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A Dual Characterization of Subdirectly Irreducible BAOs

Abstract. We give a characterization of the simple, and of the subdirectly irreducible boolean algebras with operators (including modal algebras), in terms of the dual descriptive frame, or, topological relational structure. These characterizations involve a special binary *topo-reachability* relation on the dual structure; we call a point u a topo-root of the dual structure if every ultrafilter is topo-reachable from u. We prove that a boolean algebra with operators is simple iff every point in the dual structure is a topo-root; and that it is subdirectly irreducible iff the collection of topo-roots is open and non-empty in the Stone topology on the dual structure iff this collection has non-empty interior in that topology.

Keywords: Boolean algebras with operators, subdirect irreducibility, duality.

Introduction

The duality theory between modal algebras, or more generally, boolean algebras with operators on the one hand, and (topological) frames or relational structures on the other, has been well developed, see for instance Sambin & Vaccaro [5] or Goldblatt [2]. Dualities allow the transfer, from one field to the other, of concepts, techniques and results alike. For instance, when it comes to the fundamental algebraic concept of *subdirect irreducibility*, there is a nice connection with the frame theoretical notion of *rootedness*. Without too much difficulty one can show that a Kripke frame is rooted iff its complex algebra is subdirectly irreducible (s.i.).

Unfortunately, there seems to be no such nice connection when we look at arbitrary (that is, not necessarily complex) algebras: Sambin [4] gives examples of subdirectly irreducible modal algebras of which the dual ultrafilter frame is not rooted, and conversely, of non-s.i. modal algebras with a rooted dual frame. In the same paper, Sambin brings the Stone topology of the dual structure into the picture, showing that for any modal algebra $\mathbb{A}$:

$$\text{if } I_{\mathbb{A}_*} \text{ has non-empty interior, then } \mathbb{A} \text{ is s.i.} \qquad \text{(i)}$$

Here $I_{\mathbb{A}_*}$ denotes the collection of roots, or initial points, of the dual frame $\mathbb{A}_*$. Sambin also proves that for **K4**-algebras the converse of (i) holds as well.

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In the closely related field of intuitionistic logic, similar characterizations of s.i. Heyting algebras in terms of their dual structures had been known for some time, cf. ESAKIA [1]. It is straightforward to verify that the converse of (i) goes through for all *ω-transitive* logics, but the general picture, unfortunately, is not so nice: KRACHT [3] provides an example of a subdirectly irreducible (in fact: simple) algebra $\mathbb{A}$ with an *empty* set of roots. Kracht's result indicates that in the general case there is no simple characterization of s.i. algebras in terms of the *roots* of their dual frames.

In this paper we will show that a fairly transparent characterization of s.i. algebras is possible once we consider a new kind of relation in the dual structure. Given a modal algebra $\mathbb{A} = (A, \wedge, -, \bot, \Diamond)$, we define the following *topo-reachability* relation $R^\star$ on ultrafilters:

$$R^\star uv \text{ iff for all } a \in v \text{ there is some } n \in \omega \text{ such that } \Diamond^n a \in u. \qquad \text{(ii)}$$

One may give various alternative characterizations of $R^\star$; for instance, it is not hard to prove that $R^\star uv$ iff v belongs to the *topological closure* of the subframe generated from u. This may explain the name 'topo-reachability', and also why we call an ultrafilter a *topo-root* of $\mathbb{A}_*$ if every ultrafilter is topo-reachable from it.

Our characterization of subdirect irreducibility will be in terms of the collection $T_{\mathbb{A}_*}$ of topo-roots of the dual frame associated with an algebra $\mathbb{A}$. In fact, we have two distinct dual characterizations, and we believe that the equivalence between these is of some independent interest. For modal algebras, we can formulate the main result of this paper, Theorem 2 below, as follows:

$$\begin{array}{rll} \mathbb{A} \text{ is s.i.} & \text{iff} & T_{\mathbb{A}_*} \text{ is open and non-empty} \\ & \text{iff} & T_{\mathbb{A}_*} \text{ has non-empty interior.} \end{array} \qquad \text{(iii)}$$

For readers that prefer a formulation in terms of descriptive general frames, the following is a reformulation of the second characterization in (iii):

$$\mathbb{A} \text{ is s.i. iff there is a non-empty admissible set of topo-roots in } \mathbb{A}_*.$$

