Chapter 6: 3-D Computation by Stereo Vision

Kanatani Reading Club

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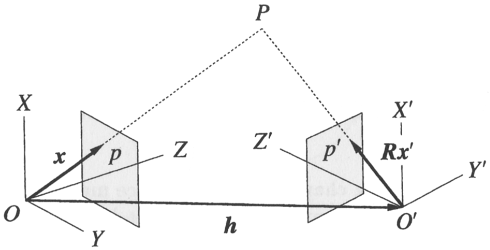

Fig. 6.1. Geometry of stereo vision.

Two cameras at positions $\mathbf{O}$ and $\mathbf{O}'$. The *base-line* vector $\mathbf{h} = \mathbf{O}' - \mathbf{O}$. Relative orientation is specified by rotation matrix $\mathbf{R}$. $\{\mathbf{h}, \mathbf{R}\}$ are the *motion parameters*. Scale is based on focal length. The *positive* Z-axis runs into the scene. Note the choice of Y-axis.

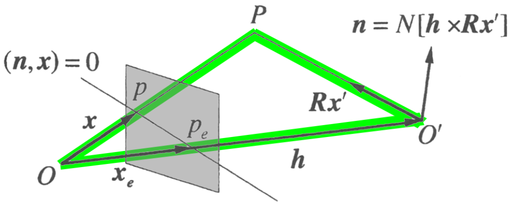

Fig. 6.2. Epipole and epipolar.

Epipolar equation: $|\mathbf{x}, \mathbf{h}, \mathbf{Rx}'| = 0$.
The volume spanned by **x**, **h** and **Rx**' is 0.
The three vectors lie in an *epipolar plane*.

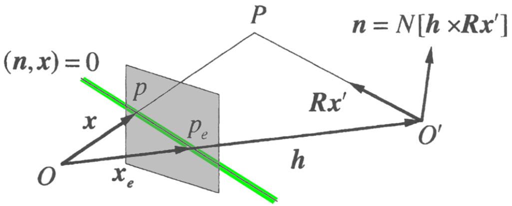

Fig. 6.2. Epipole and epipolar.

The intersection of an epipolar plane with an imaging plane is called an *epipolar* or *epipolar line*.

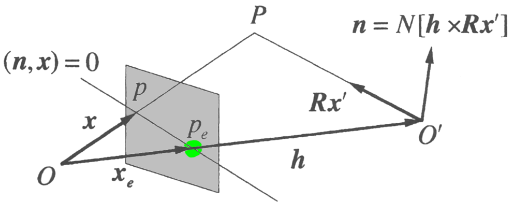

Fig. 6.2. Epipole and epipolar.

The *epipole* $\mathbf{x}_e = \frac{\mathbf{h}}{(\mathbf{k},\mathbf{h})}$. (where $\mathbf{k} = (0, 0, 1)^T$).
It is the perspective projection of one camera's viewpoint onto the image plane of another.
Epipolar lines are *concurrent*: they all run through the epipole (since the epipole is the projection of their collective 'origin').

If we define the *essential matrix*

$$\mathbf{G} = \mathbf{h} \times \mathbf{R}, \tag{6.7}$$

the epipolar equation can be written as

$$(\mathbf{x}, \mathbf{G}\mathbf{x}'). \tag{6.8}$$

Side note: the *fundamental* matrix is similar to the essential matrix but pre- and post-multiplied with the respective internal calibration (projection) matrix of each camera.

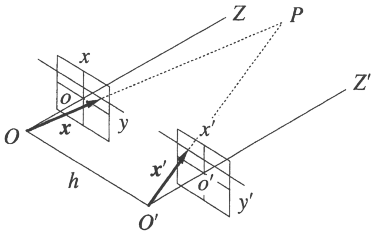

Fig. 6.3. Parallel stereo system.

In a *parallel* stereo system the optical axes (Z-axes) are parallel and the base line runs along the Y-axes of both cameras.

Any stereo system can be regarded as parallel by a change of camera coordinate systems.

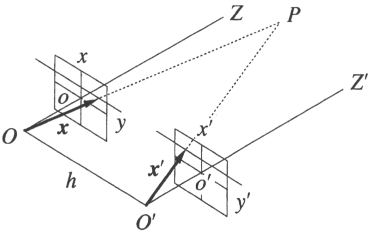

Fig. 6.3. Parallel stereo system.

Using parallel stereo systems simplifies the geometry:
Base line: $\mathbf{h} = (0, h, 0)$.
Relative orientation: $\mathbf{R} = \mathbf{I}$.
Essential matrix: $\mathbf{G} = \begin{pmatrix} 0 & 0 & h \\ 0 & 0 & 0 \\ -h & 0 & 0 \end{pmatrix}$.
Epipolar equation: $\mathbf{x} - \mathbf{x}' = 0$.

Detecting image points for 3-D Reconstruction:

- Edge based
  - + Many image points can be used.
  - - Hard to match exactly if edge is parallel to epipolar line.
  - - Precise camera calibration must be known.
- Feature based
  - + Can be used with approximate camera calibration.
  - + Can be used to perform/optimize camera calibration.
  - - Not as many image points can be used as with edge based.

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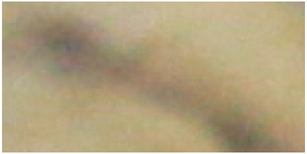
Question: how to compute variance from matching image features?



