A modal logic of relations

(with an appendix containing remarks on mosaics and step-by-step by R. Hirsch, I. Hodkinson, M. Marx, Sz. Mikulás and M. Reynolds)

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Abstract

Treating existential quantifiers as modal diamonds, we study the n-variable fragment L_n of first-order logic, as if it were a modal formalism. In order to deal with atomic formulas adequately, to the modal version of the language we add operators corresponding to variable substitution.

Since every modal language comes with an abstract Kripke-style semantics, this modal viewpoint on L_n provides an alternative, far more general semantics for the latter. One may impose conditions on the Kripke models, for instance approximating the standard Tarskian semantics. In this way one finds that some theorems of first-order logic are 'more valid' than others.

As an example, we consider a class of generalized assignment frames called *local cubes*; here the basic idea is that only certain assignments are admissible. We show that the theory of this class is finitely axiomatizable and decidable.

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1 Introduction

It is the aim of this paper to give an introduction to the modal perspective on first-order logic that has been investigated in recent years by some Dutch and Hungarian researchers. This modal point of view was first developed in (Venema, 1989), but it builds on insights developed in the algebraic theory of relations, cf. (Henkin et al., 1971 1985) or (Németi, 1986). The research has lead to a number of dissertations (Marx, 1995; Mikulás, 1995; Venema, 1992) and articles, of which we only mention (Andréka et al., 1995; van Benthem, to appear; Venema, 1995a). Although we prove some new results here (concerning the finite axiomatizability and decidability of modal versions of first-order logic), the intended contribution of the paper lies in a perspicuous *presentation* of the modal perspective on first-order logic and the proof techniques rather than in the importance of these results.

The starting point of this research is the observation that modal operators resemble quantifiers in the predicate calculus. These resemblances were already noticed early in the modern development of modal logic, and in fact form one of the basic insights underlying correspondence theory of modal logic. In general, correspondence theory studies the relation between modal and classical languages as formalisms for describing the same classes of relational (Kripke) structures, cf. (van Benthem, 1984). The usual direction in correspondence theory is to start with a modal language, and then search for a fragment of first-order logic which is expressively equivalent to it. The aim that we set ourselves is the converse, namely to devise and study modal formalisms that are as expressive as first-order logic itself.

The basic idea underlying our approach is that we will simply treat the existential quantifier as if it *were* a modal diamond. The motivation for this is that in the standard Tarskian truth definition of first-order logic, the statement '$\exists v_i \varphi$ is true in model $\mathfrak{M}$ under assignment s' holds if and only if there is an assignment s' which differs from s at most for the variable v_i, such that φ is true in $\mathfrak{M}$ under s'. The semantic connection with modal logic lies in the fact, that we can see this relation $\equiv_i$ between assignments of 'differing at most with respect to the variable v_i' as an *accessibility relation* between assignments. This presupposes that the assignments are the possible worlds in our intended semantics; hence, we arrive at multi-dimensional modal logic by identifying assignments with tuples over the base set U of the model. Since n-ary relations over a set U are sets of n-tuples over U, in our n-dimensional modal formalisms, such relations are precisely the extensions of the propositional variables — this explains the title of the paper.

First-order logic has been discussed from such a modal perspective before — as an example we mention the system PREDBOX of (Kuhn, 1980) (this paper also discusses some other approaches from the literature). PREDBOX is a *sorted* modal logic: the language contains disjoint sets of propositional variables for every natural number n, corresponding to n-adic predicate symbols of first-order logic. Kuhn's system is closely related to Quine's *predicate-functor logic*, cf. (Quine, 1971). We will refrain from developing such sorted formalisms; as a consequence of this decision, the counterpart of our multi-modal formalisms are slightly non-standard versions of first-order logic (we come back to this issue in section 6). The reason for this does not lie in a dislike of sorted modal logic on our side, but rather from our wish to stay close to the standard *algebraic* treatment of relations and first-order logic. For an overview of the algebraic approach towards relations we refer to (Németi, 1991); the most influential algebraic approaches (which actually predates the development of first-order logic as we know it), are those of Tarski and his pupils in the theory of *cylindric algebras*, cf. (Henkin et al., 1971 1985), and of Halmos, cf. (Halmos, 1962) in the theory of Polyadic

Algebras. For lack of space, the connection with these algebras will not be made precise in this paper; the interested reader will find details in (Venema, 1995a; Venema, 1995b).

The modal perspective on first-order logic has some nice features. We mention the inspiring insight, stemming from (Németi, 1986) and advocated in (van Benthem, to appear), that it allows us to play more freely with the semantics for first-order logic. For, recall that *any* modal language comes with a Kripke semantics, in which the frames are arbitrary sets endowed with an $n+1$-ary accessibility relation for each n-adic operator in the language. From this viewpoint, the traditional multi-dimensional 'assignment frames' are only a special *subclass* of the class of all possible frames for our modal formalisms. In other interesting semantic classes we may consider assignment frames in which not necessarily each tuple over the base set U is 'available' as an assignment. The advantage of these *generalized assignment* frames is that their logic may have much nicer properties (such as decidability, orthodox finite axiomatizability and Craig interpolation property) than the logic of the original full assignment frames. It is the second aim of the paper to give accessible, detailed proofs of two new results in this direction: we define, for a 'modal predicate language' *MLR* (with so-called substitution operators), a class of generalized assignment frames called 'local cubes', and we will prove that the arising logic is finitely axiomatizable (Theorem 4.2) and decidable (Theorem 5.1).

In the first sections of the paper, we confine ourselves to *finite-dimensional* modal logics, these being the modal versions of finite-variable fragments of first-order logic. The reason for this restriction is that the finite-dimensional case allows a quite perspicuous exposition, whereas in the 'modalization' of ordinary first-order logic (in an infinite-dimensional formalism) there are some issues involved (like the rank of predicate symbols) that are better discussed when the basic picture is clear.

Convention 1.1 *Throughout this paper, n denotes an arbitrary but fixed natural number. Unless stated otherwise, we assume that $n > 2$.*

Overview of paper. In the next section we introduce the syntax and the multi-dimensional semantics of the modal predicate languages that will be studied. In section 3 we discuss the more general modal semantics of our modal system, defining various classes of generalized assignment frames. In the sections 4 and 5 we single out one of these classes and prove a completeness theorem and a decidability theorem for it. Finally, in section 6 we show how to extend the finite-dimensional set-up, that we have been working in till then, to the ω-dimensional case.

Appendix containing remarks on mosaics and step-by-step. During the life of this article as a preprint, new results on the central technique in this paper –the mosaic method– where obtained at the algebraic logic group in London. We have included a research note from this group containing remarks on this technique as an appendix to this paper. The appendix is largely selfcontained but is best read in conjunction with the present paper. It contains alternative proofs of our main results featuring two novelties:

- the use of "step-by-step" arguments is largely (in the case of a single formula, completely) avoided;
- it provides a simple proof of the finite base property for locally cubic modal logic of relations, a result due to Andréka and Németi (cf. Remark 5.2).

Preliminaries We assume familiarity with some basic notions in the model theory and axiomatics of modal logic. In order to avoid misunderstandings, we briefly review some of these.

We consider modal similarity types which contain constants and diamonds, i.e., unary modalities. For a similarity type S, the *basic derivation system* K_S contains as its axioms: some finite set CT of Classical Tautologies that is complete with respect to classical propositional logic, and the axiom of Distribution:

$(DB_{\Box_i})$ $\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$,

for every $\Box_i$ of the similarity type S; the derivation rules of K_S are Modus Ponens, Universal Generalization and Substitution:

(MP) from φ and $\varphi \to \psi$ infer ψ

$(UG_{\Box_i})$ from φ infer $\Box_i\varphi$

(SUB) from φ infer $\sigma\varphi$.

(for every $\Box_i \in S$ and every substitution σ of formulas for propositional variables in formulas).

A *derivation* in a derivation system D is a finite sequence $\varphi_1, \ldots, \varphi_n$ such that every φ_i is either an axiom of D or obtainable from earlier φ_j's by application of a rule in D. A D-theorem is any formula that can appear as the last item of a derivation in D; theoremhood of a formula φ in D is denoted as $\vdash_D \varphi$. Derivation systems are sometimes identified with their sets of theorems. A formula φ is *derivable* in a derivation system D from a set of formulas Σ, notation: $\Sigma \vdash_D \varphi$ if there are $\sigma_1, \ldots, \sigma_n$ in Σ with $\vdash_D (\sigma_1 \wedge \ldots \wedge \sigma_n) \to \varphi$.

A formula φ is D-consistent if its negation is not a theorem; a set of formulas is consistent if the conjunction of any finite subset is D-consistent and maximal D-consistent if it is consistent while it has no proper D-consistent extension (in the same language). We abbreviate 'maximal consistent set' by 'MCS'.

Given a class K of frames for a modal similarity type S, we denote the S-theory of K, i.e., the set of formulas valid in K, with $\Theta_S(\mathsf{K})$. We say that a formula φ is a semantic K-consequence of a set of formulas Σ, notation: $\Sigma \models_{\mathsf{K}} \varphi$, if for every model based on a frame in K, and for every state s in this model we have that φ is true at s if every $\sigma \in \Sigma$ is true at s.

A derivation system D is strongly sound with respect to a class of frames K if $\Sigma \vdash_D \varphi$ implies $\Sigma \models_{\mathsf{K}} \varphi$ for every Σ and φ, and strongly complete if the converse implication holds.

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2 Modalizing first-order logic

In order to explain how the n-variable fragment of the predicate calculus of n-ary relations can be treated as a modal formalism, let us start with an intuitive exposition, and defer precise definitions to the end of this section; until then, the reader can think of a first–order language without constants or function symbols, in which all predicates are n-adic, and which only uses the first n variables $\{v_0, \ldots, v_{n-1}\}$; we have the standard Tarskian semantics in mind. Consider the basic declarative statement in first-order logic concerning the truth of a formula in a model under an assignment s:

$$\mathfrak{M} \models \varphi\ [s]. \tag{1}$$

The basic observation underlying our approach, is that we can read (1) from an abstract modal perspective as: "the formula φ is true in $\mathfrak{M}$ *at the possible world s*". Note that as we have only

n variables at our disposal, we can identify assignments with maps: $n(=\{0,\ldots,n-1\}) \to U$, or equivalently, with n-tuples over the domain U of the structure $\mathfrak{M}$ — we will denote the set of such n-tuples with ${}^nU$. Thus in this setting of multi-dimensional modal logic, the universe of a modal model will be of the form ${}^nU$ for some base set U. Now recall that the truth definition of the quantifiers reads as follows:

$$\mathfrak{M} \models \exists v_i\varphi\ [s] \iff \text{there is an } u \in U \text{ such that } \mathfrak{M} \models \varphi\ [s^i_u], \tag{2}$$

where s^i_u is the assignment defined by

$$s^i_u(k) = \begin{cases} u & \text{if } k = i \\ s(k) & \text{otherwise.} \end{cases} \tag{3}$$

We can replace (2) with the more 'modal' equivalent,

$$\mathfrak{M} \models \exists v_i\varphi\ [s] \iff \text{there is an assignment } s' \text{ with } s \equiv_i s' \text{ and } \mathfrak{M} \models \varphi\ [s'],$$

where $\equiv_i$ is given by

$$s \equiv_i s' \iff \text{for all } j \neq i,\ s_j = s'_j. \tag{4}$$

In other words: existential quantification behaves like a modal *diamond*, having $\equiv_i$ as its *accessibility relation*.

As the semantics of the boolean connectives in the predicate calculus is the same as in modal logic, this shows that the inductive clauses in the truth definition of first-order logic fit neatly in a modal approach. So let us now concentrate on the atomic formulas. To start with, we observe that *identity* formulas do not cause any problem: a formula $v_i = v_j$, with truth definition

$$\mathfrak{M} \models v_i = v_j\ [s] \iff s \in Id_{ij},$$

can be seen as a modal *constant*. Here Id_{ij} is defined by

$$s \in Id_{ij} \iff s_i = s_j. \tag{5}$$

The case of the other atomic formulas is more involved, however; since we confined ourselves to the calculus of n-adic relations and do not have constants or function symbols, an atomic predicate formula is of the form $Pv_{\sigma(0)}\ldots v_{\sigma(n-1)}$, where σ is a map: $n \to n$. Such maps will be called *transformations* in the sequel. In the model theory of first-order logic the predicate P will be interpreted as a subset of ${}^nU$; precisely how the propositional variables are treated in n-dimensional modal logic by a valuation. Therefore, we will identify the set of propositional variables of the modal formalism with the set of predicate symbols of our first-order language. In this way, we obtain a modal reading of (1) for the case where φ is the atomic formula $Pv_0\ldots v_{n-1}$: $\mathfrak{M} \models Pv_0\ldots v_{n-1}[s]$ iff s belongs to the interpretation of P. However, as a consequence of this approach our set-up will not enjoy a one-to-one correspondence between atomic first-order formulas and atomic modal ones: the atomic formula $Pv_{\sigma(0)}\ldots v_{\sigma(n-1)}$ will correspond to the modal atom p only if σ is the identity function on n. For the cases where σ is not the identity map we have to find some kind of a modal solution. There are many options here.

