Completeness for Flat Modal Fixpoint Logics (Extended Abstract)

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Abstract. Given a set Γ of modal formulas of the form $\gamma(x, \boldsymbol{p})$, where x occurs positively in γ, the language $\mathcal{L}_\sharp(\Gamma)$ is obtained by adding to the language of polymodal logic **K** connectives $\sharp_\gamma$, $\gamma \in \Gamma$. Each term $\sharp_\gamma$ is meant to be interpreted as the parametrized least fixed point of the functional interpretation of the term $\gamma(x)$. Given such a Γ, we construct an axiom system $\mathbf{K}_\sharp(\Gamma)$ which is sound and complete w.r.t. the concrete interpretation of the language $\mathcal{L}_\sharp(\Gamma)$ on Kripke frames. If Γ is finite, then $\mathbf{K}_\sharp(\Gamma)$ is a finite set of axioms and inference rules.
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1 Introduction

Suppose that we extend the language of basic (poly-)modal logic with a set $\{\sharp_\gamma \mid \gamma \in \Gamma\}$ of so-called *fixpoint connectives*, which are defined as follows. Each connective $\sharp_\gamma$ is indexed by a modal formula $\gamma(x, \boldsymbol{p})$ in which x occurs only positively, and the intended meaning of the formula $\sharp_\gamma(\boldsymbol{p})$ in a labelled transition system (Kripke model) is the least fixpoint of the formula $\gamma(x, \boldsymbol{p})$,

$$\sharp_\gamma(\boldsymbol{p}) \equiv \mu x.\gamma(x, \boldsymbol{p}).$$

Many logics of interest in computer science are of this kind, such fixpoint connectives can be found for instance in **PDL** [6]: $\langle a^* \rangle = \mu x.p \vee \langle a \rangle x$, in **CTL** [4]: $EU(p, q) = \mu x.p \vee (q \wedge \Diamond x)$ and $AFp = \mu x.p \vee \Box x$, and in **LTL**. Generalizing these examples we arrive at the notion of flat modal fixpoint logic. Let $\mathcal{L}_\sharp(\Gamma)$ denote the language we obtain if we extend the syntax of (poly-)modal logic with a connective $\sharp_\gamma$ for every $\gamma \in \Gamma$. The flat modal fixpoint logic induced by Γ is the set of $\mathcal{L}_\sharp(\Gamma)$-validities, i.e., the collection of formulas in the language $\mathcal{L}_\sharp(\Gamma)$ that are true at every state of every Kripke model.

Clearly, every fixpoint connective of this kind can be seen as a macro over the language of the modal μ-calculus. Because the associated formula γ of a

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fixpoint connective is itself a basic modal formula (which explains our name *flat*), it is easy to see that every flat modal fixpoint language corresponds to a relatively simple alternation-free fragment of the modal μ-calculus [7]. Despite this restrictive expressive power, flat modal fixpoint logics such as **CTL** and **LTL** are often preferred by end users, because of their transparency and simpler semantics. In fact, most verification tools implement some flat fixpoint logic rather than the full μ-calculus.

Up to now however, general investigations of flat modal fixpoint logics have not been pursued. Our research is driven by the wish to understanding the combinatorics of fixpoint logics in their wider algebraic and order theoretic setting. As such it continues earlier work by the first author [9, 10]. In this paper we move on in this direction by addressing the problem of *uniformly* axiomatizing flat fixpoint logics. Concretely, our main contribution concerns an algorithm that, when given as input a (finite) set of positive formulas Γ, produces a (finite) axiom system $\mathbf{K}_\sharp(\Gamma)$ which is sound and complete w.r.t. the standard interpretation of the language $\mathcal{L}_\sharp(\Gamma)$ in Kripke frames. Note that this result does not follow from Walukiewicz' completeness result for the modal μ-calculus [11]. Rather, it should be interpreted by saying that we add to Walukiewicz' theorem the observation that, modulo a better choice of axioms, proofs of validities in a given flat fragments of the modal μ-calculus can be carried out *inside* this fragment.

Let us summarize the construction of and the ideas behind the axiom system $\mathbf{K}_\sharp(\Gamma)$. Mimicking Kozen's axiomatization of the modal μ-calculus, an intuitive axiomatization would be to add to a standard axiomatization $\mathbf{K}$ for (poly-)modal logic, the axiom and the derivation rule

$$\vdash \gamma(\sharp_\gamma(\boldsymbol{p}), \boldsymbol{p}) \to \sharp_\gamma(\boldsymbol{p})\,, \qquad (\sharp_\gamma\text{-prefix})$$

$$\text{from } \vdash \gamma(\varphi, \boldsymbol{p}) \to \varphi \text{ infer } \vdash \sharp_\gamma(\boldsymbol{p}) \to \varphi\,, \qquad (\sharp_\gamma\text{-least})$$

for each $\gamma \in \Gamma$. These axioms and rules express that $\sharp_\gamma(\boldsymbol{p})$ is the least prefixpoint of $\gamma(-, \boldsymbol{p})$. The proof we present reveals that this is already a complete axiomatization if all the formulas in Γ are *disjunctive* or *aconjunctive* in the sense of [11, 7]. However, as soon as arbitrary formulas are considered, the usual problems on the use of conjunction within fixpoints arise obstructing the way to completeness.

