7 One-step logic

In this chapter we will zoom in on a micro-version of coalgebraic modal logic that we call *one-step logic*. For this purpose we introduce *one-step models*; intuitively, these are *windows* over a T-model, only allowing access to the coalgebraic unfolding of one single state. Formally one-step models are simply defined as triples (S, σ, m) with $\sigma \in TS$ and m a marking of states in S with a set of variables. In particular, no coalgebra map is assumed.

Then, given a modal signature Λ, we will introduce a very simple modal language 1ML_Λ for describing properties of one-step models, the *one-step language* associated with Λ. The so-called *one-step formulas* of 1ML_Λ will be based on a set A of propositional variables, which should be seen as place-holders for proper ML_Λ-formulas. In particular, they are *different* from the proposition letters of the languages — to avoid confusion the latter could be encoded as (nullary) predicate liftings, see Remark 6.11. Characteristic of one-step formulas is the syntactic restriction that all occurrences of variables are in the scope of exactly one modality; thus the one-step formalism can be seen as a very simple fragment of the full modal language.

The point of focussing on this 'local', one-step version of coalgebraic modal logic is that many properties of coalgebraic modal logic are in fact already determined at this one-step level. As we will see in this chapter and further on, this applies both to model-theoretic properties and to derivation systems.

Throughout this chapter we fix a set functor T and a modal signature Λ.

7.1 One-step syntax and semantics

We can now turn to the formal definitions of one-step logic. We start with syntax. Given any set E of expressions, we let $\text{PL}(E)$ denote the set of *(boolean) propositional* or *rank-0* formulas over E, given by the following grammar:

$$\pi \ ::= \ e \in E \mid \bot \mid \top \mid \pi_0 \vee \pi_1 \mid \pi_0 \wedge \pi_1 \mid \neg \pi.$$

Definition 7.1 Fix a set Λ of predicate liftings, and a set A of propositional variables. Given any set Φ of formulas, we define

$$\Lambda(\Phi) := \{\heartsuit_\lambda(\varphi_0, \ldots, \varphi_{n-1} \mid \lambda \in \Lambda, \mathsf{ar}(\lambda) = n, \varphi_0, \ldots, \varphi_{n-1} \in \Phi\}.$$

We shall call

$$1\text{ML}_\Lambda(A) := \text{PL}(\Lambda(\text{PL}(A))).$$

the *one-step language* for Λ over A, and refer to formulas $\alpha, \beta \in 1\text{ML}_\Lambda$ as *one-step* or *rank-1* Λ-formulas. For both $\text{PL}(A)$ and $1\text{ML}_\Lambda(A)$ we may use standard propositional abbreviations such as the connectives $\to$ and $\leftrightarrow$. $\lhd$

A more direct characterisation of the language $1\text{ML}_\Lambda(A)$ is that it consists of those $\text{ML}_\Lambda(A)$-formulas in which every variable $a \in A$ occurs in the scope of *exactly one modality*. As yet another alternative, the set $1\text{ML}_\Lambda(A)$ can be characterised by the following grammar:

$$\alpha \ ::= \ \heartsuit_\lambda(\pi_0, \ldots, \pi_{n-1}) \mid \bot \mid \top \mid \alpha_0 \vee \alpha_1 \mid \alpha_0 \wedge \alpha_1 \mid \neg \alpha,$$

where $\lambda \in \Lambda$ and $\pi_i \in \text{PL}(A)$.

Example 7.2 (1) With $A = \{a, b, c\}$ and $\Lambda = \{\Box, \Diamond\}$, examples of one-step formulas are $\Diamond(a \wedge b)$, $(\Box((a \vee a) \wedge \bot) \vee \top$. The formulas $\Diamond\Diamond a$ and $a \vee \Box\bot$ are not one-step formulas.

One-step formulas are naturally interpreted in *one-step models*, which consist of a one-step frame together with a marking. As mentioned, a one-step T-model can be seen as a window over a T-model, or as a potential unfolding of one state in a T-model.

Definition 7.3 A *one-step T-frame* is a pair (S, σ) with $\sigma \in TS$. A *one-step T-model* over a set A of variables is a triple (S, σ, m) such that (S, σ) is a one-step T-frame and $m : S \to PA$ is a A-marking on S. $\lhd$

Definition 7.4 Given a marking $m : S \to PA$, we define the 0-*step interpretation* $[\![\pi]\!]^0_m \subseteq S$ of $\pi \in \mathrm{PL}(A)$ by the obvious induction:

$$
\begin{array}{lcl}
[\![a]\!]^0_m & := & m^\flat(a) \quad \Big(= \{s \in S \mid a \in m(s)\}\Big) \\
[\![\top]\!]^0_m & := & S \\
[\![\bot]\!]^0_m & := & \varnothing \\
[\![\neg\pi]\!]^0_m & := & S \setminus [\![\pi]\!]^0_m \\
[\![\pi_0 \wedge \pi_1]\!]^0_m & := & [\![\pi_0]\!]^0_m \cap [\![\pi_1]\!]^0_m \\
[\![\pi_0 \vee \pi_1]\!]^0_m & := & [\![\pi_0]\!]^0_m \cup [\![\pi_1]\!]^0_m.
\end{array}
$$

If $s \in [\![\pi]\!]^0_m$, we will say that π is true or satisfied at s under m, and sometimes write $S, m, s \Vdash^0 \pi$.

Similarly, the 1-*step interpretation* $[\![\alpha]\!]^1_m$ of $\alpha \in 1\mathrm{ML}_\Lambda(A)$ is defined as a subset of TS, with

$$[\![\heartsuit_\lambda(\pi_0, \ldots, \pi_{n-1})]\!]^1_m := \lambda_S([\![\pi_0]\!]^0_m, \ldots, [\![\pi_{n-1}]\!]^0_m),$$

and standard clauses applying for $\bot, \wedge, \vee$ and $\neg$. Given a one-step model (S, σ, m), we write $S, \sigma, m \Vdash^1 \alpha$ for $\sigma \in [\![\alpha]\!]^1_m$. $\lhd$

Remark 7.5 The semantics of the one-step formulas defined above show that, indeed, one-step logic is a way of 'doing coalgebraic logic without coalgebras'. The link with the interpretation of coalgebraic modal logic is as follows.