Let us first assume that we are working in a first-order language with identity, i.e., we have all atomic formulas of the form $v_i = v_j$ $(i, j < n)$. Atomic formulas with a *multiple* occurrence of a variable can be rewritten as formulas with only 'unproblematic' atomic subformulas, for instance

$$\begin{array}{lll} Pv_1v_0v_0 & \iff & \exists v_2\,(v_2 = v_0 \;\wedge\; Pv_1v_2v_2) \\ & \iff & \exists v_2\,(v_2 = v_0 \;\wedge\; \exists v_0\,(v_0 = v_1 \;\wedge\; Pv_0v_2v_2)) \\ & \iff & \exists v_2\,(v_2 = v_0 \;\wedge\; \exists v_0\,(v_0 = v_1 \;\wedge\; \exists v_1\,(v_1 = v_2 \;\wedge\; Pv_0v_1v_2))). \end{array}$$

This leaves the case what to do with atoms of the form $Pv_{\sigma(0)} \ldots v_{\sigma(n-1)}$, where σ is a permutation of n, or in other words, atomic formulas where variables have been substituted *simultaneously*. The previous trick does not work here: for instance, to write an equivalent of the formula $Pv_1v_0v_2$ one needs *extra* variables as buffers, for instance, as in the formula

$$\exists v_3 \exists v_4 (v_3 = v_0 \wedge v_4 = v_1 \wedge \exists v_0 \exists v_1 (v_0 = v_4 \wedge v_1 = v_3 \wedge Pv_0v_1v_2)),$$

One might consider a solution where a predicate P is translated into *various* modal propositional variables p_σ, one for every permutation σ of n, but this is not very elegant. One might also forget about simultaneous substitutions and confine oneself to a *fragment* of n-variable logic where all atomic predicate formulas are of the form $Pv_0 \ldots v_{n-1}$. This fragment of *restricted* first-order logic is defined below and formed (in its modal disguise *CML*), the main subject of the paper (Venema, 1995a). A third solution is to take substitution seriously, so to speak, by adding special 'substitution operators' to the language. The crucial observation is that for any transformation $\sigma \in {}^n n$, we have that

$$\mathfrak{M} \models Pv_{\sigma(0)} \ldots v_{\sigma(n-1)}\;[s] \iff \mathfrak{M} \models Pv_0 \ldots v_{n-1}\;[s \circ \sigma], \tag{6}$$

where $s \circ \sigma$ is the composition of σ and s (recall that s is a map: $n \to U$). So, if we define the relation $\bowtie_\sigma \subseteq {}^n U \times {}^n U$ by

$$s \bowtie_\sigma t \iff t = s \circ \sigma, \tag{7}$$

we have rephrased (6) in terms of an accessibility relation (in fact, a function):

$$\mathfrak{M} \models Pv_{\sigma(0)} \ldots v_{\sigma(n-1)}\;[s] \iff \text{there is a } t \text{ with } s \bowtie_\sigma t \text{ and } \mathfrak{M} \models Pv_0 \ldots v_{n-1}\;[t].$$

So if we add an operator $\bigcirc_\sigma$ to the modal language for every map σ in ${}^n n$, with $\bowtie_\sigma$ as its intended accessibility relation, we have found the desired modal equivalent for a formula $Pv_{\sigma(0)} \ldots v_{\sigma(n-1)}$ in the form $\bigcirc_\sigma p$.

Finally, one could somewhat simplify a language with these substitution operators. To explain the idea, we need the notions of a transposition and a replacement. A *transposition* on n is a permutation on n swapping two elements and leaving every other element on its place. To be more precise, if $i, j < n$, then the transposition $[i, j]^n : n \to n$ is defined by:

$$[i, j]^n(k) = \begin{cases} j & \text{if } k = i \\ i & \text{if } k = j \\ k & \text{otherwise.} \end{cases} \tag{8}$$

A *replacement* of n is a function leaving all elements in their place, except for one: if $i, j < n$, then we define the replacement $[i/j]^n : n \to n$ by

$$[i/j]^n(k) = \begin{cases} j & \text{if } k = i \\ k & \text{otherwise.} \end{cases} \tag{9}$$

(Both for transpositions and replacements, we may drop the superscript n if no confusions arises.)

It is easily verified that every transformation σ of a (finite) ordinal n is a composition of transpositions and replacements. As we may infer from (6) that

$$\mathfrak{M} \models Pv_{\tau\circ\sigma(0)} \dots v_{\tau\circ\sigma(n-1)} \ [s] \iff \mathfrak{M} \models Pv_0 \dots v_{n-1} \ [(s \circ \tau) \circ \sigma],$$

we only need modal operators for transpositions and replacements: if $\sigma = \tau_0 \circ \ldots \circ \tau_k$, we may consider $\bigcirc_\sigma$ as an *abbreviated* operator:

$$\bigcirc_\sigma \varphi \ := \ \bigcirc_{\tau_0} \cdots \bigcirc_{\tau_k} \varphi.$$

Although we will not discuss such simplified languages here in detail, it is convenient to have a special notation for the substitution operators of transpositions. We define

$$\otimes_{ij}\varphi \ := \ \bigcirc_{[i,j]}\varphi \tag{10}$$

$$\bigcirc_{ij}\varphi \ := \ \bigcirc_{[i/j]}\varphi. \tag{11}$$

The modal similarity type $CMML_n$, discussed in (Venema, 1995b), contains as operators precisely these replacement and transposition operators (besides the diagonal constants and the cylindrification diamonds). Here we will mainly treat the extension of $CMML_n$ with all substitution operators.

Definition 2.1 *Let n be an arbitrary but fixed natural number. The alphabet of L_n and of L_n^r consists of a set of variables $\{v_i \mid i < n\}$, it has a countable set of n-adic relation symbols $(P_0, P_1, \ldots)$, identity (=), the boolean connectives $\neg, \vee$ and the quantifiers $\exists v_i$. Formulas of L_n and L_n^r are defined as usual in first-order logic, with the restriction that the atomic formulas of L_n^r are of the form $v_i = v_j$ or $P_l(v_0 \ldots v_{n-1})$; for L_n, we allow all atomic formulas of the form $P_l(v_{\sigma(0)} \ldots v_{\sigma(n-1)})$, where σ is a transformation of n.*

A first-order structure for $L_n^{(r)}$ is a pair $\mathfrak{M} = (U, V)$ such that U is a set called the domain of the structure and V is an interpretation function mapping every P to a subset of ${}^nU$. Truth of a formula in a model is defined as usual: let s be in ${}^nU$, then

$\mathfrak{M} \models v_i = v_j \ [s]$	*if*	$s_i = s_j$,
$\mathfrak{M} \models P(v_0 \ldots v_{n-1}) \ [s]$	*if*	$s \in V(P)$,
$\mathfrak{M} \models P(v_{\sigma 0} v_{\sigma 1} \ldots v_{\sigma n}) \ [s]$	*if*	$s \circ \sigma \ (= (s_{\sigma(0)}, \ldots, s_{\sigma(n-1)})) \in V(P)$,
$\mathfrak{M} \models \exists v_i \varphi \ [s]$	*if*	*there is a* t *with* $s \equiv_i t$ *and* $\mathfrak{M} \models \varphi \ [t]$,
etc.		

An L_n-formula φ is valid in $\mathfrak{M}$ (notation: $\mathfrak{M} \models \varphi$) if $\mathfrak{M} \models \varphi \ [s]$ for all $s \in {}^nU$, first-order valid (notation: $\models_{fo} \varphi$) if it is valid in every first-order structure of L_n. The same definition applies to L_n^r.

From now on, we will concentrate on the *modal* versions of L_n^r and L_n, which are given in the following definition:

Definition 2.2 *Let n be an arbitrary but fixed natural number. MLR_n (short for:* **modal language of relations***) is the modal similarity type having constants $\iota\delta_{ij}$ and diamonds $\diamond_i$, $\bigcirc_\sigma$ (for all $i, j < n, \sigma \in {}^n n$). Given a countable set of propositional variables $p_0, p_1, \ldots,$ the language of n-dimensional MLR-formulas, or shortly, MLR_n-formulas, is built up as usual: the atomic formulas are the (modal or boolean) constants and the propositional variables, and a formula is either atomic or of the form $\neg\varphi$, $\varphi \vee \psi$, $\diamond_i\varphi$ or $\bigcirc_\sigma\varphi$, where φ, ψ are formulas. We abbreviate $\Box_i\varphi \equiv \neg\diamond_i\neg\varphi$.*

CML_n is the similarity type of **cylindric modal logic***, i.e., the fragment of MLR_n-formulas in which no substitution operator $\bigcirc_\sigma$ occurs.*

CML_n and MLR_n are interpreted in first-order structures in the obvious way; for instance we have

$$\begin{array}{rcl} \mathfrak{M}, s \Vdash \iota\delta_{ij} & \iff & s_i = s_j \\ \mathfrak{M}, s \Vdash \bigcirc_\sigma\varphi & \iff & \mathfrak{M}, s \circ \sigma \Vdash \varphi \\ (& \iff & \text{there is a } t \text{ with } s \bowtie_\sigma t \text{ and } \mathfrak{M}, t \Vdash \varphi) \\ \mathfrak{M}, s \Vdash \diamond_i\varphi & \iff & \text{there is a } t \text{ with } s \equiv_i t \text{ and } \mathfrak{M}, t \Vdash \varphi. \end{array}$$

A MLR_n-formula φ is valid*, notation: $\mathsf{C}_n \models \varphi$, if it valid in all first-order structures*[1]*.*

The modal disguise of L_n in MLR_n and L_n^r in CML is so thin, that we feel free to give the details below without proof, or too many comments. Also, we confine ourselves to the translation mapping first-order formulas to modal ones; the reader is invited to provide a translation in the other direction.

Definition 2.3 *Let $(\cdot)^t$ be the following translation from L_n to MLR_n:*

$$\begin{array}{lcl} (Pv_0 \ldots v_{n-1})^t & = & p \\ (Pv_{\sigma(0)} \ldots v_{\sigma(n-1)})^t & = & \bigcirc_\sigma p \\ (v_i = v_j)^t & = & \iota\delta_{ij} \\ (\neg\varphi)^t & = & \neg\varphi^t \\ (\varphi \vee \psi)^t & = & \varphi^t \vee \psi^t \\ (\exists v_i\varphi)^t & = & \diamond_i\varphi^t. \end{array}$$

Proposition 2.4 *Let φ be a formula in L_n, then*

$$\models_{fo} \varphi \iff \mathsf{C}_n \models \varphi^t.$$

The above translation allows us to see L_n^r and CML_n as syntactic variants: $(\cdot)^t$ is easily seen to be an *isomorphism* between the formula algebras of L_n^r and CML_n. Note that in the case of L_n versus MLR_n, we face a different situation: where in MLR the simultaneous substitution of two variables for each other is a *primitive* operator, in first-order logic it can only be defined by induction. Nevertheless, we could easily define a translation mapping MLR_n-formulas to equivalent L_n^r-formulas.

[1] This definition is slightly ambiguous for the *CML*-fragment of *MLR*, i.e., the formulas without substitution operators. For, let m, n be two natural numbers with $m < n$, and φ a CML_m-formula. Then by definition φ is also a CML_n-formula. So, when checking the validity of φ, in principle it might make a difference whether one considers assignments in ${}^mU$ or in ${}^nU$. Our definition can be justified by a fairly straightforward proof showing this not to be the case.

3 Abstract and generalized assignment frames

In this section, we will discuss the *modal* semantics of MLR_n (and hence, of L_n^r and L_n). As we mentioned in the introduction, one of the motivations for 'modalizing' the predicate calculus is that it allows a generalization of the semantics of first-order logic, thus offering a wider perspective on the standard Tarskian semantics.

Recall that in our intended n-dimensional structures for MLR_n, the accessibility relations Id_{ij}, $\equiv_i$ and $\bowtie_\sigma$ for $\iota\delta_{ij}$, $\Diamond_i$ and $\bigcirc_\sigma$, respectively, are given by

$$\begin{array}{lll} s \in Id_{ij} & \iff & s_i = s_j \\ s \equiv_i t & \iff & \text{for all } j \neq i\text{: } s_j = t_j \\ s \bowtie_\sigma t & \iff & t = s \circ \sigma. \end{array}$$

We abstract away from this background in the relational or Kripke semantics for our languages, and define models in which the universe is an *arbitrary* set and the accessibility relations are *arbitrary* relations (of the appropriate arity).

