in\tau\cap\tilde\tau$. On the other hand, property (P2) shows that $\varphi_{k,x}$ vanishes on all faces of τ that do not contain x. We conclude that $\varphi_{k,x}$ is continuous on $\overline{\Omega}$, vanishes on Γ_D, and so that $\varphi_{k,x}$ belongs to $S(\tilde\ell,k)\cap\mathcal{H}^1$.

Remark 4.2. In our applications, we will construct $\{\check\varphi_{\check k,\check x}:\check x\in\check I_{\check k}\}\subset\check S(\tilde\ell,\check k)$ which on the *whole* of $\check\tau$, thus not only on $\partial\check\tau$ are invariant under all permutations of the barycentric coordinates that, for $n > 2$, leave the mesh $\check{\mathcal{T}}_{\check{k}-\tilde{\ell}}$ unchanged. So $\varphi_{k,x}$ defined in (4.2) will be independent of the particular choice of T_τ. Note that this construction of global bases is commonly used for affinely equivalent finite elements.

The analysis in the remainder of this subsection will be based on the following trivial relation

$$(u,v)_{L_2} = \sum_{\tau \in \mathcal{T}_k} \tfrac{\mathrm{vol}(\tau)}{\mathrm{vol}(\check{\tau})}(u \circ T_\tau, v \circ T_\tau)_{L_2(\check{\tau})} \qquad (u, v \in L_2,\, k \geq -m), \tag{4.3}$$

where $T_\tau : \check{\tau} \to \tau$ are affine bijections. In particular, for $\varphi_{k,x}$ defined in (4.2) this yields

$$\|\varphi_{k,x}\|_{L_2}^2 = \sum_{\{\tau \in \mathcal{T}_{k-\check{k}-m} : x \in \tau\}} \tfrac{\mathrm{vol}(\tau)}{\mathrm{vol}(\check{\tau})} \|\check{\varphi}_{\check{k},T_\tau^{-1}(x)}\|_{L_2(\check{\tau})}^2. \tag{4.4}$$

Lemma 4.3. *For some $\tilde{\ell} \leq m$ and fixed $\check{k} \geq \tilde{\ell} - m$, let $\{\check{\varphi}_{\check{k},\check{x}} : \check{x} \in \check{I}_{\check{k}}\}$ be a basis of $\check{S}(\tilde{\ell},\check{k})$ satisfying* (P1), (P2). *Then the collections $\{\varphi_{k,x} : x \in I_k\}$, defined by* (4.2), *form L_2-stable bases of the spaces $S(\tilde{\ell},k) \cap \mathcal{H}^1$, uniformly in $k \geq \check{k}$.*

Proof. From the construction of $\varphi_{k,x}$ and the definition of a finite element space, it follows directly that $\{\varphi_{k,x} : x \in I_k\}$ is a basis of $S(\tilde{\ell},k) \cap \mathcal{H}^1$. On account of (4.4), one has

$$\begin{aligned} \|\sum_{x \in I_k} c_{k,x}\varphi_{k,x}\|_{L_2}^2 &= \sum_{\tau \in \mathcal{T}_{k-\check{k}-m}} \tfrac{\mathrm{vol}(\tau)}{\mathrm{vol}(\check{\tau})} \|\sum_{x \in I_k \cap \tau} c_{k,x}\check{\varphi}_{\check{k},T_\tau^{-1}(x)}\|_{L_2(\check{\tau})}^2 \\ &\eqsim \sum_{\tau \in \mathcal{T}_{k-\check{k}-m}} \tfrac{\mathrm{vol}(\tau)}{\mathrm{vol}(\check{\tau})} \sum_{x \in I_k \cap \tau} |c_{k,x}|^2 \|\check{\varphi}_{\check{k},T_\tau^{-1}(x)}\|_{L_2(\check{\tau})}^2 \\ &= \sum_{x \in I_k} |c_{k,x}|^2 \|\varphi_{k,x}\|_{L_2}^2, \end{aligned} \tag{4.5}$$

confirms L_2-stability. $\square$

Definition 4.4. *Let $\tilde{\ell} \leq m$ be given. For $\check{k} \geq \tilde{\ell} - m$, with $\check{\Phi}_{\check{k}}^{\tilde{\ell},N} = \{\check{\phi}_{\check{k},\check{x}}^{\tilde{\ell},N} : \check{x} \in \check{I}_{\check{k}}\}$ we denote the* nodal *basis of $\check{S}(\tilde{\ell},\check{k})$ defined by $\check{\phi}_{\check{k},\check{x}}^{\tilde{\ell},N}(\check{y}) = \delta_{\check{x}}(\check{y})$ $(\check{y} \in \check{I}_{\check{k}})$. Clearly, it satisfies* (P1) *and* (P2). *The corresponding* global *nodal bases of $S(\tilde{\ell},k)$ $(k \geq \check{k})$, induced by* (4.2), *will be denoted by $\Phi_k^{\tilde{\ell},N} = \{\phi_{k,x}^{\tilde{\ell},N} : x \in I_k\}$.*

Note that the definition of $\Phi_k^{\tilde{\ell},N}$ is independent of $\check{k} \leq k$, so no ambiguity can arise. As "single-scale" basis Φ_k of $\mathcal{S}_k = S(\ell,k)$, we will always employ the nodal basis $\Phi_k^{\ell,N}$.

Proposition 4.5. *Let $\check{\Xi}_0 = \{\check{\xi}_{0,\check{x}} : \check{x} \in \check{I}_0\}$ and $\check{\Xi}_0^* = \{\check{\xi}_{0,\check{x}}^* : \check{x} \in \check{I}_0\}$ be bases of $\check{S}(\ell,0)$ and $\check{S}(\ell^*,0)$, respectively, satisfying* (P1), (P2). *Moreover, suppose that with respect to the Euclidean scalar product $(\cdot,\cdot)_{\ell_2(\check{I}_0)}$, the real part of the matrix $\check{\mathbf{M}}_0 := \big((\check{\xi}_{0,\check{y}}, \check{\xi}_{0,\check{x}}^*)_{L_2(\check{\tau})}\big)_{\check{x} \in \check{I}_0, \check{y} \in \check{I}_0}$ is strictly positive, i.e. $\Re(\check{\mathbf{M}}_0) := \frac{1}{2}(\check{\mathbf{M}}_0 + \check{\mathbf{M}}_0^*) > 0$. Then, the pair $(\mathcal{S}_k, \mathcal{S}_k^*)$ satisfies* (C1) *and* $(\mathrm{C1})^*$.

Proof. Let $\Xi_k = \{\xi_{k,x} : x \in I_k\}$ and $\Xi_k^* = \{\xi_{k,x}^* : x \in I_k\}$ be defined by (4.2) according to $\check{\Xi}_0$ and $\check{\Xi}_0^*$, respectively. By Lemma 4.3, Ξ_k and Ξ_k^* are L_2-stable bases of $\mathcal{S}_k$ and $\mathcal{S}_k^*$.

Define the (possibly infinite) matrix $\tilde{\mathbf{M}}_k = \Big(\frac{(\xi_{k,y},\xi_{k,x}^*)_{L_2}}{\|\xi_{k,y}\|_{L_2}\|\xi_{k,x}\|_{L_2}}\Big)_{x \in I_k, y \in I_k}$. Since by (4.4) we have $\|\xi_{k,x}\|_{L_2} \eqsim \|\xi_{k,x}^*\|_{L_2}$, for Lemma 2.3 it is sufficient to check whether

$\|\tilde{\mathbf{M}}_k\mathbf{c}_k\|_{\ell_2(I_k)} \gtrsim \|\mathbf{c}_k\|_{\ell_2(I_k)}$ and $\|\tilde{\mathbf{M}}_k^T\mathbf{c}_k\|_{\ell_2(I_k)} \gtrsim \|\mathbf{c}_k\|_{\ell_2(I_k)}$ ($\mathbf{c}_k = (c_{k,x})_{x\in I_k}$). Since $\Re(\check{\mathbf{M}}_0) > 0$ one has

$$\begin{aligned}
&\Re\left(\sum_{x,y\in I_k}(\xi_{k,y},\xi^*_{k,x})_{L_2}c_{k,y}\overline{c_{k,x}}\right)\\
&= \sum_{\tau\in\mathcal{T}_{k-m}}\tfrac{\mathrm{vol}(\tau)}{\mathrm{vol}(\check\tau)}\,\Re\left(\sum_{x,y\in I_k\cap\tau}(\check\xi_{0,T_\tau^{-1}(y)},\check\xi^*_{0,T_\tau^{-1}(x)})_{L_2(\check\tau)}\,c_{k,y}\overline{c_{k,x}}\right)\\
&\gtrsim \sum_{\tau\in\mathcal{T}_{k-m}}\tfrac{\mathrm{vol}(\tau)}{\mathrm{vol}(\check\tau)}\sum_{x\in I_k\cap\tau}|c_{k,x}|^2 \eqsim \sum_{x\in I_k}|c_{k,x}|^2\|\xi_{k,x}\|^2_{L_2},
\end{aligned}$$

so that $\Re(\tilde{\mathbf{M}}_k) \gtrsim \mathbf{I}$ with respect to $(\cdot\,,\,\cdot)_{\ell_2(I_k)}$, which implies the desired properties of $\tilde{\mathbf{M}}_k$. ☐

PROPOSITION 4.6. *Let*

$$\check J_{\check k} := \check I_{\check k}\backslash\check I_{\check k-1},\;\; J_k := I_k\backslash I_{k-1}.$$

Suppose that the three collections $\check\Phi^*_0 = \{\check\phi^*_{0,\check y} : \check y\in\check I_0\}$, $\check\Upsilon_1 = \{\check\upsilon_{1,\check x} : \check x\in\check J_1\}$ *and* $\check\Theta_1 = \{\check\theta_{1,\check y} : \check y\in\check I_0\}$ *have been chosen such that* $\check\Phi^*_0$ *and* $\check\Upsilon_1\cup\check\Theta_1$ *are bases of* $\check S(\ell^*,0)$ *and* $\check S(\ell,1)$*, respectively, both having properties* (P1), (P2). *Furthermore, assume that*

$$(\check\theta_{1,\check y},\check\phi^*_{0,\check z})_{L_2(\check\tau)} = \delta_{\check y,\check z},\qquad \check y,\check z\in\check I_0,$$

that is, $\check\Theta_1$ *is a "local" biorthogonal basis to* $\check\Phi^*_0$.

By (4.2)*, define for* $k\geq 0$ *the corresponding "global" bases* $\Phi^*_k := \{\phi_{k,y} : y\in I_k\}$, *and for* $k\geq 1$, $\Upsilon_k := \{\upsilon_{k,x}; x\in J_k\}$ *as well as* $\Theta_k := \{\theta_{k,y}; y\in I_{k-1}\}$. *Then the triple* $(\Phi^*_k,\Upsilon_{k+1},\Theta_{k+1})_{k\geq 0}$ *satisfies the conditions of Theorem* 3.1.

Proof. For $y,\,z\in I_k$, we have

$$\begin{aligned}
(\theta_{k+1,y},\phi^*_{k,z})_{L_2} &= \sum_{\{\tau\in\mathcal{T}_{k-m}:y,z\in\tau\}}\tfrac{\mathrm{vol}(\tau)}{\mathrm{vol}(\check\tau)}(\check\theta_{1,T_\tau^{-1}(y)},\check\phi^*_{0,T_\tau^{-1}(z)})_{L_2(\check\tau)}\\
&\eqsim \delta_{y,z}\|\theta_{k+1,y}\|_{L_2}\|\phi^*_{k,z}\|_{L_2},
\end{aligned}$$

i.e., the basis Θ_{k+1} is biorthogonal to Φ^*_k. The L_2-stability of the bases Φ^*_k and $\Upsilon_{k+1}\cup\Theta_{k+1}$ follows from Lemma 4.3. ☐

Recall from formula (3.1) that for computing the wavelets we need the quantities $\frac{(\upsilon_{k+1,x},\phi^*_{k,y})_{L_2}}{(\theta_{k+1,y},\phi^*_{k,y})_{L_2}}$. To this end, note that for $x\in J_{k+1}$, $y\in I_k$,

$$\frac{(\upsilon_{k+1,x},\phi^*_{k,y})_{L_2}}{(\theta_{k+1,y},\phi^*_{k,y})_{L_2}} = \frac{\displaystyle\sum_{\{\tau\in\mathcal{T}_{k-m}:x,y\in\tau\}}\mathrm{vol}(\tau)\,(\check\upsilon_{1,T_\tau^{-1}(x)},\check\phi^*_{0,T_\tau^{-1}(y)})_{L_2(\check\tau)}}{\displaystyle\sum_{\{\tau\in\mathcal{T}_{k-m}:y\in\tau\}}\mathrm{vol}(\tau)}. \tag{4.6}$$

In summary, we conclude that we have to check $\Re(\check{\mathbf{M}}_0) > 0$ (cf. Proposition 4.5) *once and for all* on the reference element. Moreover, we have to provide the collections $\check\Upsilon_1$, $\check\Theta_1$ in terms of $\check\Phi_1^{\ell,N}$ as well as the values $(\check\upsilon_{1,\check x},\check\phi^*_{0,\check y})_{L_2(\check\tau)}$ ($\check x\in\check J_1$, $\check y\in\check I_0$). Apart from that, in order to compute the wavelets, that is the basis functions of

$\mathcal{W}_{k+1} = \mathcal{S}_{k+1} \cap (\mathcal{S}_k^*)^{\perp_{L_2}}$, we only need the geometry of the initial mesh which, in particular, yields the volumes of the simplices.

