APPENDIX A

Ultraproducts

Similarly to colimits of directed systems, ultraproducts provide an important tool for creating new models from old. Roughly, the ultraproduct construction shows how to glue together a collection of structures to form a new structure satisfying any formula which holds in 'most' of the original structures. Mathematically, the notion 'most of' is formulated in terms of an ultrafilter over the collection's index set.

1. Filters and Ultrafilters

DEFINITION A.1. A*filter* over a non-empty set I is a collection F of subsets of I such that

(1) $I \in F$,

(2) if $X, Y \in F$ then $X \cap Y \in F$,

(3) If $X \in F$ and $X \subseteq Y \subseteq I$, then $Y \in F$.

A filter is *proper* if it is distinct from the full power set of I. An *ultrafilter* over I is a proper filter U such that for every subset X of I, either X or $I \setminus X$ belongs to U (but not both).

EXAMPLE A.2. (1) The full powerset $\mathcal{P}(I)$ of I is a filter over I. Given a subset $X \subseteq I$, the collection

$$\uparrow X := \{Y \subseteq I \mid X \subseteq Y\}$$

is a filter over I; this filter is called the *principal* filter generated by F. It is proper iff $X \neq \emptyset$.

It is easy to see that an ultrafilter is principal iff it is of the form $\pi_i := \{Y \subseteq I \mid i \in Y\}$ for some object $i \in I$.

(2) An example of a non-principal filter over an infinite set I is the so-called *Fréchet filter* consisting of the collection of co-finite subsets of I. (A subset of I is co-finite if its complement is finite.)

(3) A large supply of filters is provided by the following definition. Given a collection E of subsets of I, the set

$$F_E := \{Y \subseteq I \mid Y \supseteq X_1 \cap \ldots \cap X_n, \text{ with } X_1, \ldots, X_n \in E\}$$

is a filter over I, the filter that is *generated* by E. It is easy to see that this is the smallest filter over I such that $E \subseteq F$. The filter F_E is proper if and only if the set E has the *finite intersection property*, that is, the intersection of any finite subcollection of E is nonempty.

An alternative definition of an ultrafilter states that ultrafilters are filters that have no proper extension.

PROPOSITION A.3. *Let F be a filter over the set I. Then F is an ultrafilter iff F is a maximal proper filter.*

A key result on ultrafilters is the following theorem, which will provide our standard tool for proving the existence of a ultrafilter containing certain sets.

THEOREM A.4. *(Ultrafilter Theorem) Let I be a nonempty set. Then every proper filter over I can be extended to an ultrafilter over I.*

PROOF. Fix a proper filter F over I and apply Zorn's Lemma to the partial order of proper filters extending F. The result then follows by Proposition A.3. □

In applications the point is often to find ultrafilters satisfying additional properties such as the following.

DEFINITION A.5. An ultrafilter U over an infinite set I is called *regular* if there is a collection $E \subseteq U$, of the same cardinality as I, and such that each $i \in I$ belongs to only finitely many sets X in E.

2. Ultraproducts and ultrapowers

In this section we define the ultraproduct construction. Fix a language L, let $\{A_i \mid i \in I\}$ be a collection of L-structures, and let U be some ultrafilter over the index set I.

Starting with the definition of the carrier of the ultraproduct of this family, we first consider the product $\prod_{i\in I} A_i$ of the carriers of the structures. Recall that $\prod_{i\in I} A_i$ is the collection of maps $f\colon I \to \bigcup_{I\in I} A_i$ such that $f(i) \in A_i$ for every index i. Given two functions $f, g \in \prod_{i\in I} A_i$ we say that f and g are U-equivalent, notation $f \sim^U g$, if the set $\{i \in I \mid f(i) = g(i)\}$ belongs to U. It is straightforward to check the following proposition.

PROPOSITION A.6. *The relation $\sim^U$ is an equivalence relation on the set $\prod_{i\in I} A_i$.*

The *ultraproduct* of the sets A_i *modulo* U is the set of all equivalence classes of $\sim^U$,

$$\prod\nolimits_U A_i := \{f^U \mid f \in \prod\nolimits_{i\in I} A_i\},$$

where f^U denotes the $\sim^U$-equivalence class of f, that is: $f^U := \{g \in \prod_{i\in I} A_i \mid f \sim^U g\}$. We now apply the same idea to models.