The intuitive axiomatization – which we may well call *Kozen's* or Park's [5] axiomatization – may however be modified, and the Subset Construction [1, §9.5] suggests how to successfully do it. Roughly speaking, this is a procedure that transforms a $\gamma \in \Gamma$ into a *disjunctive* system of equations – called here $\mathcal{P}_+(T^\gamma_\sharp)$. It is shown in [1] that on *complete* lattices, the least solution of $\mathcal{P}_+(T^\gamma_\sharp)$ is constructible from the least fixed point of γ. The key idea of *our* axiomatization $\mathbf{K}_\sharp(\Gamma)$ is to *force* this relation to hold on arbitrary algebraic models, by imposing $\mathcal{P}_+(T^\gamma_\sharp)$ to have a least solution, constructible from $\sharp_\gamma$.

While our methodology is based on earlier work by the first author, we extend the results of [9] in two significant ways. First, the idea to use the subset construction of Arnold & Niwiński to *define an axiom system* for flat modal fixpoint logics, is novel. And second, the Representation Theorem presented in

Section 6 strengthens the main result of [9], which applies to complete algebras only, to a completeness result for *Kripke frames*.

2 Preliminaries

We first give a formal definition of the syntax and semantics of flat modal fixpoint logics. We then discuss the reformulation of modal logic in terms of the cover modalities ∇_i. Finally, we introduce modal $\sharp$-algebras as the key structures of the algebraic setting in which we shall prove our completeness result. For background in the algebraic perspective on modal logic, see [2].

Flat Modal Fixpoint Logic. Throughout this paper we fix a set Γ of (poly-) modal formulas/terms $\gamma(x, \boldsymbol{p})$ where x occurs only positively, i.e. under no negation. The vector $\boldsymbol{p}$ may be different for each γ, but we decided not to make this explicit in the syntax, in order not to clutter up notation. We also fix a finite set I of atomic actions.

Definition 1. *The set $\mathcal{L}_\sharp(\Gamma)$ of flat modal fixpoint formulas associated with Γ is defined by the following grammar:*

$$\varphi ::= p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Diamond_i \varphi \mid \sharp_\gamma(\boldsymbol{\varphi})$$

where i and γ range over I and Γ, respectively.

We move on to the intended semantics of this language. A labeled transition system of type I, or equivalently a Kripke model, is a structure $\mathbb{S} = \langle S, \{ R_i \mid i \in I \}\rangle$, where S is a set of states and $R_i \subseteq S \times S$ is, for each $i \in I$, a transition relation. Given a valuation $\boldsymbol{v} : P \longrightarrow \mathcal{P}(S)$ of propositional variables as subsets of states, we inductively define the semantics of flat modal fixed point formulas. Most of the inductive clauses are standard, for instance:

$$\|\Diamond_i \varphi\|_{\boldsymbol{v}} = \{ x \in S \mid \exists y \in S \text{ s.t. } xR_i y \text{ and } y \in \|\varphi\|_{\boldsymbol{v}} \}$$

In order to define $\|\sharp_\gamma(\boldsymbol{\varphi})\|_{\boldsymbol{v}}$, let x be a variable which is not free in $\boldsymbol{\varphi}$ and, for $Y \subseteq S$, let $(\boldsymbol{v}, x \to Y)$ be the valuation sending x to Y and every other variable y to $\boldsymbol{v}(y)$. We let

$$\|\sharp_\gamma(\boldsymbol{\varphi})\|_{\boldsymbol{v}} = \bigcap \{ Y \mid \|\gamma(x, \boldsymbol{\varphi})\|_{(\boldsymbol{v}, x \to Y)} \subseteq Y \}\,. \tag{1}$$

By the Knaster-Tarski theorem, Definition (1) just says that the interpretation of $\sharp_\gamma(\boldsymbol{\varphi})$ is the least fixed point of the order preserving function sending Y to $\|\gamma(x, \boldsymbol{\varphi})\|_{(\boldsymbol{v}, x \to Y)}$.