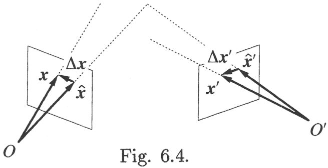


Fig. 6.4.

How to find $\Delta\mathbf{x}$ and $\Delta\mathbf{x}'$ such that $(\mathbf{x}-\Delta\mathbf{x}, \mathbf{G}(\mathbf{x}'-\Delta\mathbf{x}')) = 0$.
Minimize: $J = (\Delta\mathbf{x}, \mathbf{V}[\mathbf{x}]^{-}\Delta\mathbf{x}) + (\Delta\mathbf{x}', \mathbf{V}[\mathbf{x}']^{-}\Delta\mathbf{x}')$
under the linearized constraint $(\Delta\mathbf{x}, \mathbf{G}\mathbf{x}') + (\mathbf{x}, \mathbf{G}\Delta\mathbf{x}') = (\mathbf{x}, \mathbf{G}\mathbf{x}')$,
where $\Delta\mathbf{x}$ and $\Delta\mathbf{x}'$ must be in the image plane.

First order solution (so iterate):

$$\Delta\mathbf{x} = \frac{(\mathbf{x}, \mathbf{G}\mathbf{x}')\mathbf{V}[\mathbf{x}]\mathbf{G}\mathbf{x}'}{(\mathbf{x}', \mathbf{G}^{T}\mathbf{V}[\mathbf{x}]\mathbf{G}\mathbf{x}') + (\mathbf{x}, \mathbf{G}\mathbf{V}[\mathbf{x}']\mathbf{G}^{T}\mathbf{x})} \tag{6.20}$$

$$\Delta\mathbf{x}' = \frac{(\mathbf{x}, \mathbf{G}\mathbf{x}')\mathbf{V}[\mathbf{x}']\mathbf{G}^{T}\mathbf{x}}{(\mathbf{x}', \mathbf{G}^{T}\mathbf{V}[\mathbf{x}]\mathbf{G}\mathbf{x}') + (\mathbf{x}, \mathbf{G}\mathbf{V}[\mathbf{x}']\mathbf{G}^{T}\mathbf{x})} \tag{6.20}$$

If $\hat{\mathbf{x}} = \mathbf{x} - \Delta\mathbf{x}$ and $\hat{\mathbf{x}}' = \mathbf{x}' - \Delta\mathbf{x}'$ are the corrected image points, then variance 'propagates' as follows:

$$
\begin{aligned}
\mathbf{V}[\hat{\mathbf{x}}] &= \mathbf{V}[\mathbf{x}] - \frac{(\mathbf{V}[\mathbf{x}]\mathbf{G}\hat{\mathbf{x}}')(\mathbf{V}[\mathbf{x}]\mathbf{G}\hat{\mathbf{x}}')^T}{(\hat{\mathbf{x}}', \mathbf{G}^T\mathbf{V}[\mathbf{x}]\mathbf{G}\hat{\mathbf{x}}') + (\hat{\mathbf{x}}, \mathbf{G}\mathbf{V}[\mathbf{x}']\mathbf{G}^T\hat{\mathbf{x}})}, \\
\mathbf{V}[\hat{\mathbf{x}}'] &= \mathbf{V}[\mathbf{x}'] - \frac{(\mathbf{V}[\mathbf{x}']\mathbf{G}^T\hat{\mathbf{x}})(\mathbf{V}[\mathbf{x}']\mathbf{G}^T\hat{\mathbf{x}}')^T}{(\hat{\mathbf{x}}', \mathbf{G}^T\mathbf{V}[\mathbf{x}]\mathbf{G}\hat{\mathbf{x}}') + (\hat{\mathbf{x}}, \mathbf{G}\mathbf{V}[\mathbf{x}']\mathbf{G}^T\hat{\mathbf{x}})}, \\
\mathbf{V}[\hat{\mathbf{x}}, \hat{\mathbf{x}}'] &= -\frac{(\mathbf{V}[\mathbf{x}]\mathbf{G}\hat{\mathbf{x}}')(\mathbf{V}[\mathbf{x}']\mathbf{G}^T\hat{\mathbf{x}})^T}{(\hat{\mathbf{x}}', \mathbf{G}^T\mathbf{V}[\mathbf{x}]\mathbf{G}\hat{\mathbf{x}}') + (\hat{\mathbf{x}}, \mathbf{G}\mathbf{V}[\mathbf{x}']\mathbf{G}^T\hat{\mathbf{x}})} = \mathbf{V}[\hat{\mathbf{x}}', \hat{\mathbf{x}}]^T.
\end{aligned}
\tag{6.21}
$$

Residual of J is:

$$\hat{J} = \frac{(\mathbf{x}, \mathbf{G}\mathbf{x}')^2}{(\hat{\mathbf{x}}', \mathbf{G}^T V[\mathbf{x}]\mathbf{G}\hat{\mathbf{x}}') + (\hat{\mathbf{x}}, \mathbf{G}V[\mathbf{x}']\mathbf{G}^T\hat{\mathbf{x}})}. \tag{6.22}$$

Chi square test:
(rejects correspondance hypothesis with signifinance level of a%)

$$\hat{J} > \chi^2_{1,a}.$$

Question: how to combine this Chi square test with a measure based on the actual images (like NCC)?