Clearly, bases obtained by the above construction satisfy the locality conditions formulated in Remark 3.2. This means that the basis transformation $\mathbf{T}_j$ on $\mathcal{S}_j$ from wavelet- to nodal basis can be computed in $\mathcal{O}(\mathrm{dim}\mathcal{S}_j)$ operations.

4.3. Wavelets on compact manifolds. In this section, it will be demonstrated that the element-by-element construction of wavelets outlined above applies as well to certain types of continuous compact n-dimensional *manifolds* $\Gamma \subset \mathbb{R}^{n+1}$.

4.3.1. Function spaces. Suppose that $\Gamma = \cup_{i=1}^{p} \overline{\Gamma_i}$, where $\Gamma_i \cap \Gamma_j = \emptyset$ when $i \neq j$. We assume that each patch Γ_i is the image of a smooth regular parameterization $\kappa_i : \Omega_i \to \Gamma_i$, where $\Omega_i \subset \mathbb{R}^n$ is an open domain. For each i, we assume that $\mathcal{T}_{-m,i}$ is a collection of closed n-simplices, such that $\cup_{\tau \in \mathcal{T}_{-m,i}} \tau$ is a triangulation of $\overline{\Omega_i}$. Concerning matching of elements at the interfaces, we make the following assumption: For each maximal subset $I \subset \{1, \ldots, p\}$ for which $\cap_{i \in I} \overline{\Gamma_i} \neq \emptyset$, there exist invertible affine mappings $A_i : \mathbb{R}^n \to \mathbb{R}^n$ "gluing the Ω_i's together", such that the mapping $\cup_{i \in I} \Gamma_i \to \cup_{i \in I} A_i(\Omega_i)$ defined by

$$x \mapsto A_i(\kappa_i^{-1}(x)) \quad \text{when} \quad x \in \Gamma_i,$$

has an extension as a homeomorphic mapping between $\cup_{i \in I} \overline{\Gamma_i}$ and $\cup_{i \in I} \overline{A_i(\Omega_i)}$. We assume that $\cup_{i \in I} \cup_{\tau \in \mathcal{T}_{-m,i}} A_i(\tau)$ is a conforming triangulation of $\cup_{i \in I} \overline{A_i(\Omega_i)}$. Note that these are mild requirements that conform with typical applications, for instance, in the context of boundary element methods.

As in §4.1, by uniform, regular and dyadic refinement, on each Ω_i we define sequences of triangulations $(\mathcal{T}_{k,i})_{k \geq -m}$, nodes $(V_{k,i})_{k \geq -m}$, and for $\tilde{\ell} \leq m$, finite element spaces $(S_i(\tilde{\ell}, k))_{k \geq \tilde{\ell} - m}$. We define

$$S^{(\Gamma)}(\tilde{\ell}, k) = \{u \in C(\Gamma) : u \circ \kappa_i \in S_i(\tilde{\ell}, k),\ 1 \leq i \leq p\}.$$

Due to our assumptions, the dimension of $S^{(\Gamma)}(\tilde{\ell}, k)$ is equal to $\# I_k$, where

$$I_k = \cup_{i=1}^{p} \kappa_i(V_{k,i})$$

is now the set of nodes on the manifold.

Assuming that Γ is *globally Lipschitz continuous*, the Sobolev spaces $\mathcal{H}^s = H^s(\Gamma)$ can be defined for $|s| \leq 1$ in the usual way using a partition of unity relative to some atlas (cf. [3]).

4.3.2. Space decompositions. We define the auxiliary spaces $\mathcal{H}_s$ for $s \geq 0$ as the closure of $\{u \in C(\Gamma) : u \circ \kappa_i \in H^s(\Omega_i),\ 1 \leq i \leq p\}$ with respect to the norm $\|u\|_{\mathcal{H}_s} := \sqrt{\sum_{i=1}^{p} \|u \circ \kappa_i\|_{H^s(\Omega_i)}^2}$, and for $s < 0$ as $(\mathcal{H}_{-s})'$. The space $\mathcal{H}_s$ should not be confused with the product space $\prod_{i=1}^{p} H^s(\Gamma_i)$ that is usually associated with the norm $\| \cdot \|_{\mathcal{H}_s}$.

Fixing $\ell, \ell^* \in \mathbb{N}$ with $m = \max\{\ell, \ell^*\}$, we take $\mathcal{S}_k = S^{(\Gamma)}(\ell, k)$ and $\mathcal{S}_k^* = S^{(\Gamma)}(\ell^*, k)$. We consider space decompositions that are biorthogonal with respect to the canonical $L_2(\Gamma)$-scalar product

$$(4.7)\ (u, v)_{L_2(\Gamma)} = \int_{\Gamma} u \overline{v} d\Gamma = \sum_{i=1}^{p} \int_{\Omega_i} u(\kappa_i(\zeta)) \overline{v(\kappa_i(\zeta))} \sqrt{\det(D\kappa_i(\zeta)^T D\kappa_i(\zeta))} d\zeta.$$

Standard estimates confirm that, with respect to the $\mathcal{H}_s$-spaces, (C2), $(\mathrm{C2})^*$, (C3), $(\mathrm{C3})^*$ are valid, with $d = 2^\ell + 1$, $d^* = 2^{\ell^*} + 1$ and $\gamma = \gamma^* = \frac{3}{2}$. For this to hold true, it is sufficient that $\kappa_i \in C^{\max\{d,d^*\}-1}$ with Lipschitz continuous highest order derivatives. Assuming for the moment that also (C1), $(\mathrm{C1})^*$ are valid with $\mathcal{W}_{k+1} = \mathcal{S}_{k+1} \cap (\mathcal{S}_k^*)^{\perp_{L_2(\Gamma)}}$ we conclude that for $|s| < \frac{3}{2}$ the direct sum $\oplus_{k \geq 0} \mathcal{W}_k$ is a stable decomposition of the auxiliary space $\mathcal{H}_s$ in the sense of Theorem 2.1.

We have to clarify next how for $|s| \leq 1$ the spaces $\mathcal{H}_s$ and $\mathcal{H}^s$ are interrelated. By the denseness of smooth functions in Sobolev spaces each element in $\mathcal{H}_1$ is the limit of patchwise smooth v that are globally continuous. It is well known that such functions v belong to $\mathcal{H}^1$ and generate a dense subspace. Since $\|v \circ \kappa_i\|_{H^1(\Omega_i)} \eqsim \|v\|_{H^1(\Gamma_i)}$, one concludes the spaces $\mathcal{H}^1$ and $\mathcal{H}_1$ agree as sets and have equivalent norms. By interpolation and duality we therefore conclude that

$$\mathcal{H}^s \eqsim \mathcal{H}_s, \qquad -1 \leq s \leq 1,$$

which establishes the desired $\mathcal{H}^s$-stability for $|s| \leq 1$ also for the case of wavelet bases on Lipschitz manifolds.

4.3.3. Wavelets. Now we come to the construction of the wavelets and the verification of (C1) and $(\mathrm{C1})^*$. To reduce, as in the domain case, these questions to corresponding questions on a reference element $\check{\tau}$, we need a substitute for (4.3). We will make the assumption that the parametrizations κ_i can be chosen in such a way that the weight function

$$g : \Gamma \to I\!R : x \mapsto \sqrt{\det D\kappa_i(\kappa_i^{-1}(x))^T D\kappa_i(\kappa_i^{-1}(x))} \qquad (x \in \Gamma_i)$$

is *piecewise constant* with respect to the partition $\Gamma = \cup_{i=1}^p \overline{\Gamma_i}$. Under this assumption, for $u, v \in L_2(\Gamma)$ and $k \geq -m$ we have

$$(4.3^{(\Gamma)}) \qquad (u, v)_{L_2(\Gamma)} = \sum_i \sum_{\tau \in \mathcal{T}_{k,i}} \tfrac{\mathrm{vol}(\kappa_i(\tau))}{\mathrm{vol}(\check{\tau})} (u \circ \kappa_i \circ T_\tau, v \circ \kappa_i \circ T_\tau)_{L_2(\check{\tau})},$$

where $T_\tau : \check{\tau} \to \tau$ are affine bijections. Note that this assumption on the parametrizations allows us to handle manifolds consisting of patches that are for example parts of hyperplanes, spheres or cylinders.

In fact, since the dual norms $\sup_{0 \neq u \in \mathcal{H}^s} \frac{|(u,v)_{L_2(\Gamma)}|}{\|u\|_{\mathcal{H}^s}}$ $(s \geq 0)$ turn into equivalent ones when the weight function g is multiplied with a globally smooth, positive function, we can handle even somewhat more general cases. That is, when for some collection $\{\kappa_i\}$, the function g is a product of a piecewise constant and a globally smooth, positive function, we can remove this smooth factor from the scalar product before constructing the wavelets. In fact, recall from the discussion in §1.2 that, due to the *global* smoothness of the weight factor, the $\mathcal{H}^s$ norms of the functionals induced by the resulting modified scalar product and by the standard scalar product are equivalent so that no adverse effect on preconditioning is encountered.

If we now replace (4.2) in the domain case, by

$$(4.2^{(\Gamma)}) \quad \varphi_{k,x}(y) = \begin{cases} \check{\varphi}_{\check{k}, T_\tau^{-1}(\kappa_i^{-1}(x))}(T_\tau^{-1}(\kappa_i^{-1}(y))) & \text{if } \kappa_i^{-1}(x), \kappa_i^{-1}(y) \in \tau \in \mathcal{T}_{k-\check{k}-m,i}, \\ 0 & \text{elsewhere,} \end{cases}$$

the construction of the bases $\Phi_k = \Phi_k^{\ell,N}$ and Ξ_k, Ξ_k^*, and so the verification of (C1), $(\mathrm{C1})^*$, as well as the construction of Φ_k^*, Θ_k , Υ_k and hence of the wavelets Ψ_k follows exactly the same lines as in the domain case. Thus, after constructing the bases from Proposition 4.5 and 4.6 on a reference element for given n, ℓ and ℓ^*, we obtain an $\mathcal{H}^s$-stable wavelet basis for $|s| \leq 1$. In particular, this covers the important case of polyhedral manifolds.

4.3.4. Cancellation property. Recall that aside from the validity of norm equivalences the vanishing moment property is a corner stone of wavelet concepts being essential for compression and adaptivity. All wavelets which are supported inside a patch retain in essence these vanishing moments (with respect to functions whose pre-image is a polynomial) and thus unfold their usual compression power. However, this may no longer be the case for wavelets whose support intersects several patches Γ_i. Its part on each patch is not a wavelet and thus has no vanishing moments. Nevertheless, recall also that not the vanishing moments are important but the fact that integration of a wavelet against a smooth function produces something small which is perhaps more appropriately referred to as *cancellation property.* It is this fact that is used to derive the compression results in connection with boundary integral equations and we will point out next that the cancellation property remains valid in a form that gives rise to optimal compression effects in the context of singular integral operators.

PROPOSITION 4.7. *Under the assumptions from* §4.3.1 *and* §4.3.3*, the wavelets on manifolds obtained by the element-by-element construction have the following cancellation properties: When v is a globally continuous, patchwise smooth function on Γ one has*

$$|(v, \psi_{k+1,x})_{L_2(\Gamma)}| \lesssim 2^{-k(d^*+n/2)} |v|_{W_\infty^{d^*}(\hat{\sigma}_{k+1,x})} \|\psi_{k+1,x}\|_{L_2(\Gamma)}, \tag{4.8}$$

where $d^* := 2^{\ell^*}+1$, $\hat{\sigma}_{k+1,x}$ is some neighborhood with diameter $\eqsim 2^{-k}$ of the support of $\psi_{k+1,x}$ and $|v|_{W_\infty^{d^*}(\hat{\sigma}_{k+1,x})} := \max_{|\alpha|=d^*, 1\leq i\leq p} \sup_{\zeta\in\kappa_i^{-1}(\hat{\sigma}_{k+1,x}\cap\Gamma_i)} |D^\alpha(v\circ\kappa_i)(\zeta)|$.

