# 3 Fixpoints

The game-theoretic semantics of the modal $\mu$-calculus introduced in the previous chapter has some attractive characteristics. It is intuitive, relatively easy to understand, and, as we shall see further on, it can be used to prove some important properties of the formalism. However, it has some drawbacks as well. For instance, the evaluation games of the previous chapter have only been defined for formulas that are either clean or tidy. The game semantics can be extended to arbitrary formulas but this will make the game somewhat more involved, in particular if we want to define evaluation games for formulas that are not in negation normal form.

Furthermore, the game-theoretical semantics is not *compositional*; that is, the meaning of a formula is not defined in terms of the meanings of its subformulas. These shortcomings vanish in the *algebraic semantics* that we are about to introduce. In order to define this term, we first consider an example.

**Example 3.1** Recall that in Example 2.1, we informally introduced the formula $\mu x.p \vee \Diamond_d x$ as the smallest fixpoint or solution of the 'equation' $x \equiv p \vee \Diamond_d x$.

To make this intuition more precise, we have to look at the formula $\delta = p \vee \Diamond_d x$ as an operation. The idea is that the value (that is, the extension) of this formula is a function of the value of $x$, provided that we keep the value of $p$ constant. Varying the value of $x$ boils down to considering '$x$-variants' of the valuation $V$ of $\mathbb{S} = \langle S, R, V \rangle$. Let, for $X \subseteq S$, $V[x \mapsto X]$ denote the valuation that is exactly like $V$ apart from mapping $x$ to $X$, and let $\mathbb{S}[x \mapsto X]$ denote the $x$-variant $\langle S, R, V[x \mapsto X] \rangle$ of $\mathbb{S}$. Then $[\![\delta]\!]^{\mathbb{S}[x \mapsto X]}$ denotes the extension of $\delta$ in this $x$-variant. It follows from this that the formula $\delta$ *induces* the following function $\delta^{\mathbb{S}}_x$ on the power set of $S$:

$$\delta^{\mathbb{S}}_x(X) := [\![\delta]\!]^{\mathbb{S}[x \mapsto X]}.$$

In our example we have

$$\delta^{\mathbb{S}}_x(X) = V(p) \cup \langle R \rangle (X).$$

Now we can make precise why $\mu x.p \vee \Diamond_d x$ is a fixpoint formula: its extension, the set $[\![\mu x.p \vee \Diamond_d x]\!]$, is a fixpoint of the map $\delta^{\mathbb{S}}_x$:

$$[\![\mu x.p \vee \Diamond_d x]\!] = V(p) \cup \langle R \rangle ([\![\mu x.p \vee \Diamond_d x]\!]).$$

In fact, as we shall see in this chapter, the formulas $\mu x.p \vee \Diamond_d x$ and $\nu x.p \vee \Diamond_d x$ are such that their extensions are the *least* and *greatest* fixpoints of the map $\delta^{\mathbb{S}}_x$, respectively. $\lhd$

It is worthwhile to discuss the theory of fixpoint operators at a more general level than that of modal logic. Before we turn to the definition of the algebraic semantics of the modal $\mu$-calculus, we first discuss the general fixpoint theory of monotone operations on complete lattices.

## 3.1 General fixpoint theory

### Basics

In this chapter we assume some familiarity[2] with partial orders and lattices (see Appendix A).

**Definition 3.2** Let $\mathbb{P}$ and $\mathbb{P}'$ be two partial orders and let $f : P \to P'$ be some map. Then $f$ is called *monotone* or *order preserving* if $f(x) \leq' f(y)$ whenever $x \leq y$, and *antitone* or *order reversing* if $f(x) \geq' f(y)$ whenever $x \leq y$. ◁

**Definition 3.3** Let $\mathbb{P} = \langle P, \leq \rangle$ be a partial order, and let $f : P \to P$ be some map. Then an element $p \in P$ is called a *prefixpoint* of $f$ if $f(p) \leq p$, a *postfixpoint* of $f$ if $p \leq f(p)$, and a *fixpoint* if $f(p) = p$. The sets of prefixpoints, postfixpoints, and fixpoints of $f$ are denoted respectively as $\mathrm{PRE}(f)$, $\mathrm{POS}(f)$ and $\mathrm{FIX}(f)$.

In case the set of fixpoints of $f$ has a least (respectively greatest) member, this element is denoted as $\mathrm{LFP}.f$ ($\mathrm{GFP}.f$, respectively). These least and greatest fixpoints may also be called *extremal fixpoints.* ◁

The following theorem is a celebrated result in fixpoint theory.

**Theorem 3.4 (Knaster-Tarski)** *Let $\mathbb{C} = \langle C, \bigvee, \bigwedge \rangle$ be a complete lattice, and let $f : C \to C$ be monotone. Then $f$ has both a least and a greatest fixpoint, and these are given as*

$$\mathrm{LFP}.f = \bigwedge \mathrm{PRE}(f), \tag{23}$$

$$\mathrm{GFP}.f = \bigvee \mathrm{POS}(f). \tag{24}$$

**Proof.** We will only prove the result for the least fixpoint, the proof for the greatest fixpoint is completely analogous.

Define $q := \bigwedge \mathrm{PRE}(f)$, then we have that $q \leq x$ for all prefixpoints $x$ of $f$. From this it follows by monotonicity that $f(q) \leq f(x)$ for all $x \in \mathrm{PRE}(f)$, and hence by definition of prefixpoints, $f(q) \leq x$ for all $x \in \mathrm{PRE}(f)$. In other words, $f(q)$ is a lower bound of the set $\mathrm{PRE}(f)$. Hence, by definition of $q$ as the *greatest* such lower bound, we find $f(q) \leq q$, that is, $q$ itself is a prefixpoint of $f$.

It now suffices to prove that $q \leq f(q)$, and for this we may show that $f(q)$ is a prefixpoint of $f$ as well, since $q$ is by definition a lower bound of the set of prefixpoints. But in fact, we may show that $f(y)$ is a prefixpoint of $f$ for *every* prefixpoint $y$ of $f$ — by monotonicity of $f$ it immediately follows from $f(y) \leq y$ that $f(f(y)) \leq f(y)$. QED

Another way to obtain least and greatest fixpoints is to *approximate* them from below and above, respectively.

**Definition 3.5** Let $\mathbb{C} = \langle C, \bigvee, \bigwedge \rangle$ be a complete lattice, and let $f : C \to C$ be some map. Then by ordinal induction we define the following maps on $C$:

$$\begin{array}{rcl@{\qquad}rcl} f_\mu^0(c) &:=& c, & f_\nu^0(c) &:=& c, \\ f_\mu^{\alpha+1}(c) &:=& f(f_\mu^\alpha(c)) & f_\nu^{\alpha+1}(c) &:=& f(f_\nu^\alpha(c)), \\ f_\mu^\lambda(c) &:=& \bigvee_{\alpha<\lambda} f_\mu^\alpha(c) & f_\nu^\lambda(c) &:=& \bigwedge_{\alpha<\lambda} f_\nu^\alpha(c), \end{array}$$

[2] Readers lacking this background may take abstract complete lattices to be concrete power set algebras.

where $\lambda$ denotes an arbitrary limit ordinal. ◁

**Proposition 3.6** *Let $\mathbb{C} = \langle C, \bigvee, \bigwedge\rangle$ be a complete lattice, and let $f : C \to C$ be monotone. Then $f$ is inductive, that is, $f^{\alpha}_{\mu}(\bot) \leq f^{\beta}_{\mu}(\bot)$ for all ordinals $\alpha$ and $\beta$ such that $\alpha < \beta$.*

**Proof.** We leave this proof as an exercise to the reader. QED

Given a set $C$, we let $|C|$ denote its cardinality or size.

**Corollary 3.7** *Let $\mathbb{C} = \langle C, \bigvee, \bigwedge\rangle$ be a complete lattice, and let $f : C \to C$ be monotone. Then there is some $\alpha$ of size at most $|C|$ such that* $\mathrm{LFP}.f = f^{\alpha}_{\mu}(\bot)$.