Let $\varphi \in \mathrm{ML}_\Lambda$ be some formula of the form $\varphi = \heartsuit_\lambda(\psi_0, \ldots, \psi_{n-1})$, and let $\mathtt{Sfor}_0(\varphi) = \{\psi_0, \ldots, \psi_{n-1}\}$ denote the set of *direct* subformulas of φ. Take a set A of variables, and some substitution $\tau : A \to \mathtt{Sfor}_0(\varphi)$ mapping variables to direct subformulas of φ. Now any T-model $\mathbb{S} = (S, \sigma, U)$ induces a A-marking $m = m_{\tau,\mathbb{S}}$ given by

$$m : s \mapsto \{a \in A \mid \mathbb{S}, s \Vdash \tau(a)\}.$$

In other words, we now think of the formulas in $\mathtt{Sfor}_0(\varphi)$ as being atomic. It is then straightforward to verify that we have the following semantic link, for any formula $\varphi_0 \in 1\mathrm{ML}_\Lambda(A)$ such that $\varphi_0[\tau] = \varphi$:

$$[\![\varphi_0]\!]^1_m = [\![\varphi]\!]^{\mathbb{S}}. \tag{63}$$

Formulated in terms of one-step models, we have that, for all $s \in S$:

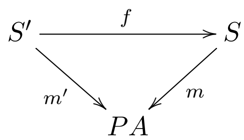
$$S, m, \sigma(s) \Vdash^1 \varphi_0 \text{ iff } \mathbb{S}, s \Vdash \varphi.$$

Notions like one-step satisfiability, validity and equivalence are defined in the obvious way.

Definition 7.6 Let α and α' be one-step formulas. The formula α is *one-step satisfiable* if there is a one-step model (S, σ, m) such that $S, \sigma, m \Vdash^1 \alpha$, and *one-step valid* if $S, \sigma, m \Vdash^1 \alpha$ for all one-step models (S, σ, m). We say that α' is a *one-step consequence* of α (written $\alpha \models^1 \alpha'$) if $S, \sigma, m \Vdash^1 \alpha$ implies $S, \sigma, m \Vdash^1 \alpha'$, for all one-step models (S, σ, m), and that α and α' are *one-step equivalent*, notation: $\alpha \equiv^1 \alpha'$, if $\alpha \models^1 \alpha'$ and $\alpha' \models^1 \alpha$. $\lhd$

We also need morphisms between one-step frames and models.

Definition 7.7 A *one-step frame morphism* between two one-step frames (S', σ') and (S, σ) is a map $f : S' \to S$ such that $(Tf)\sigma' = \sigma$. In case such a map satisfies $m' = m \circ f$, for some markings m and m' on S and S', respectively, viz.,



then we say that f is a *one-step model morphism* from (S', σ', m') to (S, σ, m). $\lhd$

The following proposition, stating that the truth of one-step formulas is invariant under one-step morphisms, is fundamental. We will occasionally refer to this proposition as *naturality*, since this invariance essentially boils down to the naturality of the predicate liftings in Λ.

Proposition 7.8 *Let $f : (S', \sigma', m') \to (S, \sigma, m)$ be a morphism of one-step models over A. Then for every formula $\alpha \in 1\mathrm{ML}_\Lambda(A)$ we have*

$$S', \sigma', m' \Vdash^1 \alpha \text{ iff } S, \sigma, m \Vdash^1 \alpha.$$

Formulating it differently, for any marking $m : S \to PA$ and any map $f : S' \to S$, we have

$$[\![\alpha]\!]^1_{m \circ f} = (\breve{P}Tf)[\![\alpha]\!])^1_m. \tag{64}$$

The following proposition states that the meaning of a one-step formula only depends on the variables occurring in it.

Proposition 7.9 *Let (S, σ, m) be a one-step model over A, and let $\alpha \in 1\mathrm{ML}_\Lambda(A)$ be a one-step formula which belongs to the set $1\mathrm{ML}_\Lambda(A')$, for some subset $A' \subseteq A$. Then we have*

$$S, \sigma, m \Vdash^1 \alpha \text{ iff } S, \sigma, m^{A'} \Vdash^1 \alpha,$$

where $m^{A'}$ is the A'-marking given by $m^{A'}(s) := m(s) \cap A'$.

Finally, an important role in one-step logic is played by the following rather special one-step models.

Definition 7.10 Given a variable set A, we define the *canonical marking for A* as the A-marking $n_A := \mathsf{id}_{PA} : PA \to PA$. We say that a one-step model is *A-canonical* if it is of the form (PA, Γ, n_A), for some $\Gamma \in TPA$. In the sequel we will occasionally denote the marking $\mathsf{id}_{PA} : PA \to PA$ as n_V. ◁

The term 'canonical' model is justified by the following proposition, the proof of which is left as an exercise.

Proposition 7.11 *Let $\alpha \in 1\mathrm{ML}_\Lambda(A)$ be a one-step formula. Then α is one-step valid iff α holds at every A-canonical one-step model.*

7.2 Monotone signatures and disjunctive bases

If Λ is a collection of *monotone* predicate liftings, we may further simplify the formulas of one-step modal logic. In this setting we can also introduce the fundamental concepts of disjunctive formulas and disjunctive bases, the latter providing us with a generic framework for obtaining *normal formals* in coalgebraic modal logic.

Definition 7.12 Let $\lambda : \breve{P}^n \dot{\to} \breve{P}T$ be a predicate lifting for the set functor T. We call λ *monotone* if it preserves the subset order in each coordinate, that is, for all sets S and all tuples of pairs of subsets $X_i, Y_i \subseteq S$, we have that

$$\text{if } X_i \subseteq Y_i, \text{ all } i, \text{ then } \lambda_S(X_0, \ldots, X_{n-1}) \subseteq \lambda_S(Y_0, \ldots, Y_{n-1}).$$

The n-ary predicate lifting λ^∂ defined by putting

$$\lambda^\partial_S(X_0, \ldots, X_{n-1}) := TS \setminus \lambda_S(S \setminus X_0, \ldots, S \setminus X_{n-1})$$

is called the *boolean dual* of λ. ◁

It is straightforward to verify that the boolean dual of a monotone predicate lifting is again mononotone.

In the setting where all predicate liftings are monotone, we may generally restrict attention to the positive fragment of one-step modal logic, provided that Λ is closed under taking boolean duals.

Definition 7.13 A *monotone modal signature* for a set functor T is a collection Λ of monotone predicate liftings which is closed under taking boolean duals. ◁

Definition 7.14 Fix a monotone modal signature Λ for the set functor T. Given any set E of expressions, we let $\mathrm{PL}^+(E)$ denote the set of *lattice formulas* over E, given by the following grammar:

$$\pi ::= e \in E \mid \bot \mid \top \mid \pi_0 \vee \pi_1 \mid \pi_0 \wedge \pi_1.$$

Where A is a set of propositional variables, we define the set $1\mathrm{ML}^+_\Lambda(A)$ of *positive one-step Λ-formulas* as $1\mathrm{ML}^+_\Lambda(A) := \mathrm{PL}^+(\Lambda(\mathrm{PL}^+(A))$. ◁

A more direct characterisation of the language $1\mathrm{ML}_\Lambda(A)$ is that it consists of those negation-free $\mathrm{ML}_\Lambda(A)$-formulas in which every variable $a \in A$ occurs in the scope of exactly one modality.

For positive one-step formulas we have the following monotonicity property.