Definition 3.1 *A* MLR_n**-frame** *is a tuple* $(W, T_i, E_{ij}, F_\sigma)_{i,j<n,\sigma\in{}^nn}$ *such that every* E_{ij} *is a subset of the universe* W*, and every* T_i *and every* F_σ *a binary relation on* W*. A* MLR_n**-model** *is a pair* $\mathfrak{M} = (\mathfrak{F}, V)$ *with* $\mathfrak{F}$ *a* MLR_n*-frame and* V *a* **valuation***, i.e., a map assigning subsets of* W *to propositional variables.*
For such models, **truth** *of a formula at a state is defined via the usual modal induction:*

$$\begin{array}{lll} \mathfrak{M}, w \Vdash p & \iff & w \in V(p), \\ \mathfrak{M}, w \Vdash \iota\delta_{ij} & \iff & w \in E_{ij}, \\ \mathfrak{M}, w \Vdash \neg\varphi & \iff & \mathfrak{M}, w \nVdash \varphi, \\ \mathfrak{M}, w \Vdash \varphi \vee \psi & \iff & \mathfrak{M}, w \Vdash \varphi \textit{ or } \mathfrak{M}, w \Vdash \psi, \\ \mathfrak{M}, w \Vdash \Diamond_i\varphi & \iff & \textit{there is a } v \textit{ with } T_iwv \;\&\; \mathfrak{M}, v \Vdash \varphi, \\ \mathfrak{M}, w \Vdash \bigcirc_\sigma\varphi & \iff & \textit{there is a } v \textit{ with } F_\sigma wv \;\&\; \mathfrak{M}, v \Vdash \varphi. \end{array}$$

We will also employ the usual notion of **validity***.*

In this very general semantics, states (i.e., elements of the universe) are no longer real assignments, but rather, abstractions thereof. First-order logic now really has become a polymodal logic, with quantification and substitution diamonds. It is interesting and instructive to see how familiar laws of the predicate calculus behave in this new set-up (cf. also (van Benthem, to appear)). For instance, the axiom schema

$$\varphi \to \exists v_i\varphi \tag{12}$$

will be valid only in n-frames where T_i is a reflexive relation (this follows from modal correspondence theory, cf. (van Benthem, 1984): the modal formula $p \to \Diamond_i p$ corresponds to the frame condition $\forall x T_i xx$). Likewise, the axiom schemes

$$\exists v_i \exists v_i \varphi \to \exists v_i \varphi \tag{13}$$

and

$$\varphi \to \forall v_i \exists v_i \varphi \tag{14}$$

will be valid only in frames where the relation T_i is transitive and symmetric, respectively.

Later on we will see more of such correspondences, the point to be made here is that the abstract perspective on the semantics of first-order logic imposes a certain 'degree of validity' on well-known theorems of the predicate calculus: some theorems are valid in *all* abstract assignment frames, like distribution:

$$\forall v_i(\varphi \to \psi) \to (\forall v_i\varphi \to \forall v_i\psi),$$

whereas others, like the ones mentioned above, are only valid in *some* classes of frames. Narrowing down the class of frames means increasing the set of valid formulas, and vice versa. In particular, we now have the option to look at classes of frames that are only slightly more general than the standard first-order structures, but have much nicer computational properties. This new perspective on first-order logic was developed in the literature on algebraic logic, cf. (Németi, 1986). As (van Benthem, to appear) points out, it provides us with an enormous freedom to play with the semantics for first-order logic. In particular, consider the fact that first-order structures can be seen as frames $({}^nU, \equiv_i, Id_{ij}, \bowtie_\sigma)_{i,j<n,\sigma\in{}^nn}$ where *all* assignments $s \in {}^nU$ are available. But why not study a semantics where states are still real assignments on the base set U, but *not all such assignments are available?* In other words, the proposal is to study frames of the form $(W, \equiv_i\restriction_W, Id_{ij} \cap W, \bowtie_\sigma\restriction_W)_{i,j<n,\sigma\in{}^nn}$, where W is some subset of the set ${}^nU$, and $\equiv_i\restriction_W$ (resp., $\bowtie_\sigma\restriction_W$) are the relations $\equiv_i$ (resp., $\bowtie_\sigma$) restricted to W.

Definition 3.2 *Let U be some set, and W a set of n-tuples over U, i.e., $W \subseteq {}^nU$. The* **cube over** *U is defined as the frame*

$$\mathfrak{C}_n(U) = ({}^nU, \equiv_i, Id_{ij}, \bowtie_\sigma)_{i,j<n,\sigma\in{}^nn}.$$

The W-**relativized cube over** *U or W-assignment frame on U is defined as the frame*

$$\mathfrak{C}_n^W(U) = (W, \equiv_i\restriction_W, Id_{ij} \cap W, \bowtie_\sigma\restriction_W)_{i,j<n,\sigma\in{}^nn}.$$

There are at least two good reasons to make such a move. First, it turns out that the logic of such generalized assignment frames has much nicer meta-properties than the logic of the cubes, such as decidability or finite axiomatizability — cf. our results below. These logics will provide less laws than the usual predicate calculus, but their supply of theorems may be sufficient for particular applications. Note for instance, that the schemes (12), (13) and (14) are still valid in every generalized assignment frame, since $\equiv_i\ \restriction_W$ is always an equivalence relation.

In some situations it may even be *useful* not to have all familiar validities. Consider for instance the schema

$$\exists v_i\exists v_j\varphi \to \exists v_j\exists v_i\varphi. \tag{15}$$

It follows from correspondence theory that (15) is valid in an abstract assignment frame $\mathfrak{F}$ iff (16) below holds in $\mathfrak{F}$.

$$\forall xz\,(\exists y\,(T_ixy \wedge T_jyz) \to \exists u\,(T_jxu \wedge T_iuz)). \tag{16}$$

The point is that the schema (15) disables us to make the dependency of variables explicit in the language (i.e., whether v_j is dependent of v_i or the other way around), while these dependencies play a very important role in some proof-theoretical approaches, cf. (Fine, 1985; van Lambalgen, 1991). This approach can be applied to general quantifiers as well, cf. (Alechina and van Lambalgen, 1995; van Benthem and Alechina, to appear). So, the second motivation for generalizing the semantics of first-order logic is that it gives us a finer sieve on the notion of equivalence between first-order formulas. Note for instance that (15) is not valid in frames with assignment gaps: take $n = 2$. In a square (2-cubic) frame we have $(a, b) \equiv_0 (a', b) \equiv_1 (a', b')$, but if (a, b') is not an available tuple, then there is no s such that $(a, b) \equiv_1 s \equiv_0 (a', b')$ — hence this frame will not satisfy (16). So, the schema (15) will not be valid in this frame.

In this new paradigm, a whole landscape of frame classes and corresponding logics arises. In the most general approach, *any* subset of ${}^nU$ may serve as the universe of a multi-dimensional frame, but it seems natural to impose restrictions on the set of available assignments. The strongest conditions, that is, demanding that $W = {}^nU$, would take us back to the cube semantics. A nice intermediate class consists of multi-dimensional frames that are *locally cube*: if $s \in {}^nU$ is an available tuple, then any tuple drawing its coordinates from the set $\{s_i \mid i < n\}$ should be available as well. An even weaker requirement would demand that if $s \in {}^nU$ is present in the universe, then all projections $s \circ [i/j]$ of s onto some diagonal set should be available as well. To be more precise, consider the definitions below:

Definition 3.3 *We define four classes of assignment frames; each of these is given as the class of frames of the form* $\mathfrak{C}_n^W(U)$ *satisfying the conditions listed below.*

R_n	W *is an arbitrary subset of* ${}^nU$
D_n	$\forall s \in W \forall i, j < n \;\; s \circ [i/j] \in W$
LC_n	$\forall s \in W \forall \sigma \in {}^nn \;\; s \circ \sigma \in W$
C_n	$W = {}^nU$.

Frames in R_n*,* LC_n *and* C_n *are called* **generalized assignment frames***,* **local cubes** *and* **cubes***, respectively.*

For the similarity type CML_n, many results are known concerning the logic of these frame classes. We list some of the meta-properties in Table 1. The results in this table hold for all finite n larger than two. The sources of these results (originally in algebraic form, but easily transformable into our modal-logical format) are provided in notes immediately below the theorem.

	$\Theta_{CML_n}(\mathsf{R}_n)$	$\Theta_{CML_n}(\mathsf{D}_n)$	$\Theta_{CML_n}(\mathsf{LC}_n)$	$\Theta_{CML_n}(\mathsf{C}_n)$
finitely axiomatizable	no[1]	yes[2]	yes[3]	no[4]
decidable	yes[5]	yes[5]	yes[5]	no[6]

1 Németi (Henkin et al., 1971 1985)
2 Resek–Thompson, Andréka (Monk, 1993), (Andréka and Thompson, 1988)
3 Thompson, Andréka (Andréka, 1995)
4 Monk (Monk, 1969)
5 Németi (Németi, 1986)
6 Tarski (Henkin et al., 1971 1985)

Table 1: Properties of (generalized) assignment frames

The technical results in this paper concern the similarity type MLR_n. Again, the starting point is given by negative results stating that the MLR_n-logic for the cubes is neither finitely axiomatizable nor decidable:

Fact 3.4 *For any finite number $n > 2$, $\Theta_{MLR_n}(\mathsf{C}_n)$ is neither finitely axiomatizable nor decidable.*

The decidability part of Fact 3.4 follows immediately from the corresponding result for CML_n (due to Tarski, cf. Theorem 4.2.18 in (Henkin et al., 1971 1985)), since the decidability problem of $\Theta_{CML_n}(\mathsf{C}_n)$ can be trivially reduced to that of $\Theta_{MLR_n}(\mathsf{C}_n)$. The non-finite axiomatizability part follows from results in (Johnson, 1969). The restriction to a number n *larger* than 2 is essential in both cases.

In the next two sections we will show that, completely analogous to the case of CML_n, we can turn these negative results into positive ones by concentrating on the class of locally cube assignment frames. The motivation for studying precisely this combination of frame class (LC_n) and modal language (MLR_n), lies in the 'nice fit' of syntax and semantics. In order to describe this in detail, it is convenient to have the following notation and terminology. For $s \in {}^nU$, we define

$$\mathcal{R}(s) = \{s_i \mid i < n\} \tag{17}$$

$$\mathcal{C}(s) = {}^n\mathcal{R}(s). \tag{18}$$

$\mathcal{R}(s)$ is simply called the *range* of s, and $\mathcal{C}(s)$ the *cubicle* of s. Note that a generalized assignment frame $\mathfrak{C}_n^W(U)$ is locally cube iff for all $s \in W$, $\mathcal{C}(s) \subseteq W$.
We will frequently use the following fact:

$$(\forall t \in \mathcal{C}(s))(\exists \sigma \in {}^nn)\ \ t = s \circ \sigma, \tag{19}$$

the verification of which is left as an exercise to the reader. Now, the 'nice fit' of the similarity type MLR_n to the locally cube semantics lies in the fact that from any n-tuple s, we can see *directly* which formulas are true in every tuple of its cubicle, via

$$s \Vdash \bigcirc_\sigma \varphi \iff s \circ \sigma \Vdash \varphi.$$

This 'nice fit' allows us to give relatively easy and perspicuous proofs for the two main results of this paper, viz. a very simple and intuitive finite axiomatization for $\Theta_{MLR_n}(\mathsf{LC}_n)$ and a decidability result for the same logic.

4 Axiomatizing the local cubes

In this section, we will prove a completeness result for the class of local cubes in the similarity type MLR. An equally important aim is to give an introduction to the so-called mosaic method. This method was developed in the context of algebraic logic (cf. (Németi, 1986)) and has of late been used intensively in decidability (and completeness) proofs for multi-dimensional modal logics. This method is complementary to the well-known step by step method for building (relativized) multi-dimensional modal models, in the following sense. In the step by step method one constructs a model for a MCS by building better, but also bigger and bigger approximations to the model. The mosaic method concentrates on gluing together two extremely small parts of the final model; if applicable, it reduces the step by step construction of the final model to a rather trivial exercise.

The complete axiom system looks as follows:

Definition 4.1 *Let QAX_n be the extension of the basic derivation system K_{MLR_n} of the similarity type MLR_n with the following axioms*[2]

(*CM1*)	$p \to \Diamond_i p$	
(*CM2*)	$p \to \Box_i \Diamond_i p$	
(*CM3*)	$\Diamond_i \Diamond_i p \to \Diamond_i p$	
(*CM5*)	$\iota\delta_{ii}$	
(*Q1*)	$\bigcirc_\sigma \neg p \leftrightarrow \neg \bigcirc_\sigma p$	
(*Q2*)	$\bigcirc_{Id_n} p \leftrightarrow p$	
(*Q3*)	$\bigcirc_\sigma \bigcirc_\tau p \leftrightarrow \bigcirc_{\sigma\circ\tau} p$	
(*Q4*)	$\bigcirc_\sigma \iota\delta_{ij} \leftrightarrow \iota\delta_{\sigma i, \sigma j}$	
(*Q5*)	$\bigcirc_{[i/j]} p \leftrightarrow \Diamond_i (p \land \iota\delta_{ij})$	*(provided $i \neq j$).*
(*Q6*)	$\bigcirc_\sigma \Diamond_i p \leftrightarrow \bigcirc_\tau \Diamond_i p$	*(provided $\sigma \equiv_i \tau$).*

Most of these axioms speak very much for themselves; all of them are in Sahlqvist form, so let us discuss what they have to say about MLR_n-frames. To start with, in any relativized assignment frame, the relation T_i is an equivalence relation — this explains *CM1* – *CM3*. Axiom *CM5* states that $\iota\delta_{ii}$ is true everywhere; it is sound in any generalized assignment frame, since for any n-tuple s, $s_i = s_i$. In a local cube, the relation $\bowtie_\sigma$ is a (total) function: $s \bowtie_\sigma t \iff t = s \circ \sigma$. Now *Q1* is the standard axiom expressing functionality of the accessibility relation. *Q2* and *Q3* then relate the monoid structure $({}^n n, \circ, Id_n)$ on the transformation set ${}^n n$ to the structure on the accessibility functions. *Q4* is needed to capture the effect of transformations on n-tuples that belong to some diagonal set, and *Q5* defines the replacement operator $\bigcirc_{[i/j]}$ in terms of $\Diamond_i$ and $\iota\delta_{ij}$.