**Proof.** By Proposition 3.6, $f$ is inductive, that is, $f^{\alpha}_{\mu}(\bot) \leq f^{\beta}_{\mu}(\bot)$ for all ordinals $\alpha$ and $\beta$ such that $\alpha < \beta$. It follows from elementary set theory that there must be two ordinals $\alpha, \beta$ of size at most $|C|$ such that $f^{\alpha}_{\mu}(\bot) = f^{\beta}_{\mu}(\bot)$. From the definition of the approximations it then follows that there must be an ordinal $\alpha$ such that $f^{\alpha}_{\mu}(\bot) = f^{\alpha+1}_{\mu}(\bot)$, or, equivalently, $f^{\alpha}_{\mu}(\bot)$ is a fixpoint of $f$. To show that it is the *smallest* fixpoint, one may prove that $f^{\beta}_{\mu}(\bot) \leq \mathrm{LFP}.f$ for every ordinal $\beta$. This follows from a straightforward ordinal induction. QED

**Definition 3.8** Let $\mathbb{C} = \langle C, \bigvee, \bigwedge\rangle$ be a complete lattice, and let $f : C \to C$ be monotone. The least ordinal $\alpha$ such that $f^{\alpha}_{\mu}(\bot) = f^{\alpha+1}_{\mu}(\bot)$ is called the *unfolding ordinal* of $f$. ◁

## 3.2 Boolean algebras

In the special case that the complete lattice is in fact a (complete) *boolean algebra*, there is more to be said.

### Dual maps

In the case of monotone maps on complete boolean algebras, the least and greatest fixed points become interdefinable, using the notion of (boolean) *duals* of maps.

**Definition 3.9** A *complete* boolean algebra is a structure $\mathbb{B} = \langle B, \bigvee, \bigwedge, -\rangle$ such that $\langle B, \bigvee, \bigwedge\rangle$ is a complete lattice and $\langle B, \vee, \wedge, -, \bot, \top\rangle$ is a boolean algebra, where $\vee$ and $\wedge$ are the binary versions of $\bigvee$ and $\bigwedge$, respectively, and $\bot := \bigvee \varnothing$, $\top := \bigwedge \varnothing$. ◁

In a boolean algebra $\mathbb{B}$, the complementation operation $- : B \to B$ is an antitone (order-reversing) map such that $x \wedge -x = \bot$ and $x \vee -x = \top$ for all $x \in B$. If $\mathbb{B}$ is complete it holds that $-\bigvee X = \bigwedge\{-x \mid x \in X\}$ and $-\bigwedge X = \bigvee\{-x \mid x \in X\}$.

**Definition 3.10** Let $\mathbb{B} = \langle B, \bigvee, \bigwedge, -\rangle$ be a complete boolean algebra. Given a map $f : B \to B$, the function $f^{\partial} : B \to B$ given by

$$f^{\partial}(b) := -f(-b).$$

is called the *(boolean) dual* of $f$. ◁

**Proposition 3.11** *Let $\mathbb{B} = \langle B, \bigvee, \bigwedge, -\rangle$ be a complete boolean algebra, and let $g : B \to B$ be monotone. Then $g^\partial$ is monotone as well, $(g^\partial)^\partial = g$, and*

$$\begin{aligned} \mathrm{LFP}.g^\partial &= -\mathrm{GFP}.g, \\ \mathrm{GFP}.g^\partial &= -\mathrm{LFP}.g. \end{aligned}$$

**Proof.** We only prove that $\mathrm{LFP}.g^\partial = -\mathrm{GFP}.g$, leaving the other parts of the proof as exercises to the reader.

First, note that by monotonicity of $g^\partial$, the Knaster-Tarski theorem gives that

$$\mathrm{LFP}.g^\partial = \bigwedge \mathrm{PRE}(g^\partial).$$

But as a consequence of the definitions, we have that

$$b \in \mathrm{PRE}(g^\partial) \iff -b \in \mathrm{POS}(g).$$

From this it follows that

$$\begin{aligned} \mathrm{LFP}.g^\partial &= \bigwedge\{b \mid -b \in \mathrm{POS}(g)\} \\ &= \bigwedge\{-a \mid a \in \mathrm{POS}(g)\} \\ &= -\bigvee \mathrm{POS}(g) \\ &= -\mathrm{GFP}.g \end{aligned}$$

which finishes the proof of the Theorem. QED

Further on we will see that Proposition 3.11 allows us to define negation as an abbreviated operator in the modal $\mu$-calculus.

### Games

In case the boolean algebra in question is in fact a *power set algebra*, a nice game-theoretic characterization of least and greatest fixpoint operators can be given.

**Definition 3.12** Let $S$ be some set and let $F : \wp(S) \to \wp(S)$ be a monotone operation. Consider the *unfolding games* $\mathcal{U}^\mu(F)$ and $\mathcal{U}^\nu(F)$. The positions and admissible moves of these two graph games are the same, see Table 6.

| Position | Player | Admissible moves |
|---|---|---|
| $s \in S$ | $\exists$ | $\{A \in \wp(S) \mid s \in F(A)\}$ |
| $A \in \wp(S)$ | $\forall$ | $A$ |

Table 6: Unfolding games for $F : \wp(S) \to \wp(S)$

The *winning conditions* of finite matches are standard (the player that got stuck loses the match). The difference between $\mathcal{U}^\mu(F)$ and $\mathcal{U}^\nu(F)$ shows up in the winning conditions of infinite matches: $\exists$ wins the infinite matches of $\mathcal{U}^\nu(F)$, but $\forall$ those of $\mathcal{U}^\mu(F)$. $\lhd$

Observe that the positions in a match of the unfolding game alternate between 'state positions' $s$, where $\exists$ needs to pick a subset $A \subseteq S$ such that $s$ belongs to $F(A)$, and 'subset positions' $A$, of which $\forall$ has to pick an element.

**Example 3.13** In fact, we have already seen an example of the unfolding game $\mathcal{U}^\nu$ in the *bisimilarity game* of Definition 1.26. Given two Kripke models $\mathbb{S}$ and $\mathbb{S}'$, consider the map $F : \wp(S \times S') \to \wp(S \times S)$ given by

$$F(Z) := \{(s, s') \in S \times S' \mid Z \text{ is a local bisimulation for } s \text{ and } s'\},$$

then it is straightforward to verify that $\mathcal{B}(\mathbb{S}, \mathbb{S}')$ is nothing but the unfolding game $\mathcal{U}^\nu(F)$. $\lhd$

The following proposition substantiates the slogan that '$\nu$ means unfolding, $\mu$ means finite unfolding'.

**Theorem 3.14** *Let $S$ be some set and let $F : \wp(S) \to \wp(S)$ be a monotone operation. Then*

1. $\mathrm{GFP}.F = \{s \in S \mid s \in \mathrm{Win}_\exists(\mathcal{U}^\nu(F))\}$,
2. $\mathrm{LFP}.F = \{s \in S \mid s \in \mathrm{Win}_\exists(\mathcal{U}^\mu(F))\}$,

**Proof.** For the inclusion $\supseteq$ of part 1, it suffices to prove that $W := S \cap \mathrm{Win}_\exists(\mathcal{U}^\nu(F))$ is a postfixpoint of $F$:

$$W \subseteq F(W). \tag{25}$$

Let $s$ be an arbitrary point in $W$, and suppose that $\exists$'s winning strategy tells her to choose $A \subseteq S$ at position $s$. Then no matter what element $s_1 \in A$ is picked by $\forall$, $\exists$ can continue the match and win. Hence, all elements of $A$ are winning positions for $\exists$. But from $A \subseteq W$ it follows that $F(A) \subseteq F(W)$, and by the legitimacy of $\exists$'s move $A$ at $s$ it holds that $s \in F(A)$. We conclude that $s \in F(W)$, which proves (25).

For the converse inclusion $\subseteq$ of part 1 of the proposition, take an arbitrary point $s \in \mathrm{GFP}.F$. We need to provide $\exists$ with a winning strategy in the unfolding game $\mathcal{U}^\nu(F)$ starting at $s$. This strategy is actually as simple as can be: $\exists$ should always play $\mathrm{GFP}.F$. Since $\mathrm{GFP}.F = F(\mathrm{GFP}.F)$, this strategy prescribes legitimate moves for $\exists$ at every point in $\mathrm{GFP}.F$. And, if she sticks to this strategy, $\exists$ will stay alive forever and thus win the match, no matter what $\forall$'s responses are.

For the second part of the theorem, let $W$ denote the set $W := S \cap \mathrm{Win}_\exists(\mathcal{U}^\mu(F))$ of states in $S$ that are winning positions for $\exists$ in $\mathcal{U}^\mu(F)$. We first prove the inclusion $W \subseteq \mathrm{LFP}.F$. Clearly it suffices to show that all points outside the set $\mathrm{LFP}.F$ are winning positions for $\forall$.

Consider a point $s \notin \mathrm{LFP}.F$. If $s \notin F(A)$ for any $A \subseteq S$ then $\exists$ is stuck, hence loses immediately, and we are done. Otherwise, suppose that $\exists$ starts a match of $\mathcal{U}^\mu(F)$ by playing some set $B \subseteq S$ with $s \in F(B)$. We claim that $B$ is not a subset of $\mathrm{LFP}.F$, since otherwise we would have $F(B) \subseteq F(\mathrm{LFP}.F) \subseteq \mathrm{LFP}.F$; which would contradict the fact that $s \notin \mathrm{LFP}.F$. But if $B \not\subseteq \mathrm{LFP}.F$ then $\forall$ may continue the match by choosing a point $s_1 \in B \setminus \mathrm{LFP}.F$. Now $\forall$ can use the same strategy from $s_1$ as he used from $s$, and so on. This strategy guarantees that either $\exists$ gets stuck after finitely many rounds (in case $\forall$ manages to pick an $s_n$ for which there is no $A$ such that $s_n \in F(A_n)$), or else the match will last forever. In both cases $\forall$ wins the match.