In a similar way, we can characterize *simplicity*. For a modal algebra $\mathbb{A}$, Theorem 1 below states that $\mathbb{A}$ is simple iff each of its ultrafilters is a topo-root of the dual structure:

$$\mathbb{A} \text{ is simple iff } T_{\mathbb{A}_*} = A_*. \qquad \text{(iv)}$$

Our results will be formulated in the more general setting of boolean algebras with operators; for this purpose we will give a definition of the relation $R^\star$ which generalizes (ii). Before going into the details, let us mention that a number of well known dual characterizations of subdirect irreducibility for special algebras, such as for ω-transitive ones, can easily be seen as special cases of our results.

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1. Preliminaries

Notation and terminology

We assume that the reader is familiar with basic concepts from universal algebra, with *boolean algebras with operators*, or BAOs for short, with *frames* and *Kripke frames*, and in particular, with the duality between the categories of BAOs and algebraic homomorphisms on the one hand, and descriptive frames with bounded morphisms on the other hand. Nevertheless, we review some of the terminology and notation.

First, let R be a binary relation on a set S. Inductively we define $R^0 = \{(s, s) \mid s \in S\}$ and $R^{n+1} = R \circ R^n$. The reflexive-transitive closure of R is denoted as R^ω. For a point s, define $R[s] = \{t \in S \mid Rst\}$. For a subset T of S, define $[R]T = \{s \in S \mid R[s] \subseteq T\}$ and $\langle R \rangle T = \{s \in S \mid Rst \text{ for some } t \in T\}$.

A boolean algebras with operators is denoted as $\mathbb{A} = (\mathbb{B}_\mathbb{A}, (f_i)_{i\in I})$ where $\mathbb{B}_\mathbb{A}$ is the boolean reduct of the algebra and the f_i are the operators; unary operators are called *diamonds*. We speak of *modal* algebras in the case that there is exactly one operator, and this operator is a diamond; such algebras are denoted $\mathbb{A} = (\mathbb{B}_\mathbb{A}, \Diamond)$. Inductively we define the notion of a *compound diamond*; first, given an n-ary operator ∇, define (for $1 \leq k \leq n$) its k-th *induced diamond* as the operation $\lambda x.\nabla(\top, \dots, \top, x, \top, \dots, \top) : A \to A$; (that is, all arguments are $\top$ except for the i-th). The collection $CD(\mathbb{A})$ of compound diamonds is then defined as the smallest collection of operations containing the identity map, such that if $\blacklozenge_1$ and $\blacklozenge_2$ are in $CD(\mathbb{A})$, and $\Diamond$ is an induced diamond, then $\lambda x.\Diamond\blacklozenge_1 x$ and $\lambda x.\blacklozenge_1 x \vee \blacklozenge_2 x$ are compound diamonds. (For instance, in the case of a modal algebra $(\mathbb{B}, \Diamond)$, the compound diamonds are the maps of the form $\lambda x. \bigvee_{k\in K} \Diamond^k x$ for some finite $K \subseteq \omega$.) It is easy to see that compound diamonds are indeed diamonds, i.e., unary operations preserving all finite joins. The boolean dual map of a compound diamond $\blacklozenge$ is denoted by $\blacksquare$; that is, $\blacksquare a = -\blacklozenge - a$.

Given a BAO $\mathbb{A} = (\mathbb{B}_{\mathbb{A}}, (f_i)_{i\in I})$, the dual structure of $\mathbb{A}$ is denoted as $\mathbb{A}_* = (A_*, (R_i)_{i\in I}, \widehat{A})$. Here A_* is the set of ultrafilters of $\mathbb{A}$ (that is, of the boolean reduct $\mathbb{B}_{\mathbb{A}}$ of $\mathbb{A}$), the relations R_i are defined as usual, and $\widehat{A}$ is the image of the domain A of $\mathbb{A}$ under the Stone isomorphism $a \mapsto \widehat{a} := \{u \in A_* \mid a \in u\}$. We may use the notation $\mathbb{A}_+$ when referring to the underlying Kripke frame $(A_*, (R_i)_{i\in I})$ of $\mathbb{A}_*$; when it comes to concepts involving the accessibility relations we may be sloppy concerning the difference between $\mathbb{A}_+$ and $\mathbb{A}_*$. The set $\widehat{A}$ is the set of clopens of, and thus forms a basis for, the Stone topology, and we will use standard facts concerning this topology without warning. A subset $X \subseteq A_*$ is said to have *non-empty interior* if it has a non-empty open subset; in our context this is equivalent to saying that X contains a non-empty clopen/admissible subset. The *closure* of a set $X \subseteq A_*$ is denoted as $\overline{X}$; recall that $\overline{X} = \bigcap\{\widehat{a} \mid a \in A, X \subseteq \widehat{a}\}$.