Given the camera geometry $\{\mathbf{R}, \mathbf{h}\}$ and the optimally corrected image points $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}'$, depths Z and Z' can be computed such that

$$Z\hat{\mathbf{x}} = \mathbf{h} + Z'\mathbf{R}\hat{\mathbf{x}}'. \tag{6.35}$$

Algebraic manipulation results in

$$Z = \frac{(\mathbf{h} \times \mathbf{R}\hat{\mathbf{x}}', \hat{\mathbf{x}} \times \mathbf{R}\hat{\mathbf{x}}')}{\|\hat{\mathbf{x}} \times \mathbf{R}\hat{\mathbf{x}}'\|^2}, \tag{6.38}$$

$$Z' = \frac{(\mathbf{h} \times \hat{\mathbf{x}}, \hat{\mathbf{x}} \times \mathbf{R}\hat{\mathbf{x}}')}{\|\hat{\mathbf{x}} \times \mathbf{R}\hat{\mathbf{x}}'\|^2}. \tag{6.38}$$

For parallel stereo systems this simplifies to

$$Z = Z' = \frac{h}{\hat{y} - \hat{y}'}. \tag{6.40}$$

Define $\hat{\mathbf{m}} = N[\mathbf{h} \times \hat{\mathbf{x}}] \times \mathbf{R}\hat{\mathbf{x}}'$ ($N[.]$ means normalize)

Then the variance of Z and reconstructed space points $\mathbf{V}[Z\hat{\mathbf{x}}]$ is:

$$\mathbf{V}[Z] = \frac{Z^2(\hat{\mathbf{m}}, \mathbf{V}[\hat{\mathbf{x}}]\hat{\mathbf{m}}) - 2ZZ'(\hat{\mathbf{m}}, \mathbf{V}[\hat{\mathbf{x}}, \hat{\mathbf{x}}']\mathbf{R}^T\hat{\mathbf{m}}) + Z'^2(\hat{\mathbf{m}}, \mathbf{R}\mathbf{V}[\hat{\mathbf{x}}']\mathbf{R}^T\hat{\mathbf{m}})}{\|\hat{\mathbf{x}} \times \mathbf{R}\hat{\mathbf{x}}'\|^2} \tag{6.48}$$

$$\mathbf{V}[\hat{\mathbf{x}}, Z] = -\frac{(Z\mathbf{V}[\hat{\mathbf{x}}] - Z'\mathbf{V}[\hat{\mathbf{x}}, \hat{\mathbf{x}}']\mathbf{R}^T)\hat{\mathbf{m}}}{(\hat{\mathbf{m}}, \hat{\mathbf{x}})} \tag{6.49}$$

$$\mathbf{V}[\mathbf{R}] = \mathbf{V}[Z\hat{\mathbf{x}}] = \mathbf{V}[Z'\hat{\mathbf{x}}'] = Z^2\mathbf{V}[\hat{\mathbf{x}}] + 2ZS[\mathbf{V}[\hat{\mathbf{x}}, Z]\hat{\mathbf{x}}^T] + \mathbf{V}[Z]\hat{\mathbf{x}}\hat{\mathbf{x}}^T \tag{6.50}$$

($S[.]$ is the symmetrization operator (Equation 2.205)).

For parallel systems the depth/reconstruction variances simplify to:

$$\mathbf{V}[Z] = \frac{2\epsilon^2 Z^4}{h^2} \tag{6.51}$$

$$\mathbf{V}[\hat{\mathbf{x}}, Z] = -\frac{\epsilon^2 Z^2}{h^2}(0, 1, 0)^T \tag{6.51}$$

$$\mathbf{V}[\mathbf{R}] = \mathbf{V}[Z\hat{\mathbf{x}}] = \mathbf{V}[Z'\hat{\mathbf{x}}'] = \frac{\epsilon^2 Z^2}{2}\left(\mathbf{P_k} + \frac{(\hat{\mathbf{x}} + \hat{\mathbf{x}}')(\hat{\mathbf{x}} + \hat{\mathbf{x}}')^T}{y - y'}\right) \tag{6.53}$$

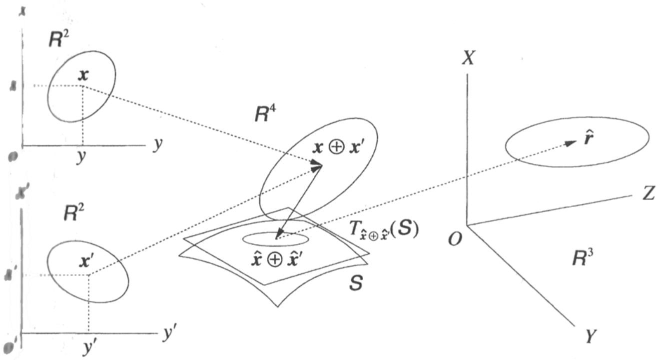


Fig. 6.10. Geometry of 3-D reconstruction by stereo.

Not very realistic since measurement noise depends on the features (local image properties) itself?

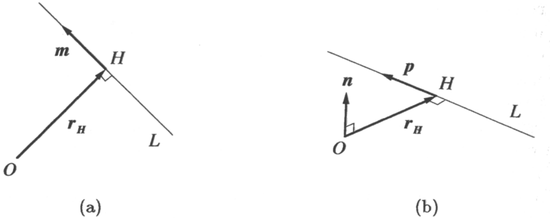


Fig. 4.6. (a) The $\{\mathbf{m}, \mathbf{r}_H\}$-representation. (b) The $\{\mathbf{p}, \mathbf{n}\}$-representation.