The most interesting axiom is *Q6*. To show why it is valid in the class of local cubes, let $\mathfrak{M} = (\mathfrak{C}_n^W(U), V)$ be a locally cubic model, and s an available assignment of this model such that $\mathfrak{M}, s \Vdash \bigcirc_\sigma \Diamond_i \varphi$. Then there is an u in U such that $(s \circ \sigma)^i_u \in W$ and

$$\mathfrak{M}, (s \circ \sigma)^i_u \Vdash \varphi. \tag{20}$$

Now let τ be such that $\sigma \equiv_i \tau$, i.e., σ and τ differ at most on i. We claim that

$$(s \circ \sigma)^i_u = (s \circ \tau)^i_u. \tag{21}$$

In fact, (21) is easily derived from the following two identities:

$$(s \circ \sigma)^i_u(i) = u = (s \circ \tau)^i_u(i),$$

and, for $j \neq i$:

$$(s \circ \sigma)^i_u(j) = s(\sigma(j)) = s(\tau(j)) = (s \circ \tau)^i_u(j).$$

[2] We adopt the convention that an axiom like *CM1* is the *conjunction* of the formulas $p \to \Diamond_i p$. In case we want to refer to a specific conjunct, we will use subscripts. So, $Q6_{\sigma,\tau,i}$ refers to the formula $\bigcirc_\sigma \Diamond_i p \leftrightarrow \bigcirc_\tau \Diamond_i p$, provided $\sigma \equiv_i \tau$. (If $\sigma \not\equiv_i \tau$, then $Q6_{\sigma,\tau,i}$ is not defined). The odd numbering of the *CM*-axioms stems from our wish to keep the numbering consistent with that of the axiom systems in the paper (Venema, 1995a).

To see why the latter holds, recall that its middle identity follows from $\sigma(j) = \tau(j)$, which in its turn holds by the assumption $\sigma \equiv_i \tau$. Now from (20) and (21) it is immediate that $\mathfrak{M}, (s \circ \tau)^i_u \Vdash \varphi$. Finally — and here we use the fact that W is a locally cubic subset of ${}^nU$ — $s \circ \tau$ is available as well. Therefore, $\mathfrak{M}, (s \circ \tau) \Vdash \Diamond_i \varphi$, so $\mathfrak{M}, s \Vdash \bigcirc_\tau \Diamond_i \varphi$.

So let us state the main result of this section. In its proof and everywhere else in this section, '(maximal) consistent' always means '(maximal) consistent with respect to QAX_n'.

Theorem 4.2 *QAX_n is strongly sound and complete with respect to LC_n, i.e., for every set $\Sigma \cup \{\varphi\}$ of MLR_n-formulas, it holds that*

$$\Sigma \vdash_{QAX_n} \varphi \iff \Sigma \models_{\mathsf{LC}_n} \varphi.$$

Proof. The proof of this theorem follows from the observation that

for any set Γ of MLR_n-formulas, Γ is consistent iff Γ is satisfiable in a local cube. (22)

This observation is immediate from the Lemmas 4.5 and 4.6 below. QED

In order to prove (22) we will use the mosaic method we mentioned before. In fact, our choice to concentrate on the combination of the similarity type *MLR* and the semantics of local cubes was guided partly by the wish to give a clear and simple presentation of this mosaic method.

Let us start with a short, intuitive explanation. There are two points of view on mosaics: an axiomatic and a semantic one. From the axiomatic perspective, mosaics are generalizations of maximal consistent sets in the following sense. Where a single MCS (of a complete axiom system) is supposed to provide all true formulas of *one state* in a model, mosaics will deliver the true formulas of every tuple in a *cubicle* of a model. Thus a mosaic will be a pair $\mu = (X, \ell)$, where X is a non-empty set of size at most n, and ℓ is a labeling function, i.e., a map assigning maximal consistent sets of formulas to every n-tuple over X. Of course, a 'good', or *coherent* mosaic will relate the label sets of distinct n-tuples over X; for instance, if $\varphi \in \ell(s)$, then $\Diamond_i \varphi$ should be in the label set $\ell(s^i_x)$ for every $x \in X$.

In the semantic perspective, mosaics should be seen as *windows* allowing us a small glimpse of a model — each mosaic enabling the sight over one cubicle. Thus a locally cubic model can be decomposed into a number of mosaics (this is in fact the soundness part of the proof), but conversely we also want to construct models from certain sets of mosaics. Then, such 'good' sets of mosaics should satisfy certain saturation conditions: where a single mosaic can be 'hungry' in the sense that it contains a tuple s with a formula $\Diamond_i \psi$ in its label set but no tuple s^i_x providing ψ, in a 'good' or *saturated* set of mosaics this hunger should be relieved by a second mosaic, in a sense to be made precise in Definition 4.4 below.

The aim of the completeness proof is then to connect the semantic and the axiomatic viewpoint on saturated sets of mosaics. This will be done in the Lemmas 4.5 and 4.6 below.

Definition 4.3 *A* **mosaic** *is a pair $\mu = (X, \ell)$, where X is a non-empty set of size at most n called the* **base (set)** *of the mosaic, and ℓ is a labeling function, i.e., a map assigning maximal consistent sets of formulas to every n-tuple over X. Such a mosaic is called* **coherent** *if it satisfies the following coherence conditions:*

(CH1) $\iota\delta_{ij} \in \ell(s) \iff s_i = s_j$ *for all $s \in {}^nX$,*

(CH2) $\varphi \in \ell(s \circ \sigma) \iff \bigcirc_\sigma \varphi \in \ell(s)$ *for all $\bigcirc_\sigma \varphi \in MLR_n$ and $s \in {}^nX$,*

(CH3) $\Diamond_i \varphi \in \ell(s) \iff \Diamond_i \varphi \in \ell(t)$ *for all $\Diamond_i \varphi \in MLR_n$ and $s, t \in {}^nX$ such that $s \equiv_i t$.*

A mosaic $\mu = (X, \ell)$ is a mosaic for a set Γ of formulas if $\Gamma \subseteq \ell(s)$ for some $s \in {}^nX$.

Definition 4.4 *Let Y be some set. A* **saturated set of mosaics** *(short: SSM) over Y is a set M of coherent mosaics with each base set contained in Y and satisfying the following condition. If $\mu = (X, \ell) \in M$, $s \in {}^nX$ and $\Diamond_i\varphi \in \ell(s)$, then either $\varphi \in \ell(s^i_x)$ for some $x \in X$, or there exists a $y \in Y \setminus X$, and a mosaic $\mu' = (X', \ell')$ such that*

(S1) $X' = \mathcal{R}(s^i_y)$

(S2) $\varphi \in \ell'(s^i_y)$

(S3) $\ell(t) = \ell'(t)$ *for every* $t \in \mathcal{C}(s) \cap \mathcal{C}(s^i_y)$.

For a set Γ of formulas, M is a saturated set of mosaics for Γ if M is an SSM over some set Y and M contains a mosaic for Γ.

Note that (in the terminology of Definition 4.4), the triple (s, i, φ) constitutes some sort of *defect* of the mosaic $\mu = (X, \ell)$ if $\Diamond_i\varphi \in \ell(s)$, while there is no $x \in X$ such that $\varphi \in \ell(s^i_x)$. The mosaic μ' can be seen to *repair* this defect by adding a new element y to the base X of μ and demanding (S2) that $\varphi \in \ell'(s^i_y)$. The consequence of adding a new element to the base set might be that the new base set $X \cup \{y\}$ would become too big for μ' to be a mosaic (namely if X itself would already be of size n). Therefore we delete the old s_i from the base set, unless $s_i \in \mathcal{R}(s^i_y)$. This explains condition (S1). Finally, we want μ and μ' to agree on their common part; this is precisely the content of (S3).

Now we can prove the two main lemmas concerning SSMs. The first connects them to semantics, the second one to axiomatics.

Lemma 4.5 *Let Γ be a set of formulas. Then Γ is satisfiable in a local cube iff there is a saturated set of mosaics for Γ.*

Proof. The direction from left to right is straightforward: assume that Γ is satisfiable in a local cube, i.e., there is a local cube model $\mathfrak{M} = (\mathfrak{C}^W_n(U), V)$, and an element r in W such that $\mathfrak{M}, r \Vdash \Gamma$.

The required SSM for Γ will be 'cut out' of $\mathfrak{M}$. For any available assignment $s \in W$, let X_s and ℓ_s be defined by

$$\begin{array}{rcl} X_s & = & \mathcal{R}(s) \\ \ell_s(t) & = & \{\varphi \mid \mathfrak{M}, t \models \varphi\}. \end{array}$$

Since all the axioms are valid, each set $\ell_s(t)$ will be maximal consistent (this is in fact the soundness direction of the completeness proof). Hence the pair

$$\mu_s = (X_s, \ell_s)$$

is a mosaic. It is straightforward to verify that every μ_s is coherent, and equally easy to show that the set

$$M = \{\mu_s \mid s \in W\}$$

is a saturated set of mosaics over the base set of W. Since $\Gamma \subseteq \ell_r(r)$, $\mu_r \in M$ is a mosaic for Γ.

For the other direction, assume that M is an SSM for Γ over Y. Without loss of generality we may assume that Γ is *maximal* consistent and that M is closed under taking submosaics[3]. We will construct a model for Γ via a step by step method.

[3]The submosaics of a mosaic $\mu = (X, \ell)$ are of the form (X', ℓ') with $X' \subseteq X$ and ℓ' being the restriction of ℓ to ${}^nX'$.

The idea of the construction is to glue together isomorphic copies of mosaics in M. At each step of the construction we are dealing with a labeled local cube on a finite base set. A *labeled local cube* (short: LLC) is a triple of the form $\mathcal{G} = (U, W, \ell)$ where U is some set called the base set of the LLC, W is a locally cubic subset of ${}^nU$, and ℓ is a labeling map assigning maximal consistent sets to each element of W. This map ℓ should satisfy the condition that each of its restrictions to cubicles in W is isomorphic[4] to a mosaic in M. Formally:

$$\text{for all } s \in W \text{ there is a } \mu \in M \text{ such that } (\mathcal{R}(s), \ell{\restriction}_{\mathcal{C}(s)}) \simeq \mu. \tag{23}$$

In the base step of the construction, we define the LLC $\mathcal{G}_0$ as some isomorphic copy of the mosaic for Γ that M contains by assumption.

In the induction step of the construction, we are dealing with a finite LLC $\mathcal{G}_m = (U_m, W_m, \ell_m)$ and a *defect* of it, i.e., a triple (s, i, φ) with $s \in W_m$, $i < n$ and $\varphi \in MLR_n$ such that $\Diamond_i\varphi \in \ell_m(s)$ while there is no $u \in U_m$ such that $s^i_u \in W_m$ and $\varphi \in \ell_m(s^i_u)$.

Let $\mu = (X, \partial)$ be a mosaic in M which is isomorphic to $(\mathcal{R}(s), \ell_m{\restriction}_{\mathcal{C}(s)})$. For the sake of a smooth presentation, we simply pretend that $(\mathcal{R}(s), \ell_m{\restriction}_{\mathcal{C}(s)})$ is *itself* a mosaic of M. Since M is saturated, there are $y \in Y \setminus \mathcal{R}(s)$ and a mosaic $\mu' = (\mathcal{R}(s^i_y), \ell')$ satisfying (S1), (S2) and (S3) of Definition 4.4. Now let u be a *new* element, i.e., $u \notin U_m$. We define the new LLC $\mathcal{G}_{m+1} = (U_{m+1}, W_{m+1}, \ell_{m+1})$ as follows:

$$\begin{array}{rcl} U_{m+1} & = & U_m \cup \{u\}, \\ W_{m+1} & = & W_m \cup \mathcal{C}(s^i_u), \\ \ell_{m+1}(t) & = & \begin{cases} \ell_m(t) & \text{if } t \in W_m \\ \ell'(s^i_y \circ \sigma) & \text{if } t \in \mathcal{C}(s^i_u),\ t = s^i_u \circ \sigma. \end{cases} \end{array}$$

In other words, $\mathcal{G}_{m+1}$ consists of gluing an isomorphic copy of μ' to $\mathcal{G}_m$. Note that the domains W_m and $\mathcal{C}(s^i_u)$ of the two constituting parts are not disjoint; fortunately, by (S3) this will cause no problems for the well-definedness of ℓ_{m+1}.

It is not difficult to see that we have *repaired* the defect (s, i, φ) of $\mathcal{G}_m$: by (S2), $\varphi \in \ell_{m+1}(s^i_u)$. Note that the new LLC satisfies condition (23), because we assumed that M is closed under submosaics.

Using standard techniques, we can set up the construction in such a way that *every* defect of every LLC $\mathcal{G}_n$ eventually gets repaired. We then define the limit $\mathcal{G} = (U, W, \ell)$ of the construction as follows:

$$U = \bigcup_{n\in\omega} U_n, \qquad W = \bigcup_{n\in\omega} W_n, \qquad \ell = \bigcup_{n\in\omega} \ell_n.$$

It follows almost immediately that this $\mathcal{G}$ is an LLC *without* any defects. Hence, $\mathcal{G}$ satisfies, for all $s, t \in W$ and formulas φ, the following:

(L1)	$s_i = s_j$	iff	$\iota\delta_{ij} \in \ell(s)$,	
(L2)	$\bigcirc_\sigma\varphi \in \ell(s)$	iff	$\varphi \in \ell(s \circ \sigma)$,	
(L3)	$\varphi \in \ell(s)$	implies	$\Diamond_i\varphi \in \ell(t)$,	provided $s \equiv_i t$
(L4)	$\Diamond_i\varphi \in \ell(s)$	implies	$\exists u \in U\ (\varphi \in \ell(s^i_u))$.	

