APPENDIX B

Skolemisation

It is often convenient to work in a first-order signature with enough function symbols to witness every existential statement.

Definition B.1. *Let T be some theory in a first-order language L. We say that T has* Skolem functions*, or is a* Skolem theory*, if for every L-formula $\varphi(x, y_1, \ldots, y_n)$ there is an n-ary function symbol f such that*

$$T \models \forall y_1 \ldots \forall y_n \big(\exists x\, \varphi(x, y_1, \ldots, y_n) \to \varphi(f(y_1, \ldots, y_n)), y_1, \ldots, y_n\big)$$

Observe that, as a special case, any theory with Skolem functions must contain a constant (witnessing the formula $\exists x\,(x = x)$). It is not hard to see that, in order to check that a theory has Skolem functions, it suffices to consider formulas $\varphi(x, y_1, \ldots, y_n)$ that are quantifier-free.

Skolem theories have some nice properties.

Proposition B.2. *Let T be a theory with Skolem functions. Then T admits quantifier elimination.*

The second property, which is easily proved using the Tarski-Vaught test, is often used to establish the downward Löwenheim-Skolem Theorem. We leave its proof as an exercise.

Proposition B.3. *Let A be a model of some Skolem theory T, and let $X \subseteq A$. Then the substructure of A that is generated by X is in fact an elementary substructure of A.*

The substructure mentioned in Proposition B.3 is called the *Skolem hull* of X.

The following proposition states that one may always extend a theory to a Skolem theory (in an enriched language).

Theorem B.4 (Skolemisation). *Let T be an L-theory in some first-order language L. Then there is a Skolem theory $T' \supseteq T$ in some language $L' \supseteq L$ such that $|L'| \leq |L| + \aleph_0$ and every model of T can be expanded to a model of T'.*

Proof. We build up L' as the union of an increasing chain $L_0 \subseteq L_1 \subseteq \cdots$ of languages, and T' as the union of a corresponding increasing chain $T_0 \subseteq T_1 \subseteq \cdots$ of theories.

For $n = 0$ we define $L_0 = L$ and $T_0 = T$. Inductively, to define L_{k+1} from L_k, we add a fresh function symbol f_φ to L_k, for every L_k-formula $\varphi(x, y_1, \ldots, y_n)$. We then define T_{k+1} by adding to T_k all sentences of the form

$$\forall y_1 \ldots \forall y_n \big(\exists x\, \varphi(x, y_1, \ldots, y_n) \to \varphi(f_\varphi(y_1, \ldots, y_n)), y_1, \ldots, y_n\big),$$

where φ is an L_k-formula. Finally, we put $L' = \bigcup_n L_n$ and $T' = \bigcup_n T_n$.

It is obvious from this definition that $|L'| \leq |L| + \aleph_0$, and that T' has Skolem functions. To show that any model of T can be expanded to a model of T', it suffices to prove that, for every n, any model of T_n can be expanded to a model of T_{n+1}. The latter proof is an easy exercise. □

Definition B.5. *If T and T' are as described as in Theorem B.4, we call T' a* Skolemisation *of T.*

The following is a straightforward consequence of Theorem B.4.

Proposition B.6. *Let T be an L-theory in some first-order language L, and let T' be a Skolemisation of T. Then T' is a conservative extension of T, that is, for every L-sentence φ:*

$$T' \models \varphi \text{ iff } T \models \varphi.$$

1. Exercises

Exercise 1. Show that Skolem theories admit quantifier elimination (i.e., prove Proposition B.2).

Exercise 2. Let T be a Skolem theory. Show that T has a universal axiomatisation.