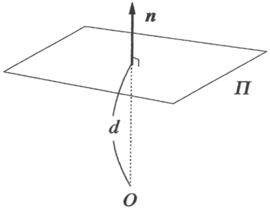


Fig. 4.8. Representation of a space plane.

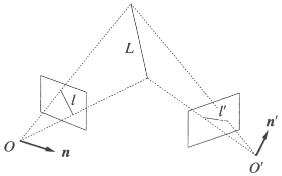


Fig. 6.11. 3-D reconstruction of a space line.

2-D image lines: $(\mathbf{n}, \mathbf{x}) = 0 \qquad (\mathbf{n}', \mathbf{x}') = 0 \qquad (6.56)$

3-D space planes: $(\mathbf{n}, \mathbf{r}) = 0 \qquad (\mathbf{R}\mathbf{n}', \mathbf{r} - \mathbf{h}) = 0 \qquad (6.57/58)$

The intesection of both space planes is $\mathbf{r} \times \mathbf{p} = \mathbf{n}$, computed using:

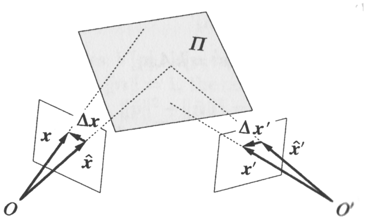
$$\mathbf{p} = \frac{\mathbf{n} \times \mathbf{R}\mathbf{n}'}{(\mathbf{h}, \mathbf{R}\mathbf{n}')} \qquad (6.59)$$

Normalize the the space line such that $\|\mathbf{p}\|^2 + \|\mathbf{n}\|^2 = 1$.

$$\begin{aligned}
\mathbf{V}[\mathbf{p}] &= \frac{1}{(\mathbf{h}, \mathbf{R}\mathbf{n}')^2}((\mathbf{R}\mathbf{n}') \times \mathbf{V}[\mathbf{n}] \times (\mathbf{R}\mathbf{n}') + (\mathbf{h}, \mathbf{R}\mathbf{V}[\mathbf{n}']\mathbf{R}^T\mathbf{h})\mathbf{p}\mathbf{p}^T \\
&\quad -2S[\mathbf{n} \times \mathbf{R}\mathbf{V}[\mathbf{n}']\mathbf{R}^T\mathbf{h}\mathbf{p}^T] + \mathbf{n} \times \mathbf{R}\mathbf{V}[\mathbf{n}']\mathbf{R}^T \times \mathbf{n}), \\
\mathbf{V}[\mathbf{p}, \mathbf{n}] &= -\frac{(\mathbf{R}\mathbf{n}') \times \mathbf{V}[\mathbf{n}]}{(\mathbf{h} m \mathbf{R}\mathbf{n}')} \qquad (6.61)
\end{aligned}$$

Normalization:

$$\begin{pmatrix} \mathbf{V}[\tilde{\mathbf{p}}] & \mathbf{V}[\tilde{\mathbf{p}}, \tilde{\mathbf{n}}] \\ \mathbf{V}[\tilde{\mathbf{n}}, \tilde{\mathbf{p}}] & \mathbf{V}[\tilde{\mathbf{n}}] \end{pmatrix} = \frac{1}{\|\mathbf{p}\|^2 + \|\mathbf{n}\|^2} \mathbf{P}_{\tilde{\mathbf{p}} \oplus \tilde{\mathbf{n}}} \begin{pmatrix} \mathbf{V}[\mathbf{p}] & \mathbf{V}[\mathbf{p}, \mathbf{n}] \\ \mathbf{V}[\mathbf{n}, \mathbf{p}] & \mathbf{V}[\mathbf{n}] \end{pmatrix} \mathbf{P}_{\tilde{\mathbf{p}} \oplus \tilde{\mathbf{n}}} \qquad (6.62)$$

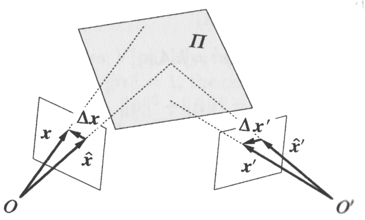


Projecting an image point onto space plane $\{\mathbf{n}_\Pi, d\}$:

$$\mathbf{r} = \frac{d\mathbf{x}}{(\mathbf{n}_\Pi, \mathbf{x})} \tag{6.64}$$

Using algebraic manipulation to find an expression for $\mathbf{x}'$ in terms of $\mathbf{x}$ and the plane:

$$\mathbf{x}' = -\frac{Z}{Z'}\mathbf{R}^T(\mathbf{h}\mathbf{n}_\Pi^T - d\mathbf{I})\mathbf{x}. \tag{6.65}$$



Thus for each *specific* plane $\{\mathbf{n}_\Pi, d\}$, we can find a matrix $\mathbf{A}$ which represents a projective transformation:

$$\mathbf{x} = k\mathbf{A}\mathbf{x}, \tag{6.66}$$

where

$$\mathbf{A} = \mathbf{R}^T(\mathbf{h}\mathbf{n}_\Pi^T - d\mathbf{I}). \tag{6.67}$$

Using $\mathbf{A}$ we can correct the back-projected point as follows:

$$(\mathbf{x}' - \Delta\mathbf{x}') \times \mathbf{A}(\mathbf{x} - \Delta\mathbf{x}), \tag{6.71}$$