The Cover Modalities ∇_i. We will frequently work in a reformulation of the modal language based on the *cover modalities* ∇_i, $i \in I$. These connectives, taking a *set* of formulas as their argument, can be defined in terms of the box and diamond operators:

$$\nabla_i \Phi := \Box_i \bigvee \Phi \wedge \bigwedge \Diamond_i \Phi,$$

where $\Diamond_i \Phi$ denotes the set $\{\Diamond_i \varphi \mid \varphi \in \Phi\}$. Conversely, the standard diamond and box modalities can be defined in terms of the cover modality:

$$\Diamond_i \varphi \equiv \nabla_i\{\varphi, \top\} \qquad\qquad \Box_i \varphi \equiv \nabla_i \varnothing \vee \nabla_i\{\varphi\},$$

from which it follows that we may equivalently base our modal language on ∇_i as a primitive symbol. What makes the cover modality so useful is the distributive law:

$$\nabla_i \Phi \wedge \nabla_i \Phi' \equiv \bigvee_{Z \in \Phi \bowtie \Phi'} \nabla_i \{\, \varphi \wedge \varphi' \mid (\varphi, \varphi') \in Z \,\}\,, \tag{2}$$

where $\Phi \bowtie \Phi'$ denotes the set of relations $R \subseteq \Phi \times \Phi'$ that are *full* in the sense that for all $\varphi \in \Phi$ there is a $\varphi' \in \Phi'$ with $(\varphi, \varphi') \in R$, and vice versa. We mention two key corollaries of (2), but first we need some definitions.

Definition 2. *Let X, Y be sets of variables. Then we define the following sets of formulas/terms:*

1. *$Lit(X)$ is the set $\{x, \neg x \mid x \in X\}$ of literals over X,*
2. *$SC(X;Y)$ is the set of special conjunctions of the form $\bigwedge \Lambda \wedge \bigwedge_{j \in J} \nabla_j \Phi_j$, where $\Lambda \subseteq Lit(X)$, $J \subseteq I$, and $\Phi_j \subseteq Y$ for each $j \in J$.*
3. *$D(X)$ is the set of disjunctive terms over X given by the following grammar:*
$$\varphi ::= x \mid \bot \mid \varphi \vee \varphi$$
4. *$DT(X)$ is the set of distributive terms over X given by the following grammar:*
$$\varphi ::= x \mid \bot \mid \varphi \vee \varphi \mid \top \mid \varphi \wedge \varphi$$
5. *$MT(X)$ is the set of modal terms over X given by the following grammar:*
$$\varphi ::= x \mid \neg x \mid \bot \mid \varphi \vee \varphi \mid \top \mid \varphi \wedge \varphi \mid \nabla_i \Phi$$
where $i \in I$ and $\Phi \subseteq MT(X)$.
6. *$MT_\nabla(X)$ is the set of terms in* ∇-normal form *given by the following grammar:*
$$\varphi ::= \bot \mid \varphi \vee \varphi \mid \bigwedge \Lambda \wedge \bigwedge_{j \in J} \nabla_j \Phi_j,$$
where $\Lambda \subseteq Lit(X)$, $J \subseteq I$, and $\Phi_j \subseteq MT_\nabla(X)$ for each $j \in J$.

Note the restricted use of conjunction in terms in ∇-normal form.

Proposition 3. *Let P and Φ be sets of proposition letters, and define $Y := \{y_\Psi \mid \Psi \subseteq \Phi\}$. There is an effective procedure associating with each modal formula $\varphi \in DT(SC(P;\Phi))$ a formula $\varphi^\vee \in D(SC(P;Y))$ such that φ is equivalent to the formula obtained from $\varphi^\vee$ by uniformly substituting each variable y_Ψ by the conjunction $\bigwedge \Psi$.*

Proposition 4. *Let X be a set of proposition letters. There is an effective procedure associating with each modal formula $\varphi \in MT(X)$ an equivalent formula $\varphi^- \in MT_\nabla(X)$.*

Modal Algebras. We now move on to the algebraic perspective on flat modal fixpoint logic. Recall that a *modal algebra* (of type I) is a structure of the form $A = \langle A, \bot, \top, \neg, \wedge, \vee, \{\Diamond_i^A \mid i \in I\}\rangle$, where each $\Diamond_i : A \to A$ preserves all finite joins of the Boolean algebra $\langle A, \bot, \top, \neg, \wedge, \vee\rangle$.

Definition 5. *Let $A = \langle A, \leq\rangle$ and $B = \langle B, \leq\rangle$ be two partial orders. Suppose that $f : A \to B$ and $g : B \to A$ are order-preserving maps such that $fa \leq b$ iff $a \leq gb$, for all $a \in A$ and $b \in B$. Then we call (f, g) an* adjoint pair*, and say that f is the* left adjoint *of, or* residuated *by, g, and that g is the* right adjoint*, or residual, of f. We say that f is an* $\mathcal{O}$-adjoint *if it satisfies the weaker property that for every $b \in B$ there is a finite set $G_f(b) \subseteq A$ such that*

$$fa \leq b \text{ iff } a \leq a' \text{ for some } a' \in G_f(b),$$

for all $a \in A$ and $b \in B$.

It is well known that left adjoint maps preserve all existing joins of a poset. Similarly, one may prove that $\mathcal{O}$-adjoints preserve all existing joins of *directed* sets.