[4]We use the obvious notion of isomorphism here: two mosaics $\mu = (X, \ell)$ and $\mu' = (X', \ell')$ are isomorphic, notation: $\mu \simeq \mu'$, if there is a bijection $f : X \to X'$ such that $\ell(s) = \ell'(f^+ \circ s)$, where $f^+ : {}^nX \to {}^nX'$ is the bijection induced by f.

Of these properties, (L3) is the only one which is not trivially true, so let us prove it. Assume $s, t \in W$ are such that $s \equiv_i t$ and $\varphi \in \ell(s)$; since $\ell(s)$ is a MCS, it is immediate by *CM1* that $\Diamond_i\varphi \in \ell(s)$. Now let j be an arbitrary index different from i; then $s \equiv_i t$ implies that $s \circ [i/j] = t \circ [i/j]$. It follows from (23) and (CH3) that $\Diamond_i\varphi \in \ell(s \circ [i/j])$. Likewise, it follows from $\Diamond_i\varphi \in \ell(t \circ [i/j])$ that $\Diamond_i\varphi \in \ell(t)$. This proves (L3).
Finally, defining a valuation V by

$$V(p) = \{s \in W \mid p \in \ell(s)\},$$

we obtain a model $\mathfrak{M} = (\mathfrak{C}^W_n(U), V)$ for which we can prove the truth lemma

$$\forall\varphi \in MLR_n \forall s \in W \ \ (\mathfrak{M}, s \Vdash \varphi \iff \varphi \in \ell(s)).$$

In the rather easy proof of this truth lemma, we use conditions (L1) – (L4) for the inductive steps.

Now, since we *based* our construction on the LLC $\mathcal{G}_0$ which was an (isomorphic copy of) a mosaic for Γ, we have indeed satisfied Γ in a local cube. QED

Lemma 4.6 *Let Σ be a set of formulas. Then Σ is consistent iff there is a saturated set of mosaics for Σ.*

Proof. The direction from right to left is immediate by the definition of an SSM: recall that mosaics label with maximal consistent sets only. So, if μ is a mosaic for Σ, then Σ is contained in a maximal consistent set, and hence, consistent.

For the other direction, let Y be some set. We will define an SSM on Y which is good for every MCS (and hence, for every consistent set). For reasons that will become clear later on, we will take Y to be a finite set of size $n + 1$.

Define, for Γ a maximal consistent set, S_Γ as the set $\{s \in {}^nY \mid s_i = s_j \text{ iff } \iota\delta_{ij} \in \Gamma\}$. Now, for any $s \in S_\Gamma$, $\mu_{\Gamma,s}$ will denote the pair $(\mathcal{R}(s), \ell_{\Gamma,s})$ where $\ell_{\Gamma,s} : \mathcal{C}(s) \to MCS$ is defined by

$$\ell_{\Gamma,s}(s \circ \sigma) = \bigcirc_\sigma^{-1}\Gamma,$$

where $\bigcirc_\sigma^{-1}\Gamma$ is given by

$$\bigcirc_\sigma^{-1}\Gamma = \{\varphi \in MLR_n \mid \bigcirc_\sigma\varphi \in \Gamma\}.$$

It follows from Lemma 4.7 below, that $\mu_{\Gamma,s}$ is a well-defined and coherent mosaic for Γ. Now define

$$M = \{\mu_{\Gamma,s} \mid \Gamma \in \text{MCS}, s \in S_\Gamma\}.$$

We will show that M is saturated. Let $\mu = (X, \ell)$ be a mosaic in M, $s \in {}^nX$ and $\Diamond_i\varphi \in \ell(s)$. If there is an $x \in X$ such that $\varphi \in \ell(s^i_x)$, we are finished, so suppose otherwise. Fix an index j different from i. Note that it follows from (CH3) and (CH1) that $\iota\delta_{ij} \wedge \Diamond_i\varphi \in \ell(s \circ [i/j])$. Now consider the following Claim.

Claim 1 Let φ be a MLR_n-formula and Δ a maximal consistent set such that $\iota\delta_{ij} \wedge \Diamond_i\varphi \in \Delta$. Then there is a maximal consistent set Γ such that $\varphi \in \Gamma$ and $\bigcirc_{ij}^{-1}\Gamma = \Delta$.

Proof of Claim Let φ and Δ be as in the Lemma. By general modal (*S5*)-considerations, $\Diamond_i\varphi \in \Delta$ implies the existence of an MCS Γ such that $\varphi \in \Gamma$ and for all formulas ψ, $\psi \in \Delta$ implies $\Diamond_i\psi \in \Gamma$. We will prove that $\bigcirc_{ij}^{-1}\Gamma = \Delta$.

Assume that $\psi \in \Delta$, then by assumption, $\iota\delta_{ij} \wedge \psi \in \Delta$. It follows that $\Diamond_i(\iota\delta_{ij} \wedge \psi) \in \Gamma$. By the axiom *Q5*, this implies $\bigcirc_{ij}\psi \in \Gamma$. Hence $\Delta \subseteq \bigcirc_{ij}^{-1}\Gamma$. For the other direction, assume that $\psi \notin \Delta$; then $\neg\psi \in \Delta$, since Δ is a maximal consistent set. We have just seen that this implies $\bigcirc_\sigma\neg\psi \in \Gamma$, so by axiom *Q1*, $\neg\bigcirc_\sigma\psi \in \Gamma$; hence $\bigcirc_\sigma\psi \notin \Gamma$. But then we have proved that $\bigcirc_{ij}^{-1}\Gamma \subseteq \Delta$, and we are finished. ◀

So by this Claim, there is a MCS Γ such that $\varphi \in \Gamma$ and

$$\bigcirc_{ij}^{-1}\Gamma = \ell(s \circ [i/j]). \tag{24}$$

Since Y has more than n elements, and X at most n, there is an element y in $Y \setminus X$; denote $s' = s^i_y$; it follows immediately that

$$s \circ [i/j] = s' \circ [i/j]. \tag{25}$$

We want to use Lemma 4.7 to show that $\mu_{\Gamma,s'}$ is a well-defined and coherent mosaic. Thus we have to show that, for all $k, l < n$, $s'(k) = s'(l)$ iff $\iota\delta_{kl} \in \Gamma$. But this follows easily from axiom *Q4* (and, in the case that $i \in \{k, l\}$, by the fact that $s'(i) = y \notin X$).

We claim that μ and $\mu_{\Gamma,s'}$ satisfy the saturation conditions of Definition 4.4. (S1) and (S2) are immediate by the definition of $\mu_{\Gamma,s'}$. In order to prove (S3), let t be an element of $\mathcal{C}(s) \cap \mathcal{C}(s')$. We have to prove that

$$\ell(t) = \ell_{\Gamma,s'}(t). \tag{26}$$

Our first observation is that we have, for our fixed $j \neq i$:

$$\mathcal{C}(s) \cap \mathcal{C}(s') = \mathcal{C}(s \circ [i/j]).$$

Hence, by (19), we may assume that there is a transformation σ such that

$$t = (s \circ [i/j]) \circ \sigma.$$

By the coherency of μ it follows that

$$\ell(t) = \ell(s \circ [i/j] \circ \sigma) = \bigcirc_\sigma^{-1}\ell(s \circ [i/j]).$$

On the other hand, it follows from (25) and the definition and coherency of $\ell_{\Gamma,s'}$ that

$$\ell_{\Gamma,s'}(t) = \ell_{\Gamma,s'}(s \circ [i/j] \circ \sigma) = \ell_{\Gamma,s'}((s' \circ [i/j]) \circ \sigma) = \bigcirc_\sigma^{-1}\bigcirc_{ij}^{-1}\Gamma.$$

So (26) follows from the last two observations and (24). QED

Lemma 4.7 *Let Γ be a maximal consistent set of formulas, and s an n-tuple such that $s_i = s_j$ iff $\iota\delta_{ij} \in \Gamma$. Then (i), (ii) and (iii) below hold:*

(i) *If $\sigma, \sigma' \in {}^n n$ are transformations such that $s \circ \sigma = s \circ \sigma'$, then $\bigcirc_\sigma^{-1}\Gamma = \bigcirc_{\sigma'}^{-1}\Gamma$.*

(ii) *Hence, the pair $\mu = (\mathcal{R}(s), \ell)$ given by $\ell(t) = \bigcirc_\sigma^{-1}\Gamma$ if $t = s \circ \sigma$, is a well-defined mosaic.*

(iii) *The mosaic μ defined in (ii) is coherent, and a mosaic for Γ.*

Proof. Let Γ, s, σ and σ' be as in the Lemma. In order to prove **(i)**, we have to show that for all formulas $\varphi \in MLR_n$, $\bigcirc_\sigma \varphi \in \Gamma$ iff $\bigcirc_{\sigma'} \varphi \in \Gamma$.

First note that $s_{\sigma i} = s \circ \sigma(i) = s \circ \sigma'(i) = s_{\sigma' i}$ for all i. Since, by assumption on s, we have $\iota\delta_{kj} \in \Gamma$ iff $s_k = s_j$, this gives $\iota\delta_{\sigma i, \sigma' i} \in \Gamma$ for all i.

We define a relation R on n as follows: jRj' if there is an i such that $j = \sigma i$ and $j' = \sigma' i$. We let $\sim_R$ denote the equivalence relation generated by R, and $[j]_R$ denotes the equivalence class of j under $\sim_R$. We need the following two claims:

Claim 1 For all i, j, $i \sim_R j$ implies that $\iota\delta_{ij} \in \Gamma$.

Proof of Claim The proof is by induction on the generation of $\sim_R$. If $i \sim_R j$ because iRj, then $\iota\delta_{ij} \in \Gamma$ because of the observation above; otherwise, we use QAX-theorems like $\iota\delta_{ii}$, $\iota\delta_{ij} \to \iota\delta_{ji}$ and $\iota\delta_{ik} \wedge \iota\delta_{kj} \to \iota\delta_{ij}$ — we leave it to the reader to verify that these formulas are indeed derivable in QAX. ◀

Call a transformation $\rho : n \longrightarrow n$ *compatible* if $i \sim_R \rho(i)$ for all $i < n$.

Claim 2 For every formula ψ, and every compatible transformation ρ, $\psi \in \Gamma$ iff $\bigcirc_\rho \psi \in \Gamma$.

Proof of Claim It is not difficult to see that any compatible transformation is generated by compatible transpositions and compatible replacements. Therefore, it suffices to prove the claim for such transformations.

For such transformations, the Claim follows from Claim 1 and the QAX-theorems $\iota\delta_{ij} \to (p \leftrightarrow \otimes_{ij} p)$ and $\iota\delta_{ij} \to (p \leftrightarrow \bigcirc_{ij} p)$, which are easily seen to be derivable from the axioms. ◀

Now we are ready to prove the final claim:

Claim 3 For all formulas φ, $\bigcirc_\sigma \varphi \in \Gamma$ iff $\bigcirc_{\sigma'} \varphi \in \Gamma$.

Proof of Claim Let φ be a formula such that $\bigcirc_\sigma \varphi \in \Gamma$. We define $\tau : n \longrightarrow n$ as the transformation mapping any j to the *smallest* element of $[j]_R$. About τ we need the following:

$$\tau \circ \sigma = \tau \circ \sigma'. \tag{27}$$

To prove (27), we will show that $\tau \circ \sigma(i) = \tau \circ \sigma'(i)$ for each $i < n$. If $\sigma(i) = \sigma'(i)$, we are done. If $\sigma(i) \neq \sigma'(i)$, it follows immediately from the definitions that $\sigma(i) R \sigma'(i)$; hence, $[\sigma(i)]_R = [\sigma'(i)]_R$, so $\tau \circ \sigma(i) = \tau(\sigma(i)) = \tau(\sigma'(i)) = \tau \circ \sigma'(i)$ by definition of τ. *This proves (27).*

Note that τ is compatible by definition; therefore, it follows from Claim 2 that $\bigcirc_\tau \bigcirc_\sigma \varphi \in \Gamma$. Then by axiom *Q3*, $\bigcirc_{\tau\circ\sigma} \varphi \in \Gamma$. But (27) implies that $\bigcirc_{\tau\circ\sigma} \varphi$ is the very same formula as $\bigcirc_{\tau\circ\sigma'} \varphi$! Again, we use axiom *Q3* to derive that $\bigcirc_\tau \bigcirc_{\sigma'} \varphi \in \Gamma$, and the compatibility of τ to establish that $\bigcirc_{\sigma'} \varphi \in \Gamma$.
The reverse implication is of course proved in the same way. ◀

(ii) is now immediate, since **(i)** implies that the definition of $\ell(t)$ does not depend on the σ such that $t = s \circ \sigma$,

Finally, we prove **(iii)**. Condition (CH1) is immediate by *Q4*. For (CH2), let $t, t' \in \mathcal{C}(s)$ be such that $t' = t \circ \tau$ for some $\tau \in {}^n n$. We have to prove that $\bigcirc_\tau^{-1} \ell(t) = \ell(t')$.