For the opposite inclusion $\subseteq$ of part 2, it suffices to show that $W$ is a prefixpoint of $F$, that is, $F(W) \subseteq W$. For that purpose, let $s \in S$ be such that $s \in F(W)$. In order to show that $s \in W$ we need to provide $\exists$ with a winning strategy in $\mathcal{U}^\mu(F)$, starting at $s$. But this is straightforward: since $s \in F(W)$, the set $W$ itself is a legitimate move for $\exists$ at position $s$. Then, after $\forall$ picks some element $t \in W$, she can simply continue with her strategy in $\mathcal{U}^\mu(F)$ that is winning when starting at position $t$. QED

## 3.3 Vectorial fixpoints

Suppose that we are given a finite family $\{\mathbb{C}_1, \ldots, \mathbb{C}_n\}$ of complete lattices, and put $\mathbb{C} = \prod_{1 \leq i \leq n} \mathbb{C}_i$. Given a finite family of monotone maps $f_1, \ldots, f_n$ with $f_i : C \to C_i$, we may define the map $f : C \to C$ given by $f(c) := (f_1(c), \ldots, f_n(c))$. Monotonicity of $f$ is an easy consequence of the monotonicity of the maps $f_i$ separately, and so by completeness of $\mathbb{C}$, $f$ has a least and a greatest fixpoint. In this context we will also use vector notation, for instance writing

$$\mu \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} . \begin{pmatrix} f_1(x_1, \ldots, x_n) \\ f_2(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{pmatrix}$$

for LFP.$f$. An obvious question is whether one may express these multi-dimensional fixpoints in terms of one-dimensional fixpoints of maps that one may associate with $f_1, \ldots, f_n$.

The answer to this question is positive, and the basic observation facilitating the computation of multi-dimensional fixpoints is the following so-called *Bekič principle*.

**Proposition 3.15** *Let $\mathbb{D}_1$ and $\mathbb{D}_2$ be two complete lattices, and let $f_i : D_1 \times D_2 \to D_i$ for $i = 1, 2$ be monotone maps. Then*

$$\eta \begin{pmatrix} x \\ y \end{pmatrix} . \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} \eta x. f_1(x, \eta y. f_2(x,y)) \\ \eta y. f_2(\eta x. f_1(x,y), y) \end{pmatrix}$$

*where $\eta$ uniformly denotes either $\mu$ or $\nu$.*

**Proof.** Define $\mathbb{D} := \mathbb{D}_1 \times \mathbb{D}_2$, and let $f : D \to D$ be given by putting $f(d) := (f_1(d), f_2(d))$. Then $f$ is clearly monotone, and so it has both a least and a greatest fixpoint.

By the order duality principle it suffices to consider the case $\eta = \mu$ of least fixed points only. Suppose that $(a_1, a_2)$ is the least fixpoint of $f$, and let $b_1$ and $b_2$ be given by

$$\begin{cases} b_1 & := & \mu x. f_1(x, \mu y. f_2(x,y)), \\ b_2 & := & \mu y. f_2(\mu x. f_1(x,y), y). \end{cases}$$

Then we need to show that $a_1 = b_1$ and $a_2 = b_2$.

By definition of $(a_1, a_2)$ we have

$$\begin{cases} a_1 & = & f_1(a_1, a_2), \\ a_2 & = & f_2(a_1, a_2), \end{cases}$$

whence we obtain

$$\begin{cases} \mu x. f_1(x, a_2) & \leq & a_1 & \text{and} \\ \mu y. f_2(a_1, y) & \leq & a_2, \end{cases}$$

From this we obtain by monotonicity that

$$f_2(\mu x. f_1(x, a_2), a_2) \leq f_2(a_1, a_2) = a_2,$$

so that we find $b_2 \leq a_2$ by definition of $b_2$. Likewise we may show that $b_1 \leq a_1$.

Conversely, by definition of $b_1$ and $b_2$ we have

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} f_1(b_1, \mu y. f_2(b_1, y)) \\ f_2(\mu x. f_1(x, b_2), b_2) \end{pmatrix}.$$

Then with $c_2 := \mu y. f_2(b_1, y)$, we have $b_1 = f_1(b_1, c_2)$. Also, by definition of $c_2$ as a fixpoint, $c_2 = f_2(b_1, c_2)$. Putting these two identities together, we find that

$$\begin{pmatrix} b_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} f_1(b_1, c_2) \\ f_2(b_1, c_2) \end{pmatrix} = f \begin{pmatrix} b_1 \\ c_2 \end{pmatrix}.$$

Hence by definition of $(a_1, a_2)$, we find that $a_1 \leq b_1$ (and that $a_2 \leq c_2$, but that is of less interest now). Analogously, we may show that $a_2 \leq b_2$. QED

Proposition 3.15 allows us to compute the least and greatest fixpoints of any monotone map $f$ on a finite product of complete lattices in terms of the least and greatest fixpoints of operations on the factors of the product, through a *elimination method* that is reminiscent of Gaussian elimination in linear algebra.

To see how it works, suppose that we are dealing with lattices $\mathbb{C}_1, \ldots, \mathbb{C}_{n+1}, \mathbb{C}$ and maps $f_1, \ldots, f_{n+1}, f$, just as described above, and that we want to compute $\eta\vec{x}.f$, that is, find the elements $a_1, \ldots, a_{n+1}$ such that

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n+1} \end{pmatrix} = \eta \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n+1} \end{pmatrix} . \begin{pmatrix} f_1(x_1, \ldots, x_n, x_{n+1}) \\ f_2(x_1, \ldots, x_n, x_{n+1}) \\ \vdots \\ f_{n+1}(x_1, \ldots, x_n, x_{n+1}) \end{pmatrix}$$

We may define

$$g_{n+1}(x_1, \ldots, x_n) := \eta x_{n+1}. f_{n+1}(x_1, \ldots, x_{n+1}),$$

and then use Proposition 3.15, with $\mathbb{D}_1 = \mathbb{C}_1 \times \cdots \times \mathbb{C}_n$, and $\mathbb{D}_2 = \mathbb{C}_{n+1}$, to obtain

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \eta \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} . \begin{pmatrix} f_1(x_1, \ldots, x_n, g_{n+1}(x_1, \ldots, x_n)) \\ f_2(x_1, \ldots, x_n, g_{n+1}(x_1, \ldots, x_n)) \\ \vdots \\ f_n(x_1, \ldots, x_n, g_{n+1}(x_1, \ldots, x_n)) \end{pmatrix}$$

We may then inductively assume to have obtained the tuple $(a_1, \ldots, a_n)$. Finally, we may compute $a_{n+1} := g_{n+1}(a_1, \ldots, a_n)$.

Observe that in case $\mathbb{C}_i = \mathbb{C}_j$ for all $i, j$ and the operations $f_i$ are all term definable in some formal fixpoint language, then each of the components $a_i$ of the extremal fixpoints of $f$ can also be expressed in this language.

## 3.4 Algebraic semantics for the modal $\mu$-calculus

### Basic definitions

In order to define the algebraic semantics of the modal $\mu$-calculus, we need to consider formulas as *operations* on the power set of the (state space of a) transitions system, and we have to prove that such operations indeed have least and greatest fixpoints. In order to make this precise, we need some preliminary definitions.