Minimize

$$J = (\Delta\mathbf{x}, \mathbf{V}[\mathbf{x}]^{-}\Delta\mathbf{x}) + (\Delta\mathbf{x}', \mathbf{V}[\mathbf{x}']^{-}\Delta\mathbf{x}') \tag{6.72}$$

under the linearized constraint

$$\mathbf{x}' \times \mathbf{A}\Delta\mathbf{x} - (\mathbf{A}\mathbf{x}) \times \Delta\mathbf{x}' = \mathbf{x}' \times \mathbf{A}\mathbf{x}, \tag{6.73}$$

where $\Delta\mathbf{x}$ and $\Delta\mathbf{x}'$ must be in the image plane.
First order solution:

$$\begin{aligned} \Delta\mathbf{x} &= (\mathbf{V}[\mathbf{x}]\mathbf{A}^{T} \times \mathbf{x}')\mathbf{W}(\mathbf{x}' \times \mathbf{A}\mathbf{x}), \\ \Delta\mathbf{x}' &= -(\mathbf{V}[\mathbf{x}']\mathbf{A}^{T} \times \mathbf{x})\mathbf{W}(\mathbf{x}' \times \mathbf{A}\mathbf{x}), \end{aligned} \tag{6.74}$$

where

$$\mathbf{W} = (\mathbf{x}' \times \mathbf{A}\mathbf{V}[\mathbf{x}]\mathbf{A}^{T} \times \mathbf{x}' + (\mathbf{A}\mathbf{x}) \times \mathbf{V}[\mathbf{x}'] \times (\mathbf{A}\mathbf{x}))_2^{-}. \tag{6.75}$$

Variance propagation:

$$
\begin{aligned}
\mathbf{V}[\hat{\mathbf{x}}] &= \mathbf{V}[\mathbf{x}] - (\mathbf{V}[\mathbf{x}]\mathbf{A}^T \times \hat{\mathbf{x}}')\hat{\mathbf{W}}(\mathbf{V}[\mathbf{x}]\mathbf{A}^T \times \hat{\mathbf{x}}')^T \\
\mathbf{V}[\hat{\mathbf{x}}'] &= \mathbf{V}[\mathbf{x}'] - (\mathbf{V}[\mathbf{x}'] \times (\mathbf{A}\hat{\mathbf{x}}))\hat{\mathbf{W}}(\mathbf{V}[\mathbf{x}'] \times (\mathbf{A}\hat{\mathbf{x}}))^T \\
\mathbf{V}[\hat{\mathbf{x}}, \hat{\mathbf{x}}'] &= (\mathbf{V}[\mathbf{x}]\mathbf{A}^T \times \hat{\mathbf{x}}')\hat{\mathbf{W}}(\mathbf{V}[\mathbf{x}'] \times (\mathbf{A}\hat{\mathbf{x}}))^T = \mathbf{V}[\hat{\mathbf{x}}', \hat{\mathbf{x}}]^T
\end{aligned} \tag{6.76}
$$

Residual:

$$
\hat{J} = (\mathbf{x}' \times \mathbf{A}\mathbf{x}, \hat{\mathbf{W}}\mathbf{x}' \times \mathbf{A}\mathbf{x})) \tag{6.77}
$$

Chi squared test: (rejects point-is-on-plane hypothesis with signifinance level of a%)

$$
\hat{J} > \chi^2_{2,a}.
$$

Noise level estimation:

$$
\hat{\epsilon}^2 = \frac{1}{2}(\mathbf{x}' \times \mathbf{A}\mathbf{x}, \hat{\mathbf{W}}_0(\mathbf{x}' \times \mathbf{A}\mathbf{x})), \tag{6.80}
$$

$$
\hat{\mathbf{W}}_0 = (\hat{\mathbf{x}}' \times \mathbf{A}\mathbf{V}_0[\mathbf{x}]\mathbf{A}^T \times \hat{\mathbf{x}}' + (\mathbf{A}\hat{\mathbf{x}}) \times \mathbf{V}_0[\mathbf{x}'] \times (\mathbf{A}\hat{\mathbf{x}}))^-_2 \qquad (6.80)
$$

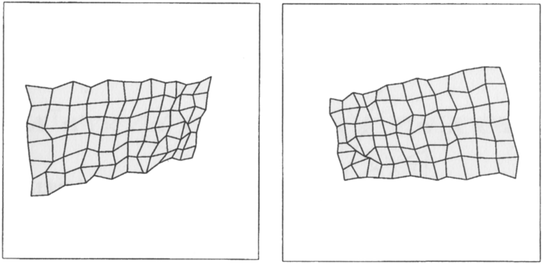
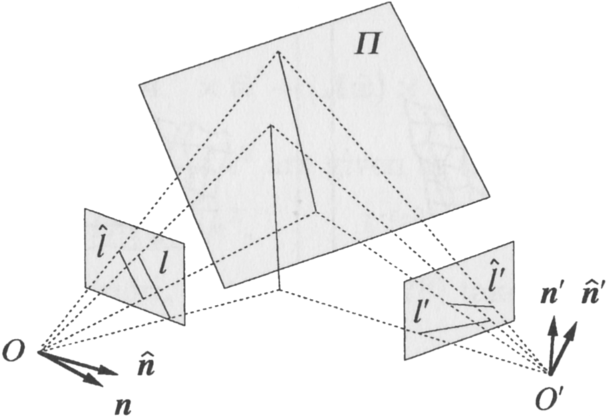


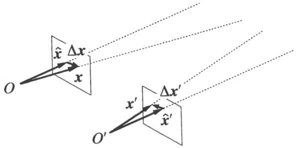
Fig. 6.13. Stereo images of a planar grid.