Proof. Consider the projector

$$P_k v := \sum_{y\in I_k} \frac{(v, \phi^*_{k,y})_{L_2(\Gamma)}}{(\theta_{k+1,y}, \phi^*_{k,y})_{L_2(\Gamma)}} \theta_{k+1,y},$$

and note that, by construction, $P_k \psi_{k+1,x} = 0$. Thus

$$(v, \psi_{k+1,x})_{L_2(\Gamma)} = (v, (I - P_k)\psi_{k+1,x})_{L_2(\Gamma)} = ((I - P_k^*)v, \psi_{k+1,x})_{L_2(\Gamma)}.$$

In addition let

$$P_k^{(N)} v := \sum_{y\in I_k} v(y) \phi_{k,y}^{\ell^*,N}$$

denote the nodal projector associated with the nodal basis $\Phi_k^{\ell^*,N}$ of $\mathcal{S}_k^*$. Because of $P_k^* P_k^{(N)} = P_k^{(N)}$, one obtains

$$(I - P_k^*)v = (I - P_k^*)(I - P_k^{(N)})v, \tag{4.9}$$

so that

$$|(v, \psi_{k+1,x})_{L_2(\Gamma)}| \leq \|(I - P_k^*)(I - P_k^{(N)})v\|_{L_2(\sigma_{k+1,x})} \|\psi_{k+1,x}\|_{L_2(\Gamma)}, \tag{4.10}$$

where $\sigma_{k+1,x} := \operatorname{supp}(\psi_{k+1,x})$. Moreover,

$$\|(I - P_k^*)(I - P_k^{(N)})v\|_{L_2(\sigma_{k+1,x})} \lesssim \|(I - P_k^{(N)})v\|_{L_2(\hat{\sigma}_{k+1,x})},$$

where $\hat{\sigma}_{k+1,x}$ is the union of the supports of the $\theta_{k+1,y}$ for which the support of $\phi^*_{k,y}$ intersects $\sigma_{k+1,x}$. Thus one still has that $\operatorname{diam}(\hat{\sigma}_{k+1,x}) \eqsim 2^{-k}$ and thus $\hat{\sigma}_{k+1,x}$ is comprised of a uniformly bounded number of $\kappa_i(\tau)$, for $\tau \in \mathcal{T}_{k-\ell^*,i}$, $1 \leq i \leq p$.

To estimate $\|(I - P_k^{(N)})v\|_{L_2(\kappa_i(\tau))}$ for such a $\kappa_i(\tau)$, note for any polynomial p of at most degree $d^* - 1$, $(p \circ \kappa_i^{-1})|_{\kappa_i(\tau)}$ can be extended to a function $q \in \mathcal{S}_k^*$, which is therefore reproduced by $P_k^{(N)}$. Writing $v\,|_{\kappa_i(\tau)} = w \circ \kappa_i^{-1}$, for any of such p and q one has

$$\begin{aligned} 2^{nk/2}\|(I - P_k^{(N)})v\|_{L_2(\kappa_i(\tau))} &\lesssim \|(I - P_k^{(N)})v\|_{L_\infty(\kappa_i(\tau))} \\ &\leq \|p - w\|_{L_\infty(\tau)} + \|P_k^{(N)}(q - v)\|_{L_\infty(\kappa_i(\tau))}. \end{aligned}$$

Since

$$\begin{aligned} \|P_k^{(N)}(q - v)\|_{L_\infty(\kappa_i(\tau))} &\leq \sum_{y \in \kappa_i(\tau \cap V_{k,i})} |(q - v)(y)|\|\phi_{k,y}^{\ell^*,N}\|_{L_\infty(\Gamma)} \\ &\lesssim \|p - w\|_{L_\infty(\tau)} \end{aligned}$$

by taking p as a Taylor polynomial of w we conclude that

$$\|(I - P_k^{(N)})v\|_{L_2(\kappa_i(\tau))} \lesssim 2^{-nk/2} 2^{-kd^*} |v|_{W_\infty^{d^*}(\kappa_i(\tau))},$$

whence the assertion follows. $\square$

Let us now briefly point out the relevance of the estimate (4.8) in the context of *matrix compression*. Suppose that the bilinear form $a(\cdot\,,\cdot\,)$ from (1.1) has the form

$$a(u, v) = \int_\Gamma \int_\Gamma K(s, t)u(s)v(t)dsdt,$$

where the kernel K has *global support*. We wish to estimate the size of the entries

$$a(\psi_{\tilde{k}+1,y}, \psi_{k+1,x}) = (K, \psi_{\tilde{k}+1,y} \otimes \psi_{k+1,x})_{L_2(\Gamma \times \Gamma)}. \tag{4.11}$$

To this end, consider for any function $v : \Gamma \times \Gamma \to \mathbb{R}$ and any two projectors P, Q, acting on functions defined on Γ, the *Boolean sum*

$$(P \oplus Q)v := (P \otimes I)v + (I \otimes Q)v - (P \otimes Q)v, \tag{4.12}$$

where $(P \otimes I)v(s, t) := P(x \mapsto v(x, t))(s)$, $(I \otimes Q)v$ is defined analogously, and $(P \otimes Q) := (P \otimes I)(I \otimes Q)$. The Boolean sum is designed to ensure that coordinate errors multiply, i.e.

$$(I - (P \oplus Q)) = (I - P \otimes I)(I - Q \otimes I). \tag{4.13}$$

From (4.12), one readily infers that

$$(P_{\tilde{k}} \oplus P_k)(\psi_{\tilde{k}+1,y} \otimes \psi_{k+1,x}) = 0, \tag{4.14}$$

whereas, on account of (4.9), it is straightforward to verify that

$$I-(P_{\tilde{k}}\oplus P_k)^*=(I-(P^*_{\tilde{k}}\oplus P^*_k))(I-(P^{(N)}_{\tilde{k}}\oplus P^{(N)}_k)). \tag{4.15}$$

Hence, on account of (4.11), (4.14) and (4.15), the same reasoning as in the proof of Proposition 4.7 yields

$$\begin{aligned}
&2^{(\tilde{k}+k)n/2}\frac{|a(\psi_{\tilde{k}+1,y},\psi_{k+1,x})_{L_2}|}{\|\psi_{\tilde{k}+1,y}\|_{L_2}\|\psi_{k+1,x}\|_{L_2}} \qquad (4.16)\\
&\lesssim \|(I-P^{(N)}_{\tilde{k}}\otimes I)(I-I\otimes P^{(N)}_k)K\|_{L_\infty(\hat{\sigma}_{\tilde{k}+1,y}\times\hat{\sigma}_{k+1,x})}\\
&\lesssim 2^{-d^*\tilde{k}}\max_{|\alpha|=d^*,1\le i,j\le p}\ \sup_{(\zeta,t)\in\kappa_i^{-1}(\hat{\sigma}_{\tilde{k}+1,y}\cap\Gamma_i)\times(\hat{\sigma}_{k+1,x}\cap\Gamma_j)}|D^\alpha_\zeta(I-I\otimes P^{(N)}_k)K(\kappa_i(\zeta),t)|\\
&=2^{-d^*\tilde{k}}\max_{|\alpha|=d^*,1\le i,j\le p}\ \sup_{(\zeta,t)\in\kappa_i^{-1}(\hat{\sigma}_{\tilde{k}+1,y}\cap\Gamma_i)\times(\hat{\sigma}_{k+1,x}\cap\Gamma_j)}|(I-I\otimes P^{(N)}_k)D^\alpha_\zeta K(\kappa_i(\zeta),t)|\\
&\lesssim 2^{-d^*(\tilde{k}+k)}\max_{|\alpha|,|\beta|=d^*,1\le i,j\le p}\ \sup_{(\zeta,\eta)\in\kappa_i^{-1}(\hat{\sigma}_{\tilde{k}+1,y}\cap\Gamma_i)\times\kappa_j^{-1}(\hat{\sigma}_{k+1,x}\cap\Gamma_j)}|D^\beta_\eta D^\alpha_\zeta K(\kappa_i(\zeta),\kappa_j(\eta))|,
\end{aligned}$$

where we have used that taking derivatives with respect to the variable ζ commutes with the action of the nodal interpolant on the variable t.

Now in many applications the kernel K is actually smooth away from its diagonal. In fact, for the previously mentioned double-, single-layer or hypersingular operators one has estimates of the form

$$|D^\alpha_s D^\beta_t K(s,t)|\lesssim \operatorname{dist}(s,t)^{-(n+|\alpha|+|\beta|+2r)},$$

when $2r$ is the order of the equation. Combining this with (4.16), assuming that the wavelets are normalized to $\|\psi_{k,x}\|_{L_2}\eqsim 1$, one finally obtains for wavelets with disjoint supports $\Gamma_{k,x}=\operatorname{supp}\psi_{k,x}$ the typical decay estimate

$$|(a(\psi_{\tilde{k}+1,y},\psi_{k+1,x})_{L_2}|\lesssim\frac{2^{-(\tilde{k}+k)(d^*+n/2)}}{\operatorname{dist}(\Gamma_{\tilde{k}+1,y},\Gamma_{k+1,x})^{n+2d^*+2r}},$$

which all the above mentioned compression approaches are based upon, see [12, 25, 20].

4.4. Some comments on implementation. Before we turn to the discussion of several concrete realizations of the above concepts we will make some preparatory comments concerning *implementation*. We will focus on the basis transformation $\mathbf{T}_j$ on $\mathcal{S}_j$ from wavelet basis $\Psi^j=\cup_{k=0}^j\Psi_k$ to nodal basis $\Phi^{\ell,N}_j$. As we already have noted, $\mathbf{T}_j$ can be applied at the cost of $\mathcal{O}(\mathrm{dim}\mathcal{S}_j)$ operations. Here we will make some comments which our *quantitative* complexity analysis in each case will be based upon. Before starting, we briefly explain our interest in the implementation of $\mathbf{T}_j$ and its adjoint.

Recall problem (1.3): Given some $f\in\mathcal{H}^{-r}$, find $u_j\in\mathcal{S}_j$ such that

$$a(u_j,v_j)=f(v_j)\qquad(v_j\in\mathcal{S}_j), \tag{4.17}$$

where a is a scalar product satisfying $a(v,v)\eqsim\|v\|^2_{\mathcal{H}^r}$. Let us denote the matrix-vector equations corresponding to this problem with respect to $\Phi^{\ell,N}_j$ and Ψ^j respectively by $\mathbf{A}_{\Phi^{\ell,N}_j}\mathbf{U}_{\Phi^{\ell,N}_j}=\mathbf{F}_{\Phi^{\ell,N}_j}$ and $\mathbf{A}_{\Psi^j}\mathbf{U}_{\Psi^j}=\mathbf{F}_{\Psi^j}$. When r is in the stability range $(-\gamma^*,\gamma)$

of the wavelet bases, the stiffness matrix $\mathbf{A}_{\Psi^j}$, preconditioned by its diagonal, is uniformly well conditioned. On the other hand, since the wavelets on lower levels have large supports, $\mathbf{F}_{\Psi^j} = (f(\psi_{k,x}))_{k\in\{0,\dots,j\},x\in J_k}$ cannot directly be computed in $\mathcal{O}(\dim \mathcal{S}_j)$ operations, and $\mathbf{U}_{\Psi^j}^T \Psi^j$ is frequently not the representation of u_j that one likes to have. However, both problems can be solved by using the relations $\mathbf{F}_{\Psi^j} = \overline{\mathbf{T}}_j^T \mathbf{F}_{\Phi_j^{\ell,N}}$ and $\mathbf{U}_{\Phi_j^{\ell,N}} = \mathbf{T}_j \mathbf{U}_{\Psi^j}$. Of course, the situation requires a different appraisal when employing *adaptive* methods so that the subspace spanned by those wavelets needed to approximate the solution within a desired accuracy might have a significantly smaller dimension than the full trial space $\mathcal{S}_j$ where j is the highest level of resolution appearing in the wavelet subspace. We will, however, not discuss this issue in this paper.

Secondly, when $a(\cdot,\cdot)$ stems from a differential equation the stiffness matrix $\mathbf{A}_{\Psi^j}$ will, in contrast to $\mathbf{A}_{\Phi_j^{\ell,N}}$, not be (fully) sparse. Nevertheless, upon using the relation $\mathbf{A}_{\Psi^j} = \overline{\mathbf{T}}_j^T \mathbf{A}_{\Phi_j^{\ell,N}} \mathbf{T}_j$, the application of $\mathbf{A}_{\Psi^j}$ to a vector can be computed in $\mathcal{O}(\dim\mathcal{S}_j)$ operations. Note that in case $a(\cdot,\cdot)$ stems from an integral equation, the situation is reversed. Although both $\mathbf{A}_{\Psi^j}$ and $\mathbf{A}_{\Phi_j^{\ell,N}}$ are generally densely populated, it is $\mathbf{A}_{\Psi^j}$ that, thanks to the cancellation property, which will be close to a sparse matrix.