**Definition 3.16** Given an LTS $\mathbb{S} = \langle S, V, R \rangle$ and subset $X \subseteq S$, define the valuation $V[x \mapsto X]$ by putting

$$V[x \mapsto X](y) := \begin{cases} V(y) & \text{if } y \neq x, \\ X & \text{if } y = x. \end{cases}$$

Then, the LTS $\mathbb{S}[x \mapsto X]$ is given as the structure $\langle S, V[x \mapsto X], R \rangle$. $\lhd$

Now inductively assume that $[\![\varphi]\!]^{\mathbb{S}}$ has been defined for all LTSs. Given a labelled transition system $\mathbb{S}$ and a propositional variable $x \in \mathsf{P}$, each formula $\varphi$ induces a map $\varphi^{\mathbb{S}}_x : \wp(S) \to \wp(S)$ defined by

$$\varphi^{\mathbb{S}}_x(X) := [\![\varphi]\!]^{\mathbb{S}[x \mapsto X]}$$

**Example 3.17** a) Where $\varphi_a = p \vee x$ we have $(\varphi_a)^{\mathbb{S}}_x(X) = [\![p \vee x]\!]^{\mathbb{S}[x \mapsto X]} = V(p) \cup X$.

b) Where $\varphi_b = \overline{x}$ we have $(\varphi_b)^{\mathbb{S}}_x(X) = [\![\overline{x}]\!]^{\mathbb{S}[x \mapsto X]} = S \setminus X$.

c) Where $\varphi_c = p \vee \Diamond_d x$ we find $(\varphi_c)^{\mathbb{S}}_x(X) = [\![p \vee \Diamond_d x]\!]^{\mathbb{S}[x \mapsto X]} = V(p) \cup \langle R_d \rangle X$.

d) Where $\varphi_d = \Diamond_d \overline{x}$ we find $(\varphi_d)^{\mathbb{S}}_x(X) = [\![\Diamond_d \overline{x}]\!]^{\mathbb{S}[x \mapsto X]} = \langle R_d \rangle (S \setminus X)$. $\lhd$

**Remark 3.18** Clearly, relative to a model $\mathbb{S}$, $X$ is a fixpoint of $\varphi^{\mathbb{S}}_x$ iff $X = \varphi^{\mathbb{S}}_x(X)$; a prefixpoint iff $\varphi^{\mathbb{S}}_x(X) \subseteq X$ and a postfixpoint iff $X \subseteq \varphi^{\mathbb{S}}_x(X)$.

Writing $\mathbb{S} \Vdash \varphi$ for $S = [\![\varphi]\!]^{\mathbb{S}}$, an alternative but equivalent way of formulating this is to say that in $\mathbb{S}$, $X$ is a *prefixpoint of a formula* $\varphi(x)$ iff $\mathbb{S}[x \mapsto X] \Vdash \varphi \to x$, a *postfixpoint* iff $\mathbb{S}[x \mapsto X] \Vdash x \to \varphi$, and a *fixpoint* iff $\mathbb{S}[x \mapsto X] \Vdash x \leftrightarrow \varphi$. $\lhd$

**Example 3.19** Consider the formulas of Example 3.17.

a) The sets $V(p)$ and $S$ are fixpoints of $\varphi_a$, as is in fact any $X$ with $V(p) \subseteq X \subseteq S$.

b) Since we do not consider structures with empty domain, the formula $\overline{x}$ has no fixpoints at all. (Otherwise $X$ would be identical to its own complement relative to some nonempty set $S$.)

c) Two fixpoints of $\varphi_c$ were already given in Example 2.1.

d) Consider any model $\mathbb{Z} = \langle Z, S, V \rangle$ based on the set $Z$ of integers, where $S = \{(z, z+1) \mid z \in Z\}$ is the successor relation. Then the only two fixpoints of $\varphi_d$ are the sets of even and odd numbers, respectively. $\lhd$

In particular, it is not the case that every formula has a least fixpoint. If we can guarantee that the induced function $\varphi^{\mathbb{S}}_x$ of $\varphi$ is monotone, however, then the Knaster-Tarski theorem (Theorem 3.4) provides both least and greatest fixpoints of $\varphi^{\mathbb{S}}_x$. Precisely for this reason, in the definition of fixpoint formulas, we imposed the condition in the clauses for $\eta x.\varphi$, that $x$ may only occur positively in $\varphi$. As we will see, this condition on $x$ guarantees monotonicity of the function $\varphi^{\mathbb{S}}_x$.

**Definition 3.20** Given a $\mu\mathtt{ML}_{\mathsf{D}}$-formula $\varphi$ and a labelled transition system $\mathbb{S} = \langle S, V, R\rangle$, we define the *meaning* $[\![\varphi]\!]^{\mathbb{S}}$ of $\varphi$ in $\mathbb{S}$, together with the map $\varphi^{\mathbb{S}}_x : \wp(S) \to \wp(S)$ by the following simultaneous formula induction:

$$
\begin{array}{lcllcl}
[\![\bot]\!]^{\mathbb{S}} & = & \varnothing & [\![\top]\!]^{\mathbb{S}} & = & S \\
[\![p]\!]^{\mathbb{S}} & = & V(p) & [\![\overline{p}]\!]^{\mathbb{S}} & = & S \setminus V(p) \\
[\![\varphi \vee \psi]\!]^{\mathbb{S}} & = & [\![\varphi]\!]^{\mathbb{S}} \cup [\![\psi]\!]^{\mathbb{S}} & [\![\varphi \wedge \psi]\!]^{\mathbb{S}} & = & [\![\varphi]\!]^{\mathbb{S}} \cap [\![\psi]\!]^{\mathbb{S}} \\
[\![\Diamond_d \varphi]\!]^{\mathbb{S}} & = & \langle R_d \rangle [\![\varphi]\!]^{\mathbb{S}} & [\![\Box_d \varphi]\!]^{\mathbb{S}} & = & [R_d][\![\varphi]\!]^{\mathbb{S}} \\
[\![\mu x.\varphi]\!]^{\mathbb{S}} & = & \bigcap \mathrm{PRE}(\varphi^{\mathbb{S}}_x) & [\![\nu x.\varphi]\!]^{\mathbb{S}} & = & \bigcup \mathrm{POS}(\varphi^{\mathbb{S}}_x)
\end{array}
$$

The map $\varphi^{\mathbb{S}}_x$, for $x \in \mathsf{Prop}$, is given by $\varphi^{\mathbb{S}}_x(X) = [\![\varphi]\!]^{\mathbb{S}[x \mapsto X]}$. ◁

**Theorem 3.21** *Let $\varphi$ be an $\mu\mathtt{ML}_{\mathsf{D}}$-formula, in which $x$ occurs only positively, and let $\mathbb{S}$ be a labelled transition system. Then $[\![\mu x.\varphi]\!]^{\mathbb{S}} = \mathrm{LFP}.\varphi^{\mathbb{S}}_x$, and $[\![\nu x.\varphi]\!]^{\mathbb{S}} = \mathrm{GFP}.\varphi^{\mathbb{S}}_x$.*

**Proof.** This is an immediate consequence of the Knaster-Tarski theorem, provided we can prove that $\varphi^{\mathbb{S}}_x$ is monotone in $x$ if all occurrences of $x$ in $\varphi$ are positive. We leave the details of this proof to the reader (see Exercise 3.2). QED

### Negation in the modal $\mu$-calculus

It follows from the definitions that the set $\mu\mathtt{ML}_{\mathsf{D}}$ is closed under taking *negations*. Informally, let $\sim\varphi$ be the result of simultaneously replacing all occurrences of $\top$ with $\bot$, of $p$ with $\overline{p}$ and vice versa (for *free* variables $p$), of $\wedge$ with $\vee$, of $\Box_d$ with $\Diamond_d$, of $\mu x$ with $\nu x$, and vice versa, while leaving occurrences of bound variables unchanged. As an example, $\sim(\mu x.p \vee \Diamond x) = \nu x.\overline{p} \wedge \Box x$. Formally, it is easiest to define $\sim\varphi$ via the *boolean dual* of $\varphi$.

**Definition 3.22** Given a modal fixpoint formula $\varphi$, we define its *boolean dual* $\varphi^{\partial}$ inductively as follows:

$$
\begin{array}{lcllcl}
\bot^{\partial} & := & \top & \top^{\partial} & := & \bot \\
p^{\partial} & := & p & (\overline{p})^{\partial} & := & \overline{p} \\
(\varphi \vee \psi)^{\partial} & := & \varphi^{\partial} \wedge \psi^{\partial} & (\varphi \wedge \psi)^{\partial} & := & \varphi^{\partial} \vee \psi^{\partial} \\
(\Diamond_d \varphi)^{\partial} & := & \Box_d \varphi^{\partial} & (\Box_d \varphi)^{\partial} & := & \Diamond_d \varphi^{\partial} \\
(\mu x.\varphi)^{\partial} & := & \nu x.\varphi^{\partial} & (\nu x.\varphi)^{\partial} & := & \mu x.\varphi^{\partial}
\end{array}
$$

Based on this definition, we define the formula $\sim\varphi$ as the formula $\varphi^{\partial}[p \rightleftharpoons \overline{p} \mid p \in FV(\varphi)]$ that we obtain from $\varphi^{\partial}$ by replacing all occurrences of $p$ with $\overline{p}$, and vice versa, for all free proposition letters $p \in FV(\varphi)$. ◁

**Example 3.23** Here are two examples:

$$
\begin{array}{lcllcl}
\varphi & := & \mu x.p \vee \Diamond(x \wedge \overline{q}) & \psi & := & \nu p\, \mu x.p \vee \Diamond(x \wedge \overline{q}) \\
\varphi^{\partial} & := & \nu x.p \wedge \Box(x \vee \overline{q}) & \psi^{\partial} & := & \mu p\, \nu x.p \wedge \Box(x \vee \overline{q}) \\
\sim\varphi & := & \nu x.\overline{p} \wedge \Box(x \vee q) & \sim\psi & := & \mu p\, \nu x.p \wedge \Box(x \vee q)
\end{array}
$$

Note the difference between $\sim\varphi$ and $\sim\psi$ with respective to the propositional variable $p$, which is free in $\varphi$ but bound in $\psi$. ◁

The following proposition states that the operation $\sim$ functions as a standard boolean negation. We let $\sim_S X := S \setminus X$ denote the complement of $X$ in $S$.