DEFINITION A.7. Given a collection $\{A_i \mid i \in I\}$ of L-structures, and an ultrafilter U over the index set I, the *ultraproduct* $\prod_U A_i$ is the L-structure A described as follows:

(1) the domain of A is the set $\prod_U A_i$ just defined,

(2) for a constant c in L, we define $c^A := \big(\lambda i.c^{A_i}\big)^U$,

(3) for an n-ary function symbol h in L and an n-tuple $(f_1^U, \dots, f_n^U)$ in $\prod_U A_i$, we define

$$h^A(f_1^U, \dots, f_n^U) := \big(\lambda i.h^{A_i}(f_1(i), \dots, f_n(i))\big)^U,$$

(4) for an n-ary relation symbol R in L and an n-tuple $(f_1^U, \dots, f_n^U)$ in $\prod_U A_i$, we define

$$R^A(f_1^U, \dots, f_n^U) :\Leftrightarrow \big\{i \in I \mid R^{A_i}(f_1(i), \dots, f_n(i))\big\} \in U.$$

In the case where the structures A_i are all the same, say, $A_i = B$ for all i, we speak of the *ultrapower of* B, notation: $\prod_U B$ or B^I/U.

Clearly, one should check that the clauses (3) and (4) of the above construction are well-defined, i.e., that the definitions do not depend on the particular representants of the equivalence classes.

The key result on ultraproducts, fundamental to any use of the construction in model theory, is the folllowing.

THEOREM A.8. *(Łos' Theorem) Let U be an ultrafilter over the index set I, and let A_i be an L-structure for each $i \in I$. Then for any first-order formula $\varphi(x_1, \ldots, x_n)$ and for any n-tuple $(f_1, \ldots, f_n)$ of elements in $\prod_{i \in I} A_i$ we have*

$$\prod_U A_i \models \varphi(f_1^U, \ldots, f_n^U) \text{ iff } \{i \in I \mid A_i \models \varphi(f_1(i), \ldots, f_n(i))\} \in U.$$

PROOF. Exercise. □

As some immediate corollaries of Łos' Theorem we mention the following.

COROLLARY A.9. *For any theory T the class of models for T is closed under taking ultraproducts.*

COROLLARY A.10. *Let U be some ultrafilter over an index set I. For any structure A, the diagonal map d given by $d(a) := (\lambda i.d)^U$ is an elementary embedding of A into A^I/U.*

3. Applications

In this section we show some application of ultraproducts, starting with compactness.

THEOREM A.11. *Any finitely satisfiable theory T is satisfiable.*

PROOF. Assume that every finite subset of T has a model; we will construct a model for T as some ultraproduct of these models.

For this purpose let I be the collection of finite subsets of T, and for each $i \in I$ let A_i be a model of i. Given a formula $\varphi \in T$, let $I_\varphi \subseteq I$ be given by $i \in I_\varphi \Leftrightarrow \varphi \in i$. It is not hard to prove that the set

$$F := \{I_\varphi \mid \varphi \in T\}$$

has the *finite intersection property*, so that F can be extended to an ultrafilter U. (Note that this ultrafilter is *regular*, see Definition A.5

We claim that the ultraproduct $A := \prod_U A_i$ is the required model of T. To see this, take an arbitrary formula $\varphi \in T$. It follows from our assumption on the structures A_i that $\varphi \in i$ implies $A_i \models \varphi$. But then by definition of I_φ we find $I_\varphi \subseteq \{i \in I \mid A_i \models \varphi\}$, and since $I_\varphi \in U$ by our choice of U we find $\{i \in I \mid A_i \models \varphi\} \in U$ by upward closure of (ultra)filters. From this, an application of Łos' Theorem reveals that $A \models \varphi$ as required. □

A very strong result on ultrapowers is the Keisler-Shelah theorem.

THEOREM A.12. *(Keisler-Shelah) Let A and B be structures for some language L. Then $A \equiv B$ iff there is an index set I and an ultrafilter U over I such that $A^I/U \cong B^I/U$.*

The proof of this theorem lies beyond the scope of this note, but we can show how the following characterisation result follows from it.

Corollary A.13. *Let, for some language L, K be a class of L-structures which is closed under isomorphic copies. Then K is axiomatisable by an L-theory iff it is closed under ultraproducts, while its complement is closed under taking ultrapowers.*

Proof. Leaving the direction from left to right as an exercise for the reader, we focus on the opposite direction. Let K be a class of L-structures which is closed under isomorphic copies and ultraproducts, while its complement $\overline{K}$ is closed under taking ultrapowers. We claim that $K = \mathrm{Mod}(\mathrm{Th}(K))$, i.e., K is axiomatised by $\mathrm{Th}(K)$.

Clearly it suffices to show that $\mathrm{Mod}(\mathrm{Th}(K)) \subseteq K$, so assume that $A \models \mathrm{Th}(K)$. From this it easily follows that the theory of A is finitely satisfiable in K, so that by the (proof of) Theorem A.11 $\mathrm{Th}(A)$ is satisfied in some ultraproduct B of structures in K. This B then belongs to K by assumption, while $B \models \mathrm{Th}(A)$ simply means that $A \equiv B$. By the Keisler-Shelah Theorem we may infer from the latter observation that A and B have isomorphic ultrapowers. But, given the closure conditions on K and $\overline{K}$, this implies that A actually belongs to K, as required. □