In the following, we order the wavelets in such a way that the wavelet basis Ψ^j, again viewed as a column vector, can be written as $(\Psi^j)^T = (\Psi_j^T, \cdots, \Psi_0^T)^T$. From $(\Psi^j)^T = (\Phi_j^{\ell,N})^T \mathbf{T}_j$, $\Psi_0 = \Phi_0$ and $[\Psi_{k+1}^T \quad (\Phi_k^{\ell,N})^T] = (\Phi_{k+1}^{\ell,N})^T [\mathbf{P}_{k+1,1} \quad \mathbf{P}_{k+1,0}]$, we obtain the basic version of the pyramid scheme

$$\mathbf{T}_0 = \mathbf{I}, \ \ \mathbf{T}_j = [\mathbf{P}_{j,1} \quad \mathbf{P}_{j,0}\mathbf{T}_{j-1}] \ (j = 1, 2, \dots).$$

We will refer to this version as the “naive” implementation.

We discuss two improvements of this implementation. Theorem 3.1 shows that

$$\Psi_{k+1}^T = \Upsilon_{k+1}^T - \Theta_{k+1}^T \mathbf{Z}_{k+1},$$

where the $\#I_k \times \#J_{k+1}$ matrix $\mathbf{Z}_{k+1}$ is defined by $(\mathbf{Z}_{k+1})_{y,x} = \frac{(\upsilon_{k+1,x},\phi_{k,y}^*)_{L_2}}{(\theta_{k+1,y},\phi_{k,y}^*)_{L_2}}$. In our concrete realizations discussed below in §4.5, we will always choose

$$\Upsilon_{k+1} = \{\phi_{k+1,x}^{\ell,N} : x \in J_{k+1}\} \subset \Phi_{k+1}^{\ell,N}, \tag{4.18}$$

that is, $\Upsilon_{k+1}^T = (\Phi_{k+1}^{\ell,N})^T E_{k+1}$, with the $\#I_{k+1} \times \#J_{k+1}$ matrix $\mathbf{E}_{k+1}$ defined by $(\mathbf{E}_{k+1})_{x,y} = \delta_{x,y}$. With $\mathbf{G}_{k+1}$ the $\#I_{k+1} \times \#I_k$ matrix defined by $\Theta_{k+1}^T = (\Phi_{k+1}^{\ell,N})^T \mathbf{G}_{k+1}$, we obtain

$$\mathbf{P}_{k+1,1} = \mathbf{E}_{k+1} - \mathbf{G}_{k+1}\mathbf{Z}_{k+1}.$$

Thanks to the fact that the sparse matrices $\mathbf{G}_{k+1}$ and $\mathbf{Z}_{k+1}$ have smaller sizes than $\mathbf{P}_{k+1,1}$, at least for $n > 1$ making use of this “factorized” form of $\mathbf{P}_{k+1,1}$ results in a more efficient implementation.

In our realizations we will construct $\theta_{k+1,y}$ $(y \in I_k)$ as linear combinations of $\phi_{k+1-i,x}^{\ell,N}$ $(x \in I_{k+1-i})$ for $0 \le i \le m + 1 - \ell$, where $m = \max\{\ell, \ell^*\}$. As a second improvement, instead of expressing for $i > 0$, $\phi_{k+1-i,x}^{\ell,N}$ directly in terms of $\phi_{k+1,z}^{\ell,N}$, we can exploit the fact that the prolongation operations needed for this purpose have to be executed anyway. This observation has been also made in [23, 17].

More specifically, let

$$\Theta_{k+1}^T = \sum_{i=0}^{m+1} (\Phi_{k+1-i}^{\ell,N})^T \mathbf{G}_{k+1,i}. \tag{4.19}$$

Then setting $(\mathbf{d}^j)^T = (\mathbf{d}_j^T, \cdots, \mathbf{d}_0)^T$ $(\mathbf{d}_k \in \mathbb{C}^{\# J_k})$, the arrays $\mathbf{c}_j = \mathbf{T}_j \mathbf{d}^j$ can be computed by performing the following steps:

```
$\mathbf{d}_{j+1} = \cdots = \mathbf{d}_{j+m-\ell} := 0$
$\mathbf{c}_{-1} = \cdots = \mathbf{c}_{-m+\ell} := 0$
$\mathbf{c}_0 := \mathbf{d}_0$
for $k = 1$ to $j + m - \ell$ do
    $\mathbf{q}_k := \mathbf{Z}_k \mathbf{d}_k$
    $\mathbf{c}_k := \mathbf{E}_k \mathbf{d}_k - \mathbf{G}_{k,0} \mathbf{q}_k$
    for $i = 1$ to $m + \ell + 1$ do
        $\mathbf{c}_{k-i} := \mathbf{c}_{k-i} - \mathbf{G}_{k,-i} \mathbf{q}_k$
    end
    $\mathbf{c}_{k-m+\ell} := \mathbf{c}_{k-m+\ell} + \mathbf{P}_{k-m+\ell,0} \mathbf{c}_{k-m+\ell-1}$
end
```

We will refer to this implementation as the "advanced" implementation. Ignoring, for the case $\ell \leq m-1$, a few computations on levels with negative indices, we note that besides the costs of applying the prolongation operators $\mathbf{P}_{k,0}$ $(1 \leq k \leq j)$, that are determined by the multiresolution analysis $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots$, and the costs of applying the $\mathbf{Z}_k$ $(1 \leq k \leq j)$, determined by the Φ_{k-1}^*, the costs of the "advanced" implementation of the pyramid scheme depend on the total number of of non-zero entries in the matrices $\mathbf{G}_{k,i}$. In our realizations we will minimize this number of non-zero entries in the non-unique representation (4.19). The resulting version of "advanced" implementation will turn out to be significantly more economic than the " naive" one.

4.5. Applications, realization of concrete cases. We will confine the discussion to *piecewise linear* wavelets, i.e., $\ell = 0$, in combination with the following cases:

- $\ell^* = 0$, $n \in \{1, 2, 3\}$,
- $\ell^* = 1$, $n \in \{1, 2\}$,
- $\ell^* = 2$, $n \in \{1, 2\}$.

In all cases one has $m = \max\{\ell, \ell^*\} = \ell^*$, so that $\check{\mathcal{T}}_{-\ell^*} = \{\check{\tau}\}$. We choose

$$\check{\tau} = \{x \in \mathbb{R}^n : 0 \leq x_1 \leq \cdots \leq x_n \leq 2^{\ell^*+1}\},$$

so that $\check{I}_{\check{k}} = \check{\tau} \cap 2^{1-\check{k}} \mathbb{N}^n$ $(-\ell^* \leq \check{k} \leq 1)$. In particular, $\check{I}_{-\ell^*}$ is the set of the $n+1$ vertices of $\check{\tau}$.

Concerning (C1) and (C1)*, when $\ell^* = 0$ we have $\mathcal{S}_k = \mathcal{S}_k^*$, and so both conditions are trivially satisfied. For the other cases we have verified the sufficient condition $\Re(\check{\mathbf{M}}_0) > 0$ from Proposition 4.5, where we took $\check{\Xi}_0 = \Phi_0^{0,N}$ and $\check{\Xi}_0^* = \Phi_0^{\ell^*,N}$.

As we have already announced in (4.18), in all cases we fix

$$\check{\Upsilon}_1 = \{\check{\phi}_{1,\check{x}}^{0,N} : \check{x} \in \check{J}_1\},$$

and thus

$$\Upsilon_{k+1} = \{\phi_{k+1,x}^{0,N} : x \in J_{k+1}\}.$$

For the above mentioned cases, in the following we define the remaining ingredients of the wavelet construction, viz. a basis $\check{\Phi}_0^* = \{\check{\phi}_{0,\check{y}}^* : \check{y} \in \check{I}_0\}$ of $\check{S}(\ell^*, 0)$, and a biorthogonal set $\check{\Theta}_1 = \{\check{\theta}_{1,\check{y}} : \check{y} \in \check{I}_0\} \subset \check{S}(0,1)$ such that $\check{\Upsilon}_1 \cup \check{\Theta}_1$ is a basis of $\check{S}(0,1)$.

4.5.1. The case $\ell^* = 0$, $n \in \{1,2,3\}$. In this case $\check{\mathcal{T}}_0 = \{\check{\tau}\}$, $\check{I}_0$ is the set of vertices of $\check{\tau}$, and $\check{J}_1$ is the set of midpoints of the edges of $\check{\tau}$.

We set

$$\check{\Phi}_0^* = \check{\Phi}_0^{0,N}$$

and search for $\check{\theta}_{1,\check{y}}$ $(\check{y} \in \check{I}_0)$ as a linear combination of the fine- and coarse-grid nodal basis functions $\check{\phi}_{1,\check{y}}^{0,N}$ and $\check{\phi}_{0,\check{y}}^{0,N}$. Some computations reveal that

$$\check{\theta}_{1,\check{y}} = 2(n+1)!\,(\check{\phi}_{1,\check{y}}^{0,N} - (\tfrac{1}{2})^{n+1} \check{\phi}_{0,\check{y}}^{0,N})$$

satisfies $(\check{\theta}_{1,\check{y}}, \check{\phi}_{0,\check{z}}^{0,N})_{L_2(\check{\tau})} = \delta_{\check{y},\check{z}}$ $(\check{y}, \check{z} \in \check{I}_0)$, for any $n \in I\!\!N$.

The union $\check{\Upsilon}_1 \cup \check{\Theta}_1$ is a basis of $\check{S}(0,1)$. Both $\check{\Phi}_0^*$ and $\check{\Upsilon}_1 \cup \check{\Theta}_1$ satisfy the condition (P1) of invariance on $\partial\check{\tau}$ under permutations of the barycentric coordinates, as well as the condition (P2) which says that basis functions labeled with $\check{x}$ vanish on faces that do not contain $\check{x}$. Taking $\Psi_0 = \Phi_0^{0,N}$, the application of Proposition 4.6 and Theorem 3.1 now yields a basis $\Psi^j = \cup_{k=0}^j \Psi_k$ of $\mathcal{S}_j$ that is uniformly stable in $\mathcal{H}^s$ for $|s| < \frac{3}{2}$ ($|s| \leq 1$ for Lipschitz manifolds). Moreover, its elements, except those from the coarsest level, have vanishing moments of order $2^{\ell^*} + 1 = 2$ (respectively satisfy estimates of the type (4.8)).

By construction, we have

$$\psi_{k+1,x} = \phi_{k+1,x}^{0,N} - \sum_{y \in I_k} \frac{(\phi_{k+1,x}^{0,N}, \phi_{k,y}^{0,N})_{L_2}}{(\theta_{k+1,y}, \phi_{k,y}^{0,N})_{L_2}} \theta_{k+1,y} \qquad (k \geq 0,\ x \in J_{k+1}),$$

with $\theta_{k+1,y} = 2(n+1)!\,(\phi_{k+1,y}^{0,N} - (\frac{1}{2})^{n+1} \phi_{k,y}^{0,N})$. As shown in (4.6), the computation of the coefficients $\frac{(\phi_{k+1,x}^{0,N}, \phi_{k,y}^{0,N})_{L_2}}{(\theta_{k+1,y}, \phi_{k,y}^{0,N})_{L_2}}$ can be reduced to the computation of $(\check{\phi}_{1,\check{x}}^{0,N}, \check{\phi}_{0,\check{y}}^{0,N})_{L_2(\check{\tau})}$ for $\check{x} \in \check{J}_1$, $\check{y} \in \check{I}_0$.

For $n \leq 2$, the involved basis functions on the reference element, and so the corresponding scalar products are invariant under permutations of the barycentric coordinates (cf. Remark 4.2). For $n = 1$,

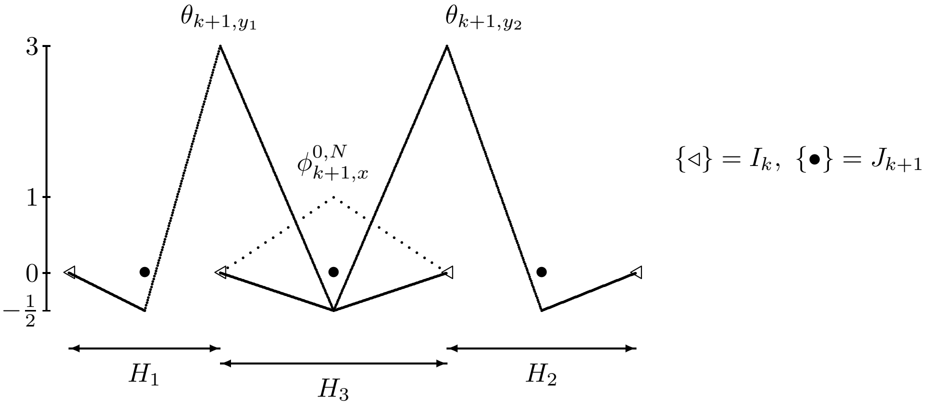
$$(\check{\phi}_{1,\check{x}}^{0,N}, \check{\phi}_{0,\check{y}}^{0,N})_{L_2(\check{\tau})} = \tfrac{1}{2} \qquad (\check{x} \in \check{J}_1,\ \check{y} \in \check{I}_0).$$

An illustration of the wavelet construction in this case is given in Fig. 4.1.