**Proposition 3.24** *Let $\varphi$ be a modal fixpoint formula. Then $\sim\varphi$ corresponds to the negation of $\varphi$, that is,*

$$[\![\sim\varphi]\!]^{\mathbb{S}} = \sim_S [\![\varphi]\!]^{\mathbb{S}} \tag{26}$$

*for every labelled transition system $\mathbb{S}$.*

**Proof.** We first show, by induction on $\varphi$, that $\varphi^{\partial}$ corresponds to the boolean dual of $\varphi$. For this purpose, given a labelled transition system $\mathbb{S} = (S, R, V)$, we let $\mathbb{S}^{\sim}$ denote the *complemented* model, that is, the structure $(S, R, V^{\sim})$, where $V^{\sim}(p) := \sim_S V(p)$. Then we claim that

$$[\![\varphi^{\partial}]\!]^{\mathbb{S}} = \sim_S [\![\varphi]\!]^{\mathbb{S}^{\sim}}, \tag{27}$$

and we prove this statement by induction on the complexity of $\varphi$. Leaving all other cases as exercises for the reader, we concentrate on the inductive case where $\varphi$ is of the form $\mu x.\psi$. In this case the left hand side of (27) evaluates to

$$\begin{aligned} [\![(\mu x.\psi)^{\partial}]\!]^{\mathbb{S}} &= [\![\nu x.\psi^{\partial}]\!]^{\mathbb{S}} && \text{(Definition } (\mu x.\psi)^{\partial}) \\ &= \mathrm{GFP}.(\psi^{\partial})^{\mathbb{S}}_x && \text{(Theorem 3.21)} \end{aligned}$$

while for the right hand side we find

$$\begin{aligned} \sim_S [\![\mu x.\psi]\!]^{\mathbb{S}^{\sim}} &= \sim_S \mathrm{LFP}.\psi^{\mathbb{S}^{\sim}}_x && \text{(Theorem 3.21)} \\ &= \mathrm{GFP}.(\psi^{\mathbb{S}^{\sim}}_x)^{\partial} && \text{(Proposition 3.11)} \end{aligned}$$

In other words, to prove (27) it suffices to show that

$$(\psi^{\partial})^{\mathbb{S}}_x = (\psi^{\mathbb{S}^{\sim}}_x)^{\partial}. \tag{28}$$

To this aim, take an arbitrary subset $U$ of $S$. Applying the map on the left hand side of (28) to $U$, we find

$$(\psi^{\partial})^{\mathbb{S}}_x(U) = [\![\psi^{\partial}]\!]^{\mathbb{S}[x \mapsto U]},$$

while the map on the right hand side yields

$$(\psi^{\mathbb{S}^{\sim}}_x)^{\partial}(U) = \sim_S \psi^{\mathbb{S}^{\sim}}_x(\sim_S U) = \sim_S [\![\psi]\!]^{(\mathbb{S}^{\sim}[x \mapsto \sim_S U])} = \sim_S [\![\psi]\!]^{(\mathbb{S}[x \mapsto U])^{\sim}},$$

so that by the inductive hypothesis we find that $(\psi^{\partial})^{\mathbb{S}}_x(U) = (\psi^{\mathbb{S}^{\sim}}_x)^{\partial}(U)$, as required to prove (28), and thus (27).

In other words, we have shown that the formula $\varphi^{\partial}$ indeed behaves as the boolean dual of $\varphi$. To see that, likewise, the formula $\sim\varphi$ behaves as the negation of $\varphi$, we now show how to derive (26) from (27). First observe that for any formula $\chi$ we have

$$[\![\chi[p \rightleftharpoons \overline{p} \mid p \in FV(\chi)]]\!]^{\mathbb{S}} = [\![\chi]\!]^{\mathbb{S}^{\sim}}. \tag{29}$$

But then, taking $\varphi^{\partial}$ for $\chi$, we find that

$$[\![\sim\varphi]\!]^{\mathbb{S}} = [\![\varphi^{\partial}[p \rightleftharpoons \overline{p} \mid p \in FV(\varphi)]]\!]^{\mathbb{S}} = [\![\varphi^{\partial}]\!]^{\mathbb{S}^{\sim}} = \sim_S [\![\varphi]\!]^{(\mathbb{S}^{\sim})^{\sim}} = \sim_S [\![\varphi]\!]^{\mathbb{S}},$$

where the first equality holds by the definition of $\sim\varphi$, the second by (29), the third equality is (27), and the fourth equality follows from the trivial observation that $(\mathbb{S}^{\sim})^{\sim} = \mathbb{S}$. QED

**Remark 3.25** It follows from the Proposition above that we could indeed have based the language of the modal $\mu$-calculus on a smaller alphabet of primitive symbols. Given a set $\mathsf{D}$ of atomic actions, we could have defined the set of modal fixpoint formulas using the following induction:

$$\varphi \ ::= \ \bot \ | \ p \ | \ \neg\varphi \ | \ \varphi \vee \varphi \ | \ \Diamond_d \varphi \ | \ \mu x.\varphi$$

where $p$ and $x$ are propositional variables, $d \in \mathsf{D}$, and in $\mu x.\varphi$, all free occurrences of $x$ must be positive (that is, under an even number of negation symbols). Here we define $FV(\neg\varphi) = FV(\varphi)$ and $BV(\neg\varphi) = BV(\varphi)$.

In this set-up, the constant $\top$ and the connectives $\wedge$ and $\Box_d$ are defined using the standard abbreviations, while for the greatest fixpoint operator we may put

$$\nu x.\varphi := \neg\mu x.\neg\varphi(\neg x).$$

Note the *triple* use of the negation symbol here, which can be explained by Proposition 3.11 and the observation that we may think of $\neg\varphi(\neg x)$ as the formulas $\varphi^{\partial}$. ◁

### Other immediate consequences

Earlier on we defined the notions of *clean* and *guarded* formulas.

**Proposition 3.26** *Every fixpoint formula is equivalent to a clean formula, and hence, to a tidy one.*

**Proof.** We leave this proof as an exercise for the reader. QED

**Proposition 3.27** *Every fixpoint formula is equivalent to a guarded formula.*

**Proof.**(Sketch) We prove this proposition by formula induction. Clearly the only nontrivial case to consider concerns the fixpoint operators. Consider a formula of the form $\eta x.\delta(x)$, where $\delta(x)$ is guarded and clean, and suppose that $x$ has an unguarded occurrence in $\delta$.

First consider an unguarded occurrence of $x$ in $\delta(x)$ inside a fixpoint subformula, say, of the form $\theta y.\gamma(x, y)$. By induction hypothesis, all occurrences of $y$ in $\gamma(x, y)$ are guarded. Obtain the formula $\overline{\delta}$ from $\delta$ by replacing the subformula $\theta y.\gamma(x, y)$ with $\gamma(x, \theta y.\gamma(x, y))$. Then clearly $\overline{\delta}$ is equivalent to $\delta$, and all of the unguarded occurrences of $x$ in $\overline{\delta}$ are outside of the scope of the fixpoint operator $\theta$.

Continuing like this we obtain a formula $\eta x.\overline{\delta}(x)$ which is equivalent to $\eta x.\delta(x)$, and in which none of the unguarded occurrences of $x$ lies inside the scope of a fixpoint operator. That leaves $\wedge$ and $\vee$ as the only operation symbols in the scope of which we may find unguarded occurrences of $x$.

From now on we only consider the case where $\eta = \mu$, leaving the very similar case where $\eta = \nu$ as an exercise. Clearly, using the laws of classical propositional logic, we may bring the formula $\overline{\delta}$ into conjunctive normal form

$$(x \vee \alpha_1(x)) \wedge \cdots \wedge (x \vee \alpha_n(x)) \wedge \beta(x), \tag{30}$$

where all occurrences of $x$ in $\alpha_1, \ldots, \alpha_n$ and $\beta$ are guarded. (Note that we may have $\beta = \top$, or $\alpha_i = \bot$ for some $i$.)