Given a frame $\mathbb{S} = (S, (R_i)_{i\in I})$, we define the *one step reachability relation* $R_{\mathbb{S}}$ as follows:

$$R_{\mathbb{S}}st \quad \text{iff} \quad \text{for some } i \in I \text{ there are } t_1, \ldots, t_n \text{ such that } R_i st_1 \ldots t_n \text{ and } t \text{ occurs among } t_1, \ldots, t_n.$$

When no confusion arises we will write R instead of $R_{\mathbb{S}}$. The reflexive transitive closure $R^{\omega}_{\mathbb{S}}$ of $R_{\mathbb{S}}$ will be called the *reachability* relation of $\mathbb{S}$. A subset $X \subseteq S$ is *hereditary* if $s \in X$ and $R_{\mathbb{S}}st$ imply $t \in X$; obviously, $X \subseteq S$ is hereditary iff $R^{\omega}_{\mathbb{S}}[s] \subseteq X$ for all $s \in X$. Notice that the hereditary subsets of S correspond to the *generated subframes* of $\mathbb{S}$. An point s is called a *root* of $\mathbb{S}$ if every point is reachable from s; that is, if $R^{\omega}_{\mathbb{S}}[u] = S$. Given an algebra $\mathbb{A}$, the collection of roots of the dual structure is denoted as $I_{\mathbb{A}_*}$.

Without warning we will also employ the correspondence between boolean filters of $\mathbb{A}$ and closed subsets of A_*, and the correspondence between (i) congruences on $\mathbb{A}$, (ii) closed, hereditary subsets of A_* and (iii) modal filters on $\mathbb{A}$ (that is, boolean filters which are closed under induced boxes).

It is well known that subdirect irreducibility of a BAO can be characterized nicely using the notion of an *opremum* introduced by Rautenberg. An opremum of a BAO $\mathbb{A} = (\mathbb{B}_{\mathbb{A}}, (f_i)_{i\in I})$ is an element $o \in A$ such that $o < \top$ while for all $a \in A$ such that $a < \top$ we can find a compound diamond $\blacklozenge$ such that $o \geq \blacksquare a$. The characterization of s.i. BAOs in terms of oprema is given by the fact below (cf. KRACHT [3] or SAMBIN [4] for proofs and further information).

FACT 1.1. *The following are equivalent for any boolean algebra with operators* $\mathbb{A}$*:*

1. $\mathbb{A}$ *is subdirectly irreducible;*
2. $\mathbb{A}$ *has an opremum;*
3. $\mathbb{A}_*$ *has a largest nontrivial, closed and hereditary subset.*

Topo-reachability

We can pretend that each compound diamond $\blacklozenge$ of a BAO $\mathbb{A}$ comes with its own accessibility relation $R_\blacklozenge \subseteq A_* \times A_*$ given by $R_\blacklozenge uv$ iff $\blacklozenge a \in u$ for all $a \in v$. It is then easy to verify that the reachability relation R^ω is the union of the accessibility relations of all compound diamonds:

$$R^\omega = \bigcup_{\blacklozenge \in CD(\mathbb{A})} R_\blacklozenge.$$

In other words, R^ω can be characterized as follows:

$$R^\omega uv \text{ iff there is a compound diamond } \blacklozenge \text{ with } \blacklozenge a \in u \text{ for all } a \in v. \quad \text{(v)}$$

Our definition of the *topo*-reachability relation is obtained by swapping the universal and the existential quantifier in (v).