For $n = 2$, $\check{x} \in \check{J}_1$, $\check{y} \in \check{I}_0$ one has

$$(\check{\phi}_{1,\check{x}}^{0,N}, \check{\phi}_{0,\check{y}}^{0,N})_{L_2(\check{\tau})} = \begin{cases} \frac{5}{24} & \text{if } \check{x}, \check{y} \text{ share an edge} \\ \frac{1}{12} & \text{if } \check{x}, \check{y} \text{ do not share an edge} \end{cases}.$$

Recall that for $n = 3$ the dyadic subdivision of tetrahedra is not unique. More precisely, the convex hull of the midpoints of the edges has yet to be decomposed into four tetrahedra. This can be done by making any two midpoints of edges that have no vertex in common the end points of a new edge. There are three such possibilities to form a dyadic triangulation $\check{\mathcal{T}}_1$. Once such a decomposition has been fixed the set



$$\psi_{k+1,x} = \phi^{0,N}_{k+1,x} - \sum_{i\in\{1,2\}} \tfrac{1}{2}\tfrac{H_i}{H_i+H_3}\theta_{k+1,y_i}$$

FIG. 4.1. *Wavelet construction for $n = 1$, $\ell = \ell^* = 0$.*

$\check{J}_1$ can be splitted into the set A containing these two midpoints connected with the new edge, and the set B of the remaining midpoints. Note that the elements of A are vertices of six tetrahedra, while each element of B is shared by four tetrahedra. One can verify that for $\check{x} \in \check{J}_1$, $\check{y} \in \check{I}_0$ one has

$$(\check{\phi}^{0,N}_{1,\check{x}}, \check{\phi}^{0,N}_{0,\check{y}})_{L_2(\check{\tau})} = \begin{cases} \frac{4}{48} & \text{if } \check{x} \in A \text{ and } \check{x}, \check{y} \text{ share an edge} \\ \frac{2}{48} & \text{if } \check{x} \in A \text{ and } \check{x}, \check{y} \text{ do not share an edge} \\ \frac{3}{48} & \text{if } \check{x} \in B \text{ and } \check{x}, \check{y} \text{ share an edge} \\ \frac{1}{48} & \text{if } \check{x} \in B \text{ and } \check{x}, \check{y} \text{ do not share an edge.} \end{cases}$$

As an application of the complexity analysis carried out in §4.4, we give an asymptotic operation count (for $j \to \infty$) per unknown for performing the basis transformation on $\mathcal{S}_j$. For the case $n > 1$ we assume that the mesh is uniform, i.e., $\Omega = [0,1]^n$ and

$$\mathcal{T}_k = \{2^{-k}\alpha + \{x \in \mathbf{R}^n; 0 \le x_{\sigma(1)} \le \cdots \le x_{\sigma(n)} \le 2^{-k}\} : \alpha \in \{1,\ldots,2^k-1\}^n, \\ \sigma \text{ a permutation of } \{1,\ldots,n\}\}.$$

We do not count scalar operations when they are combined with vector operations.

For $n = 1$, $\psi_{k+1,x}$ equals $\phi^{0,N}_{k+1,x}$ minus some linear combination of two $\theta_{k+1,y}$'s, i.e., the matrices $\mathbf{Z}_{k+1}$ from §4.4 contain only two nonzero entries per column. By expanding each $\theta_{k+1,y}$ in terms of $\Phi^{0,N}_{k+1}$, we see that each wavelet is a linear combination of 5 nodal basis functions on its level. For both the "naive" and the "advanced" implementation, the operation count is **8** operations per unknown.

For $n = 2$, $\psi_{k+1,x} - \phi^{0,N}_{k+1,x}$ is a linear combination of four $\theta_{k+1,y}$'s, in the uniform mesh case yielding a wavelet that is a linear combination of 23 nodal basis functions on its level (see [22]). The operation count for the advanced implementation is **8** operations, compared to $25\frac{1}{3}$ for the naive implementation. Thus, the advanced implementation realizes the basis transformation at the same expense as the naive implementation of the basis transformation for a wavelet basis where the wavelets are linear combinations of an average of $5\frac{2}{3}$ nodal basis functions.

For $n = 3$, in the uniform mesh case $\psi_{k+1,x} - \phi^{0,N}_{k+1,x}$ is a linear combination of six (3 of 7 cases) or eight (4 of 7 cases) $\theta_{k+1,y}$'s, or each wavelet is a linear combination of 77 or 101 nodal basis functions on its level (see [22]). The operation count for the advanced implementation is $\mathbf{10\frac{2}{7}}$ operations, compared to $92\frac{4}{7}$ operations for the naive implementation.

To compare this, with $\psi_{k+1,x} = \phi^{0,N}_{k+1,x}$, i.e., with the hierarchical basis (cf. [26]), an analogous operation count for the basis transformation to nodal basis yields 4, $3\frac{1}{3}$ and $3\frac{1}{7}$ operations per unknown, for $n = 1$, 2 and 3, respectively. Of course, the hierarchical basis neither has any cancellation properties nor gives rise to norm equivalences within the relevant scope of Sobolev regularity.

4.5.2. The case $\ell^* = 1$, $n = 1$. In this case $\check{\mathcal{T}}_{-1} = \{\check{\tau}\}$, $\check{\tau} = [0,4]$, $\check{I}_{-1} = \{0,4\}$, $\check{J}_0 = \{2\}$ and $\check{J}_1 = \{1,3\}$. Whenever this is relevant, we order elements from the index sets with increasing sizes.

An obvious choice for $\check{\Phi}^*_0$ would be to take $\check{\Phi}^{1,N}_0$, i.e., the nodal basis of second order polynomials with respect to the nodes $\check{I}_0$. However, note that we may add a multiple of $\check{\phi}^{1,N}_{0,2}$ to $\check{\phi}^{1,N}_{0,0}$ and $\check{\phi}^{1,N}_{0,4}$, since then (P2) is retained. We use this freedom to minimize the number of non-zero coefficients $(\check{\phi}^{0,N}_{1,\check{x}}, \check{\phi}^*_{0,\check{y}})_{L_2(\check{\tau})}$ ($\check{x} \in \check{J}_1$, $\check{y} \in \check{I}_0$), and thus the number of non-zero coefficients in (3.1). We take

$$\check{\Phi}^{*\,T}_0 = (\check{\Phi}^{1,N}_0)^T \begin{bmatrix} 1 & 0 & 0 \\ \frac{5}{24} & 1 & \frac{5}{24} \\ 0 & 0 & 1 \end{bmatrix}.$$

This choice yields

$$\Big((\check{\phi}^{0,N}_{1,\check{x}}, \check{\phi}^*_{0,\check{y}})_{L_2(\check{\tau})}\Big)_{\check{x}\in\check{J}_1,\check{y}\in\check{I}_0} = \begin{bmatrix} \frac{1}{2} & \frac{17}{24} & 0 \\ 0 & \frac{17}{24} & \frac{1}{2} \end{bmatrix}.$$

We wish to represent $\check{\theta}_{1,\check{y}}$ for $\check{y} \in \{0,4\}$ as a linear combination of $\check{\phi}^{0,N}_{-1,\check{y}}$, $\check{\phi}^{0,N}_{0,\check{y}}$ and $\check{\phi}^{0,N}_{1,\check{y}}$, and $\check{\theta}_{1,2}$ as a linear combination of $\check{\phi}^{0,N}_{0,2}$ and $\check{\phi}^{0,N}_{1,2}$. Another option would be to select $\check{\theta}_{1,0}$ e.g. as a linear combination of $\check{\phi}^{0,N}_{1,0}$, $\check{\phi}^{0,N}_{1,1}$ and $\check{\phi}^{0,N}_{1,2}$, and analogously for $\check{\theta}_{1,4}$. This choice would minimize the supports of the resulting wavelets. However, note that with the latter $\check{\theta}_{1,0}$ and $\check{\theta}_{1,4}$, global biorthogonal basis functions $\theta_{k+1,y}$ with $y \in I_k$ a boundary point of a "macro element", i.e. $y \in I_{k-1}$, would be linear combinations of 5 nodal basis functions. With our choice, global biorthogonal basis functions $\theta_{k+1,y}$ are linear combinations of 3 or 2 ($y \in I_{k-1}$ or $y \in J_k$) nodal basis functions, however, belonging to different levels. Nevertheless, as explained in §4.4, this latter fact is harmless as far as the cost of implementation is concerned.

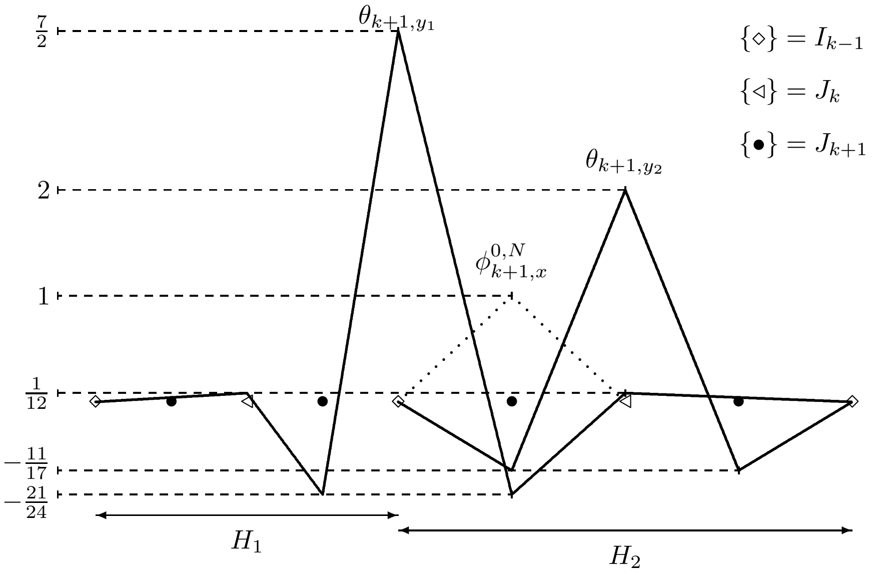
At this point we stress that the favorable effect on the costs of the implementation of selecting $\check{\theta}_{1,\check{y}}$ as a linear combination of nodal basis functions from different levels is more enhanced in more dimensions. On the other hand, it will appear that in more dimensions, there is less freedom in selecting a clever $\check{\Phi}^*_0$ such that $\Big((\check{\phi}^{0,N}_{1,\check{x}}, \check{\phi}^*_{0,\check{y}})_{L_2(\check{\tau})}\Big)_{\check{x}\in\check{J}_1,\check{y}\in\check{I}_0}$ has possibly many zeros.

Some computations show that

$$\begin{aligned} \check{\theta}_{1,\check{y}} &= \tfrac{1}{6}\check{\phi}^{0,N}_{-1,\check{y}} - 2\check{\phi}^{0,N}_{0,\check{y}} + \tfrac{16}{3}\check{\phi}^{0,N}_{1,\check{y}}, \qquad \check{y} \in \{0,4\}, \\ \check{\theta}_{1,2} &= -\tfrac{22}{17}\check{\phi}^{0,N}_{0,2} + \tfrac{56}{17}\check{\phi}^{0,N}_{1,2}, \end{aligned}$$

satisfy $(\check{\theta}_{1,\check{y}}, \check{\phi}^*_{0,\check{z}})_{L_2(\check{\tau})} = \delta_{\check{y},\check{z}}$ ($\check{y}, \check{z} \in \check{I}_0$), and $\check{\Upsilon}_1 \cup \check{\Theta}_1$ is a basis of $\check{S}(0,1)$.

Since $\check{\Phi}_0^*$ and $\check{\Upsilon}_1 \cup \check{\Theta}_1$ satisfy (P2), and are invariant under permutations of the barycentric coordinates (cf. Remark 4.2), Proposition 4.6 and Theorem 3.1 yield an $\mathcal{H}^s$-stable wavelet basis for $|s| < \frac{3}{2}$ ($|s| \leq 1$), now consisting of wavelets having $2^{\ell^*} + 1 = 3$ vanishing moments. An illustration is given in Fig. 4.2.



$$\psi_{k+1,x} = \phi^{0,N}_{k+1,x} - \tfrac{17}{24}\theta_{k+1,y_2} - \tfrac{1}{2}\tfrac{H_2}{H_1+H_2}\theta_{k+1,y_1}$$

FIG. 4.2. *Wavelet construction for $n = 1$, $\ell = 0$ and $\ell^* = 1$.*

As in the case $\ell = 0$, $n = 1$, $\psi_{k+1,x} - \phi^{0,N}_{k+1,x}$ is a linear combination of only two $\theta_{k+1,y}$'s, which itself are linear combinations of 3 and 2 nodal basis functions from different levels. With the "advanced" implementation as exposed in §4.4, for $j \to \infty$ the basis transformation $\mathbf{T}_j$ on $\mathcal{S}_j$ from wavelet- to nodal basis costs $\mathbf{8\frac{1}{2}}$ operations per unknown.