Clearly (30) is equivalent to the formula

$$\delta'(x) := (x \vee \alpha(x)) \wedge \beta(x),$$

where $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_n$. Thus we are done if we can show that

$$\mu x.\delta'(x) \equiv \mu x.\alpha(x) \wedge \beta(x). \tag{31}$$

Since $\alpha \wedge \beta$ implies $\delta'$, it is easy to see (and left for the reader to prove) that $\mu x.\alpha \wedge \beta$ implies $\mu x.\delta'$. For the converse, it suffices to show that $\varphi := \mu x.\alpha(x) \wedge \beta(x)$ is a prefixpoint of $\delta'(x)$. But it is not hard to derive from $\varphi \equiv \alpha(\varphi) \wedge \beta(\varphi)$ that

$$\delta'(\varphi) = (\varphi \vee \alpha(\varphi)) \wedge \beta(\varphi) \equiv ((\alpha(\varphi) \wedge \beta(\varphi)) \vee \alpha(\varphi)) \wedge \beta(\varphi) \equiv \alpha(\varphi) \wedge \beta(\varphi) \equiv \varphi,$$

which shows that $\varphi$ is in fact a fixpoint, and hence certainly a prefixpoint, of $\delta'(x)$. QED

Combining the proofs of the previous two propositions one easily shows the following.

**Proposition 3.28** *Every fixpoint formula is equivalent to a clean, guarded formula, and hence, to a tidy, guarded one.*

**Remark 3.29** The equivalences of the above propositions are in fact *effective* in the sense that there are algorithms for computing an equivalent clean and/or guarded equivalent to an arbitrary formula in $\mu$ML. It is an interesting question what the complexity of these algorithms is, and what the minimum *size* of the equivalent formulas is. We will return to this issue later on, but already mention here that there are formulas that are exponentially smaller than any of their clean equivalents. The analogous question for guarded transformations, i.e., constructions that provide guarded equivalents to an arbitrary formula, is open. ◁

## 3.5 Adequacy

In this section we prove the *equivalence* of the two semantic approaches towards the modal $\mu$-calculus. Since the algebraic semantics is usually taken to be the more fundamental notion, we refer to this result as the *Adequacy Theorem* stating, informally, that games are an adequate way of working with the algebraic semantics.

We first consider the subformula game.

**Theorem 3.30 (Adequacy of the subformula game)** *Let $\xi$ be a clean $\mu\mathtt{ML}_{\mathsf{D}}$-formula. Then for all labelled transition systems $\mathbb{S}$ and all states $s$ in $\mathbb{S}$:*

$$s \in [\![\xi]\!]^{\mathbb{S}} \iff (\xi, s) \in \mathrm{Win}_{\exists}(\mathcal{E}(\xi, \mathbb{S})). \tag{32}$$

**Proof.** The theorem is proved by induction on the complexity of $\xi$. We only discuss the inductive steps where $\xi$ is of the form $\eta x.\delta$ (with $\eta$ denoting either $\mu$ or $\nu$), leaving the other cases as exercises to the reader.

**Preparatory observations** Our proof for these inductive cases will involve *three* games: the unfolding game for $\delta_x^{\mathbb{S}}$, and the evaluation games for $\xi$ and $\delta$, respectively. It is based on two key observations: One concerns the nature of the unfolding game for $\delta_x^{\mathbb{S}}$ and its role in the semantics for $\eta x.\delta$; the other observation concerns the similarity between the evaluation games for $\xi$ and for $\delta$.

1. Starting with the first observation, note that by definition of the algebraic semantics of the fixpoint operators, the set $[\![\eta x.\delta]\!]^{\mathbb{S}}$ is the least/greatest fixed point of the map $\delta_x^{\mathbb{S}} : \wp(S) \to \wp(S)$, and that by our earlier Theorem 3.14 on unfolding games, we have

   $$[\![\eta x.\delta]\!]^{\mathbb{S}} = \mathrm{Win}_\exists(\mathcal{U}^\eta(\delta_x^{\mathbb{S}})) \cap S. \tag{33}$$

   Hence, in order to prove (32), it suffices to show that, for any state $s_0$:

   $$s_0 \in \mathrm{Win}_\exists(\mathcal{U}^\eta(\delta_x^{\mathbb{S}})) \iff (\xi, s_0) \in \mathrm{Win}_\exists(\mathcal{E}(\xi, \mathbb{S})). \tag{34}$$

   In other words, the crucial tasks in the proof of this inductive step concern the transformation of a winning strategy for $\exists$ in the unfolding game $\mathcal{U}^\eta(\delta_x^{\mathbb{S}})@s_0$ to a winning strategy for her in the evaluation game $\mathcal{E}(\xi, \mathbb{S})@(\xi, s_0)$, and vice versa.

   Given the importance of the unfolding game for $\delta_x^{\mathbb{S}}$ then, let us look at it in a bit more detail. Note that a round of this game, starting at position $s \in S$, consists of $\exists$ picking a subset $A \subseteq S$ that is subject to the constraint that $s \in \delta_x^{\mathbb{S}}(A) = [\![\delta]\!]^{\mathbb{S}[x \mapsto A]}$. But here the inductive hypothesis comes into play: it implies that, for all $A \subseteq S$, we have

   $$s \in \delta_x^{\mathbb{S}}(A) \iff (\delta, s) \in \mathrm{Win}_\exists(\mathcal{E}(\delta, \mathbb{S}[x \mapsto A])). \tag{35}$$

   In other words, each round of the unfolding game for the map $\delta_x^{\mathbb{S}}$ crucially involves the evaluation game for the formula $\delta$, played on some $x$-variant $\mathbb{S}[x \mapsto A]$ of $\mathbb{S}$.

2. This leads us to the comparison between the games $\mathcal{G} := \mathcal{E}(\xi, \mathbb{S})$ and $\mathcal{G}_A := \mathcal{E}(\delta, \mathbb{S}[x \mapsto A])$. The second key observation in the inductive step for the fixpoint operators is that these games are *very* similar indeed. For a start, the positions of the two games are essentially the same. Positions of the form $(\xi, t)$, which exist in the first game but not in the second, are the only exception — but in $\mathcal{G}$, any position $(\xi, t)$ is immediately and automatically succeeded by the position $(\delta, t)$ which does exist in the second game. What is important is that the positions for $\exists$ are exactly the same in the two games, and thus we may apply her positional strategies for the one game in the other game as well. The only real difference between the games shows up in the rule concerning positions of the form $(x, u)$. In $\mathcal{G}_A$, $x$ is a *free* variable ($x \in FV(\delta)$), so in a position $(x, u)$ the game is over, the winner being determined by $u$ being a member of $A$ or not. In $\mathcal{G}$ however, $x$ is *bound*, so in position $(x, u)$, the variable $x$ will get unfolded to $\delta$.

Combining these two observations, the key insight in the proof of (34) will be to think of $\mathcal{E}(\xi, \mathbb{S})$ as a variant of the unfolding game $\mathcal{U} := \mathcal{U}^\eta(\delta_x^{\mathbb{S}})$ where each round of $\mathcal{U}$ corresponds to a version of the game $\mathcal{G}_T$, with $T$ being the subset of $S$ picked by $\exists$ in $\mathcal{U}$. We are now ready for the details of the proof of (34).

**For the direction from left to right of (34),** suppose that $\exists$ has a winning strategy in the game $\mathcal{U}$ starting at some position $s_0$. Without loss of generality (see Exercise 3.7) we may assume that this strategy is *positional*. Thus we may represent it as a map $T : S \to \wp(S)$, where we will write $T_s$ rather than $T(s)$. By the legitimacy of this strategy, for every $s \in \mathrm{Win}_\exists(\mathcal{U})$ it holds that $s \in \delta_x^{\mathbb{S}}(T_s)$. So by the inductive hypothesis (35), for each such $s$ we may assume the existence of a winning strategy $f_s$ for $\exists$ in the game $\mathcal{G}_{T_s}@(\delta, s)$. Given the similarities between the games $\mathcal{G}$ and $\mathcal{G}_{T_s}$ (see the discussion above), this strategy is also applicable in the game $\mathcal{G}@(\delta, s)$, at least, until a new position of the form $(x, t)$ is reached. This suggests the following strategy $g$ for $\exists$ in $\mathcal{G}@(\xi, s_0)$:

1. after the initial automatic move, the position of the match is $(\delta, s_0)$; $\exists$ first plays her strategy $f_{s_0}$;
2. each time a position $(x, s)$ is reached, the match automatically moves to position $(\delta, s)$, where we distinguish cases:
   (a) if $s \in \mathrm{Win}_\exists(\mathcal{U})$ then $\exists$ continues with $f_s$;
   (b) if $s \notin \mathrm{Win}_\exists(\mathcal{U})$ then $\exists$ continues with a random strategy.