(a) (b)

Fig. 6.14. (a) Simple back projection. (b) Optimal back projection.

More of the same, so skip?





Condition for being parallel:

$$\mathbf{x} = k\mathbf{R}\mathbf{x}' \tag{6.100}$$

$$(\mathbf{x} + \Delta\mathbf{x}) \times \mathbf{R}(\mathbf{x}' + \Delta\mathbf{x}') = 0 \tag{6.101}$$

So minimize

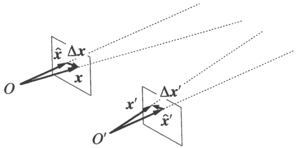
$$\mathbf{J} = (\Delta\mathbf{x}, \mathbf{V}[\mathbf{x}]^{-}\Delta\mathbf{x}) + (\Delta\mathbf{x}', \mathbf{V}[\mathbf{x}']^{-}\Delta\mathbf{x}'), \tag{6.102}$$

under the linearized constraint

$$\mathbf{x} \times \mathbf{R}\Delta\mathbf{x}' - (\mathbf{R}\mathbf{x}') \times \Delta\mathbf{x} = \mathbf{x} \times \mathbf{R}\mathbf{x}', \tag{6.102}$$

where $\Delta\mathbf{x}$ and $\Delta\mathbf{x}'$ must be in the image plane.

Space Points Infinitely Far Away



First order solution:

$$\begin{aligned} \Delta \mathbf{x} &= -(\mathbf{V}[\mathbf{x}] \times (\mathbf{R}\mathbf{x}'))\mathbf{W}(\mathbf{x} \times (\mathbf{R}\mathbf{x}')), \\ \Delta \mathbf{x}' &= (\mathbf{V}[\mathbf{x}]\mathbf{R}^T \times \mathbf{x})\mathbf{W}(\mathbf{x} \times (\mathbf{R}\mathbf{x}')), \end{aligned} \tag{6.104}$$

where

$$\mathbf{W} = ((\mathbf{R}\mathbf{x}') \times \mathbf{V}[\mathbf{x}] \times (\mathbf{R}\mathbf{x}') + \mathbf{x} \times \mathbf{R}\mathbf{V}[\mathbf{x}']\mathbf{R}^T \times \mathbf{x})_2^- . \tag{6.105}$$

Variance propagation:

$$\begin{aligned} \mathbf{V}[\hat{\mathbf{x}}] &= \mathbf{V}[\mathbf{x}] - (\mathbf{V}[\mathbf{x}] \times (\mathbf{R}\hat{\mathbf{x}}'))\hat{\mathbf{W}}(\mathbf{V}[\mathbf{x}] \times (\mathbf{R}\hat{\mathbf{x}}'))^T \\ \mathbf{V}[\hat{\mathbf{x}}'] &= \mathbf{V}[\mathbf{x}'] - (\mathbf{V}[\mathbf{x}']\mathbf{R}^T \times \hat{\mathbf{x}})\hat{\mathbf{W}}(\mathbf{V}[\mathbf{x}']\mathbf{R}^T \times \hat{\mathbf{x}})^T \qquad (6.106) \\ \mathbf{V}[\hat{\mathbf{x}}, \hat{\mathbf{x}}'] &= (\mathbf{V}[\mathbf{x}'] \times (\mathbf{R}\hat{\mathbf{x}}'))\hat{\mathbf{W}}(\mathbf{V}[\mathbf{x}']\mathbf{R}^T \times \hat{\mathbf{x}})^T = \mathbf{V}[\hat{\mathbf{x}}', \hat{\mathbf{x}}]^T \end{aligned}$$

Chi square test: (rejects point-is-infinitely-far-away hypothesis with signifinance level of a%)

$$\hat{\mathbf{J}} = (\mathbf{x} \times \mathbf{R}\mathbf{x}', \hat{\mathbf{W}}(\mathbf{x} \times \mathbf{R}\mathbf{x}')) > \chi^2_{2,a}. \qquad (6.108)$$

Camera calibration errors and image noise are considered to be independent.

The following errors are considered:

- Base line $\mathbf{h}$
- Rotation $\mathbf{R}$
- Focal length f

General method used:
-Find out how image points are affected by the error.
-Adjust existing equations to incorporate this change.

Error which are not considered: optical center, distortion coefficients.

$$\Delta\hat{\mathbf{e}} = (\hat{\mathbf{x}}, \mathbf{G}'\hat{\mathbf{x}}') = (\hat{\mathbf{x}}, \Delta\mathbf{h} \times \mathbf{R}\hat{\mathbf{x}}') = -(\hat{\mathbf{x}} \times \mathbf{R}\hat{\mathbf{x}}', \Delta\mathbf{h}) = -(\hat{\mathbf{a}}, \Delta\mathbf{h}) \tag{6.127}$$

where $\hat{\mathbf{a}} = \hat{\mathbf{x}} \times \mathbf{R}\hat{\mathbf{x}}'$. (missing primes in 6.127?)