Proposition 7.15 *Assume that Λ is a monotone modal signature for a set functor T, let (S, σ) be a one-step frame, and let $m, m' : S \to PA$ be two markings such that $m(s) \subseteq m'(s)$, for all $s \in S$. Then for any formula $\alpha \in 1\mathrm{ML}^+_\Lambda(A)$ we have that*

$$S, \sigma, m \Vdash^1 \alpha \text{ implies } S, \sigma, m' \Vdash^1 \alpha.$$

We now introduce the fundamental concept of a disjunctive basis. First we consider the notion of a disjunctive formula.

Definition 7.16 A one-step formula $\alpha \in 1\mathrm{ML}^+_\Lambda(A)$ is called *disjunctive* if for every one-step model (S, σ, m) such that $S, \sigma, m \Vdash^1 \alpha$ there is a one-step frame (S', σ') together with a one-step frame morphism $h : (S', \sigma') \to (S, \sigma)$ and a marking $m' : S' \to PA$, such that:

1. $S', \sigma', m' \Vdash^1 \alpha$;
2. $m'(s') \subseteq m(h(s'))$, for all $s' \in S'$;
3. $|m'(s')| \leq 1$, for all $s' \in S'$.

We sometimes refer to the one-step frame (S', σ') together with the map h as a *cover* of (S, σ), and to the one-step model (S', σ', m') together with the map h as a *separating cover* of (S, σ, m) for α. ◁

The intuition behind disjunctive formulas is that, in a certain sense, they never 'force' two distinct propositional variables to be true together, that is, any one-step model for a disjunctive formula δ in $1\mathrm{ML}^+_\Lambda(A)$ is covered by one in which every point satisfies at most one propositional variable from A.

Example 7.17 (1) A trivial example of a disjunctive formula is $\bigcirc a$ for $a \in A$, where we recall that $\bigcirc$ was the next-time modality viewed as a predicate lifting for the identity functor Id. A one-step model for this functor is a triple S, s, m consisting of a set S, an element $s \in S$ and a marking $m : S \to PA$. Then $S, s, m \Vdash^1 \bigcirc a$ if, and only if, $a \in m(s)$. But then, no elements in S besides s are relevant to the evaluation of $\bigcirc a$, and for s we can just forget about all other variables: set $m'(s) = \{a\}$ and $m'(v) = \varnothing$ for all $s \in S \setminus \{v\}$. We have $S, s, m' \Vdash^1 \bigcirc a$, $m'(v) \subseteq m(v)$ and $|m'(v)| \leq 1$ for all $v \in S$.

(2) For an example of a one-step formula that is *not* disjunctive, consider $\Box a \wedge \Diamond b$ (where $\Diamond$ and $\Box$ are the standard modalities for Kripke structures, that is,, coalgebras for the power set functor P). Observe that a one-step model for this functor is a triple (S, σ, m) with $\sigma \subseteq S$. It should be obvious that for the formula $\Box a \wedge \Diamond b$ to hold at such a structure, σ needs to have an element s where b holds, while at the same time *every* element of σ, including s, must satisfy a. There is no escape here: we can only have $(S, \sigma, m) \Vdash^1 \Box a \wedge \Diamond b$ if there is an element s making *both* a and b true.

(3) This is very different if we consider the typical disjunctive formulas for basic modal logic, which are of the form $\nabla\{a_1, \ldots, a_n\}$, or, in the language with $\Box$ and $\Diamond$:

$$\Diamond a_1 \wedge \cdots \wedge \Diamond a_n \wedge \Box(a_1 \vee \cdots \vee a_n)$$

For example, consider the formula $\nabla\{a, b, c\}$, which is true in a one-step model (S, s, m) with $S = \{u, v\}$, $m(u) = \{a, b\}$ and $m(v) = \{b, c\}$. Clearly there is no way to simply shrink the marking m to a separating marking m' so that $(S, s, m) \Vdash^1 \nabla\{a, b, c\}$: there are too many pigeons and too few pigeon holes — so the obvious solution is to make more pigeon holes! One way to do this is to "split" the points in $\{u, v\}$ so that we make room for each variable to be witnessed at a separate point. More formally, let $S' = \{(u, a), (u, b), (v, b), (v, c)\}$ and define $m' : S' \to \{a, b, c\}$ via the projection on $\{a, b, c\}$, i.e., $m'(u, a) = \{a\}$, etc. This one-step model satisfies $\nabla\{a, b, c\}$ and it has the obvious covering map h being the projection π_S on S.

(4) In fact, all nabla formulas are disjunctive. That is, let T be a standard and smooth set functor; given a set A and an element $\alpha \in T_\omega A$ we may think of $\nabla\alpha$ as a one-step formula. It is not hard to show that every one-step formula of this form is disjunctive.

As we will see later on, disjunctive formulas have very attractive properties, and so modal signatures with a rich supply of disjunctive formulas are of interest. Example 7.17(3) shows that not *every* one-step formula is disjunctive. It does make sense to consider modal signatures with a 'critical' amount of disjunctive formulas in the form of a *disjunctive basis*. Roughly speaking, the existence of a *disjunctive basis* means that the coalgebraic modal logic satisfies some version of the *modal distributive laws* that we proved for the ∇-logic in Chapter 5. For more details, we need some definitions.

Definition 7.18 We will consider, next to a (finite) collection A of variables, every element $B \in P_\omega A$ as a (formal) variable $\underline{B}$, and we shall be interested in the substitution

$$\bigwedge_A : P_\omega A \to A$$

that replaces each formal variable $\underline{B}$ with its corresponding conjunction $\bigwedge B \in \mathrm{PL}^+(A)$. $\lhd$

Intuitively then, Λ having a disjunctive basis means that there are sufficiently many disjunctive formulas so that for every formula $\alpha \in 1\mathrm{ML}_\Lambda(A)$ there is a disjunctive formula $\delta \in 1\mathrm{ML}_\Lambda(P_\omega A)$ such that

$$\alpha \equiv \delta[\textstyle\bigwedge_A]. \tag{65}$$

Observe that if we take α to be a conjunction of disjunctive formulas (65) expresses a law stating that conjunctions distribute over disjunctive formulas.

Example 7.19 ▶ `example 1: next time operator`

▶ `example 2:` $\Box a \wedge \Diamond b$

We formulate the definition of a disjunctive basis in such a way that we only need to check the existence of disjunctive formulas satisfying (65) in a small but representative number of cases.