Choose $\sigma, \sigma' \in {}^n n$ such that $t = s \circ \sigma$ and $t' = s \circ \sigma'$. By associativity of function composition, it follows that $t' = s \circ (\sigma \circ \tau)$. Now let φ be an arbitrary formula such that $\bigcirc_\tau \varphi \in \ell(t)$. By definition of ℓ, it follows that $\bigcirc_\sigma \bigcirc_\tau \varphi \in \ell(s)$, so using axiom *Q3* we find $\bigcirc_{\sigma \circ \tau} \varphi \in \ell(s)$. But then $\varphi \in \ell(t')$, again by definition of ℓ — here we use part **(i)** of this Lemma. It follows that $\bigcirc_\tau^{-1} \ell(t) \subseteq \ell(t')$. But since $\bigcirc_\tau^{-1} \ell(t)$ and $\ell(t')$ are both maximal consistent sets, this implies that $\bigcirc_\tau^{-1} \ell(t) = \ell(t')$.

Our last task is to show that μ satisfies CH3. Let $t, t' \in \mathcal{C}(s)$ be such that $t \equiv_i t'$ for some index i, and let φ be an arbitrary formula. We have to prove that

$$\Diamond_i \varphi \in \ell(t) \text{ iff } \Diamond_i \varphi \in \ell(t'). \tag{28}$$

It is not difficult to see that there are transformations σ, σ' such that $t = s \circ \sigma$, $t' = s \circ \sigma'$ and $\sigma \equiv_i \sigma'$. Using axiom *Q6*, we observe that $\bigcirc_\sigma \Diamond_i \varphi \leftrightarrow \bigcirc_{\sigma'} \Diamond_i \varphi$ is a theorem of QAX. Thus (28) follows from the following equivalences:

$$\begin{array}{rcll} \Diamond_i \varphi \in \ell(t) & \iff & \bigcirc_\sigma \Diamond_i \varphi \in \ell(s) & \text{(definition } \ell,\ t = s \circ \sigma) \\ & \iff & \bigcirc_{\sigma'} \Diamond_i \varphi \in \ell(s) & \text{(the observation above)} \\ & \iff & \Diamond_i \varphi \in \ell(t') & \text{(definition } \ell,\ t' = s \circ \sigma'). \end{array}$$

QED

5 Deciding the local cubes

In this section we turn to the issue of decidability. As we saw in section 3, the MLR_n-theory of the cubes is undecidable. The multi-dimensional modal theories of various classes of *relativized* cubes behave much nicer. For the similarity type CML_n, decidability results concerning the classes R_n, D_n and LC_n follow from Theorem 10 in (Németi, 1995). Here we give a short proof showing that (at least for LC_n) these results go through if we extend the similarity type with the substitution operators. The result below was obtained independently of (Mikulás, 1995), to whom it is due.

Theorem 5.1 *For $n > 1$, the modal logic $\Theta_{MLR_n}(\mathsf{LC}_n)$ is decidable.*

Proof. The Theorem will be proved by 'finitizing' the completeness proof of Theorem 4.2. The key idea is to prove that a MLR_n-formula ξ is satisfiable in a local cube if and only if there is a finite saturated set of finite mosaics for ξ. In order to make this idea precise, we need some definitions.

First, the cylindric depth $d(\varphi)$ of a formula φ is defined by the following induction:

$$\begin{array}{rcl} d(p) & = & 0 \\ d(\iota\delta_{ij}) & = & 0 \\ d(\neg\varphi) & = & d(\varphi) \\ d(\varphi \wedge \psi) & = & \max(d(\varphi), d(\psi)) \\ d(\bigcirc_\sigma \varphi) & = & d(\varphi) \\ d(\Diamond_i \varphi) & = & d(\varphi) + 1. \end{array}$$

Now, for a formula φ, we define $For_{(\varphi)}$ as the set of MLR_n-formulas that have a cylindric depth not bigger than $d(\varphi)$ and that only use propositional variables from φ.

Claim 1 For every formula ξ, $For_{(\xi)}$ is finite modulo LC_n-equivalence.

Proof of Claim Let $For^m(\xi)$ be the set of MLR_n-formulas of cylindric depth at most m which only use proposition letters from ξ. By natural induction on m we prove that

$$\text{for every } m,\ For^m(\xi) \text{ is finite modulo } \mathsf{LC}_n\text{-equivalence.} \tag{29}$$

The key observations in both cases are that the substitution diamonds distribute over booleans, and that any formula of the form $\bigcirc_{\sigma_1} \ldots \bigcirc_{\sigma_n} \varphi$ is equivalent to $\bigcirc_{\sigma_1 \circ \ldots \circ \sigma_n} \varphi$. In the case $m = 0$, we prove that any $\Diamond_i$-free formula is equivalent to a boolean combination of formulas of the form $\bigcirc_\sigma \psi$ with ψ a propositional variable or a constant. In the inductive step ($m > 0$) we prove that any formula in $For^m(\xi)$ is equivalent to a boolean combination of For^{m-1}-formulas and formulas of the form $\bigcirc_\sigma \Diamond_i \chi$ (with χ a For^{m-1}-formula). ◀

Our next step is to finitize the notions of a mosaic and a saturated set of mosaics. For a MLR_n-formula ξ, a ξ-mosaic is a pair $\mu = (X, \ell)$ such that its base X is a set of size at most n and ℓ labels elements of ${}^nX$ with subsets of $For_{(\xi)}$. (We will not require such sets to be consistent — this would be begging the question of the decidability problem.) A ξ-mosaic is coherent if it satisfies the following conditions, for all $s, t \in {}^nX$ and $\varphi, \psi \in For_{(\xi)}$:

(FC1)	$\iota\delta_{ij} \in \ell(s)$	iff	$s_i = s_j$,	
(FC2)	$\neg\varphi \in \ell(s)$	iff	$\varphi \notin \ell(s)$	
(FC3)	$\varphi \wedge \psi \in \ell(s)$	iff	$\varphi \in \ell(s)$ & $\psi \in \ell(s)$	
(FC4)	$\bigcirc_\sigma \varphi \in \ell(s)$	iff	$\varphi \in \ell(s \circ \sigma)$	
(FC5)	$\Diamond_i \varphi \in \ell(s)$	iff	$\Diamond_i \varphi \in \ell(t)$	provided $s \equiv_i t$ and $\Diamond_i \varphi \in For_{(\xi)}$
(FC6)	$\Diamond_i \varphi \in \ell(s)$	if	$\varphi \in \ell(s)$	provided $\Diamond_i \varphi \in For_{(\xi)}$.

Now fix a set Y of size $n + 1$. A set M of coherent ξ-mosaics is called a saturated set of ξ-mosaics, short: a ξ-SSM, if the base of every ξ-mosaic in M is a subset of Y, M satisfies the saturation conditions of Definition 4.4 and there is a $\mu = (X, \ell)$ in M such that $\xi \in \ell(s)$ for some $s \in {}^nX$.

Now we are ready to prove the Theorem. First we need the following claim:

Claim 2 Any MLR_n-formula ξ is satisfiable in a local cube iff there is a saturated set of ξ-mosaics.

Proof of Claim Fix ξ, and first assume that ξ is satisfiable in a local cube. By the completeness Theorem 4.2, ξ is QAX-consistent. By (the proof of) Lemma 4.6, there is an SSM M for ξ, based on a set Y of size $n + 1$. Now define

$$M' = \{\mu' \mid \mu \in M\},$$

where, for $\mu = (X, \ell) \in M$, $\mu' = (X, \ell')$ is given by

$$\ell'(s) = \ell(s) \cap For_{(\xi)}.$$

It is straightforward to verify that M' is a ξ-SSM.

The other direction is proved via a step-by-step construction, analogous to the one in the proof of Lemma 4.5. Note that here we need the fact that the set $For_{(\xi)}$ is closed under the boolean operations and the substitution operators. ◀

So, to prove the decidability of $\Theta_{MLR_n}(\mathsf{C}_n)$, it suffices to show that it is decidable whether a given formula ξ has a ξ-SSM. But this follows from the observations 1–4 below:

1. there is a bounded number of ξ-mosaics over Y (here we use Claim 1).
2. hence, there is a bounded number of sets of ξ-mosaics over Y.
3. given a ξ-mosaic, it is decidable whether it is coherent: there are finitely many finitary relations between finite sets of formulas to be checked.
4. given a finite set M of ξ-mosaics over Y, it is decidable whether M is a ξ-SSM.

QED

Remark 5.2 Theorem 5.1 can also be proved using filtrations. The crucial point, here captured in Claim 1, is that the closure of a finite set of formulas under the booleans and the substitution operators remains finite (modulo LC_n-equivalence). The advantage of the filtration method is that it shows the *finite frame property* as well: every consistent formula can be satisfied in a model over a *finite* MLR_n-frame validating $\Theta_{MLR_n}(\mathsf{LC}_n)$. In fact, a recent result of H. Andréka and I. Németi shows that the logic $\Theta_{MLR_n}(\mathsf{LC}_n)$ even has the *finite base property*, i.e., every consistent formula can be satisfied in a local cube $\mathfrak{C}_n^W(U)$ over a finite *base* set U.

6 Infinite dimensions

There are no a priori reasons why we should confine ourselves to finite dimensions in setting up a multi-dimensional framework for modal logic. In this section we will see how to extend the formalisms developed so far to the ω-dimensional case, leaving it to the reader to supply the details for the α-dimensional case where α is an arbitrary ordinal. More than in the finite-dimensional case, there are a number of choices to be made, in the areas of syntax, intended structures and semantics. It is our intention here to indicate the various options rather than delve into the technical details. We start with the syntax, i.e., the choice of operators in the similarity type MLR_ω; then, we introduce two new classes of generalized assignment frames; in the third subsection we return to the connection between first-order logic and multi-dimensional modal logic; and finally, in the last subsection we generalize the results from the previous sections to the infinite case.

6.1 ω-dimensional syntax

For the sake of a smooth presentation, assume that we have a semantics of local cubes in mind here. Now, concerning the cylindrifications and the diagonals there does not seem to be so much of a choice: the similarity type will have diamonds $\Diamond_i$ and constants $\iota\delta_{ij}$ for all $i, j < \omega$, completely analogous to the finite-dimensional case. The situation is different for the transformation operators, however.

The obvious approach seems to be to admit the diamond $\bigcirc_\sigma$ for *every* transformation $\sigma : \omega \to \omega$, but there are problems involved with this choice. One objection to the proposed option is that the cardinality of the language would become uncountable. To circumvent this, one could decide to include an operator $\bigcirc_\sigma$ in our ω-dimensional similarity type only if σ is a *finite transformation on* ω, i.e., $\sigma : \omega \to \omega$ is a map such that $\sigma(i) \neq i$ for *finitely* many indices i only. We denote the set of all finite transformations on ω by FT_ω.

In the definition below we see that in fact, we can study a similarity type for *every* set of transformations on the index set ω.

Definition 6.1 *CML_ω is the modal similarity type having constants $\iota\delta_{ij}$ and diamonds $\Diamond_i$ for every $i, j < \omega$. Now let H be a set of transformations, i.e., $H \subseteq {}^\omega\omega$. Then MLR^H_ω is the extension of CML_ω with a diamond $\bigcirc_\sigma$ for each transformation $\sigma \in H$. MLR_ω and MLR'_ω are defined as $MLR^{{}^\omega\omega}_\omega$ and $MLR^{FT_\omega}_\omega$, respectively.*

A more serious drawback of the similarity type MLR_ω, which also applies to MLR'_ω, is that we will never be able to find a finite schema axiomatization in it for *any* class of generalized assignment frames. Let us explain the problem in some detail. First there is the point to consider, what the notion of a finite axiomatization could *mean* for an infinite similarity type. Clearly, already the *minimal* modal derivation system K_{MLR_ω} of the similarity type CML_ω is infinite, since one needs infinitely many versions of the distribution axiom. Hence, one needs to generalize the notion of finite axiomatizability; a good approach is developed in (Monk, 1969), namely that of finite *schema* axiomatizability. We will explain this notion by example, not by definition; our exposition is based on the introduction of (Sain and Thompson, 1991), for more precise definitions we refer the reader to (Henkin et al., 1971 1985). Consider the axiom

$$(Q5) \qquad \bigcirc_{ij} p \leftrightarrow \Diamond_i (p \wedge \iota\delta_{ij}) \text{ (provided } i \neq j),$$

which we defined in the finite-dimensional case as the *conjunction* of all versions of *Q5* in which i and j are replaced by actual ordinals, cf. footnote 2. In the infinite-dimensional case we will view *Q5* as a *schema* of axioms; to be precise, it abbreviates the *set* of axioms

$$\{Q5_{ij} \mid i, j < \omega\}.$$

This schema is a Monk-type schema, in the following sense. Let $\xi \in {}^\omega\omega$ be a permutation. Then the ξ-substitution instance $\xi\varphi$ of a formula φ is obtained by replacing every index $i \in \omega$ with its ξ-image. For instance, $\xi(\bigcirc_{01} p \leftrightarrow \Diamond_0(p \wedge \iota\delta_{01}))$ is the formula $\bigcirc_{\xi 0 \xi 1} p \leftrightarrow \Diamond_{\xi 0}(p \wedge \iota\delta_{\xi 0 \xi 1})$ Now the idea of a finite schema axiomatization is that its axiom set can be brought in the form $\{\xi\varphi \mid \varphi \in A \ \& \ \xi \text{ is a permutation on } \omega\}$, for some *finite* set A. So let us see what the problem is in casu the similarity type MLR_ω and the class LC_ω of local cubes[5]. Suppose that we want to extend our completeness result, Theorem 4.2, to the class of ω-dimensional local cubes. Consider the axiom schema

$$(Q3) \qquad \bigcirc_\sigma \bigcirc_\tau p \leftrightarrow \bigcirc_{\sigma\circ\tau} p,$$

which we now would like to stand for the *set* of formulas

$$\{Q3_{\sigma,\tau} \mid \sigma, \tau \in {}^\omega\omega\}$$

[5]We will be a bit sloppy concerning the notation (and terminology) of frames and frame classes. For instance, we use the term 'local cubes' for all similarity types MLR^H_ω alike. Of course, the local cubes for MLR_ω differ from those for MLR'_ω in that the first contain accessibility relations for *every* transformation in ${}^\omega\omega$.