DEFINITION 1.2. *Given a boolean algebra with operators* $\mathbb{A}$*, define the* topo-reachability *relation* $R^\star \subseteq A_* \times A_*$ *as follows:*

$$R^\star uv \textit{ iff for all } a \in v \textit{ there is a compound diamond } \blacklozenge \textit{ with } \blacklozenge a \in u. \quad \text{(vi)}$$

We let $T_{\mathbb{A}_*}$ *denote the set of* topo-roots *of* $\mathbb{A}_*$*; that is, the collection of those ultrafilters* u *such that* $R^\star[u] = A_*$.

We leave it for the reader to verify that for a modal algebra $\mathbb{A} = (\mathbb{B}_\mathbb{A}, \diamond)$, the above definition boils down to

$$R^\star uv \text{ iff for all } a \in v \text{ there is some } n \in \omega \text{ such that } \diamond^n a \in u. \quad \text{(ii)}$$

EXAMPLE 1.3. It is instructive to have a somewhat closer look at Sambin's example of an s.i. algebra of which the dual Kripke frame is not rooted. First consider the frame $\mathbb{Z} = (Z, N, C)$ with Z the set of integers, N the neighbour relation (sNt iff $s = t + 1$ or $s = t - 1$), and C the collection of finite and cofinite subsets of Z. We leave it to the reader to verify that the algebra $\mathbb{C} = (C, \cup, -, \varnothing, \langle N \rangle)$ is subdirectly irreducible (a proof can be found in SAMBIN [4]). The dual structure $\mathbb{C}_+$, based on the collection C_* of ultrafilters of $\mathbb{C}$, consists of (an isomorphic copy of) the structure (Z, N) together with a single ultrafilter ∞ containing all and only the cofinite subsets of Z. This point ∞ is reflexive, but not related to any other point, and hence, $\mathbb{C}_+$ has no roots at all.

On the other hand, using (ii) the reader can easily verify that every principal ultrafilter of $\mathbb{C}$ is a *topo-root* of $\mathbb{C}_*$.

The following proposition shows that the relation $R^\star$ has some interesting properties. In particular, it follows from item 5 below that for any ultrafilter u, the set $R^\star[u]$ is the topological closure of the subframe generated by u.

PROPOSITION 1.4. *Let $\mathbb{A}$ be a boolean algebra with operators. The operation $R^\star$ satisfies the following properties:*

1. *$R^\star$ is transitive;*
2. *$R^\omega \subseteq R^\star$;*
3. *$R^\star[u]$ is hereditary for all ultrafilters u;*
4. *$R^\star[u]$ is closed for all ultrafilters u;*
5. *$R^\star[u] = \overline{R^\omega[u]}$ for all ultrafilters u.*

PROOF. The first two items are almost immediate from the definitions and (v); taken together, they readily imply the third one. Concerning item 4, we leave it for the reader to verify that

$$R^\star[u] \;=\; \bigcap\{\widehat{a} \mid \blacksquare a \in u \text{ for all } \blacklozenge \in CD(\mathbb{A})\}. \tag{vii}$$

Finally, in order to prove part 5 of the Proposition it suffices to show that $R^\star[u] \subseteq \overline{R^\omega[u]}$, since the opposite inclusion is immediate from the items 2 and 4.

Consider an arbitrary element $a \in A$ such that $R^\omega[u] \subseteq \widehat{a}$. We claim that $R^\star[u] \subseteq \widehat{a}$. Suppose for contradiction that $\blacksquare a \notin u$ for some compound diamond $\blacklozenge$. Hence, we obtain $\blacklozenge{-a} \in u$, so we can find some $v \in R_{\blacklozenge}[u]$ with $-a \in v$. Thus we find on the one hand, by definition of R^ω, that $R^\omega uv$ while on the other hand $-a \in v$ implies $a \notin v$ and hence, $v \notin \widehat{a}$. Taking these facts together, we obtain the desired contradiction with our assumption that $R^\omega[u] \subseteq \widehat{a}$. Hence, we may assume that $\blacksquare a$ belongs to u for all compound diamonds $\blacklozenge$, so, by (vii) we obtain $R^\star[u] \subseteq \widehat{a}$. This shows that

$$R^\star[u] \;\subseteq\; \bigcap\{\widehat{a} \mid R^\omega[u] \subseteq \widehat{a}\} \;=\overline{R^\omega[u]},$$

since a was arbitrary. ∎

Recall that, for a BAO $\mathbb{A}$, the map $[R^\star] : \mathcal{P}(A_*) \to \mathcal{P}(A_*)$ is given by $[R^\star]U = \{s \in A_* \mid R^\star[s] \subseteq U\}$. This map has some nice properties that we list for future use.