Definition 7.20 Let $\mathtt{D}$ be an assignment of a set of positive one-step formulas $\mathtt{D}(A) \subseteq 1\mathrm{ML}^{+}_{\Lambda}(A)$ for all sets of variables A. Then $\mathtt{D}$ is called a *disjunctive basis* for Λ if each formula in $\mathtt{D}(A)$ is disjunctive, and the following conditions hold:

1. $\mathtt{D}(A)$ contains $\top$ and is closed under finite disjunctions (in particular, also $\bot = \bigvee \varnothing \in \mathtt{D}(A)$).
2. $\mathtt{D}$ is *distributive over* Λ: for every one-step formula of the form $\heartsuit_{\lambda}\overline{\pi}$ in $1\mathrm{ML}^{+}_{\Lambda}(A)$ there is a formula $\delta \in \mathtt{D}(P(A))$ such that $\heartsuit_{\lambda}\overline{\pi} \equiv \delta[\bigwedge_{A}]$.
3. $\mathtt{D}$ *admits a binary distributive law*: for any two formulas $\alpha \in \mathtt{D}(A)$ and $\alpha' \in \mathtt{D}(A')$, there is a formula $\gamma \in \mathtt{D}(P^{\star 2}(A \cup A')$ such that $\alpha \wedge \alpha' \equiv \gamma[\bigwedge_{A \cup A'}]$.

Here $P^{\star 2}(A \cup A')$ is the collection of subsets C of $A \cup A'$ such that either $|C| = 1$ or $|C| = 2$ and C overlaps with both A and A'. $\lhd$

The key property of disjunctive bases is captured by the following normal form theorem, which is easy to derive from the definition.

Proposition 7.21 *Suppose* $\mathtt{D}$ *is a disjunctive basis for* Λ. *Then for every one-step formula* $\alpha \in 1\mathrm{ML}^{+}_{\Lambda}(A)$ *there is a formula* $\delta \in \mathtt{D}(P(A))$ *such that* (65) *holds:* $\alpha \equiv \delta[\bigwedge_{A}]$.

▶ `examples to be added`

7.3 One-step derivation systems: sequent style

In this section and the next we will develop some natural *logic* at the level of one-step formulas; that is, we will introduce one-step derivation systems and discuss the notions of one-step soundness and completeness pertaining to such systems. As we will see in more detail in the next Chapter, the basic idea is that coalgebraic derivation systems consist of so-called *one-step rules*, on top of a classical propositional calculus.

The derivation system in this section will be based on *sequents*, that is, finite sets of formulas. We read sequents *disjunctively*, that is, we think of a sequent Γ as representing the formula $\bigvee \Gamma$, and the idea of a derivation of the sequent Γ is to provide a formal proof that this disjunction is valid. Following standard conventions in proof theory we will often denote a singleton sequent by its unique member (that is, write φ instead of $\{\varphi\}$), and use the comma for the union symbol (that is, let Γ, Δ denote the union $\Gamma \cup \Delta$ of the two sequents Γ and Δ).

Remark 7.22 Our set-up, which is sometimes referred to as Tait-style, differs from the slightly more common approach in which sequents are *pairs* of sets (or even multi-sets) of formulas, with the pair (Γ, Δ), usually written as $\Gamma \vdash \Delta$ or $\Gamma \Rightarrow \Delta$ representing the formula $\bigwedge \Gamma \to \bigvee \Delta$. The point is that, given our classical (boolean) context we can represent the two-sided sequent $\Gamma \Rightarrow \Delta$ as $\varnothing \Rightarrow \overline{\Gamma}, \Delta$, where $\overline{\Gamma} := \{\overline{\gamma} \mid \gamma \in \Gamma\}$ is the set of negations of the formulas in Γ. We may then subsequently drop the (fixed) antecedent $\varnothing$ and focus on the Tait-style sequent $\overline{\Gamma}, \Delta$. Conversely, a Tait-style sequent Σ represents the consequent (right hand side) of the two-sided sequent $\varnothing \Rightarrow \Sigma$.

Based on these correspondences one may develop the proof theory of two-sided sequent system along the same lines as what we doing here for one-sided sequent systems.

Remark 7.23 As a matter of fact, we could also have pursued a *tableau-style* approach where the idea is to focus on the antecedent part (left hand side) of a two-sided sequent. Such sequents are then to be read conjunctively, and the idea of a derivation is to *refute* the sequent, i.e., to provide a formal proof that the sequent is not satisfiable.

7.3.1 Rules and derivations

We shall formulate derivation rules in terms of propositional and rank-1 formulas over A. Accordingly, we shall refer to finite sets of propositional formulas, (rank-1 formulas, respectively) as *propositional* and *rank-1* sequents.

The key concept in coalgebraic derivation system is that of a *one-step derivation rule*.

Definition 7.24 A *derivation rule* is a pair $\mathsf{d} = (\mathsf{Prem}(\mathsf{d}), \mathsf{conc}(\mathsf{d}))$, where $\mathsf{Prem}(\mathsf{d})$ is a set of sequents, the *premises* of d, and $\mathsf{conc}(\mathsf{d})$ is a sequent, the *conclusion* of d. If $\mathsf{Prem}(\mathsf{d}) = \varnothing$ we say that d is *axiomatic* and we call $\mathsf{conc}(\mathsf{d})$ an *axiom*. A *derivation system* or *calculus* is nothing but a set of derivation rules.

A derivation rule is *propositional* if all of its premises and its conclusion are propositional sequents, whereas a *one-step* rule is given by a set of propositional premises and a rank-1 conclusion. In particular, one-step axioms are sets of one-step formulas. ◁

A derivation rule d with $\mathsf{Prem}(\mathsf{d}) = \{\Delta_1, \ldots, \Delta_{n-1}\}$ and $\mathsf{conc}(\mathsf{d}) = \Delta_0$ is often represented as $\Delta_1, \ldots, \Delta_{n-1}/\Delta_0$ or as

$$\frac{\Delta_1 \quad \cdots \quad \Delta_{n-1}}{\Delta_0}\,\mathsf{d}$$

Example 7.25 Examples of propositional rules are Modus Ponens $(\{a, a \to b\}, b)$ and Ex Falso Quodlibet $(\{\bot\}, a)$.

In ordinary modal logic (that is, where $T = P$ and we take both the diamond and the box modality as primitives), we can present the usual modal rule as follows:

$$\frac{a, b_0, \ldots, b_{n-1}}{\Box a, \Diamond b_0, \ldots, \Diamond b_{n-1}}\,\mathsf{M}$$

Other examples of one-step rules are the congruence rule C_λ:

$$\frac{a_0 \leftrightarrow b_0 \quad \cdots \quad a_{n-1} \leftrightarrow b_{n-1}}{\heartsuit_\lambda(a_0, \ldots, a_{n-1}) \leftrightarrow \heartsuit_\lambda(b_0, \ldots, b_{n-1})}\,C_\lambda$$

and the monotonicity rule M_λ

$$\frac{a_0 \to b_0 \quad \cdots \quad a_{n-1} \to b_{n-1}}{\heartsuit_\lambda(a_0, \ldots, a_{n-1}) \to \heartsuit_\lambda(b_0, \ldots, b_{n-1})}\,M_\lambda$$

that we will associate with an n-ary modality $\heartsuit_\lambda$.