Modal $\sharp$-Algebras. Given a modal algebra A, a modal formula $\gamma(x, p_1, \ldots, p_n)$ is interpreted as a map $\gamma^A : A \times A^n \to A$, its *term function*. Given a vector $\boldsymbol{b} = (a_1, \ldots, a_n) \in A^n$, we let $\gamma_{\boldsymbol{b}}^A : A \to A$ denote the map given by

$$\gamma_{\boldsymbol{b}}^A(a) := \gamma^A(b, \boldsymbol{a}). \tag{3}$$

Definition 6. *A modal $\sharp$-algebra is a modal algebra A endowed with an operation $\sharp_\gamma^A$ for each $\gamma \in \Gamma$ such that for each $\boldsymbol{b}$, $\sharp_\gamma^A(\boldsymbol{b})$ is the least fixpoint of $\gamma_{\boldsymbol{b}}^A$ as defined in (3).*

In this paper we will be mainly interested in two kinds of modal $\sharp$-algebras: the "concrete" or "semantic" ones that encode a Kripke frame, and the "axiomatic" ones that can be seen as algebraic versions of the axiom system $\mathbf{K}_\sharp$ to be defined in the next section. We first consider the concrete ones.

Definition 7. *Let $\mathbb{S} = \langle S, \{ R_i \mid i \in I \}\rangle$ be a transition system. Define, for each $i \in I$, the operation $\langle R_i\rangle$ by putting, for each $X \subseteq S$, $\langle R_i\rangle X = \{ y \in S \mid \exists x \in X \text{ s.t. } yR_ix \}$. The $\sharp$-complex algebra $\mathbb{S}^\sharp$ is given as the structure*

$$\langle \mathcal{P}(S), \varnothing, S, \overline{(\cdot)}, \cup, \cap, \{ \langle R_i\rangle \mid i \in I \}\rangle.$$

We will also call these structures Kripke $\sharp$-algebras*.*

Definition 8. *Let $A = \langle A, \leq\rangle$ be a partial order with least element $\bot$, and let $f : A \to A$ be an order-preserving map on A. For $k \in \omega$ and $a \in A$, we inductively define $f^k a$ by putting $f^0 a := a$ and $f^{k+1} a := f(f^k a)$. If f has a least fixpoint $\mu.f$, then we say that this least fixpoint is* constructive *if $\mu.f = \bigvee_{k\in\omega} f^k(\bot)$. A modal $\sharp$-algebra is called constructive if $\sharp_\gamma^A(\boldsymbol{b})$ is a constructive least fixpoint, for each $\gamma \in \Gamma$ and each $\boldsymbol{b}$ in A.*

We explain now why $\mathcal{O}$-adjoints are relevant for the theory of the least fixed point. If $f : A^n \longrightarrow A$ is an $\mathcal{O}$-adjoint, say that $V \subseteq A$ is f-closed if $y \in V$ and $\boldsymbol{a} = (a_1, \ldots, a_n) \in G_f(y)$ implies $a_i \in V$ for $i = 1, \ldots, n$. If $\mathcal{F}$ is a family of $\mathcal{O}$-adjoints $f : A^n \longrightarrow A$, say that V is $\mathcal{F}$-closed if it is f-closed for each $f \in \mathcal{F}$.

Definition 9. *A family of $\mathcal{O}$-adjoints $\mathcal{F} = \{ f_i : A^{n_i} \longrightarrow A \mid i \in I \}$ is said to be* finitary *if, for each $x \in A$, the least set $\mathcal{F}$-closed set containing x is finite. The $\mathcal{O}$-adjoint $f : A \longrightarrow A$ is finitary if the singleton $\{ f \}$ is finitary.*

Clearly, if f belongs to a finitary family, then it is finitary.

Proposition 10. *If $f : A \longrightarrow A$ is a finitary $\mathcal{O}$-adjoint, then its least prefixed point, whenever it exists, is constructive.*

3 The axiomatization

The axiomatization we shall propose depends on what is informally called the *subset construction* [1, Theorem 9.3.4]. This transformation takes as input a set of modal terms and produces a set of modal terms in ∇-normal form that are equivalent – w.r.t. the respective least prefixed points – to the terms given in input. Since the transformation plays an essential role both in the proposed axiomatization as well as in the proof of its completeness, we recall it and, at the same time, we adapt it to the setting of flat fixpoint logic.

Before carrying on, let us fix some notation. If $t \in MT(Y \cup P)$ and $\{ s_y \mid y \in Y \} \subseteq MT(X)$ is a collection of terms indexed by Y, then we shall denote by $\boldsymbol{s}$ such a collection, and denote by $t[\boldsymbol{s}/\boldsymbol{y}]$ the result of simultaneously substituting every variable $y \in Y$ with the term s_y.

In order to obtain the axiomatization, the following steps must be performed, for each $\gamma(x, p_1, \ldots, p_n) \in \Gamma$.