PROPOSITION 1.5. *Let $\mathbb{A}$ be a boolean algebra with operators. For any $a \in A$, we have that*

$$[R^\star]\widehat{a} = \bigcap_{\blacklozenge \in CD(\mathbb{A})} \widehat{\blacksquare a}. \qquad \text{(viii)}$$

Hence, $[R^\star]C$ is closed for an arbitrary closed set $C \subseteq A_*$.

PROOF. Fix an element a of the BAO $\mathbb{A}$. First assume that u is an ultrafilter that belongs to $\widehat{\blacksquare a}$ for all compound diamonds $\blacklozenge$. That is, we have $\blacksquare a \in u$ for all $\blacklozenge$. Now consider an arbitrary ultrafilter v such that $R^\star uv$. By (vii) we find that $a \in v$, whence $v \in \widehat{a}$. Since v was arbitrary, this gives that $u \in [R^\star]\widehat{a}$, and thus shows that $[R^\star]\widehat{a} \supseteq \bigcap\{\widehat{\blacksquare a} \mid \blacklozenge \in CD(\mathbb{A})\}$.

For the other inclusion, suppose that u does *not* belong to the right hand side of (viii). Then for some compound diamond $\blacklozenge$ we have that $\blacksquare a \notin u$; that is, $\blacklozenge{-a} \in u$. Thus we can find an ultrafilter v such that $R_\blacklozenge uv$ and $-a \in v$. Clearly then, by Proposition 1.4.2 we have that $R^\star uv$ and $v \notin \widehat{a}$, revealing that u does not belong to the left hand side of (viii) either.

The second part of the Proposition is immediate from the first part and the observation that any map $[R]$ distributes over arbitrary intersections. ■

2. Results

Our first main result characterizes the simple algebras as the ones of which the dual frame is topo-generated from each point:

THEOREM 1. *Let $\mathbb{A}$ be a boolean algebra with operators. Then $\mathbb{A}$ is simple if and only if $T_{\mathbb{A}_*} = A_*$.*

PROOF. For the direction from left to right, we leave it for the reader to verify that if $s \notin T_{\mathbb{A}_*}$, then $R^\star[s] \neq A_*$ corresponds to a non-trivial modal filter of $(\mathbb{A}_*)^* \cong \mathbb{A}$. This shows that if not every ultrafilter of $\mathbb{A}$ is a topo-root of $\mathbb{A}_*$, then $\mathbb{A}$ is not simple.

For the other direction, suppose that $\mathbb{A}$ is not simple. Then $\mathbb{A}_*$ has a closed, hereditary subset $B \neq A_*$. Take an arbitrary point $s \in B$. Then $R^\omega[s] \subseteq B$ since B is hereditary, whence $R^\star[s] = \overline{R^\omega[s]} \subseteq B$ since B is closed. It follows that $R^\star[s] \neq A_*$, so s is not a topo-root of $\mathbb{A}_*$. ■

Our second result gives a similar characterization of the subdirectly irreducible algebras.

THEOREM 2. *Let $\mathbb{A}$ be a boolean algebra with operators. Then the following are equivalent:*

1. *$\mathbb{A}$ is subdirectly irreducible;*
2. *$T_{\mathbb{A}_*}$ is open and non-empty;*
3. *$T_{\mathbb{A}_*}$ has non-empty interior;*
4. *there is a clopen/admissible set of topo-roots in $\mathbb{A}_*$.*

PROOF. The equivalence of (3) and (4) is standard, and the implication (2 $\Rightarrow$ 3) is trivial; hence, it is left to prove that the first two statements are equivalent, and that the third one implies the second. In this proof we let $K_{\mathbb{A}_*} = A_* \setminus T_{\mathbb{A}_*}$ denote the complement of $T_{\mathbb{A}_*}$.

For the implication (1 $\Rightarrow$ 2), assume that $\mathbb{A}$ is s.i., then by Fact 1.1 we may assume that $\mathbb{A}$ has an opremum c.