4.5.3. The case $\ell^* = 2$, $n = 1$. In this case $\check{\mathcal{T}}_{-2} = \{\check{\tau}\}$, $\check{\tau} = [0, 8]$, $\check{I}_{-2} = \{0, 8\}$, $\check{J}_{-1} = \{4\}$, $\check{J}_0 = \{2, 6\}$ and $\check{J}_1 = \{1, 3, 5, 7\}$.

Following the ideas described in §4.5.2, we define

$$(\check{\Phi}_0^*)^T = (\check{\Phi}_0^{2,N})^T \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{29781}{315868} & 1 & \frac{439}{1132} & \frac{-417}{5807} & \frac{2551}{315868} \\ \frac{-2585}{157934} & \frac{1420}{5807} & 1 & \frac{1420}{5807} & \frac{-2585}{157934} \\ \frac{2551}{315868} & \frac{-417}{5807} & \frac{439}{1132} & 1 & \frac{29781}{315868} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The resulting $\check{\Phi}_0^*$ is a basis of $\check{M}(2, 0)$, and

$$\Big((\check{\phi}^{0,N}_{1,\check{x}}, \check{\phi}^*_{0,\check{y}})_{L_2(\check{\tau})} \Big)_{\check{x} \in \check{J}_1, \check{y} \in \check{I}_0} = \begin{bmatrix} \frac{12}{29} & \frac{5057}{5807} & 0 & 0 & 0 \\ 0 & \frac{11281}{17421} & \frac{2723}{3396} & 0 & 0 \\ 0 & 0 & \frac{2723}{3396} & \frac{11281}{17421} & 0 \\ 0 & 0 & 0 & \frac{5057}{5807} & \frac{12}{29} \end{bmatrix}.$$

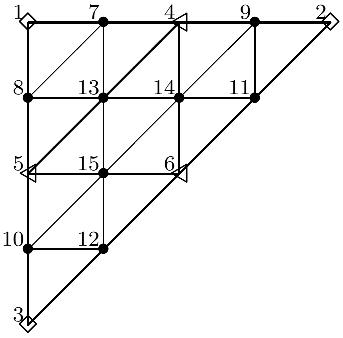


FIG. 4.3. *Numbering of* $\check{I}_{-1} = \{\Diamond\}$, $\check{J}_0 = \{\triangleleft\}$ *and* $\check{J}_1 = \{\bullet\}$.

With

$$\begin{aligned}
\check{\theta}_{1,0} &= \tfrac{2}{25}\check{\phi}^{0,N}_{-2,0} - \tfrac{93}{400}\check{\phi}^{0,N}_{-1,0} + \tfrac{1091}{900}\check{\phi}^{0,N}_{0,0} + \tfrac{137}{30}\check{\phi}^{0,N}_{1,0} - \tfrac{3799}{1800}\check{\phi}^{0,N}_{1,3}\\
\check{\theta}_{1,2} &= -\tfrac{5807}{5835}\check{\phi}^{0,N}_{1,1} + \tfrac{23228}{5835}\check{\phi}^{0,N}_{1,2} - \tfrac{2781553}{733265}\check{\phi}^{0,N}_{1,3} + \tfrac{4053286}{2199795}\check{\phi}^{0,N}_{1,4} - \tfrac{795559}{2199795}\check{\phi}^{0,N}_{1,5}\\
\check{\theta}_{1,4} &= \tfrac{16414}{40845}\check{\phi}^{0,N}_{-1,4} - \tfrac{535436}{106197}\check{\phi}^{0,N}_{0,4} + \tfrac{4874392}{530985}\check{\phi}^{0,N}_{1,4},
\end{aligned}$$

and $\check{\theta}_{1,6}$, $\check{\theta}_{1,8}$ defined as $\check{\theta}_{1,2}$, $\check{\theta}_{1,0}$ by permuting the barycentric coordinates, one has $(\check{\theta}_{1,\check{y}}, \check{\phi}^*_{0,\check{z}})_{L_2(\check{\tau})} = \delta_{\check{y},\check{z}}$ $(\check{y}, \check{z} \in \check{I}_0)$, and $\check{\Upsilon}_1 \cup \check{\Theta}_1$ is a basis of $\check{S}(0,1)$.

Both $\check{\Phi}^*_0$ and $\check{\Upsilon}_1 \cup \check{\Theta}_1$ satisfy (P2), and are invariant under permutations of the barycentric coordinates (cf. Remark 4.2), and so we obtain an $\mathcal{H}^s$-stable wavelet basis for $|s| < \frac{3}{2}$ $(|s| \leq 1)$, consisting of wavelets having $2^{\ell^*} + 1 = 5$ vanishing moments.

The above choice of $\check{\Theta}_1$ yields a minimizer for the cost of implementation. For $j \to \infty$, the basis transformation $\mathbf{T}_j$ on $\mathcal{S}_j$ for $j \to \infty$ from wavelet-to-nodal basis can be performed with $\mathbf{10\frac{1}{2}}$ operations per unknown.

4.5.4. The case $\ell^* = 1$, $n = 2$. In this case $\check{\mathcal{T}}_{-1} = \{\check{\tau}\}$, where $\check{\tau}$ is the reference triangle $\{x \in I\!R^2 : 0 \leq x_1 \leq x_2 \leq 4\}$. For convenience, we introduce a hierarchical numbering of elements of $\check{I}_{-1}$, $\check{J}_0$ and $\check{J}_1$ as indicated in Fig. 4.3, and in the following we will identify sets $\check{I}_{\check{k}}$ and $\check{J}_{\check{k}}$ with the corresponding sets of numbers.

We will define $\check{\Phi}^*_0$ and $\check{\Theta}_1$ invariant under permutations of the barycentric coordinates. This means that it is sufficient to specify $\check{\phi}^*_{0,\check{y}}$ and $\check{\theta}_{1,\check{y}}$ for $\check{y} \in \{1, 6\}$.

In view of (P2), we have to take

$$\check{\phi}^*_{0,6} = \check{\phi}^{1,N}_{0,6}.$$

We may look for $\check{\phi}^*_{0,1}$ as a linear combination of $\check{\phi}^{1,N}_{0,1}$ and $\check{\phi}^{1,N}_{0,4} + \check{\phi}^{1,N}_{0,5}$ (However, because of (P1), an arbitrary linear combination of $\check{\phi}^{1,N}_{0,1}$, $\check{\phi}^{1,N}_{0,4}$ and $\check{\phi}^{1,N}_{0,5}$ is not allowed). We use this freedom for making $(\check{\phi}^{0,N}_{1,11}, \check{\phi}^*_{0,1})_{L_2(\tau)} = (\check{\phi}^{0,N}_{1,12}, \check{\phi}^*_{0,1})_{L_2(\check{\tau})} = 0$. One may verify that it would be equally efficient to make $(\check{\phi}^{0,N}_{1,\check{x}}, \check{\phi}^*_{0,1})_{L_2(\tau)} = 0$ for $\check{x} = 14, 15$. Instead, arranging this scalar product to be zero for $\check{x} = 7, 8$ or $\check{x} = 9, 10$ or $\check{x} = 13$ would save only half the number of operations. With

$$\check{\phi}^*_{0,1} = \check{\phi}^{1,N}_{0,1} + \tfrac{3}{14}(\check{\phi}^{1,N}_{0,4} + \check{\phi}^{1,N}_{0,5}),$$

one has

$$\Big((\check\phi^{0,N}_{1,\check x}, \check\phi^*_{0,\check y})_{L_2(\check\tau)}\Big)_{\check x=7,9,11,13,14,\,\check y=1,6} = \begin{bmatrix} \frac{27}{112} & \frac{5}{336} & 0 & \frac{19}{84} & \frac{1}{21} \\ \frac{1}{32} & \frac{11}{96} & \frac{9}{32} & \frac{11}{48} & \frac{23}{48} \end{bmatrix}^T,$$

whereas the other quantities $(\check\phi^{0,N}_{1,\check x}, \check\phi^*_{0,\check y})_{L_2(\check\tau)}$ for $\check x \in \check J_1$, $\check y \in \check I_0$ follow by permuting the barycentric coordinates.

We search for $\check\theta_{1,1}$ as a linear combination of $\check\phi^{0,N}_{-1,1}$, $\check\phi^{0,N}_{0,1}$, $\check\phi^{0,N}_{1,1}$ and $\check\phi^{0,N}_{1,13}$, and $\check\theta_{1,6}$ as a linear combination of $\check\phi^{0,N}_{0,6}$, $\check\phi^{0,N}_{1,6}$, $\check\phi^{0,N}_{1,13}$ and $(\check\phi^{0,N}_{1,11} + \check\phi^{0,N}_{1,12})$. Although these choices do not minimize the supports of the resulting wavelets, they do minimize the costs of the implementation. Requiring that $(\check\theta_{1,\check y}, \phi^*_{0,\check z})_{L_2(\check\tau)} = \delta_{\check y,\check z}$ $(\check y, \check z \in \check I_0)$, we found

$$\begin{aligned}
\check\theta_{1,1} &= \tfrac{1}{8}\check\phi^{0,N}_{-1,1} - 3\check\phi^{0,N}_{0,1} + 16\check\phi^{0,N}_{1,1} + 0\check\phi^{0,N}_{1,13} \\
\check\theta_{1,6} &= \tfrac{-38}{63}\check\phi^{0,N}_{0,6} + \tfrac{376}{63}\check\phi^{0,N}_{1,6} + \tfrac{8}{63}\check\phi^{0,N}_{1,13} - \tfrac{4}{3}(\check\phi^{0,N}_{1,11} + \check\phi^{0,N}_{1,12}),
\end{aligned}$$

whereas the other $\check\theta_{1,\check y}$ are obtained by permuting the barycentric coordinates. Note that the resulting global biorthogonal basis functions $\theta_{k+1,y}$ are linear combinations of three $(y \in I_{k-1})$ or six ($\check\phi^{0,N}_{1,13}$ is doubled) $(y \in J_k)$ nodal basis functions from different levels.

The constructed $\check\Phi^*_0$ and $\check\Upsilon_1 \cup \check\Theta_1$ are bases of $\check S(1,0)$ and $\check S(0,1)$, respectively, and satisfy (P1) and (P2). By Proposition 4.6 and Theorem 3.1, an $\mathcal{H}^s$-stable wavelet basis is obtained for $|s| < \frac{3}{2}$ $(|s| \le 1)$, consisting of wavelets having vanishing moments of order $2^{\ell^*} + 1 = 3$.

For a uniform mesh and the number of levels tending to infinity, the basis transformation from wavelet to nodal basis via the "advanced" implementation as described in §4.4, costs $\mathbf{11\frac{7}{12}}$ operations per unknown.

4.5.5. The case $\ell^* = 2$, $n = 2$. In this case $\check{\mathcal{T}}_{-2} = \{\check\tau\}$, where $\check\tau$ is the reference triangle $\{x \in I\!\!R^2 : 0 \le x_1 \le x_2 \le 8\}$. We number the elements of $\check I_{-2}$, $\check J_{-1}$, $\check J_0$ and $\check J_1$ as indicated in Fig. 4.4. Based on this numbering we will identify in the following $\check I_{\check k}$ and $\check J_{\check k}$ with corresponding sets of numbers.