(or, for the set $\{Q3_{\sigma,\tau} \mid \sigma, \tau \in FT_\omega\}$ in the case that we are considering the similarity type MLR'_ω). There is a subtle but important difference between the schemas *Q5* and *Q3*; this difference lies in the fact that the index set ${}^\omega\omega$ has an *algebraic structure* on it, which is *used* to make *Q3* work. The point is that, for a permutation ξ on ${}^\omega\omega$, the translation of $Q3_{\sigma,\tau}$ will be the formula

$$\bigcirc_{\xi(\sigma)}\bigcirc_{\xi(\tau)}p \leftrightarrow \bigcirc_{\xi(\sigma\circ\tau)}p.$$

Now *this* formula will *only* be valid in the class of local cubes if ξ, in addition to being a bijection, is a *automorphism* on the algebra $({}^\omega\omega, \circ)$; it is crucially needed that $\xi(\sigma \circ \tau) = \xi(\sigma) \circ \xi(\tau)$.

In order to avoid such a trivial reason for non-finite schema axiomatizability, one could return to the suggestion to consider a similarity type with diamonds $\bigcirc_{ij}$ for *replacements* $[i/j]^\omega$ and $\otimes_{ij}$ for *transpositions* $[i,j]^\omega$ only (cf. section 2).

Definition 6.2 *Let $CMML_\omega$ be the similarity type MLR^J_ω where J is the set of transformations $[i,j]^\omega$, cf. (8), and replacements $[i/j]^\omega$, cf. (9).*

Recall from section 2 that the operators $\bigcirc_{ij}$ and $\otimes_{ij}$ have the following interpretation in multi-dimensional models (we drop the superscript ω):

$$\begin{aligned} \mathfrak{M}, s \Vdash \bigcirc_{ij}\varphi &\iff \mathfrak{M}, s \circ [i/j] \Vdash \varphi, \\ \mathfrak{M}, s \Vdash \otimes_{ij}\varphi &\iff \mathfrak{M}, s \circ [i,j] \Vdash \varphi. \end{aligned}$$

Here $s \circ [i/j]$ is the ω-tuple which is like s, with the only possible difference that $s \circ [i/j](i) = s(j)$; and $s \circ [i,j]$ is the ω-tuple s, but with the elements $s(i)$ and $s(j)$ swapped. Now for this similarity type we show that we *can* find a finite schema axiomatization for the class of ω-dimensional local cubes (cf. Theorem 6.8).

6.2 ω-dimensional assignment frames

With respect to the intended structures for these languages, we have a range of possibilities, even if we want to confine ourselves to the multi-dimensional framework. One can still opt for arbitrarily relativized assignment frames, local cubes or 'plainly full' cubes, and there are in fact some interesting new options: weak cubes and S-local cubes (where S is the similarity type). We only give a formal definition of the first class.

Definition 6.3 *Let t be some ω-tuple over some set U; we define ${}^\omega U^{(t)}$ as the set of ω-tuples over U that differ from t in at most finitely many indices, i.e.,*

$${}^\omega U^{(t)} = \{s \in {}^\omega U \mid s \sim_\omega t\},$$

where $\sim_\omega$ is the relation between ω-tuples defined by

$$s \sim_\omega t \iff \{i < \omega \mid s(i) \neq t(i)\} \text{ is finite.}$$

A relativized assignment frame $\mathfrak{C}^W_\omega(U)$ is called a 'weak cube' if W is the disjoint union of a family of sets of the form ${}^\omega U^{(t)}$, or equivalently, if W is closed under the relation $\sim_\omega$. The class of weak cubes is denoted WC_ω.

Note that if we consider finite tuples, the notions of weak cube and cube coincide.

Does it make a difference to the logic, whether we study the class C_ω or WC_ω? It is interesting to note that the answer to this question depends on the similarity type.

Proposition 6.4 $\Theta_{MLR'_\omega}(\mathsf{WC}_\omega) = \Theta_{MLR'_\omega}(\mathsf{C}_\omega)$, *while* $\Theta_{MLR_\omega}(\mathsf{WC}_\omega) \neq \Theta_{MLR_\omega}(\mathsf{C}_\omega)$.

Proof. For the first part of the proposition, the inclusion from left to right is immediate, since every cube is a weak cube. For the other inclusion we reason by contraposition. Assume that the MLR'_ω-formula φ can be satisfied in a weak cube $\mathfrak{C}^W_\omega(U)$. We will show that there is also a cube in which φ is satisfiable. By definition, there is a valuation V and an ω-tuple $s \in W$ such that $\mathfrak{C}^W_\omega(U), V, s \Vdash \varphi$. Define the valuation V' on the cube $\mathfrak{C}_\omega(U)$ as follows:

$$V'(p) = V(p)$$

for all proposition letters p. By a straightforward formula induction one proves that in fact, for *all* MLR'_ω-formuals ψ, we have, for each assignment $r \in W$, that

$$\mathfrak{C}^W_\omega(U), V, r \Vdash \psi \iff \mathfrak{C}_\omega(U), V', r \Vdash \psi. \tag{30}$$

The crucial observation in the modal-inductive steps of the proof of (30) is that the accessibility relations for the diamonds $\Diamond_i$ and $\bigcirc_\sigma$ never take us out of W, since W is closed under the relation $\sim_\omega$. (In other words, $\mathfrak{C}^W_\omega(U)$ is a *generated subframe* of $\mathfrak{C}_\omega(U)$.) The key fact here is that FT_ω only contains *finite* transformations, implying that $r \sim_\omega r \circ \sigma$ for every $\sigma \in FT_\omega$. From (30) it is immediate that φ is satisfiable in the cube $\mathfrak{C}_\omega(U)$.

In order to prove the second part of the proposition, we will show that the universe of a weak cube is *not* always closed under the accessibility relation $\bowtie_\sigma$ if σ is not a finite transformation. Consider for example the transformation $\mathsf{suc} : \omega \to \omega$ given by $\mathsf{suc}(n) = n+1$. Let $U = \{0, 1\}$, consider the tuple $t = (0, 1, 0, 1, \ldots)$ (i.e., $t(i) = i \bmod 2$), and let W be the set ${}^\omega U^{(t)}$. Clearly the tuple $t \circ \mathsf{suc} = (1, 0, 1, 0, \ldots)$ does not belong to W. Hence, for any valuation V on W, we have $\mathfrak{C}^W_\omega(U), V, t \not\Vdash \bigcirc_{\mathsf{suc}}\top$. So $\bigcirc_{\mathsf{suc}}\top \notin \Theta_{MLR_\omega}(\mathsf{WC}_\omega)$. Since obviously $\mathsf{C}_\omega \models \bigcirc_{\mathsf{suc}}\top$, we are finished. QED

The proof of Proposition 6.4 leads us to a brief outline of the second new way of relativizing assignment frames, in which the similarity type determines the class of relativized frames. The basic idea is as follows. For a set H of transformations on ω, an H-local cube is defined as a relativized cube of the form $\mathfrak{C}^W_\omega(U)$ in which W is closed under the accessibility relation $\bowtie_\sigma$ for every $\sigma \in H$, or maybe simpler: W satisfies

$$\text{for all } s \in W \text{ and } \sigma \in H\text{: } s \circ \sigma \in W.$$

The idea is that one always wants $s \circ \sigma$ to be an available assignment whenever s is available and $\bigcirc_\sigma$ is a diamond of the similarity type (cf. the proof of Proposition 6.4, where we had an example of a violation of this condition for the transformation suc).

6.3 ω-dimensional semantics

Finally, we can follow two alternative ways to define the semantics of these ω-dimensional languages in these intended structures. For convenience, let us restrict the discussion to the combination of the similarity type MLR'_ω and the frame class C_ω. To explain the alternative interpretations here, we return to section 2, where we motivated the introduction of the similarity types CML_n and MLR_n, by explaining that we were looking for *modal* versions of the n-variable fragments of first-order logic. We saw that in the finite-dimensional case, the connection between the modal and the first-order formalism are very tight, cf. Proposition 2.4. Recall that the dimension n comes from the *number of variables* in L_n: we wanted to identify assignments of the first-order semantics with tuple-states in the modal framework. Since ordinary first-order logic[6] has ω many variables, it is then natural to expect the framework of ω-dimensional modal logic to be the natural niche for a modal version of ordinary first-order logic. We will see now that this is indeed the case; but the connection between $L_{\omega\omega}$ and for instance MLR_ω is less straightforward than one might expect. The reason for this lies in the fact that in ordinary first-order logic, every predicate has a fixed *finite rank*. So, if P for instance is a dyadic predicate, the intuitive interpretation for P would be a set of pairs; but in an ω-dimensional modal semantics, we have to interpret P as a set of ω-tuples. Thus, it seems that the finite rank of the predicate symbols and the infinite dimension of the modal set-up do not go well together.

There are a number of solutions to this problem. The first and easiest solution would be to adapt the first-order syntax to the modal semantics. For, in the same way that n-dimensional cylindric modal logic corresponds to restricted first-order logic of n-adic predicate symbols (and n variables), MLR_ω would correspond to a language L_ω with ω-adic predicate symbols (and ω many variables). An atomic formula of this language is of the form $P_l v_{\sigma 0} v_{\sigma 1} \ldots$ (where σ is a (finite) transformation on ω). If one does not like infinitely long formulas, one could replace this formula with P_l^σ; this is without any loss of information, since the sequence $v_{\sigma 0} v_{\sigma 1} \ldots$ is completely determined by P_l and σ. Formalisms like L_ω have been studied in the literature under the name of *finitary logic of infinitary relations*, cf. (Sain, 1982).

A second solution would be to question the assumption that one is forced to accept an ω-dimensional modal semantics because the first-order language has ω many variables. The key observation here is that any *formula* in ordinary first-order logic has only a *finite* number of *free* variables. So, one can imagine a set-up in which the universe of an intended model does not consist of a set of ω-tuples over a base set U, but of all *finite* tuples over U. This approach naturally leads to a many-sorted modal formalism, like the one treated in (Kuhn, 1980).

The third solution is the converse of the first one, namely, one can also adapt the modal semantics to the first-order syntax by allowing only certain kinds of valuations in cube models. The basic idea underlying this solution is that m-ary relations can mimick relations of a smaller arity n via a *dummy representation*: if $R \subseteq {}^nU$, then we can represent R as the following subset of ${}^mU$:

$$Dr_n^m(R) = \{s \in {}^mU \mid s{\restriction}n \in R\}. \tag{31}$$

In other words, an m-tuple s is in $Dr_n^m(R)$ iff its 'n-cut-off' tuple (consisting of the first n coordinates of s) is an n-tuple in R.

Now our ω-dimensional semantics can be set up as follows. The universe of our assignment models will be the set ${}^\omega U$ of ω-tuples over some set U, i.e., maps $s : \omega \to U$. Then, for a first-order language having predicate symbols of arbitrary finite rank, we could interpret an

[6] With *ordinary* first-order logic we understand the following language $L_{\omega\omega}$. Its alphabet consists of countably many individual variables, countably many predicate symbols, each of a fixed, finite rank, and the usual logical symbols (boolean connectives, quantifiers and the identity symbol '='). The semantics we have in mind is the usual one.

n-adic relation symbol as a subset of ${}^{\omega}U$ of the form $Dr_n^{\omega}(R)$, where R is an n-ary relation on U. This is in accordance with the usage in first-order model theory, where the formula Pv_0v_1 holds in a model $\mathfrak{M}$ under an assignment s iff $s\lceil 2 = (s_0, s_1)$ is in the extension of P.

Definition 6.5 *Let $\mathfrak{C}$ be the ω-cube over the set U. A model $\mathfrak{M} = (\mathfrak{C}, V)$ is called a* **dummy model**[7] *if V is a dummy valuation, i.e., for every propositional variable p, there is an n such that $V(p)$ is of the form $Dr_n^{\omega}(R)$ for some n-ary relation R on U.*

It can be shown that in some straightforward sense, every dummy model is equivalent to a structure for ordinary first-order logic. Note however, that Definition 6.5 still does not give us $L_{\omega\omega}$, since we did not fix the rank of the propositional variables *beforehand*. We could make this modification in our set-up; but then we have to turn the ω-dimensional language into a many-sorted modal formalism, in which each propositional variable p_l is assigned a rank $\nu(p_l)$, and we can only consider models in which the propositional variable p_l is interpreted as the dummy representation of a $\nu(p_l)$-ary relation. We will *not* take this approach, however; the reason for this is that by fixing the rank of propositional variables, the modal language would become *too much* like first-order logic. For instance, concerning the axiomatics, we would have to add axioms like

$(LF) \qquad p_l \to \Box_i p_l \qquad (i > \nu(p_l))$

In other words, we regain much of the complex bookkeeping of variables in first-order logic.