Remark 7.26 As mentioned already, the basic idea in the development of proof theory for coalgebraic modal logic is to consider derivation systems that consist of a *fixed* set of propositional derivation rules, together with a *specific*, purpose-built collection of one-step rules. For instance, in the setting of one-sided sequent systems, we will consider the following propositional rules for the connectives $\top$, $\neg$ and $\wedge$:

$$\frac{}{a, \neg a}\,\mathsf{Ax0} \qquad \frac{}{\top}\,\mathsf{Ax1} \qquad \frac{a}{\neg\neg a}\,\mathsf{R}_{\neg\neg} \qquad \frac{a \quad b}{a \wedge b}\,\mathsf{R}_{\wedge} \qquad \frac{\neg a, \neg b}{\neg(a \wedge b)}\,\mathsf{R}_{\neg\wedge}$$

Given a set $\mathbf{G}$ of one-step rules, we obtain the system $\mathbf{G}^{+}$ by adding a suitable set of propositional rules, including the ones given above and similar ones for the other primitive propositional connectives.

To introduce *derivations*, we need to say how we can use substitutions to define instances of derivation rules.

Definition 7.27 A *substitution* is a map $\tau : A \to \mathrm{ML}_{\Lambda}(A)$. We will use the notation φ/a for the substitution that maps the variable a to the formula φ (and is the identity on the set of remaining variables). A substitution τ is *propositional* or *rank-0* if $\tau : A \to \mathrm{PL}(A)$, and *rank-1* if $\tau : A \to 1\mathrm{ML}_{\Lambda}(A)$. A *renaming* is a simple substitution $\tau : A \to A$ that replaces variables with variables.

A substitution τ naturally induces a translation $[\tau]$ mapping $\mathrm{ML}_{\Lambda}(A)$-formulas to $\mathrm{ML}_{\Lambda}(A)$-formulas. For this translation we shall use postfix notation, $\varphi[\tau]$ denoting the result of applying the substitution τ to the formula φ. For a sequent Φ we will write $\Phi[\tau] := \{\varphi[\tau] \mid \varphi \in \Phi\}$. $\lhd$

Remark 7.28 It is easy to see that if τ is a propositional substitution, we obtain $[\tau] : \mathrm{PL}(A) \to \mathrm{PL}(A)$ and $[\tau] : 1\mathrm{ML}_{\Lambda}(A) \to 1\mathrm{ML}_{\Lambda}(A)$. On the other hand, if τ is rank-1, then $[\tau]$ maps rank-0 formulas to rank-1 formulas, $[\tau] : \mathrm{PL}(A) \to 1\mathrm{ML}_{\Lambda}(A)$. In the sequel we will use these observations without explicit notice.

Roughly speaking, an *instance* of a derivation rule is a version of that rule where we uniformly replace every formula in the rule with some substitution instance, and we (possibly) add context formulas.

Definition 7.29 To define the *instances* of a derivation rule $\mathsf{d} = \Delta_1, \ldots, \Delta_{n-1}/\Delta_0$ is, we make a case distinction:

- If d is propositional, then any instance of d can be represented as a rule

  $$\frac{\Gamma, \Delta_1[\tau] \quad \cdots \quad \Gamma, \Delta_{n-1}[\tau]}{\Gamma, \Delta_0[\tau]}\,\mathsf{d}$$

  where τ is a substitution and Γ is a set of formulas.

- If d is a one-step rule, then any instance of d can be represented as a rule

$$\frac{\Delta_1[\tau] \quad \cdots \quad \Delta_{n-1}[\tau]}{\Gamma, \Delta_0[\tau]}\ \mathsf{d}$$

where τ is a rank-0 substitution.

In both cases we refer to the formulas in $\Delta_0[\tau]$ as the *principal* formulas of the rule instance, and the ones in Γ as the *context* formulas. $\triangleleft$

Note the distinction between instances of propositional and one-step rules when it comes to the role of the context formulas: in the case of propositional rules, the context formulas in the conclusion are also part of each premise, whereas in the case of a one-step rule the context only appears in the conclusion.

Definition 7.30 Let $\mathbf{D}$ be a derivation system. A *derivation* in $\mathbf{D}$ is a structure (W, C, L, r), where (W, C, r) is a well-founded tree[10] with root r, and L is a map *labelling* every node w of W with a sequent[11] $L(w)$, in such a way that for every inner node $t \in W$, the pair $\big(\{L(u) \mid u \in C(t)\}, L(t)\big)$ is an instance of a derivation rule in $\mathbf{D}$.

Given an $\mathbf{D}$-derivation $\mathcal{D} = (W, C, L, r)$, we call a sequent Σ an *assumption* of $\mathcal{D}$ if $\Sigma = L(t)$ for some *leaf* t, but Σ is not an instance of an axiom; we denote the set of assumptions of $\mathcal{D}$ by $\mathsf{Ass}(\mathcal{D})$. We refer to the sequent $L(r)$ labelling the root of the tree as the *result* of $\mathcal{D}$.

A sequent $\Sigma \in 1\mathrm{ML}_\Lambda(A)$ is *derivable* from a collection $\mathcal{A}$ of sequents in a derivation system $\mathbf{D}$, notation: $\mathcal{A} \vdash_{\mathbf{D}} \Sigma$ if Σ is the result of an $\mathbf{D}$-derivation $\mathcal{D}$ with $\mathsf{Ass}(\mathcal{D}) \subseteq \mathcal{A}$. If $\varnothing \vdash_{\mathbf{D}} \Sigma$ we simply write $\vdash_{\mathbf{D}} \Sigma$ and we say that Σ is *derivable.* $\triangleleft$

Remark 7.31 Derivations of one-step sequents in one-step derivation systems have a rather specific shape. The tree of such a derivation can be partitioned into an 'upper' and a 'lower' part, where the nodes of the uppper (lower) part are all labelled with rank-0 (respectively, rank-1) sequents. Formulated differently, on every branch $t_0 t_1 \cdots t_n$ from the root $r = t_0$ to a leaf $l = t_n$ of the tree, we either have $L(t_i) \subseteq 1\mathrm{ML}_\Lambda(A)$, for every i, or else there is a (necessarily unique) index $k \geq 0$ such that $L(t_i) \subseteq 1\mathrm{ML}_\Lambda(A)$ for all i with $0 \leq i \leq k$, while $L(t_i) \subseteq \mathrm{PL}(A)$ for all i with $k < i \leq n$. In the latter case $L(t_k)$ is the only sequent on the mentioned branch that is obtained by the application of a *one-step* rule — all other sequents are the conclusion of an application of a *propositional* rule.

[10] Trees and related notions are defined in Definition A.8.

[11] Sometimes we will also need to make explicit the actual rule that has been applied at the node.