We will define $\check\Phi^*_0$ and $\check\Theta_1$ invariant under permutations of the barycentric coordinates, and so we only have to specify $\check\phi^*_{0,\check y}$ and $\check\theta_{1,\check y}$ for $\check y \in \{1, 6, 7, 13\}$. Following the ideas developed in §4.5.1-4.5.4, we take

$$\begin{aligned}
\check\phi^*_{0,1} &= \check\phi^{2,N}_{0,1} - \tfrac{18073}{3761398}(\check\phi^{2,N}_{0,4} + \check\phi^{2,N}_{0,5}) + \tfrac{1547703}{7522796}(\check\phi^{2,N}_{0,7} + \check\phi^{2,N}_{0,8}) - \tfrac{68623}{7522796}(\check\phi^{2,N}_{0,9} + \check\phi^{2,N}_{0,10}) + \\
&\qquad \tfrac{54503}{3761398}\check\phi^{2,N}_{0,13} - \tfrac{6415}{7522796}(\check\phi^{2,N}_{0,14} + \check\phi^{2,N}_{0,15}) \\
\check\phi^*_{0,6} &= \check\phi^{2,N}_{0,6} + \tfrac{111981}{178628}(\check\phi^{2,N}_{0,11} + \check\phi^{2,N}_{0,12}) + \tfrac{5459}{178628}\check\phi^{2,N}_{0,13} + \tfrac{18549}{89314}(\check\phi^{2,N}_{0,14} + \check\phi^{2,N}_{0,15}) \\
\check\phi^*_{0,7} &= -\tfrac{32}{45}\check\phi^{2,N}_{0,4} + \check\phi^{2,N}_{0,7} - \tfrac{11}{15}\check\phi^{2,N}_{0,9} + \tfrac{1}{9}\check\phi^{2,N}_{0,13} - \tfrac{13}{45}\check\phi^{2,N}_{0,14} - \tfrac{2}{45}\check\phi^{2,N}_{0,15} \\
\check\phi^*_{0,13} &= \check\phi^{2,N}_{0,13} + \tfrac{229}{527}(\check\phi^{2,N}_{0,14} + \check\phi^{2,N}_{0,15}).
\end{aligned}$$

Then

$$\Big((\check\phi^{0,N}_{1,\check x}, \check\phi^*_{0,\check y})_{L_2(\check\tau)}\Big)_{\check x=16,18,20,22,24,26,28,29,31,33,35,37,39,41,43,45,\ \check y=1,6,13} =$$

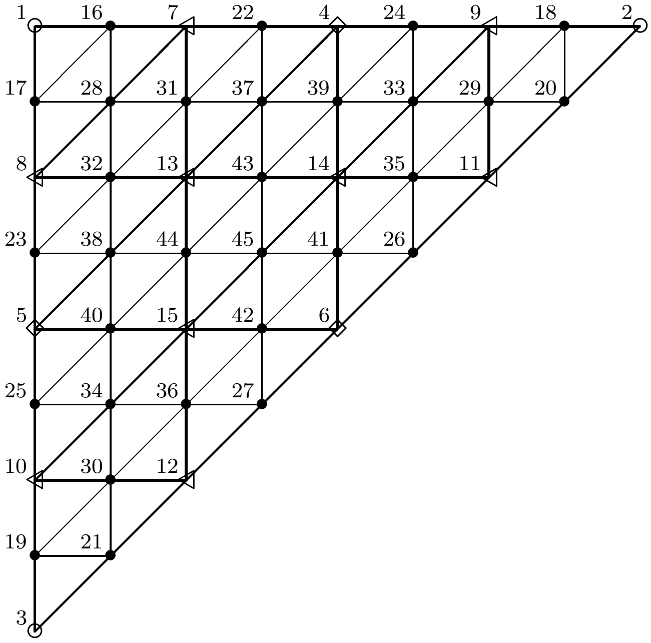


FIG. 4.4. *Numbering of* $\check{I}_{-2} = \{\bigcirc\}$, $\check{J}_{-1} = \{\Diamond\}$, $\check{J}_0 = \{\lhd\}$ *and* $\check{J}_1 = \{\bullet\}$.

$$\left[\begin{array}{ccccccccc}
\big[\frac{41058559}{180547104} & \frac{-88145}{180547104} & 0 & \frac{4733767}{180547104} & \frac{-1030601}{180547104} & 0 & \frac{427722}{1880699} & \frac{-1728}{1880699} & \\
 & \frac{289767}{3761398} & \frac{-8208}{1880699} & 0 & \frac{94659}{7522796} & \frac{-9936}{1880699} & \frac{864}{1880699} & 0 & 0\ \big] \\
\big[\quad 0 & \frac{4685}{178628} & \frac{635179}{6430608} & 0 & \frac{3263}{535884} & \frac{2337265}{6430608} & 0 & \frac{48895}{267942} & \\
 & \frac{4387}{1071768} & \frac{108937}{1071768} & \frac{139431}{357256} & \frac{39479}{2143536} & \frac{34971}{714512} & \frac{1125269}{2143536} & \frac{13127}{133971} & \frac{46597}{178628}\big] \\
\big[\ \frac{3427}{37944} & \frac{1937}{303552} & 0 & \frac{49583}{303552} & \frac{13393}{151776} & \frac{11}{12648} & \frac{25655}{50592} & \frac{115}{1984} & \\
 & \frac{24691}{33728} & \frac{4225}{16864} & \frac{641}{5952} & \frac{11697}{16864} & \frac{50365}{101184} & \frac{263}{1984} & \frac{26059}{33728} & \frac{24263}{50592}\big]
\end{array}\right]^T,$$

$$\Big((\check{\phi}^{0,N}_{1,\check{x}}, \check{\phi}^{*}_{0,7})_{L_2(\check{\tau})}\Big)_{16\le\check{x}\le45} =$$
$$\big[\tfrac{2087}{4320}\ \tfrac{163}{720}\ \tfrac{-59}{864}\ 0\ 0\ 0\ \tfrac{229}{4320}\ \tfrac{77}{1080}\ \tfrac{-583}{1440}\ \tfrac{23}{2160}\ 0\ 0\ \tfrac{1541}{2160}\ \tfrac{-91}{540}\ \tfrac{-1}{540}\ \tfrac{56}{135}$$
$$\tfrac{197}{540}\ \tfrac{-1039}{2160}\ \tfrac{1}{432}\ \tfrac{-61}{540}\ \tfrac{-7}{540}\ \tfrac{-373}{2160}\ \tfrac{329}{2160}\ \tfrac{-115}{216}\ \tfrac{23}{540}\ \tfrac{-37}{540}\ \tfrac{-19}{540}\ \tfrac{-23}{108}\ \tfrac{-23}{1080}\ \tfrac{-289}{2160}\big]^T,$$

whereas the other $(\check{\phi}^{0,N}_{1,\check{x}}, \check{\phi}^{*}_{0,\check{y}})_{L_2(\check{\tau})}$ for $\check{x} \in \check{J}_1$, $\check{y} \in \check{I}_0$ follow by permuting the barycentric coordinates.

With

$$\begin{aligned}
\check{\theta}_{1,1} = {} & \tfrac{491}{63840}\check{\phi}^{0,N}_{-2,1} - \tfrac{1447}{1140}\check{\phi}^{0,N}_{-1,1} + \tfrac{1556}{285}\check{\phi}^{0,N}_{0,1} + \tfrac{7936}{1995}\check{\phi}^{0,N}_{1,1} - \tfrac{32}{285}\check{\phi}^{0,N}_{1,13} - \tfrac{200}{57}\check{\phi}^{0,N}_{1,28} \\
& + \tfrac{24}{95}\check{\phi}^{0,N}_{1,45} + \tfrac{40}{19}(\check{\phi}^{0,N}_{1,31} + \check{\phi}^{0,N}_{1,32}) - \tfrac{8}{19}(\check{\phi}^{0,N}_{1,43} + \check{\phi}^{0,N}_{1,44}) \\
\check{\theta}_{1,6} = {} & \tfrac{39552736427842}{16415540169075}\check{\phi}^{0,N}_{-1,6} - \tfrac{115253331674198}{16415540169075}\check{\phi}^{0,N}_{0,6} - \tfrac{50086735845548}{16415540169075}\check{\phi}^{0,N}_{1,6} \\
& - \tfrac{155797658892424}{16415540169075}\check{\phi}^{0,N}_{1,13} + \tfrac{1073782164671}{16415540169075}\check{\phi}^{0,N}_{1,28} + \tfrac{163041515506583}{16415540169075}\check{\phi}^{0,N}_{1,45} \\
& - \tfrac{46695093990479}{5471846723025}(\check{\phi}^{0,N}_{1,14} + \check{\phi}^{0,N}_{1,15}) - \tfrac{75538846833187}{32831080338150}(\check{\phi}^{0,N}_{1,20} + \check{\phi}^{0,N}_{1,21}) \\
& + \tfrac{85694636471981}{10943693446050}(\check{\phi}^{0,N}_{1,26} + \check{\phi}^{0,N}_{1,27})
\end{aligned}$$

$$\begin{aligned}\check{\theta}_{1,7} = &-\tfrac{521770882944}{316151143997}\check{\phi}^{0,N}_{1,4} + \tfrac{1438064282366}{316151143997}\check{\phi}^{0,N}_{1,7} + \tfrac{228057857077}{948453431991}\check{\phi}^{0,N}_{1,13} - \tfrac{2057981328101}{1264604575988}\check{\phi}^{0,N}_{1,16}\\ &-\tfrac{652279514773}{632302287994}\check{\phi}^{0,N}_{1,22} + \tfrac{530136572039}{1264604575988}\check{\phi}^{0,N}_{1,24} - \tfrac{318838046349}{1264604575988}\check{\phi}^{0,N}_{1,31} + \tfrac{197161204745}{1896906863982}\check{\phi}^{0,N}_{1,32}\\ &-\tfrac{1432064752285}{3793813727964}\check{\phi}^{0,N}_{1,33} + \tfrac{715847976}{316151143997}\check{\phi}^{0,N}_{1,34} - \tfrac{4164167977237}{3793813727964}\check{\phi}^{0,N}_{1,37} - \tfrac{46192346505}{632302287994}\check{\phi}^{0,N}_{1,38}\\ &+\tfrac{5449028597573}{3793813727964}\check{\phi}^{0,N}_{1,39} - \tfrac{5506954693}{28312042746}\check{\phi}^{0,N}_{1,43} + \tfrac{18811161423}{632302287994}\check{\phi}^{0,N}_{1,44}\\ \check{\theta}_{1,13} = &-\tfrac{252514670598911312}{32123023372190565}\check{\phi}^{0,N}_{0,13} + \tfrac{581506357809065767}{32123023372190565}\check{\phi}^{0,N}_{1,13} + \tfrac{13466719718137331}{12849209348876226}\check{\phi}^{0,N}_{1,28}\\ &+\tfrac{36849360316746824}{32123023372190565}\check{\phi}^{0,N}_{1,45} + \tfrac{902873612494110469}{556799071784636460}(\check{\phi}^{0,N}_{1,31} + \check{\phi}^{0,N}_{1,32})\\ &-\tfrac{68672167167430439}{111359814356927292}(\check{\phi}^{0,N}_{1,33} + \check{\phi}^{0,N}_{1,34}) + \tfrac{28971942655711213}{185599690594878820}(\check{\phi}^{0,N}_{1,35} + \check{\phi}^{0,N}_{1,36})\\ &+\tfrac{20805027809585291}{9279984529743941}(\check{\phi}^{0,N}_{1,37} + \check{\phi}^{0,N}_{1,38}) + \tfrac{321019232499078934}{139199767946159115}(\check{\phi}^{0,N}_{1,39} + \check{\phi}^{(N)}_{1,40}),\end{aligned}$$

and the other $\check{\theta}_{1,\check{y}}$ for $\check{y} \in \check{I}_0$ obtained by permuting the barycentric coordinates, one has $(\check{\theta}_{1,\check{y}}, \check{\phi}^*_{0,\check{z}})_{L_2(\check{\tau})} = \delta_{\check{y},\check{z}}$ $(\check{y}, \check{z} \in \check{I}_0)$. The constructed $\check{\Phi}^*_0$ and $\check{\Upsilon}_1 \cup \check{\Theta}_1$ are bases of $\check{S}(2,0)$ and $\check{S}(0,1)$ and satisfy (P1) and (P2). Proposition 4.6 and Theorem 3.1 yield an $\mathcal{H}^s$-stable wavelet basis for $|s| < \frac{3}{2}$ ($|s| \leq 1$), consisting of wavelets having vanishing moments of order $2^{\ell^*} + 1 = 5$.

With the "advanced" implementation as described in §4.4, in case of a uniform mesh and the number of levels tending to infinity, performing the basis transformation from wavelet to nodal basis costs $\mathbf{24\frac{11}{48}}$ operations per unknown.

Appendix. Proof of Theorem 2.1. We start with proving Part (a). The assertion is closely related to a well-known criterion by Fortin [15] for the validity of the LBB-condition which is relevant in the context of saddle point problems. Let $Q_k : L_2 \to \mathcal{S}^*_k \subset L_2$ denote the orthogonal projector of L_2 onto $\mathcal{S}^*_k$, defined by

$$(u, u^*_k)_{L_2} = (Q_k u, u^*_k)_{L_2}, \qquad (u \in L_2,\, u^*_k \in \mathcal{S}^*_k), \tag{A.1}$$

and let R_k denote its restriction to $\mathcal{S}_k$. Clearly, $\|R_k\|_{L_2 \to L_2} \leq 1$, whereas (C1) is equivalent to $\|R_k u_k\|_{L_2} \gtrsim \|u_k\|_{L_2}$, $(u_k \in \mathcal{S}_k)$. Since $\mathcal{S}_k$ is closed, both properties of R_k show that $\Im(R_k)$ is closed in $\mathcal{S}^*_k$, and so $\Im(R_k) \neq \mathcal{S}^*_k$ would mean that there exists a $0 \neq u^*_k \in \mathcal{S}^*_k$ such that $u^*_k \perp_{L_2} \Im(R_k)$. However, this would contradict $\sup_{u_k \in \mathcal{S}_k} |(u_k, u^*_k)_{L_2}| > 0$, which is a consequence of (C1)*. We conclude that the inverse $R_k^{-1} : \mathcal{S}^*_k \to \mathcal{S}_k$ exists, and that it is uniformly bounded.