$$\Delta\hat{\mathbf{x}} = -\frac{\Delta\hat{\mathbf{e}}\mathbf{V}[\mathbf{x}]\mathbf{G}\mathbf{x}'}{(\hat{\mathbf{x}}', \mathbf{G}^T\mathbf{V}[\mathbf{x}]\mathbf{G}\hat{\mathbf{x}}') + (\hat{\mathbf{x}}, \mathbf{G}\mathbf{V}[\mathbf{x}']\mathbf{G}^T\hat{\mathbf{x}})}$$

$$\Delta\hat{\mathbf{x}}' = -\frac{\Delta\hat{\mathbf{e}}\mathbf{V}[\mathbf{x}']\mathbf{G}^T\hat{\mathbf{x}}}{(\hat{\mathbf{x}}', \mathbf{G}^T\mathbf{V}[\mathbf{x}]\mathbf{G}\hat{\mathbf{x}}') + (\hat{\mathbf{x}}, \mathbf{G}\mathbf{V}[\mathbf{x}']\mathbf{G}^T\hat{\mathbf{x}})} \tag{6.129}$$

Variance propagation:

$$\begin{aligned}
\mathbf{V}[\hat{\mathbf{x}}] &= \frac{\mathbf{V}[\hat{\mathbf{e}}](\mathbf{V}[\mathbf{x}]\mathbf{G}\hat{\mathbf{x}}')(\mathbf{V}[\mathbf{x}]\mathbf{G}\hat{\mathbf{x}}')^T}{((\hat{\mathbf{x}}', \mathbf{G}^T\mathbf{V}[\mathbf{x}]\mathbf{G}\hat{\mathbf{x}}') + (\hat{\mathbf{x}}, \mathbf{G}\mathbf{V}[\mathbf{x}']\mathbf{G}^T\hat{\mathbf{x}}))^2}, \\
\mathbf{V}[\hat{\mathbf{x}}'] &= \frac{\mathbf{V}[\hat{\mathbf{e}}](\mathbf{V}[\mathbf{x}']\mathbf{G}^T\hat{\mathbf{x}})(\mathbf{V}[\mathbf{x}']\mathbf{G}^T\hat{\mathbf{x}}')^T}{((\hat{\mathbf{x}}', \mathbf{G}^T\mathbf{V}[\mathbf{x}]\mathbf{G}\hat{\mathbf{x}}') + (\hat{\mathbf{x}}, \mathbf{G}\mathbf{V}[\mathbf{x}']\mathbf{G}^T\hat{\mathbf{x}}))^2}, \\
\mathbf{V}[\hat{\mathbf{x}}, \hat{\mathbf{x}}'] &= \frac{\mathbf{V}[\hat{\mathbf{e}}](\mathbf{V}[\mathbf{x}]\mathbf{G}\hat{\mathbf{x}}')(\mathbf{V}[\mathbf{x}']\mathbf{G}^T\hat{\mathbf{x}})^T}{((\hat{\mathbf{x}}', \mathbf{G}^T\mathbf{V}[\mathbf{x}]\mathbf{G}\hat{\mathbf{x}}') + (\hat{\mathbf{x}}, \mathbf{G}\mathbf{V}[\mathbf{x}']\mathbf{G}^T\hat{\mathbf{x}}))^2} = \mathbf{V}[\hat{\mathbf{x}}', \hat{\mathbf{x}}]^T.
\end{aligned} \tag{6.130}$$

where $\mathbf{V}[\hat{\mathbf{e}}] = E[\Delta\hat{\mathbf{e}}^2] = (\hat{\mathbf{a}}, \mathbf{V}[\mathbf{h}]\hat{\mathbf{a}})$.

Variance in depth:

$$\mathbf{V}[\mathbf{h}, \hat{\mathbf{e}}] = -\mathbf{V}[\mathbf{h}]\hat{\mathbf{a}}. \tag{6.133}$$

$$\mathbf{V}[\mathbf{h}, \hat{\mathbf{x}}] = -\frac{\mathbf{V}[\mathbf{h}, \hat{\mathbf{e}}](\mathbf{V}[\mathbf{x}]\mathbf{G}\hat{\mathbf{x}}')^T}{(\hat{\mathbf{x}}', \mathbf{G}^T\mathbf{V}[\mathbf{x}]\mathbf{G}\hat{\mathbf{x}}') + (\hat{\mathbf{x}}, \mathbf{G}\mathbf{V}[\mathbf{x}']\mathbf{G}^T\hat{\mathbf{x}})^2} \tag{6.132}$$

$$\begin{aligned} \mathbf{V}[Z] = \frac{1}{\|\hat{\mathbf{x}} \times \mathbf{R}\hat{\mathbf{x}}'\|^2} \quad (\quad & Z^2(\hat{\mathbf{m}}, \mathbf{V}[\hat{\mathbf{x}}]\hat{\mathbf{m}}) - 2ZZ'(\hat{\mathbf{m}}, \mathbf{V}[\hat{\mathbf{x}}, \hat{\mathbf{x}}']\mathbf{R}^T\hat{\mathbf{m}}) \\ + \quad & {Z'}^2(\hat{\mathbf{m}}, \mathbf{R}\mathbf{V}[\hat{\mathbf{x}}']\mathbf{R}^T\hat{\mathbf{m}}) - 2Z(\hat{\mathbf{m}}, \mathbf{V}[\hat{\mathbf{x}}, \mathbf{h}]\hat{\mathbf{m}}) \\ + \quad & 2Z'(\hat{\mathbf{m}}, \mathbf{R}\mathbf{V}[\hat{\mathbf{x}}, \mathbf{h}]\hat{\mathbf{m}}) + (\hat{\mathbf{m}}, \mathbf{V}[\mathbf{h}\hat{\mathbf{m}})), \end{aligned} \tag{6.136}$$

where $\hat{\mathbf{m}} = N[\mathbf{h} \times \hat{\mathbf{x}}] \times \mathbf{R}\hat{\mathbf{x}}'$.