7.3.2 One-step soundness and completeness

We now introduce the two key concepts in the one-step approach towards derivation systems for coalgebraic modal logics: *one-step soundness* and *one-step completeness*.

We may restrict attention to one-step rules in a simplified form. Recall that $\mathrm{Lit}(A)$ is the set of *literals* over A, that is, variables and negations of variables in A.

Definition 7.32 A *modal rule* is a one-step rule d such that every $\Delta \in \mathsf{Prem}(\mathsf{d})$ satisfies $\Delta \subseteq \mathrm{Lit}(A)$ while $\mathsf{conc}(\mathsf{d}) \subseteq \mathrm{Lit}(\Lambda(A))$. Furthermore we require that every variable that occurs in one the premises also appears in the conclusion. $\lhd$

Remark 7.33 The restriction to modal rules can be justified by the observation that every one-step derivation system $\mathbf{G}$ can be replaced with a modal derivation system $\mathbf{M_G}$ that is equivalent to $\mathbf{G}$ in the sense that the same set of one-step rules are derivable in the two systems (extended with propositional rules). More details of this equivalence will be provided later.

The definition of one-step soundness is not surprising.

Definition 7.34 A modal rule d is *one-step sound* if, for very marking $m : X \to PA$ such that $[\![\Delta]\!]^0_m = X$ for each $\Delta \in \mathsf{Prem}(\mathsf{d})$ we have $[\![\mathsf{conc}(\mathsf{d})]\!]^1_m = TX$. A set G is *one-step sound* if every rule in G is so. $\lhd$

Remark 7.35 We leave it for the reader that all propositional-substitution instances of sound one-step rules are sound.

The definition of one-step completeness is somewhat more involved.

Definition 7.36 A set G of modal rules is *one-step complete* if, for every $\Gamma \subseteq \mathrm{Lit}(\Lambda(A))$ and every marking $m : X \to PA$ such that $[\![\Gamma]\!]^1_m = TX$, there is some rule $\mathsf{d} = \Gamma_1, \ldots, \Gamma_n/\Gamma_0$ and some renaming $\tau : A \to A$ such that $\Gamma_0[\tau] \subseteq \Gamma$ and $[\![\Gamma_i[\tau]]\!]^0_m = X$, for all i. $\lhd$

Remark 7.37 The above definition is not the only way to make the concept of one-step completeness precise. Here are two alternative (but basically equivalent) approaches.

First of all, we could get rid of renamings in the definition. That is, we could say that a set G of modal rules is one-step complete if, for every $\Gamma \subseteq \mathrm{Lit}(\Lambda(A))$ and every marking $m : X \to PA$ such that $[\![\Gamma]\!]^1_m = TX$, there is some rule $\mathsf{d} = \Gamma_1, \ldots, \Gamma_n/\Gamma_0$ such that $\Gamma_0 \subseteq \Gamma$ and $[\![\Gamma_i]\!]^0_m = X$, for all i. The disadvantage of this definition is that for any rule in the system we would also need to have each of its renamings.

Moving in the opposite direction, we could call a set G of modal rules one-step complete if, for every $\Gamma \subseteq \mathrm{Lit}(\Lambda(A))$ and every marking $m : X \to PA$ such that $[\![\Gamma]\!]^1_m = TX$, there are rank-0 sequents $\Gamma_1, \ldots, \Gamma_n$ such that Γ is *derivable* from $\Gamma_1, \ldots, \Gamma_n$ using the rules in G, in combination with propositional logic. This definition would lead to smaller derivation systems, but a drawback is that it involves propositional reasoning and thus somewhat obscures the fact that the notion of one-step completeness is really about one-step rules.

Example 7.38 We look at standard modal logic (that is, $T = P$), taking the diamond operator $\Diamond$ as the single modality. We claim that the set of rules of the form

$$\frac{\neg a_0, a_1, \dots, a_n}{\neg\Diamond a_0, \Diamond a_1, \dots, \Diamond a_n}\,\mathsf{m}_n$$

is one-step complete.

To see this, assume that $[\![\Gamma]\!]^1_m = TX$, for some marking $m : X \to PA$ and some sequent $\Gamma \subseteq \mathrm{Lit}\big(\{\Diamond a \mid a \in A\}\big)$, say, $\Gamma = \{\neg\Diamond b_0, \dots, \neg\Diamond b_{k-1}, \Diamond c_0, \dots, \Diamond c_{n-1}\}$. To arrive at a contradiction, assume that $[\![\neg b_i, c_0, \dots, c_{n-1}]\!]^0_m \neq X$, for every i. Then for each i we may take some object $x_i \notin [\![\neg b_i, c_0, \dots, c_{n-1}]\!]^0_m$. Defining $Y := \{x_i \mid 0 \leq i < k\}$, we find $Y \notin [\![\Gamma]\!]^1_m$, contradicting our earlier assumption. This means that we must have $[\![\neg b_i, c_0, \dots, c_{n-1}]\!]^0_m = X$ for some i.

Let Δ_0 and Δ_1 be the sequents $\Delta_0 := \{\neg a_0, a_1, \dots, a_n\}$ and $\Delta_1 := \{\neg\Diamond a_0, \Diamond a_1, \dots, \Diamond a_n\}$, so that Δ_0/Δ_1 is the rule m_n. Furthermore, consider the renaming τ given by $\tau : a_0 \mapsto b_i$ and $\tau : a_j \mapsto c_{j-1}$ for $j \geq 1$. We then have $[\![\Delta_0[\tau]]\!] = X$ and $\Delta_1[\tau] \subseteq \Delta$, as required by the definition of one-step completeness.

The following Theorem states that we can *always* find a one-step complete set of one-step rules — for every set functor and every modal signature for that functor.

Theorem 7.39 *Let Λ be a modal signature for a set functor T. Then the collection S_Λ of all sound one-step rules is one-step complete.*

7.4 One-step derivation systems: Hilbert-style

In this section we continue our exploration of one-step derivation systems, but now we concentrate on so-called *Hilbert-style systems.* Characteristically, these manipulate single formulas, rather than sequents.

Definition 7.40 A *Hilbert calculus* is a derivation system consisting of one-step rules in which the premises and conclusion of every rule are all singletons. A *(Hilbert) axiomatization* is a Hilbert calculus consisting of axiomatic rules only. $\lhd$

A Hilbert-style derivation rule will usually be presented as a pair $R = (\Pi_R, \gamma_R)$ where Π_R is a set of formulas and γ_R is a formula – that is, we replace the singleton sequents with their unique formula members. Also, we will identify a Hilbert axiomatisation $\mathbf{H}$ with the set $\{\gamma_R \mid R \in \mathbf{H}\}$ of its axioms.

Remark 7.41 The restriction to singleton sequences (and thus, equivalently, to formulas) extends to derivations well. As a consequence of this, in Hilbert-style derivations we will only consider rule instances where the context is *empty.*

Convention 7.42 Throughout this text we will assume an arbitrary but fixed derivation system $\mathbf{C}$ (consisting of propositional axioms and rules) which is sound and complete for classical propositional logic.