(i) *Transform γ into an equivalent guarded formula.* We can assume that each occurrence of x is guarded in γ, that is, each occurrence of x is in the scope of some modal operator. As a matter of fact, our goal is to axiomatize the least prefixed point of $\gamma(x)$. If x is not guarded in γ, then we can find terms γ_1, γ_2 such that the equation

$$\gamma(x, \boldsymbol{p}) = (x \wedge \gamma_1(x, \boldsymbol{p})) \vee \gamma_2(x, \boldsymbol{p})\,,$$

holds on every modal algebra, and x is guarded in both γ_1 and γ_2. It is easily seen that, on every modal algebra, γ and γ_2 have the same set of prefixed points. Thus, instead of axiomatizing $\sharp_\gamma$, we can equivalently axiomatize $\sharp_{\gamma_2}$.

(ii) *Transform γ into an equivalent system of equations T^γ.* By Proposition 4, we can assume that $\gamma \in MT_\nabla(\{ x \} \cup \{ P \})$, where $P = \{ p_1, \ldots, p_n \}$. Let SC_γ denote the set of subformulas of γ that are special conjunctions, i.e., that are of the form $\bigwedge \Lambda \wedge \bigwedge_{j \in J} \nabla_j \Phi_j$ where $\Lambda \subseteq Lit(\{ x \} \cup P)$ and $J \subseteq I$. If $\psi \in SC_\gamma$, then we modify it as follows: (a) if its set of literals Λ does not contain x, $\Lambda \subseteq Lit(P)$, then we let $\widetilde{\psi} = \psi$, (b) otherwise we can write $\psi = x \wedge \psi'$, where ψ' is a special conjunction whose set of literals does not contain x, and, in this case, we let $\widetilde{\psi} = \psi'$. Moreover, we let $\widetilde{\gamma} = \gamma$. Let $Z = \{ z_\psi \mid \psi \in \{ \gamma \} \cup SC_\gamma \}$ be a set of variables, disjoint from $\{ x \}$ and P, in bijection with $\{ \gamma \} \cup SC_\gamma$. Express each $\widetilde{\psi}$, $\psi \in \{ \gamma \} \cup SC_\gamma$, as the result of substituting the modified version of the special conjunctions into a modal term t_{z_ψ}, of modal depth one, whose variables are among Z and x:

$$\widetilde{\psi} = t_{z_\psi}[\, \widetilde{\varphi}/z_\varphi \mid \varphi \in \{ \gamma \} \cup SC_\gamma \text{ and } \varphi \text{ is a proper subformula of } \psi \,]\,.$$

The reader will have no difficulties verifying that each term t_z is a disjunction of special conjunctions $\bigwedge \Lambda \wedge \bigwedge_{j \in J} \nabla_j T_j$, where $\Lambda \subseteq Lit(P)$ and $t \in DT(\{ x \} \cup Z)$ whenever $j \in J$ and $t \in T_j$. We call $T^\gamma = \langle Z, \{ t_z \mid z \in Z \} \rangle$ the *system representation of* γ.

(iii) *Construct the system* $T^\gamma_\sharp$. This system is obtained from T^γ by substituting each occurrence of x with z_γ. That is, if we let $r_z = t_z[z_\gamma/x]$, $z \in Z$, then $T^\gamma_\sharp = \langle Z, \{ r_z \mid z \in Z \} \rangle$. Observe that each term r_z is a disjunction of special conjunctions $\bigwedge \Lambda \wedge \bigwedge_{j \in J} \nabla_j T_j$ with now $t \in DT(Z)$ (instead of $t \in DT(\{ x \} \cup Z)$), for $j \in J$ and $t \in T_j$, and, as before, $\Lambda \subseteq Lit(P)$.

(iv) *Construct* $\mathcal{P}_+(T^\gamma_\sharp)$, *the powerset system of* $T^\gamma_\sharp$. Let $Y = \{ y_S \mid S \in \mathcal{P}_+(Z) \}$ be a set of new variables in bijection with $\mathcal{P}_+(Z)$, the set of non empty subsets of Z. For $S \in \mathcal{P}_+(Z)$, let

$$z_{y_S} = \bigwedge_{z \in S} z\,,$$

and denote by $\boldsymbol{z}$ the vector of terms $\{ z_y \mid y \in Y \}$.

Lemma 11. *A collection of terms* $\{ q_y \mid y \in Y \}$ *can be constructed such that*

- *each term* q_y *is a disjunction of special conjunctions* $\bigwedge \Lambda \wedge \bigwedge_{j \in J} \nabla_j D_j$, *where* $\Lambda \subseteq Lit(P)$ *and* $d \in D(Y)$ *for each* $d \in D_j$,
- *the equations*

$$\bigwedge_{z \in S} r_z = q_{y_S}[\boldsymbol{z}/\boldsymbol{y}] \tag{4}$$

hold on every modal algebra.