Example 6.1 If each coordinate is perturbed independently by Gaussian noise of mean 0 and variance ϵ^2, the covariance matrices of $\boldsymbol{x}$ and $\boldsymbol{x}'$ are $V[\boldsymbol{x}] = V[\boldsymbol{x}'] = \epsilon^2 \boldsymbol{P_k}$. The optimal correction (6.20) reduces to

$$\Delta \boldsymbol{x} = \frac{(\boldsymbol{x}, \boldsymbol{G}\boldsymbol{x}')\boldsymbol{P_k}\boldsymbol{G}\boldsymbol{x}'}{\|\boldsymbol{P_k}\boldsymbol{G}^\top \boldsymbol{x}\|^2 + \|\boldsymbol{P_k}\boldsymbol{G}\boldsymbol{x}'\|^2},$$

$$\Delta \boldsymbol{x}' = \frac{(\boldsymbol{x}, \boldsymbol{G}\boldsymbol{x}')\boldsymbol{P_k}\boldsymbol{G}^\top \boldsymbol{x}}{\|\boldsymbol{P_k}\boldsymbol{G}^\top \boldsymbol{x}\|^2 + \|\boldsymbol{P_k}\boldsymbol{G}\boldsymbol{x}'\|^2}. \qquad (6.28)$$

The a posteriori covariance matrices (6.21) become

$$V[\hat{\boldsymbol{x}}] = \epsilon^2 \left(\boldsymbol{P_k} - \frac{(\boldsymbol{P_k}\boldsymbol{G}\hat{\boldsymbol{x}}')(\boldsymbol{P_k}\boldsymbol{G}\hat{\boldsymbol{x}}')^\top}{\|\boldsymbol{P_k}\boldsymbol{G}^\top \hat{\boldsymbol{x}}\|^2 + \|\boldsymbol{P_k}\boldsymbol{G}\hat{\boldsymbol{x}}'\|^2} \right),$$

$$V[\hat{\boldsymbol{x}}'] = \epsilon^2 \left(\boldsymbol{P_k} - \frac{(\boldsymbol{P_k}\boldsymbol{G}^\top\hat{\boldsymbol{x}})(\boldsymbol{P_k}\boldsymbol{G}^\top\hat{\boldsymbol{x}})^\top}{\|\boldsymbol{P_k}\boldsymbol{G}^\top \hat{\boldsymbol{x}}\|^2 + \|\boldsymbol{P_k}\boldsymbol{G}\hat{\boldsymbol{x}}'\|^2} \right),$$

$$V[\hat{\boldsymbol{x}}, \hat{\boldsymbol{x}}'] = -\frac{\epsilon^2(\boldsymbol{P_k}\boldsymbol{G}\hat{\boldsymbol{x}}')(\boldsymbol{P_k}\boldsymbol{G}^\top\hat{\boldsymbol{x}})^\top}{\|\boldsymbol{P_k}\boldsymbol{G}^\top \hat{\boldsymbol{x}}\|^2 + \|\boldsymbol{P_k}\boldsymbol{G}\hat{\boldsymbol{x}}'\|^2} = V[\hat{\boldsymbol{x}}', \hat{\boldsymbol{x}}]^\top. \qquad (6.29)$$

An unbiased estimator of the variance ϵ^2 is obtained in the form

$$\hat{\epsilon}^2 = \frac{(\boldsymbol{x}, \boldsymbol{G}\boldsymbol{x}')^2}{\|\boldsymbol{P_k}\boldsymbol{G}^\top \hat{\boldsymbol{x}}\|^2 + \|\boldsymbol{P_k}\boldsymbol{G}\hat{\boldsymbol{x}}'\|^2}. \qquad (6.30)$$

Example 6.2 Consider the parallel stereo system described in Section 6.1.3 with the noise characteristics given in Example 6.1. Let (x, y) and (x', y') be the corresponding image points on the first and the second image planes, respectively. The optimal correction (6.28) reduces to

$$\hat{\boldsymbol{x}} = \begin{pmatrix} (x+x')/2 \\ y \\ 1 \end{pmatrix}, \qquad \hat{\boldsymbol{x}}' = \begin{pmatrix} (x+x')/2 \\ y' \\ 1 \end{pmatrix}. \tag{6.31}$$

The residual (6.22) is simply

$$\hat{J} = \frac{1}{2\epsilon^2}(x - x')^2, \tag{6.32}$$

which gives an unbiased estimator of ϵ^2 in the form

$$\hat{\epsilon}^2 = \frac{1}{2}(x - x')^2. \tag{6.33}$$

The a posteriori covariance matrices (6.29) reduce to

$$V[\hat{\boldsymbol{x}}] = V[\hat{\boldsymbol{x}}'] = \epsilon^2 \begin{pmatrix} 1/2 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \quad V[\hat{\boldsymbol{x}}, \hat{\boldsymbol{x}}'] = \epsilon^2 \begin{pmatrix} 1/2 & & \\ & 0 & \\ & & 0 \end{pmatrix}. \tag{6.34}$$