Definition 7.43 Given a derivation system $\mathbf{H}$, we let $\mathbf{H}^+$ denote the extension of $\mathbf{H}$ with (1) all axioms and rules from $\mathbf{C}$ and (2) the congruence rule (C_λ), for every $\lambda \in \Lambda$. $\triangleleft$

In Hilbert calculi the tendency is to prefer axioms to non-axiomatic rules. In fact one often fixes a minimal set of derivation rules and focuses on one-step axomatizations on top of these. This approach is justified by Theorem 7.44 below which states that in fact we may completely restrict attention to derivation systems in which the congruence rule is the *only* non-axiomatic one-step rule.

Theorem 7.44 *Let Λ be some set of predicate liftings for a set functor T, and let $\mathbf{H}$ be a one-step Hilbert-style derivation system. Then there is an axiomatization $\mathbf{A}$ such that $\mathbf{H}^+$ and $\mathbf{A}^+$ derive the same one-step formulas.*

Proof. Clearly it suffices to show that we can replace every one-step rule R with an axiom α_R which is equivalent to R in the sense that R and α_R are derivable from one another by propositional reasoning and the congruence rule. Without loss of generality we may assume the premises of R to be satisfiable; if not, we may just reply R with the axiom $\top$.

Our proof is then based on an interesting property of classical propositional logic: The claim below states that every satisfiable propositional formula has a so-called *projective unifier*.

Claim 1 Let $\pi \in \mathrm{PL}(A)$ be satisfiable. Then π has a projective unifier, that is, a substitution $\tau : A \to \mathrm{PL}(A)$ such that (1) $\models \pi[\tau]$ and (2) $\pi \models a \leftrightarrow \tau(a)$, for all $a \in A$.

Proof of Claim Given a propositional formula ρ and a valuation $U : A \to 2$ we write $[\![\rho]\!]_U \in \{0, 1\}$ to denote the truth value of ρ under U.

Since π is satisfiable, there is a valuation $V : A \to 2$ with $[\![\pi]\!]_V = 1$. Let τ be given by

$$\tau(a) := \begin{cases} \pi \wedge a & \text{if } V(a) = 1 \\ \pi \to a & \text{if } V(a) = 0. \end{cases}$$

It is straightforward to verify that $\pi \models a \leftrightarrow \tau(a)$, for all $a \in A$. To check that $\pi[\tau]$ is a tautology, consider an arbitrary valuation $U : A \to 2$. We claim that $[\![\pi[\tau]]\!]_U = 1$.

To see this, we make a case distinction. If $[\![\pi]\!]_U = 1$, we find by part (2) that $[\![a]\!]_U = [\![\tau(a)]\!]_U$, and from this an easy induction shows that $[\![\rho]\!]_U = [\![\rho[\tau]]\!]_U$, for any formula $\rho \in \mathrm{PL}(A)$. In particular we have $[\![\pi]\!]_U = [\![\pi[\tau]]\!]_U$ and since $[\![\pi]\!]_U = 1$ this gives $[\![\pi[\tau]]\!]_U = 1$, as required. On the other hand, if $[\![\pi]\!]_U = 0$ then by definition of τ, no matter whether $V(a) = 0$ or $V(a) = 1$, we always have $[\![a]\!]_V = [\![\tau(a)]\!]_U$. From this again an easy induction shows that $[\![\rho]\!]_V = [\![\rho[\tau]]\!]_U$, for any formula $\rho \in \mathrm{PL}(A)$. In particular then for $\rho = \pi$ we find that $[\![\rho[\tau]]\!]_U = [\![\rho]\!]_V = 1$. $\triangleleft$

Let R be the rule Π/γ, and let τ be the substitution we obtain by applying the claim to the formula $\bigwedge \Pi$. Then we define

$$\alpha_R := \gamma[\tau].$$

and we claim that R and α_R are mutually derivable.

To see this, by the first property of the claim the formula $\pi[\tau]$ is derivable, for each premise $\pi \in \Pi$. From this we can derive α_R by one application of R. Conversely, take the set Π as assumptions, so that we can derive $\bigwedge \Pi$ by propositional reasoning. Then by the second property of the claim we may derive $a \leftrightarrow \tau(a)$, for all propositions a. But then we may easily derive $\alpha \leftrightarrow \alpha[\tau]$ by applications of the congruence rule, and since $\alpha[\tau]$ is now an axiom we obtain α. QED

Example 7.45 Consider the monotonicity rule $a \to b/\heartsuit a \to \heartsuit b$, and observe that there are three valuations making the premise $a \to b$ true. In the table below we list these, together with the induced substitutions.

V	a	b	$\tau(a)$			$\tau(b)$		
V_0	0	0	$(a \to b) \wedge a$	$\equiv$	$a \wedge b$	$(a \wedge b) \wedge b$	$\equiv$	b
V_1	0	1	$(a \to b) \wedge a$	$\equiv$	$a \wedge b$	$(a \wedge b) \to b$	$\equiv$	$a \vee b$
V_2	1	1	$(a \to b) \to a$	$\equiv$	a	$(a \wedge b) \to b$	$\equiv$	$a \vee b$

Applying these three substitutions to the conclusion $\heartsuit a \to \heartsuit b$ we obtain three axioms which may replace the monotonicity rule: $\heartsuit(a \wedge b) \to \heartsuit b$, $\heartsuit(a \wedge b) \to \heartsuit(a \vee b)$ and $\heartsuit a \to \heartsuit(a \vee b)$.

We now move to the concepts of one-step soundness and completeness for Hilbert-style systems. The definition of soundness is the same as before.

Definition 7.46 Let Λ be a set of predicate liftings for a set functor T. A one-step rule $R = \Pi/\gamma$ for Λ is *one-step sound* if

$$\bigcap_{\pi \in \Pi} [\![\pi]\!]^0_m = S \text{ implies } [\![\gamma]\!]^1_m = TS,$$

for all sets S and all markings $m : S \to PA$. A derivation system $\mathbf{H}$ is *one-step sound* if all of its derivation rules are one-step sound. $\lhd$

The definition of one-step completeness is slightly more involved than in the case for sequent systems.

Definition 7.47 Let Λ be a set of predicate liftings for a set functor T. We say that $\pi \in \mathrm{PL}(A)$ is a *true (propositional) fact* of a marking $m : S \to PA$ if $[\![\pi]\!]^0_m = S$; we let $TPF(m)$ denote the collection of all these facts.