The pair $\mathcal{P}_+(T^\gamma_\sharp) = \langle Y, \{ q_y \mid y \in Y \} \rangle$ is what we call the powerset system of $T^\gamma_\sharp$. We remark that the terms q_y validating the equations (4) may be constructed by iteratively applying distributive laws of distributive lattices as well as the distributive law (2) of the cover modalities.

(v) *Produce the axiom system for* γ. Recall that $\widetilde{\psi}$, $\psi \in \{ \gamma \} \cup SC_\gamma$, are the modified special conjunctions of γ. Let

$$\widetilde{\psi}^\sharp_{y_S} = \bigwedge_{z_\psi \in S} \widetilde{\psi}[\sharp_\gamma/x]\,, \qquad\qquad S \in \mathcal{P}_+(Z)\,,$$

and, as usual, let $\widetilde{\boldsymbol{\psi}}^\sharp$ be the vector of terms $\{ \widetilde{\psi}^\sharp_y \mid y \in Y \}$.

Definition 12. *Let* $\mathbf{K}$ *be a standard axiomatization which completely axiomatizes the set of polymodal validities. The axiom system* $\mathbf{K}_\sharp(\gamma)$ *is obtained by adding to* $\mathbf{K}$ *the axiom* ($\sharp_\gamma$-prefix)*, the derivation rules* ($\sharp_\gamma$-least)*, the axioms:*

$$\vdash q_{y_S}[\widetilde{\boldsymbol{\psi}}^\sharp/\boldsymbol{y}] \to \widetilde{\psi}^\sharp_{y_S}, \qquad S \in \mathcal{P}_+(Z)\,,$$

as well as the following derivation rule:

$$\begin{aligned} &\textit{from}\ \{ \vdash q_{y_T}[\boldsymbol{\varphi}/\boldsymbol{y}] \to \varphi_{y_T} \ \mid\ T \in \mathcal{P}_+(Z) \,\} \\ &\textit{infer}\ \vdash q_{y_S}[\widetilde{\boldsymbol{\psi}}^\sharp/\boldsymbol{y}] \to \varphi_{y_S}\,, \qquad S \in \mathcal{P}_+(Z)\,. \end{aligned}$$

Finally, the axiom system $\mathbf{K}_\sharp(\Gamma)$ *is obtained as the union of all the axioms and inferences rules of the axiom systems* $\mathbf{K}_\sharp(\gamma)$*,* $\gamma \in \Gamma$*.*

The axioms and derivation rules of the form ($\sharp_\gamma$-prefix) and ($\sharp_\gamma$-least) may be eliminated from the axiom system. We include them mainly for clarity of exposition. On the other hand, if γ itself is already in ∇-normal form, then the simpler axiomatization, adding ($\sharp_\gamma$-prefix) and ($\sharp_\gamma$-least) to $\mathbf{K}$, already suffices. We can now formulate the main result of this paper:

Theorem 13. *The axiom system* $\mathbf{K}_\sharp(\Gamma)$ *is sound and complete with respect to the Kripke semantics of* $\mathcal{L}_\sharp(\Gamma)$*.*

Soundness will be discussed in the remainder of this section, an overview of the completeness proof will be given in the next.

Algebraic Interpretation. We elucidate now the algebraic meaning of the proposed axiomatization. To begin with, let us formally define a *modal system* (of equations) as a pair $T = \langle Z, \{\, t_z \,\}_{z \in Z} \rangle$ where Z is a finite set of variables and $t_z \in MT(Z \cup P)$ for each $z \in Z$. We say that a modal system is pointed if it comes with a specified variable $z_0 \in Z$. Given a modal system T and a modal algebra A, there exists a unique function $T^A : A^Z \times A^P \longrightarrow A^Z$ such that, for each projection $\pi_z : A^Z \longrightarrow A$, $\pi_z \circ T^A = t^A_z$. We shall say that T^A is the interpretation of T in A. Whenever it exists, we shall denote by $\mu_Z.T^A : A^P \longrightarrow A^Z$ the parametrized least prefixed point of T^A.

The (directed) graph of a system T has as vertices the variables Z and edges are of the form $z \to z'$ whenever z' occurs in t_z. We say that T is *acyclic* if the graph of T contains no cycle. Let us define the iterates of a modal system by $T^1 = T$, and $T^{n+1} = \langle Z, \{\, t_z[T^n/\boldsymbol{z}] \mid z \in Z \,\} \rangle$. If T is acyclic, then $T^{n+1} = T^n$ for some $n \geq 1$. Let n_0 be the least such integer and define $S = T^{n_0} = \langle Z, \{\, s_z \mid z \in Z \,\} \rangle$. If the terms in T are not themselves variables, then each term s_z of S is a term in $MT(P)$. If A is an arbitrary modal algebra, let $S^A : A^P \longrightarrow A^Z$ be determined by $\pi_z \circ S^A = s^A_z$, $z \in Z$. It is easily seen that $T^A(S^A(\boldsymbol{v}), \boldsymbol{v}) = S^A(\boldsymbol{v})$, and that S^A is the parameterized least prefixed point of T^A, $S^A = \mu_Z.T^A$. Let us call S the solution of T.