We will first prove that $\mathbb{A}_+$ is topo-rooted. Take an arbitrary point $u \in K_{\mathbb{A}_*}$; that is, we have that $R^\star[u] \neq A_*$, and since $R^\star[u]$ is closed, there must be a clopen $\widehat{a} \neq A_*$ with $R^\star[u] \subseteq \widehat{a}$. Then from $a \neq \top$ and the properties of the opremum c it follows that $\blacksquare a \leq c$ for some compound modality $\blacklozenge$. But as $R^\star[u] \subseteq \widehat{a}$ it holds that $u \in [R^\star]\widehat{a}$, so using (viii) we find that $u \in \widehat{\blacksquare a}$. Thus by $\widehat{\blacksquare a} \subseteq \widehat{c}$ we obtain that $u \in \widehat{c}$. Since u was an arbitrary element of $K_{\mathbb{A}_*}$, this shows that

$$K_{\mathbb{A}_*} \subseteq \widehat{c}. \tag{ix}$$

But c is an opremum, and hence, *smaller* than the top element of $\mathbb{A}$. This means that $K_{\mathbb{A}_*}$ is a *proper* subset of A_*, and thus its complement $T_{\mathbb{A}_*}$ is non-empty.

Our second aim is to prove that $T_{\mathbb{A}_*}$ is open, or, equivalently, that $K_{\mathbb{A}_*}$ is closed. We will first show that

$$K_{\mathbb{A}_*} = [R^\star]K_{\mathbb{A}_*}. \tag{x}$$

By reflexivity of $R^\star$ it is immediate that $[R^\star]K_{\mathbb{A}_*} \subseteq K_{\mathbb{A}_*}$. For the other inclusion, suppose that u belongs to $K_{\mathbb{A}_*}$, and take an arbitrary ultrafilter v such that $R^\star uv$. Suppose for contradiction that v is a topo-root of $\mathbb{A}_*$, that is, suppose that every ultrafilter is $R^\star$-reachable from v. From this and the results in Proposition 1.4 it would follow immediately that u is a topo-root of $\mathbb{A}_*$ as well, which contradicts the fact that $u \in K_{\mathbb{A}_*}$. It follows that no $R^\star$-successor of u can be a topo-root of u; in other words, we see that $R^\star[u] \subseteq K_{\mathbb{A}_*}$, and thus that $u \in [R^\star]K_{\mathbb{A}_*}$. This proves (x).

Now we claim that in fact,

$$K_{\mathbb{A}_*} \;=\; [R^\star]\widehat{c}. \tag{xi}$$

For the inclusion from left to right, first observe that it follows from (x), (ix) and the monotonicity of the operation $[R^\star]$, that $K_{\mathbb{A}_*} = [R^\star]K_{\mathbb{A}_*} \subseteq [R^\star]\widehat{c}$. In order to establish the converse inclusion of (xi), consider an arbitrary point u in $[R^\star]\widehat{c}$ and suppose, for contradiction, that u does *not* belong to $K_{\mathbb{A}_*}$. That is, u is a topo-root of $\mathbb{A}_*$, so, we have that $R^\star uv$ for *every* ultrafilter v. But then $u \in [R^\star]\widehat{c}$ implies that every ultrafilter v belongs to $\widehat{c}$. This gives the desired contradiction with the fact that $\widehat{c}$, being an opremum of $(\mathbb{A}_*)^*$, must be a *proper* subset of A_*. Thus we find that indeed, $[R^\star]\widehat{c} \subseteq K_{\mathbb{A}_*}$, and we have proved (xi).

Finally, observe that it immediately follows from (xi) and Proposition 1.5 that $K_{\mathbb{A}_*}$ is closed.

For the converse implication, i.e., $(2 \Rightarrow 1)$, assume that $T_{\mathbb{A}_*}$ is open and non-empty. It is not difficult to see that this implies that $K_{\mathbb{A}_*}$ is a nontrivial closed and hereditary subset of A_*. We claim that it is in fact the largest such set.

For, let $J \subset A_*$ be closed and hereditary. Suppose for contradiction that J is not contained in $K_{\mathbb{A}_*}$, then there is an ultrafilter $u \in J \setminus K_{\mathbb{A}_*}$. From $u \in J$ and the assumptions on J it easily follows that $R^\star[u] \subseteq J$. But then $R^\star[u]$ is a proper subset of A_*; from this we infer that u is not a topo-root of $\mathbb{A}_*$; that is, we find $u \in K_{\mathbb{A}_*}$. This shows that J is a subset of $K_{\mathbb{A}_*}$ after all. Hence, $K_{\mathbb{A}_*}$ is indeed the *largest* nontrivial, closed and hereditary subset of A_*. But then it follows from Fact 1.1 that $\mathbb{A}$ is s.i.