A one-step derivation system $\mathbf{H}$ for Λ is *one-step complete* if for every marking $m : S \to PA$, and every $\alpha \in 1\mathrm{ML}_\Lambda(A)$ we have that

$$[\![\alpha]\!]^1_m = TS \text{ implies } TPF(m) \vdash^1_{\mathbf{H}^+} \alpha,$$

i.e., all formulas that are one-step true with respect to m are derivable from the true propositional facts of m in the *extended* derivation system $\mathbf{H}^+$. $\lhd$

Remark 7.48 Note that in the case of Hilbert-style derivation systems our definition of one-step completeness uses the approach sketched in Remark 7.37, rather than the one of Definition 7.36.

Example 7.49 As an example of a one-step complete derivation system, consider the modal signature $\{\Box\}$ for the powerset functor P (i.e., we look at standard modal logic where we take the $\Box$ modality as primitive). In the sequel we will continue writing T instead of P, however,in order to clarify the role of the functor in the argument. We claim that the axiomatisation

$$\mathbf{K} := \{\Box\top, \Box(a \wedge b) \leftrightarrow \Box a \wedge \Box b\}$$

is one-step sound and complete. Leaving soundness as an exercise for the reader, we prove one-step completeness here.

Let m be a A-marking on some set S, and let α be a one-step formula such that $[\![\alpha]\!]^1_m = TS$. We will show that α is derivable from the true propositional facts of m:

$$TPF(m) \vdash^1_{\mathbf{K}} \alpha.$$

Since we have the full power of classical propositional logic at our disposal, we may without loss of generality assume that α is in conjunctive normal form, i.e., $\alpha = \bigwedge_{\beta\in B} \beta$ for some finite set B, where each β is of the form

$$\beta \;=\; \bigvee_{i\in I} \Box\pi_i \vee \bigvee_{j\in J} \neg\Box\tau_j$$

for some finite index sets I and J, and where all $\pi_i, \tau_j \in \mathrm{PL}(A)$. Clearly it suffices to prove that we can derive each conjunct of α from the true propositional facts of m, so fix such a conjunct $\beta \in B$. Obviously it follows from $[\![\alpha]\!]^1_m = TS$ that $[\![\beta]\!]^1_m = TS$.

We first claim that

$$I \neq \varnothing. \tag{66}$$

To see this, assume for contradiction that $I = \varnothing$, so that $\beta = \bigvee_{j\in J} \neg\Box\tau_j$. Consider the empty set $\varnothing \in TS$, and observe that $S, m, \varnothing \Vdash^1 \Box\pi$ for *all* rank-0 formulas $\pi \in \mathrm{PL}(A)$; so in particular, $S, m, \varnothing \Vdash^1 \Box\tau_j$ for all $j \in J$. But this means $S, m, \varnothing \not\Vdash^1 \beta$, which clearly contradicts the earlier observation that $[\![\beta]\!]^1_m = TS$. This finishes the proof of (66).

Our second claim is that

$$\text{at least one of the formulas } \xi_i \text{ is a true propositional fact of } m, \tag{67}$$

where we define

$$\xi_i \;=\; \pi_i \vee \bigvee_{j\in J} \neg\tau_j.$$

To prove (67), assume for contradiction that none of formulas ξ_i belongs to the set $TPF(m)$. Then there are states $(s_i)_{i\in I}$ such that $s_i \notin [\![\xi_i]\!]^0_m$ for each $i \in I$. So $S, m, s_i \not\Vdash^0 \pi_i$, while $S, m, s_i \Vdash^0 \tau_j$ for all j. Now consider the set $\sigma := \{s_i \mid i \in I\} \in PS$. It is immediate by the semantics of $\Box$ that $S, m, \sigma \not\Vdash^1 \Box\pi_i$ for any $i \in I$, while at the same time $S, m, \sigma \Vdash^1 \Box\tau_j$ for all $j \in J$. Clearly then we find that $S, m, \sigma \not\Vdash^1 \beta$, which provides the desired contradiction with our assumption that $S, m, \sigma \Vdash^1 \alpha$. This proves (67).

To finish the one-step completeness proof, assume $\xi_i = \pi_i \vee \bigvee_{j\in J} \neg\tau_j \in TPF(m)$, then so is the (propositionally equivalent) formula

$$\bigwedge_{j\in J} \tau_j \to \pi_i.$$

Our third claim is that from the above formula we can derive its 'boxed version'

$$\bigwedge_{j\in J} \Box\tau_j \to \Box\pi_i,$$

by some propositional reasoning, applications of the congruence rule and of the axioms $\Box(a \wedge b) \leftrightarrow \Box a \wedge \Box b$ and $\Box\top$, followed by further propositional reasoning. We leave the details for the reader.

Finally, we can use propositional reasoning to show that the formula $\bigwedge_{j\in J} \Box\tau_j \to \Box\pi_i$ is equivalent to

$$\Box\pi_i \vee \bigvee_{j\in J} \neg\Box\tau_j,$$

and therefore implies

$$\beta = \bigvee_{i\in I} \Box\pi_i \vee \bigvee_{j\in J} \neg\Box\tau_j.$$

In other words, we have established that

$$TPF(m) \vdash^1_{\mathbf{K}} \beta.$$

And since $\beta \in B$ was an arbitrary conjunct of α, this means that we can also derive $\alpha = \bigwedge_{\beta\in B} \beta$ from the true propositional facts of m, as required.

Example 7.50 Similar axiomatizations can be given to *monotone* and *graded* one-step logic.

(1) The derivation system **M**, consisting of the monotonicity rule for the modality $\Box$ of monotone modal logic:

$$\mathbf{M} := \{a \to b/\Box a \to \Box b\}$$

is one-step and complete.

(2) We write $\Diamond^k$ for the counting modality $\heartsuit_{\lambda^{\geq k}}$ associated with the predicate lifting $\lambda^{\geq k}$ for the bag functor, cf. Example 6.7. One may show that the following provides a one-step sound and complete axiomatisation for the signature consisting of all these modalities:

a. $\Diamond^{n+1} a \to \Diamond^n a$
b. $\Box^1(a \to b) \to (\Diamond^n a \to \Diamond^n b)$
c. $\neg\Diamond^1(a \wedge b) \wedge \Diamond^{k_1!} a \wedge \Diamond^{k_2!} b \to \Diamond^{(k_1+k_2)!}(a \vee b)$
d. $\Box^1\top$

Here we abbreviate $\Box^k\pi := \neg\Diamond^k\neg\pi$ and $\Diamond^{k!}\pi \equiv \Diamond^k\pi \wedge \neg\Diamond^{k+1}\pi$.

Finally, as a Hilbert-style analog of Theorem 7.39 we have the following result. It implies that there always exists *some* one-step complete Hilbert-style derivation system; and thus, a one-step complete Hilbert-style axiomatisation, for every set functor T and for every modal signature for T.

Theorem 7.51 *Let Λ be a modal signature for a set functor T. Then the set of all one-step sound one-step Hilbert-style derivation rules is in fact a one-step complete Hilbert-style derivation system for Λ and T.*