Let us now fix $\gamma \in \Gamma$. When presenting the axiomatization we have introduced three modal systems T^γ, $T^\gamma_\sharp$, and $\mathcal{P}_+(T^\gamma_\sharp)$. Since γ is fixed we shall from now on and later – whenever γ is understood – omit the superscript. T, the system representation of γ is acyclic. Also, T and $T_\sharp$ are modal systems pointed by z_γ. Let in the following A be a fixed but arbitrary modal algebra. Since T is acyclic let $S = \{ s_z \mid z \in Z \}$ be its solution. It is easily argued that $s^A_{z_\psi} = \widetilde{\psi}^A$, hence, in particular, $\pi_{z_\gamma} \circ S^A = s^A_{z_\gamma} = \gamma^A$. Recall that $T_\sharp$ is obtained from $T = \langle Z, z_\gamma, \{ t_z \}_{z \in Z} \rangle$ by substituting z_γ for x in every term t_z. This means that $T^A_\sharp$ is the following compose:

$$T^A_\sharp : \; A^Z \times A^P \xrightarrow{\langle \pi_Z, \pi_{z_\gamma} \rangle \times A^P} A^Z \times A^x \times A^P \xrightarrow{T^A} A^Z \,.$$

The following Lemma shows that, in view of our axiomatizing purposes, it is equivalent to axiomatize $\mu_x.\gamma$ or to axiomatize $\mu_Z.T_\sharp$.

Lemma 14. *For every modal algebra A and every vector $\boldsymbol{v} \in A^P$, the least prefixed point $\mu_Z.T^A_\sharp(Z, \boldsymbol{v})$ exists if and only if the least prefixed point $\mu_x.\gamma^A(x, \boldsymbol{v})$ exists. If existing, they are related as follows:*

$$\mu_Z.T^A_\sharp(Z, \boldsymbol{v}) = S^A(\mu_x.\gamma^A(x, \boldsymbol{v}), \boldsymbol{v}) \,, \qquad \mu_x.\gamma^A(x, \boldsymbol{v}) = \pi_{z_\gamma}(\mu_Z.T^A_\sharp(Z, \boldsymbol{v})) \,.$$

The proof of the Lemma is an application of the Bekič and Rolling rules, see for example [8, §2.3] and [3, §8.29]. It follows that, if A is a modal $\sharp$-algebra, then $S^A(\sharp^A(\boldsymbol{v}), \boldsymbol{v})$ is necessarily the least fixed point of $T^A_\sharp$.

Let us analyse now the role of the system $\mathcal{P}_+(T_\sharp)$. If $S \in \mathcal{P}_+(Z)$ and $\boldsymbol{v} \in A^Z$, let $\iota^A_{y_S}(\boldsymbol{v}) = \bigwedge_{z \in S} \boldsymbol{v}_z$ and $\iota^A : A^Z \longrightarrow A^{\mathcal{P}_+(Z)}$ be defined by $\pi_{y_S} \circ \iota^A = \iota^A_{y_S}(z)$, $S \in \mathcal{P}_+(Z)$. The meaning of the equations (4) is that the diagram

$$\begin{tikzcd}
A^Z \times A^P \arrow[r, "T^A_\sharp"] \arrow[d, "\iota^A \times A^P"] & A^Z \arrow[d, "\iota^A"] \\
A^Y \times A^P \arrow[r, "\mathcal{P}_+(T_\sharp)^A"] & A^Y
\end{tikzcd} \tag{5}$$

commutes. The main statement proved in [1, §9] is the following Proposition.

Proposition 15. *If A is a complete modal algebra and $\boldsymbol{v} \in A^P$, then*

$$\iota^A(\mu_Z.T^A_\sharp(\boldsymbol{v})) = \mu_Y.\mathcal{P}_+(T_\sharp)^A(\boldsymbol{v}) \,. \tag{6}$$

The proof presented in [1, §1.2.15] that equation (6) holds crucially depends on A being complete. It is not clear that this equation is derivable only on the basis that $\mu_Z.T^A_\sharp$ is the least prefixed point of $T^A_\sharp$. However equation (6) holds in every Kripke model, and if our goal is to collect the formulas valid in every Kripke model, then we can freely add to a formal system axioms and inference rules stating that the least solution of $\mathcal{P}_+(T_\sharp)$ is the image by ι^A of the least solution of $T^A_\sharp$, that is, $\iota^A(S^A(\sharp^A(\boldsymbol{v}), \boldsymbol{v}))$. This is precisely the goal of the axiom system $\mathbf{K}_\sharp(\gamma)$ as well as of the next Definition.

