Topics in Modal Logic (Fall 2025)

Tutorial Exercises 5

Exercise 1 (counting modalities for the bag functor) Convince yourself that, for each $k \in \mathbb{N}$, the family of maps given by putting, for each set S,

$$\lambda^{\geq k}_S : U \mapsto \{\mu : S \to \mathbb{N}^\infty \mid \sum_{u \in U} \mu(u) \geq k\}.$$

indeed defines a predicate lifting $\lambda^{\geq k} : \breve{P} \to \breve{P}B$ for the bag functor B.

Exercise 2 (counting in Set with modified arrows) For an arbitrary set S, we consider the map $\theta_S : PS \to PPS$ given by

$$\theta_S : U \mapsto \{A \in PS \mid |A \cap U| \geq 3\}.$$

Check whether θ makes the naturality diagram commute:

$$\begin{array}{ccccc} S' & \quad & PS' & \xrightarrow{\theta_{S'}} & PPS' \\ {\scriptstyle f}\downarrow & & {\scriptstyle \breve{P}f}\uparrow & & \uparrow{\scriptstyle \breve{P}Pf} \\ S & & PS & \xrightarrow[\theta_S]{} & PPS \end{array} \tag{1}$$

in case f is, respectively, an arbitrary, injective, surjective or bijective function.

Now assume that we consider variations Set_i, Set_s and Set_b of the standard category Set, by restricting the arrows to, respectively, injective, surjective and bijective functions. For which of the four categories is θ a predicate lifting?

Exercise 3 (operations on predicate liftings) Fix a set functor T. Show that the collection of unary predicate liftings for T forms a boolean algebra, for some naturally defined operations. Is this still the case for n-ary predicate liftings, where $n < \omega$ is arbitrary?

Exercise 4 (third modal distributive law) Let T be a smooth and standard set functor, and recall that ML_T denotes the set of formulas in the coalgebraic language associated with this functor. Let $\alpha \in T_\omega(\mathrm{ML}_T)$ and $\Psi \in T_\omega P_\omega(Base(\alpha))$ be such that $(\alpha, \Psi) \notin \overline{T}(\not\in)$ (i.e., the pair (α, Ψ) does not belong to the lifting of the relation $\not\in$).

(a) Show that the formula $\nabla\alpha \wedge \nabla T(\bigwedge \circ P\neg)\Psi$ is not satisfiable.

(b) Is smoothness really needed in your proof?

Note that this is part of the modal distributive law for the negation (cf. Definition 5.25); you may use the validity of the other coalgebraic distributive laws, but not of the one for the negation.