Finally, we prove the implication $(3 \Rightarrow 2)$. Assuming that $\widehat{a}$ is a non-empty set of topo-roots, we will prove that

$$T_{\mathbb{A}_*} \;=\; \bigcup_{\blacklozenge \in CD(\mathbb{A})} \widehat{\blacklozenge a}, \tag{xii}$$

which clearly shows that $T_{\mathbb{A}_*}$ is open. For the inclusion $\subseteq$, consider an arbitrary topo-root u. Since $\widehat{a}$ is non-empty, we may take an ultrafilter v with $a \in v$. Because u is a topo-root it follows that $R^\star uv$, so we have that $\blacklozenge a \in u$ for some compound diamond $\blacklozenge$. This immediately gives that $u \in \widehat{\blacklozenge a}$.

For the converse inclusion, take a compound diamond $\blacklozenge$ and an ultrafilter u in $\widehat{\blacklozenge a}$. Using standard reasoning in modal duality theory, we can find an ultrafilter v such that $R_\blacklozenge uv$ and $v \in \widehat{a}$. It follows from $\widehat{a} \subseteq T_{\mathbb{A}_*}$ that v is a topo-root, so by $R_{\blacklozenge} \subseteq R^{\star}$ and transitivity of $R^{\star}$ we may infer that u is a topo-root as well. This finishes the proof of (xii), and thus, the proof of (3 $\Rightarrow$ 2). ∎

As corollaries of the last theorem we obtain some well known results showing that in some special cases, nicer characterizations are indeed possible.

We call a boolean algebra with operators *ω-transitive* if there is some compound diamond $\lozenge$ such that $\blacklozenge a \leq \lozenge a$ for all compound diamonds $\blacklozenge$ and all a in $\mathbb{A}$. (With some authors, this property goes under the name of *weak* transitivity). Recall that for an algebra $\mathbb{A}$, we let $I_{\mathbb{A}_*}$ denote the collection of roots of the dual structure.

COROLLARY 1. *Let $\mathbb{A}$ be a ω-transitive boolean algebra with operators. Then $\mathbb{A}$ is simple iff $I_{\mathbb{A}_*} = A_*$, and subdirectly irreducible iff $I_{\mathbb{A}_*}$ is non-empty and open iff there is an admissible/clopen set of roots in $\mathbb{A}_*$.*

PROOF. Suppose that $\lozenge$ is a compound diamond of $\mathbb{A}$ such that $\blacklozenge a \leq \lozenge a$ for all compound diamonds $\blacklozenge$ and all a in $\mathbb{A}$. It is easy to verify that in this case we have $R^{\omega} = R_{\lozenge}$; but since $R_{\lozenge}[u]$ is closed for every ultrafilter u, it follows from Proposition 1.4.5 that $R^{\star} = R_{\lozenge} = R^{\omega}$. This means that $I_{\mathbb{A}_*} = T_{\mathbb{A}_*}$, or in words: the roots and the topo-roots of $\mathbb{A}_*$ *coincide*. Thus the results follows immediate from the Theorems 1 and 2. ∎

COROLLARY 2. *Let $\mathbb{A}$ be a finite boolean algebra with operators. Then $\mathbb{A}$ is subdirectly irreducible iff $\mathbb{A}_*$ is rooted.*

PROOF. It is easy to see that finite BAOs are ω-transitive. Hence, the result follows from Corollary 1 and the observation that if $\mathbb{A}$ is finite then any subset of A_* is open. ∎

Finally, we show how the earlier mentioned result (i) of Sambin can be obtained as a corollary to our results.

COROLLARY 3. *Let $\mathbb{A}$ be a boolean algebra with operators such that $I_{\mathbb{A}_*}$ has non-empty interior. Then $\mathbb{A}$ is subdirectly irreducible.*

PROOF. Immediate by Theorem 2 (3 $\Rightarrow$ 1) and the observation that $I_{\mathbb{A}_*} \subseteq T_{\mathbb{A}_*}$. ∎

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