First we show that this strategy guarantees that whenever a position of the form $(x, s)$ is visited, $s$ belongs to $\mathrm{Win}_\exists(\mathcal{U})$, so that case (b) mentioned above never occurs. The proof is by induction on the number of positions $(x, s)$ that have been visited already. For the inductive step, if $s$ is a winning position for $\exists$ in $\mathcal{U}$, then, as we saw, $f_s$ is a winning strategy for $\exists$ in the game $\mathcal{G}_{T_s}@(\delta, s)$. This means that if a position of the form $(x, t)$ is reached, the variable $x$ must be *true* at $t$ in the model $\mathbb{S}[x \mapsto T_s]$, and so $t$ must belong to the set $T_s$. But by assumption of the map $T : S \to \wp(S)$ being a winning strategy in $\mathcal{U}$, any element of $T_s$ is again a member of $\mathrm{Win}_\exists(\mathcal{U})$.

In fact we have shown that every unfolding of the variable $x$ in $\mathcal{G}$ marks a new round in the unfolding game $\mathcal{U}$. To see why the strategy $g$ guarantees a win for $\exists$ in $\mathcal{G}@(\xi, s_0)$, consider an arbitrary $\mathcal{G}@(\xi, s_0)$-match $\pi$ in which $\exists$ plays $g$. Distinguish cases.

First suppose that $x$ is unfolded only finitely often. Let $(x, s)$ be the last basic position in $\pi$ where this happens. Given the similarities between the games $\mathcal{G}$ and $\mathcal{G}_{T_s}$, the match from this moment on can be seen as both a $g$-guided $\mathcal{G}$-match and an $f_s$-guided $\mathcal{G}_{T_s}$-match. As we saw, $f_s$ is a winning strategy for $\exists$ in the game $\mathcal{G}_{T_s}@(\delta, s)$. But since no further position of the form $(x, t)$ is reached, and $\mathcal{G}$ and $\mathcal{G}_{T_s}$ only differ when it comes to $x$, this means that $\pi$ is also a win for $\exists$ in $\mathcal{G}$.

If $x$ is unfolded infinitely often during the match $\pi$, then by the fact that $\xi = \eta x.\delta$, it is the *highest* variable that is unfolded infinitely often. We have to distinguish the case where $\eta = \nu$ from that where $\eta = \mu$. In the first case, $\exists$ is the winner of the match $\pi$, and we are done. If $\eta = \mu$, however, $x$ is a least fixpoint variable, and so $\exists$ would lose the match $\pi$. We therefore have to show that this situation cannot occur. Suppose for contradiction that $s_1, s_2, \ldots$ are the positions where $x$ is unfolded. Then it is easy to verify that the sequence $s_0 T_{s_0} s_1 T_{s_1} \ldots$ constitutes a $\mathcal{U}$-match in which $\exists$ plays her strategy $T$. But this is not possible, since $T$ was assumed to be a *winning* strategy for $\exists$ in the *least* fixpoint game $\mathcal{U} = \mathcal{U}^\mu(\delta_x^{\mathbb{S}})$.

**For the direction from right to left of (34),** we will show how each positional winning strategies $f$ for $\exists$ in $\mathcal{G}$ induces a positional strategy for her in $\mathcal{U}$, and that this strategy $U_f$ is winning for her starting at every position $s \in W := \{s \in S \mid (\xi, s) \in \mathrm{Win}_\exists(\mathcal{G})\}$.

So fix a positional winning strategy $f$ for $\exists$ in $\mathcal{G}$; that is, $\exists$ is guaranteed to win any $f$-guided match starting at a position $(\varphi, t) \in \mathrm{Win}_\exists(\mathcal{G})$. Observe that, as discussed above, we may and will treat $f$ as a positional strategy in each of the games $\mathcal{G}_A$ as well.

Given a state $s \in W$, we let $\mathbb{T}_f(s)$ be the *strategy tree* induced by $f$ in $\mathcal{G}_A@(\delta, s)$, where $A$ is some arbitrary subset of $S$. That is, the nodes of $\mathbb{T}_f$ consist of all $f$-guided finite matches in $\mathcal{G}_A$ that start at $(\delta, s)$. In more detail, the root of this tree is the single-position match $(\delta, s)$; to define the successor relation of $\mathbb{T}_f$, let $\Sigma$ be an arbitrary $f$-guided match starting at position $\mathsf{first}(\Sigma) = (\delta, s)$. If $\mathsf{last}(\Sigma)$ is a position owned by $\exists$, then $\Sigma$ will have a single successor in $\mathbb{T}_f$, viz., the unique extension of $\Sigma$ with the position $f(\Sigma)$ picked by $f$. On the other hand, if $\mathsf{last}(\Sigma)$ is owned by $\forall$, then every possible continuation $\Sigma \cdot b$, where $b$ is an admissible position picked by $\forall$, is a successor of $\Sigma$.

We let $U_f(s)$ be the set of states $u$ such that the position $(x, u)$ occurs as the last element $(x, u) = \mathsf{last}(\Sigma)$ of some match $\Sigma$ in $\mathbb{T}_f(s)$. It is easy to see that any $\mathcal{G}_A$-match $\Sigma$ ending in a position of the form $(x, u)$, is finished immediately, and thus provides a *leaf* of the tree $\mathbb{T}_f$. It is also an easy consequence of the definitions that, whenever $t \in U_f(s)$ for some $s \in W$, then there is an $f$-guided match $\Sigma_{s,t}$ such that $\mathsf{first}(\Sigma_{s,t}) = (\delta, s)$ and $\mathsf{last}(\Sigma_{s,t}) = (x, t)$. Note that this match $\Sigma_{s,t}$ can be seen both as a (full) $\mathcal{G}_A$-match and as a (partial) $\mathcal{G}$-match.

Given our definition of a set $U_f(s) \subseteq S$ for every $s \in W$, in effect we have defined a map

$$U_f : W \to \wp(S).$$

Claim 1 Viewing this map $U_f$ as a positional strategy for $\exists$ in $\mathcal{U}$, we claim that in fact it is a *winning* strategy for her in $\mathcal{U}@s_0$.

Proof of Claim We need two auxiliary claims on $U_f$. First we observe that

$$\text{if } s \in W \text{ then } s \in \delta_x^{\mathbb{S}}(U_f(s)). \tag{36}$$

For a proof of (36), it is obvious from the definition of $U_f(s)$ that $f$ is a positional winning strategy for $\exists$ in $\mathcal{G}_{U_f(s)} = \mathcal{E}(\delta, \mathbb{S}[x \mapsto U_f(s)])$ starting at $(\delta, s)$. But then by the inductive hypothesis on $\delta$ we obtain that $\mathbb{S}[x \mapsto U_f(s)], s \Vdash \delta$, or, equivalently, $s \in \delta_x^{\mathbb{S}}(U_f(s))$.

Second, we claim that

$$\text{if } s \in W \text{ then } U_f(s) \subseteq W. \tag{37}$$

To see this, first note that if $s \in W$ then by definition $(\xi, s) \in \mathrm{Win}_\exists(\mathcal{G})$; but from this it is immediate that $(\delta, s) \in \mathrm{Win}_\exists(\mathcal{G})$, and since we assumed $f$ to be a positional winning strategy for $\exists$ in $\mathcal{G}$, it follows by definition of $U_f(s)$ that for every $u \in U_f(s)$ the position $(x, u)$ is winning for $\exists$ in $\mathrm{Win}_\exists(\mathcal{G})$. But from this it is easy to derive that both $(\delta, u)$ and $(\xi, u)$ are winning position for $\exists$ in $\mathcal{G}$ as well. The latter fact then shows that $u \in W$ and since $u$ was an arbitrary element of $U_f(s)$, (37) follows.

We can now prove that $U_f$ is a winning strategy for $\exists$ in $\mathcal{U}@s_0$. First of all, it follows from (36) that $U_f(s)$ is a legitimate move in $\mathcal{U}$ for every position $s \in W$. From this and (37) we may conclude that $\exists$ never gets stuck in an $U_f$-guided $\mathcal{U}$-match starting at $s_0$; that is, she wins every *finite* $U_f$-guided $\mathcal{U}$-match. In case $\eta = \nu$ this suffices, since in $UG^{\nu}(\delta_x^{\mathbb{S}})$ all infinite matches are won by $\exists$.