If we confine ourselves to the dummy semantics in which the propositional variables do not have a fixed rank, it is interesting to note that the restriction to dummy semantics does not give us more theorems, at least in the similarity type MLR'_{ω}.

Proposition 6.6 *Let φ be a MLR'_{ω}-formula. Then φ is satisfiable in a cube model if and only if φ is satisfiable in a dummy cube model.*

Proof. The direction from right to left is trivial; for the other direction, suppose that φ is satisfiable in C_{ω}, i.e., there is an ω-tuple s of a model $\mathfrak{M} = (\mathfrak{C}_{\omega}(U), V)$ such that $\mathfrak{M}, s \Vdash \varphi$. We will turn $\mathfrak{M}$ into a dummy model $\mathfrak{M}' = (\mathfrak{C}_{\omega}(U), V')$ in which φ holds at s as well.

Since φ only uses finitely many operators, and the language MLR'_{ω} contains only finite transformations, it follows that there is a natural number n that is (a) larger than the largest number i such that one of $\Diamond_i$, $\iota\delta_{ij}$ or $\iota\delta_{ji}$ occurs in φ, and (b) also larger than the largest number i such that there is an operator $\bigcirc_{\sigma}$ occurring in φ for which $\sigma(i) \neq i$. It will be intuitively clear that in the evaluation of φ at s, 'only the first n dimensions matter'. This intuition is used as follows; for arbitrary states $s, t \in {}^{\omega}U$, let $mix_n(t, s)$ be the following state:

$$mix_n(t,s) = \begin{cases} t(i) & \text{if } i < n \\ s(i) & \text{if } i \geq n. \end{cases}$$

Now we transform the model $\mathfrak{M}$ into the model $\mathfrak{M}' = (\mathfrak{C}_{\omega}(U), V')$, with V' given by

$$V'(p) = \{t \in {}^{\omega}U \mid mix_n(t,s) \in V(p)\}.$$

[7]A better (and more elegant) name would be 'locally finite and regular' model, since this terminology would be closer to the usage in algebraic logic, cf. (Henkin et al., 1971 1985). For an introduction to the connections between cylindric algebras and first-order model theory we refer to (Monk, 1993).

It is not very difficult to verify that that $\mathfrak{M}'$ is a dummy model. Then, by a straightforward inductive argument, one can show that for all subformulas ψ of φ, and for every $t \in \mathfrak{C}_\omega(U)$, we have

$$\mathfrak{M}', t \Vdash \psi \iff \mathfrak{M}, mix_n(t, s) \Vdash \psi.$$

From this it follows immediately that φ holds in $\mathfrak{M}'$ at the state $s = mix_n(s, s)$ and hence, φ is satisfiable in the dummy model $\mathfrak{M}'$. QED

On the other hand, we loose *compactness*: it is fairly easy to show that the set $\{p \wedge \Diamond_i \neg p \mid i \in \omega\}$ is easily satisfiable in an ω-cube, while it is not satisfiable in a dummy model. To see this, observe that the set contains the negations of the axioms LF.

6.4 ω-dimensional results

It will be clear from the previous subsections that there is really a *landscape* of ω-dimensional modal logics. In this landscape there are still a lot of things to be investigated. In particular, intriguing questions are given by the so-called finitization problem; in our multi-dimensional modal context, one of its manifestations is the question, whether we can find a 'nice' multi-dimensional modal similarity type S such that the S-theory of the class C_ω allows a 'nice' finite *schema* axiomatization. For more information and various, positive and negative results, we refer to (Németi, 1991), respectively (Sain and Gyuris, 1995).

Here we will only discuss the question, whether we can generalize the positive results in Theorems 4.2 and 5.1 to the infinite-dimensional case. In the first section we already saw, that for a finite schema axiomatization we have to consider the similarity type $CMML_\omega$ (cf. Definition 6.2). As explained in section 2, the similarity types MLR_n and $CMML_n$ are term-definably equivalent, when interpreted on local cubes. Clearly the same holds for MLR'_ω and $CMML_\omega$. Having this in mind, a completeness and a decidability result are not very difficult to obtain.

Completeness.

The proof of the following theorem (of which we only give a sketch) crucially depends on insights from M. Hollenberg and H. Andréka.

Definition 6.7 *Let QAX_ω be the extension of the basic derivation system of the similarity type $CMML_\omega$ with the following axiom schemes:*

(*CM1*)	$p \to \Diamond_i p$	
(*CM2*)	$p \to \Box_i \Diamond_i p$	
(*CM3*)	$\Diamond_i \Diamond_i p \to \Diamond_i p$	
(*CM5*)	$\iota\delta_{ii}$	
(*Q1′*)	$\bigcirc_{ij} \neg p \leftrightarrow \neg \bigcirc_{ij} p$	
	$\otimes_{ij} \neg p \leftrightarrow \neg \otimes_{ij} p$	
(*Q4′*)	$\bigcirc_{ik} \iota\delta_{ij} \leftrightarrow \iota\delta_{jk}$	
	$\bigcirc_{kl} \iota\delta_{ij} \leftrightarrow \iota\delta_{ji}$	*(provided $k \notin \{i, j\}$).*
	$\otimes_{ik} \iota\delta_{ij} \leftrightarrow \iota\delta_{kj}$	
	$\otimes_{kl} \iota\delta_{ij} \leftrightarrow \iota\delta_{ij}$	*(provided $k, l \notin \{i, j\}$).*
(*Q5*)	$\bigcirc_{ij} p \leftrightarrow \Diamond_i (p \wedge \iota\delta_{ij})$	*(provided $i \neq j$).*

Besides these, we take the following set of **Jónsson axioms***, which are all subject to the* *condition that i, j, k and l are distinct, with the possible exception of i and l.*

(*J1*) $\otimes_{ij}p \leftrightarrow \otimes_{ji}p$
(*J2*) $\otimes_{ij}\otimes_{ij}p \leftrightarrow p$
(*J3*) $\otimes_{ij}\otimes_{ik}p \leftrightarrow \otimes_{jk}\otimes_{ij}p$
(*J4*) $\otimes_{ij}\bigcirc_{ki}p \leftrightarrow \bigcirc_{kj}\otimes_{ij}p$
(*J5*) $\otimes_{ij}\bigcirc_{ji}p \leftrightarrow \bigcirc_{ij}p$
(*J6*) $\bigcirc_{ji}\bigcirc_{kl}p \leftrightarrow \bigcirc_{kl}\bigcirc_{ji}p$
(*J7*) $\bigcirc_{ji}\bigcirc_{jl}p \leftrightarrow \bigcirc_{jl}p$

The axioms *J1* – *J7* are interesting because of the following. It follows from a result by Jónsson (Jónsson, 1962) that these axioms are necessary and sufficient to prove every formula of the form

$$\bigcirc_{\sigma_1}\bigcirc_{\sigma_2}\cdots\bigcirc_{\sigma_n}p \leftrightarrow \bigcirc_{\tau_1}\bigcirc_{\tau_2}\cdots\bigcirc_{\tau_n}p$$

where the σ_i, τ_i are simple substitutions $[i/j]$ or transpositions $[i,j]$ such that $\sigma_1 \circ \sigma_2 \ldots \sigma_n = \tau_1 \circ \tau_2 \ldots \tau_n$.

Theorem 6.8 *QAX_ω is strongly sound and complete with respect to the class LC_ω of ω-dimensional local cubes.*

Proof. The basic idea of the proof is to modify the mosaic-based approach of section 4 for the infinite-dimensional case, using operators $\bigcirc_\sigma$ as *abbreviated* diamonds, for all finite transformations σ. We need to make sure that all instances of the axioms *Q1* – *Q6* are derivable in QAX_ω. *Q1* follows immediately from *Q1′*. For *Q2* – *Q3* the Jónsson axioms are sufficient; for *Q4* we also need *Q4′*. For *Q6*, M. Hollenberg observed that the formula

$$\bigcirc_{ij}\diamond_i p \leftrightarrow \diamond_i p$$

is derivable in QAX_ω. H. Andréka showed that from this, all instances of *Q6* are derivable in QAX_ω, as follows.

Let σ and τ be finite transformation such that $\sigma \equiv_i \tau$. Then $\sigma \circ [i/j] = \tau \circ [i/j]$, for any $j \neq i$. It follows that the formula $\bigcirc_\sigma\bigcirc_{ij}p \leftrightarrow \bigcirc_\tau\bigcirc_{ij}p$ is derivable from the Jónsson axioms, and hence (substitute $\diamond_i p$ for p), the formula $\bigcirc_\sigma\bigcirc_{ij}\diamond_i p \leftrightarrow \bigcirc_\tau\bigcirc_{ij}\diamond_i p$. But since $\bigcirc_{ij}\diamond_i p$ and $\diamond_i p$ are provably equivalent, this implies QAX_ω-theoremhood of $\bigcirc_\sigma\diamond_i p \leftrightarrow \bigcirc_\tau\diamond_i p$.

The main difference with respect to the finite-dimensional case is that we can and will work with the so-called weak local cubes: these are generalized assignment frames based on subsets $W \subseteq {}^\omega U$ satisfying, for $s \in W$, that $s \circ \sigma \in W$ only for the *finite* transformations $\sigma \in {}^\omega\omega$ (cf. also Definition 6.3 and Proposition 6.4). QED

Decidability.

As an infinite-dimensional analogue to the decidability result in Theorem 5.1, we prove decidability for $CMML_\omega$ over the local cubes. As a corollary we obtain decidability of $\Theta_{MLR'_\omega}(\mathsf{LC}_\omega)$.

Theorem 6.9 *The theory $\Theta_{CMML_\omega}(\mathsf{LC}_\omega)$ is decidable.*

Proof. The crucial claim in the proof of the theorem is an adaptation of lemma 10.10(ii) in (Németi, 1995) to the similarity type *CMML* (the mentioned lemma concerns the type *CML*).

Claim 1 Let φ be a *CMML*$_n$-formula. Then, for $1 < n < \alpha$, $\alpha \leq \omega$,

$$\mathsf{LC}_{n+1} \models \varphi \iff \mathsf{LC}_\alpha \models \varphi.$$

The theorem follows immediately from Claim 1, since Theorem 5.1 provides an algorithm deciding whether $\mathsf{LC}_{n+1} \models \varphi$.

Németi's paper (Németi, 1995) also contains an example showing that the '+1-part' in the formulation is necessary: he gives a formula φ such that φ is LC_n-satisfiable but not LC_ω-satisfiable.

Proof of Claim Let φ, α and n be as in the claim. We show that

$$\varphi \text{ is } \mathsf{LC}_\alpha\text{-satisfiable iff } \varphi \text{ is } \mathsf{LC}_n\text{-satisfiable,}$$

from which the claim follows immediately.

We start with the unproblematic direction (from left to right): here we actually have the stronger statement that

$$\mathsf{LC}_\alpha\text{-satisfiability implies } \mathsf{LC}_n\text{-satisfiability.} \tag{32}$$

Let $\mathfrak{M}$ be a LC_α-model. Assume $\mathfrak{M}, s \Vdash \varphi$. Let $\mathfrak{M}_s$ be the submodel which is "n-generated" from s (i.e., we only take the accessibility relations $\equiv_i, \bowtie_{[i,j]}$ and $\bowtie_{[i/j]}$ into account for which $i, j < n$). Then $\mathfrak{M}_s, s \Vdash \varphi$. It is easy to see that $\mathfrak{M}_s$ is isomorphic to a LC_n-model (hint: cut off all "tails" from the sequences; the model is locally cube because of the relations $\bowtie_{[i,j]}$ and $\bowtie_{[i/j]}$). But then φ is LC_n-satisfiable.

For the other direction, let $\mathfrak{M} = (\mathfrak{C}^W_\omega(U), V)$ be a LC_{n+1}-model, and $\mathfrak{M}, s \Vdash \varphi$. Since W is locally cube, we may assume the existence of sets U_i such that $W = \bigcup_{i\in I} {}^{n+1}U_i$. Define

$$\begin{array}{lcl} W^* & = & \bigcup_{i\in I} {}^{\omega}U_i \\ B & = & \{(x,y) \in W \times W^* : y{\restriction}_{n+1} = x,\ x(n) = s(n),\ \text{and } (\forall i > n)\, y(i) = s(n)\} \end{array}$$

$\mathfrak{M}^* = (\mathfrak{C}^{W^*}_\omega(U), V^*)$ is the LC_α-model with V^* defined by $V^*(p) = \{y \in W^* : \exists x((x,y) \in B$ and $x \in V(p)\}$. We leave it to the reader to verify that V^* is well-defined and that s is an element of the domain of B. Finally, a relatively easy inductive argument[8] shows that for all *CMML*$_n$-formulas ψ and for all $(x,y) \in B$:

$$\mathfrak{M}, x \Vdash \psi \iff \mathfrak{M}^*, y \Vdash \psi. \tag{33}$$

But then we can satisfy φ in $\mathfrak{M}^*$, actually $\mathfrak{M}^*, s * (s(n), s(n), \ldots) \Vdash \varphi$. ◀

QED

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[8] An easy way to see that (33) holds is by showing that B is an 'n-bisimulation' between $\mathfrak{M}$ and $\mathfrak{M}^*$ (i.e., we only take the relations $D_{ij}, \equiv_i, \bowtie_{[i,j]}$ and $\bowtie_{[i/j]}$, for $i, j < n$ into account).

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