By substituting $u = R_k^{-1} Q_k v$ in (A.1), we find

$$(R_k^{-1} Q_k v, u^*_k)_{L_2} = (Q_k v, u^*_k)_{L_2} = (v, u^*_k)_{L_2} \qquad (v \in L_2,\, u^*_k \in \mathcal{S}^*_k).$$

This equation shows that $\Im(I - R_k^{-1} Q_k) \subset (\mathcal{S}^*_k)^{\perp_{L_2}}$ and, by using (C1), that $R_k^{-1} Q_k$ is the identity operator on $\mathcal{S}_k$. We conclude that $\mathcal{Q}_k := R_k^{-1} Q_k$ is a uniformly bounded projector satisfying $\Im(\mathcal{Q}_k) = \mathcal{S}_k$ and, because $v \in (\mathcal{S}^*_k)^{\perp_{L_2}}$ implies $Q_k v = 0$, furthermore $\Im(I - \mathcal{Q}_k) = (\mathcal{S}^*_k)^{\perp_{L_2}}$. From these properties one infers that the adjoint projector $\mathcal{Q}^*_k$ satisfies $\Im(\mathcal{Q}^*_k) = \mathcal{S}^*_k$ and $\Im(I - \mathcal{Q}^*_k) = (\mathcal{S}_k)^{\perp_{L_2}}$. Note that $\mathcal{S}^*_k \subset \mathcal{S}^*_{k+1}$ implies $\mathcal{Q}^*_{k+1} \mathcal{Q}^*_k = \mathcal{Q}^*_k$, which, in turn yields $\mathcal{Q}_k \mathcal{Q}_{k+1} = \mathcal{Q}_k$. Analogously, we have $\mathcal{Q}^*_k \mathcal{Q}^*_{k+1} = \mathcal{Q}^*_k$, which proves the first part of the assertion.

Now we come to the proof of (2.1) and (2.1)*. Suppose $s \in \mathbb{R}$ is such that for some $\epsilon > 0$ one has

$$\|v_k\|_{\mathcal{H}^{s \pm \epsilon}} \lesssim \rho^{(s \pm \epsilon)k} \|v_k\|_{L_2} \qquad (v_k \in \Im(\mathcal{Q}_k - \mathcal{Q}_{k-1})). \tag{A.2}$$

Then

$$\begin{aligned}\|\sum_k v_k\|_{\mathcal{H}^s}^2 &= (\sum_k v_k, \sum_\ell v_\ell)_{\mathcal{H}^s} \lesssim \sum_k \sum_{\ell\geq k} \|v_k\|_{\mathcal{H}^{s+\epsilon}} \|v_\ell\|_{\mathcal{H}^{s-\epsilon}} \\ &\lesssim \sum_k \sum_{\ell \geq k} \rho^{\epsilon k} \rho^{-\epsilon \ell} (\rho^{sk} \|v_k\|_{L_2})(\rho^{s\ell}\|v_\ell\|_{L_2}) \lesssim \sum_k \rho^{2sk} \|v_k\|_{L_2}^2.\end{aligned}$$

Condition (C3) shows inequality (A.2) for $s \pm \epsilon \in [0, \gamma)$. Now let $t = s \pm \epsilon \in [-d^*, 0]$. Then for $v_k \in \Im(\mathcal{Q}_k - \mathcal{Q}_{k-1})$, there holds

$$\|v_k\|_{\mathcal{H}^t} = \sup_{0 \neq w \in \mathcal{H}^{-t}} \frac{|(v_k, w)_{L_2}|}{\|w\|_{\mathcal{H}^{-t}}} = \sup_{0 \neq w \in \mathcal{H}^{-t}} \frac{|(v_k, (\mathcal{Q}_k^* - \mathcal{Q}_{k-1}^*)w)_{L_2}|}{\|w\|_{\mathcal{H}^{-t}}}.$$

From $\|(\mathcal{Q}_k^* - \mathcal{Q}_{k-1}^*)w\|_{L_2} \leq \|\mathcal{Q}_k^* - \mathcal{Q}_{k-1}^*\|_{L_2 \to L_2} \inf_{u_{k-1}^* \in \mathcal{S}_{k-1}^*} \|w - u_{k-1}^*\|_{L_2}$ combined with (C2)* we obtain inequality (A.2) for this case. We conclude that for $s \in (-d^*, \gamma)$, the mapping

$$G_{\mathcal{Q}} : \ell_{2,s}(\mathcal{Q}) \to \mathcal{H}^s : (v_k)_k \mapsto \sum_{k=0}^{\infty} v_k$$

is bounded, i.e., (2.1) is valid. The same proof shows (2.1)*.

We will now show that (2.1)* implies (2.2) and, analogously that (2.1) implies (2.2)*. To this end we will make use of the following fact.

LEMMA A.1. *With respect to the dual pairing* $\langle (v_k^*)_k, (v_k)_k \rangle = \sum_k (v_k^*, v_k)_{L_2}$ *on* $\ell_{2,-s}(\mathcal{Q}^*) \times \ell_{2,s}(\mathcal{Q})$*, one has* $\ell_{2,s}(\mathcal{Q})^* = \ell_{2,-s}(\mathcal{Q}^*)$ *with equivalent norms. The analogous result is valid with interchanged roles of* $\mathcal{Q}$ *and* $\mathcal{Q}^*$.

Proof. The above assertion is a consequence of similar more general results from [**?**], §1.11.1, see also [**?**], Theorem 4.5.1 or [8], Theorem 5.1. Since the proof for the above version is relatively short we include it for the convenience of the reader. For $(v_k^*)_k \in \ell_{2,-s}(\mathcal{Q}^*)$, term-wise application of the Cauchy-Schwarz inequality shows that the functional $(v_k)_k \mapsto \sum_k (v_k^*, v_k)_{L_2}$ is bounded on $\ell_{2,s}(\mathcal{Q})$ with norm less than or equal to $\|(v_k^*)_k\|_{\ell_{2,-s}(\mathcal{Q}^*)}$.

If $(v_k^*)_k \neq 0$, then $v_\ell^* \neq 0$ for some ℓ. Defining $v_k = \begin{cases} (\mathcal{Q}_l - \mathcal{Q}_{l-1}) v_\ell^* & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell \end{cases}$, we obtain $\langle (v_k^*)_k, (v_k)_k \rangle = \|v_\ell^*\|_{L_2}^2 \neq 0$, i.e., the mapping from $\ell_{2,-s}(\mathcal{Q}^*)$ to $\ell_{2,s}(\mathcal{Q})^*$ is injective.

Now let $f \in \ell_{2,s}(\mathcal{Q})^*$. By Riesz' representation theorem, a $(w_k)_k \in \ell_{2,s}(\mathcal{Q})$ exists with $\|(w_k)_k\|_{\ell_{2,s}(\mathcal{Q})} = \|f\|$, satisfying

$$\begin{aligned} f((v_k)_k) &= ((w_k)_k, (v_k)_k)_{\ell_{2,s}(\mathcal{Q})} = \sum_k \rho^{2sk} (w_k, v_k)_{L_2} \\ &= \langle (\rho^{2sk} (\mathcal{Q}_k^* - \mathcal{Q}_{k-1}^*) w_k)_k, (v_k)_k \rangle, \qquad (v_k)_k \in \ell_{2,s}(\mathcal{Q}). \end{aligned}$$

The uniform boundedness of $\|\mathcal{Q}_k^*\|_{L_2 \to L_2}$ shows that

$$\sum_k \rho^{-2sk} \|\rho^{2sk} (\mathcal{Q}_k^* - \mathcal{Q}_{k-1}^*) w_k\|_{L_2}^2 \lesssim \sum_k \rho^{2sk} \|w_k\|_{L_2}^2,$$

i.e., $(\rho^{2sk}(\mathcal{Q}_k^* - \mathcal{Q}_{k-1}^*) w_k)_k \in \ell_{2,-s}(\mathcal{Q}^*)$, its norm being bounded by some multiple of $\|f\|$. ▯

Now let $s \in (-\gamma^*, d)$. Then by (2.1)*, $G_{\mathcal{Q}^*} : \ell_{2,-s}(\mathcal{Q}^*) \to \mathcal{H}^{-s} : (v_k^*)_k \mapsto \sum_{k=0}^{\infty} v_k^*$ is bounded. For $u \in \mathcal{H}^s$ and $(v_k^*)_k \in \ell_{2,-s}(\mathcal{Q}^*)$, we have

$$(u, G_{\mathcal{Q}^*}(v_k^*)_k)_{L_2} = \sum_k (u, v_k^*)_{L_2} = \langle ((\mathcal{Q}_k - \mathcal{Q}_{k-1})u)_k, (v_k^*)_k \rangle,$$

i.e., the adjoint operator $G_{\mathcal{Q}^*}^* : \mathcal{H}^s \to \ell_{2,s}(\mathcal{Q})$ is defined by $G_{\mathcal{Q}^*}^* u = ((\mathcal{Q}_k - \mathcal{Q}_{k-1})u)_k$. The boundedness of $G_{\mathcal{Q}^*}^*$ is equivalent to (2.2).

Finally, let $s \in (-\gamma^*, \gamma)$. We have to show that for the bounded mappings $G_{\mathcal{Q}} : \ell_{2,s}(\mathcal{Q}) \to \mathcal{H}^s$ and $G_{\mathcal{Q}^*}^* : \mathcal{H}^s \to \ell_{2,s}(\mathcal{Q})$ there holds $G_{\mathcal{Q}^*}^* G_{\mathcal{Q}} = I$ and $G_{\mathcal{Q}} G_{\mathcal{Q}^*}^* = I$.

Let $(v_k)_k \in \ell_{2,s}(\mathcal{Q})$, then $\sum_k v_k$ is convergent in $\mathcal{H}^s$, so that

$$G_{\mathcal{Q}^*}^* G_{\mathcal{Q}}(v_k)_k = \lim_{k\to\infty}(v_0, v_1, \ldots, v_k, 0, \ldots) = (v_k)_k.$$

Since for $u \in \mathcal{H}^s$ one has $G_{\mathcal{Q}} G_{\mathcal{Q}^*}^* u = \sum_{k=0}^{\infty}(\mathcal{Q}_k - \mathcal{Q}_{k-1})u = \lim_{k\to\infty} \mathcal{Q}_k u$, we have to show that this limit equals u. For $\tilde{s} \in (-\gamma^*, \gamma)$, we define $F_{\mathcal{Q}^*,k} : \mathcal{H}^{\tilde{s}} \to \ell_{2,\tilde{s}}(\mathcal{Q})$ by $(F_{\mathcal{Q}^*,k}u)_\ell = \begin{cases} (G_{\mathcal{Q}^*}^* u)_\ell & \text{if } \ell \le k \\ 0 & \text{if } \ell > k \end{cases}$. Clearly, $F_{\mathcal{Q}^*,k}$ is bounded uniformly in k, and so is $\mathcal{Q}_k = G_{\mathcal{Q}} F_{\mathcal{Q}^*,k} : \mathcal{H}^{\tilde{s}} \to \mathcal{H}^{\tilde{s}}$. Now, let $t \in (0, \gamma)$ and $t^* \in (0, \gamma^*)$ such that $s \in (-t^*, t)$. By $\|\mathcal{Q}_k\|_{L_2 \to L_2} \lesssim 1$ and (C2), (C2)* we obtain that

$$\|I - \mathcal{Q}_k\|_{\mathcal{H}^t \to \mathcal{H}^{-t^*}} \le \|I - \mathcal{Q}_k^*\|_{\mathcal{H}^{t^*} \to L_2} \|I - \mathcal{Q}_k\|_{\mathcal{H}^t \to L_2} \lesssim \rho^{-(t+t^*)k}.$$

This estimate together with $\|I - \mathcal{Q}_k\|_{\mathcal{H}^t \to \mathcal{H}^t} \lesssim 1$ shows that $\|I - \mathcal{Q}_k\|_{\mathcal{H}^t \to \mathcal{H}^s} \lesssim \rho^{-(t-s)k}$ by interpolation. Since $\mathcal{H}^t \subset \mathcal{H}^s$ is dense, and $\|\mathcal{Q}_k\|_{\mathcal{H}^s \to \mathcal{H}^s} \lesssim 1$, from

$$\|u - \mathcal{Q}_k u\|_{\mathcal{H}^s} \le \inf_{v \in \mathcal{H}^t} \{\|\mathcal{Q}_k(v - u)\|_{\mathcal{H}^s} + \|(I - \mathcal{Q}_k)v\|_{\mathcal{H}^s} + \|u - v\|_{\mathcal{H}^s}\},$$

we conclude that $\lim_{k\to\infty} \mathcal{Q}_k u = u$ on $\mathcal{H}^s$, which completes the proof.

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