Where $\eta = \mu$ we have a bit more work to do, since in this case all infinite matches of $\mathcal{U}^{\mu}(\delta_x^{\mathbb{S}})$ are won by $\forall$. Suppose for contradiction that $\Sigma = s_0 U_f(s_0) s_1 U_f(s_1) \cdots$ would be an infinite $U_f$-guided match of $\mathcal{U}^{\mu}(\delta_x^{\mathbb{S}})$. Then for every $i \in \omega$ we have that $s_{i+1} \in U_f(s_i)$, so that there is a partial $f$-guided match $\Sigma_i = \Sigma_{s_i s_{i+1}}$ with $\mathsf{first}(\Sigma_i) = (\delta, s_i)$ and $\mathsf{last}(\Sigma_i) = (x, s_{i+1})$. But then it is straightforward to verify that the infinite match $\Sigma_{\mathcal{G}} := \Sigma_0 \cdot \Sigma_1 \cdot \Sigma_2 \cdots$ we obtain by concatenating the individual $f$-guided matches $\Sigma_i$, constitutes an infinite $f$-guided $\mathcal{G}$-match with $\mathsf{first}(\Sigma_{\mathcal{G}}) = \mathsf{first}(\Sigma_0) = (\xi, s_0)$. Since the highest fixpoint variable unfolded infinitely often during $\Sigma_{\mathcal{G}}$ obviously would be $x$, this match would be lost by $\exists$. Here we arrive at the desired contradiction, since $(\xi, s_0) \in \mathrm{Win}_{\exists}(\mathcal{G})$, and $f$ was assumed to be a positional winning strategy in $\mathcal{G}$. ◀

QED

**Convention 3.31** In the sequel we will use the Adequacy Theorem without further notice. Also, we will write $\mathbb{S}, s \Vdash \varphi$ in case $s \in [\![\varphi]\!]^{\mathbb{S}}$, or, equivalently, $\mathbb{S}, s \Vdash_g \varphi$.

▶ `Adequacy of the closure game to be discussed and proved.`

**Theorem 3.32 (Adequacy of the closure game)** *Let $\xi$ be a tidy $\mu\mathsf{ML}_{\mathsf{D}}$-formula. Then for all labelled transition systems $\mathbb{S}$ and all states $s$ in $\mathbb{S}$:*

$$s \in [\![\xi]\!]^{\mathbb{S}} \iff (\xi, s) \in \mathrm{Win}_{\exists}(\mathcal{E}^c(\xi, \mathbb{S})). \tag{38}$$

**Proof.**

▶ ...

By induction on the length of formulas, we will show that every tidy formula $\varphi$ has the following property:

$$\text{all formulas } \xi \in Cl(\varphi) \text{ satisfy (32).} \tag{39}$$

We only consider the case where $\varphi$ is a fixpoint formula, say, it of the form $\varphi = \eta x\, \psi$. Note that the formula $\psi$ may not be tidy, but we can work with a formula $\psi' := \psi[x'/x]$, where $x'$ is a fresh variable. This formula *is* tidy, and shorter than $\varphi$, so that the inductive hypothesis applies to it.

We now distinguish cases. If $\varphi$ is a derivative of $\psi'$ then we have $Cl(\varphi) \subseteq Cl(\psi')$ so that (**??**) is immediate by the inductive hypothesis on $\psi'$.

▶ `In the sequel we will therefor assume that` $\varphi \notin Cl(\psi')$.

QED

## Notes

What we now call the Knaster-Tarski Theorem (Theorem 3.4) was first proved by Knaster [9] in the context of power set algebras, and subsequently generalized by Tarski [21] to the setting of complete lattices. The Bekič principle (Proposition 3.15) stems from an unpublished technical report.

▶ `more notes and references to be supplied`

As far as we know, the results in section 3.2 on the duality between the least and the greatest fixpoint of a monotone map on a complete boolean algebra, are folklore. The characterization of least and greatest fixpoints in game-theoretic terms is fairly standard in the theory of (co-)inductive definitions, see for instance Aczel [1]. The equivalence of the algebraic and the game-theoretic semantics of the modal $\mu$-calculus (here formulated as the Adequacy Theorem **??**) was first established by Emerson & Jutla [6].

## Exercises

**Exercise 3.1** Prove Proposition 3.6: show that monotone maps on complete lattices are inductive.

**Exercise 3.2** Prove Theorem 3.21.
(Hint: given complete lattices $\mathbb{C}$ and $\mathbb{D}$, and a monotone map $f : C \times D \to C$, show that the map $g : D \to C$ given by

$$g(d) := \mu x.f(x, d)$$

is monotone. Here $\mu x.f(x, d)$ is the least fixpoint of the map $f_d : C \to C$ given by $f_d(c) = f(c, d)$.)

**Exercise 3.3** Let $F : \wp(S) \to \wp(S)$ be some monotone map. A collection $\mathcal{D} \in \wp\wp(S)$ of subsets of $S$ is *directed* if for every two sets $D_0, D_1 \in \mathcal{D}$, there is a set $D \in \mathcal{D}$ with $D_i \subseteq D$ for $i = 0, 1$. Call $F$ *(Scott) continuous* if it preserves directed unions, that is, if $F(\bigcup \mathcal{D}) = \bigcup_{D \in \mathcal{D}} F(D)$ for every directed $\mathcal{D}$.
Prove the following:

(a) $F$ is Scott continuous iff for all $X \subseteq S$: $F(X) = \bigcup\{F(Y) \mid Y \subseteq_\omega X\}$.
(Here $Y \subseteq_\omega X$ means that $Y$ is a finite subset of $X$.)

(b) If $F$ is Scott continuous then the unfolding ordinal of $F$ is at most $\omega$.

(c) Give an example of a Kripke frame $\mathbb{S} = \langle S, R \rangle$ such that the operation $[R]$ is not continuous.

(d) Give an example of a Kripke frame $\mathbb{S} = \langle S, R \rangle$ such that the operation $[R]$ has closing/unfolding ordinal $\omega + 1$.

**Exercise 3.4** By a mutual induction we define, for every finite set $\mathsf{P}$ of propositional variables, the fragment $\mu\mathtt{ML}^C_{\mathsf{P}}$ by the following grammar:

$$\varphi ::= p \mid \psi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \mu q.\varphi',$$

where $p \in \mathsf{P}$, $\psi \in \mu\mathtt{ML}$ is a $\mathsf{P}$-free formula, and $\varphi' \in \mu\mathtt{ML}^C_{\mathsf{P}\cup\{q\}}$.

Prove that for every Kripke model $\mathbb{S}$, every formula $\varphi \in \mu\mathtt{ML}^C_{\mathsf{P}}$, and every proposition letter $p \in \mathsf{P}$, the map $\varphi^{\mathbb{S}}_p : \wp(S) \to \wp(S)$ is continuous.

**Exercise 3.5** Let $F : \wp(S) \to \wp(S)$ be a monotone operation, and let $\gamma_F$ be its unfolding ordinal. Sharpen Corollary 3.7 by proving that the cardinality of $\gamma_F$ is bounded by $|S|$ (rather than by $|\wp(S)|$).

**Exercise 3.6** The proof of Theorem 3.14 is based on the characterisation of least fixed points as the intersection of all prefixpoints, and similarly, of greatest fixpoints as the union of all postfixpoints. Can you also prove the theorem using the characterisation of least- and greatest fixpoints via ordinal approximations?

**Exercise 3.7** Prove that the unfolding game of Definition 3.12 satisfies *positional determinacy*. That is, let $\mathcal{U}^{\mu}(F)$ be the least fixpoint unfolding game for some monotone map $F : \wp(S) \to \wp(S)$. Prove the existence of two *positional* strategies $f_{\exists} : S \to \wp(S)$ and $f_{\forall} : \wp(S) \to S$ such that for every position $p$ of the game, either $f_{\exists}$ is a winning strategy for $\exists$ in $\mathcal{U}^{\mu}(F)@p$, or else $f_{\forall}$ is a winning strategy for $\forall$ in $\mathcal{U}^{\mu}(F)@p$.

**Exercise 3.8** Let $\mathbb{C}$ be a complete boolean algebra and let $f : C \to C$ be a monotone map. Pick an element $d \in C$ and let $\mu x.f(x)$ be the least fixpoint of $f$.

(a) Show that $d \wedge \mu x.f(x) = \bot$ iff $d \wedge \mu x.f(x \wedge \neg d) = \bot$, where $\mu x.f(x \wedge \neg d)$ denotes the smallest fixpoint of the map sending any element $x \in C$ to $f(x \wedge \neg d)$.

(b) Conclude that, for any formula of the form $\mu x.\varphi$ and an arbitrary formula $\gamma$: the formula $\gamma \wedge \mu x.\varphi$ is satisfiable iff the formula $\gamma \wedge \mu x.\varphi[x \wedge \neg\gamma/x]$ is satisfiable. (A formula $\varphi$ is called satisfiable if there exists a pointed Kripke model such that $\mathbb{S}, s \Vdash \varphi$.)

▶ `add exercise on the closure ordinal of a formula`

▶ `add exercise on (complete) additivity`