Definition 16. *A modal $\sharp$-algebra A is* regular *if, for each $\gamma \in \Gamma$ and each $\boldsymbol{v} \in A^P$, $\iota^A(S^{\gamma A}(\sharp_\gamma^A(\boldsymbol{v}), \boldsymbol{v}))$ is the least prefixed point of $\mathcal{P}_+(T_\sharp^\gamma)_{\boldsymbol{v}}^A$:*

$$\iota^A(S^{\gamma A}(\sharp_\gamma^A(\boldsymbol{v}), \boldsymbol{v})) = \mu_Y.\mathcal{P}_+(T_\sharp^\gamma)^A(\boldsymbol{v}) .$$

From Proposition 15, we obtain the following corollary, implying *soundness.*

Corollary 17. *Every Kripke $\sharp$-algebra is regular.*

4 Overview of the Completeness Proof

Let us recall that $f : A \longrightarrow B$ is a modal algebra morphism if the operations $\langle \bot, \top, \neg$
$\wedge, \{ \Diamond_i \mid i \in I \}\rangle$ are preserved by f. If A and B are also modal $\sharp$-algebras then f is a *modal $\sharp$-algebra morphism* if moreover each $\sharp_\gamma$, $\gamma \in \Gamma$, is preserved by f. This means that

$$f(\sharp_\gamma^A(\boldsymbol{v})) = \sharp_\gamma^B(f \circ \boldsymbol{v}) ,$$

for each $\boldsymbol{v} \in A^P$ and $\gamma \in \Gamma$. A $\sharp$-algebra morphism is an embedding if it is injective. We say that A embeds into B if there exists an embedding $f : A \longrightarrow B$.

Let X be a set of variables. The elements of the *Lindenbaum algebra* $\mathcal{L}(X)$ are equivalence classes of terms whose variables are contained in X, where two terms t, s are declared to be equivalent if $\vdash t \leftrightarrow s$ is derivable in the system $\mathbf{K}_\sharp(\Gamma)$. This is a standard construction of an algebra from the syntax of the logic [2], for example we shall have $\sharp_\gamma^{\mathcal{L}(X)}([t_1], \ldots, [t_n]) = [\sharp_\gamma(t_1, \ldots, t_n)]$. By construction, $\mathcal{L}(X)$ is a regular modal $\sharp$-algebra and there is a canonical interpretation of the variables in X as elements of $\mathcal{L}(X)$, sending the variable x to the equivalence class $[x]$ of the term x. Moreover, $\mathcal{L}(X)$ has the following property: whenever A is a regular modal $\sharp$-algebra and $\boldsymbol{v} : X \longrightarrow A$ is a valuation of the variables in x as elements of A, then there exists a unique modal $\sharp$-algebra morphism $f : \mathcal{L}(X) \longrightarrow A$ such that $f[x] = \boldsymbol{v}(x)$ for all $x \in X$. In universal algebraic, or categorical terms, $\mathcal{L}(X)$ is the *free regular $\sharp$-algebra over X*. We recall that this property, freeness, determines $\mathcal{L}(X)$ up to isomorphism of modal $\sharp$-algebras. In the sequel we shall use the words 'free regular $\sharp$-algebra' as a synonym of the Lindenbaum algebra.

The key to the completeness of the system $\mathbf{K}_\sharp(\Gamma)$ is the following Theorem:

Theorem 18. *If X is countable, then $\mathcal{L}(X)$ embeds in a Kripke $\sharp$-algebra.*

The theorem implies completeness as follows. Let X be the set of variables of a term/
formula t. If the formula t is valid in every Kripke frame, then the equation $t = \top$ holds in every Kripke $\sharp$-algebra, and thus certainly in the one that $\mathcal{L}(X)$ embeds into. Consequently, the equation $t = \top$ holds in the Lindenbaum algebra $\mathcal{L}(X)$. This in particular implies $[\top] = [t]$, that is $\vdash \top \leftrightarrow t$ is derivable in $\mathbf{K}_\sharp(\Gamma)$. As usual, this implies that $\vdash t$ is derivable in $\mathbf{K}_\sharp(\Gamma)$.

In turn, the proof of Theorem 18 is subdivided in many steps, which we here collect into two main results, to be proved successively in the next two sections.

Theorem 19. *The modal operators $\Diamond_i^{\mathcal{L}(X)}$, $i \in I$, of a Lindenbaum algebra $\mathcal{L}(X)$ are residuated. Moreover, $\mathcal{L}(X)$ is constructive.*

Theorem 20. *If a countable $\sharp$-algebra A is constructive and its modal operators $\Diamond_i^A$, $i \in I$, are residuated, then A has an embedding into a Kripke $\sharp$-algebra.*

Since $\mathcal{L}(X)$ is countable whenever X is countable, Theorem 18 follows.

5 Properties of the Lindenbaum Algebra

The goal of this section is to prove that the Lindenbaum algebra $\mathcal{L}(X)$ is constructive, cf. Definition 8. We shall obtain this result by subsequently analyzing properties of this algebra. Let us first say that a modal algebra A generated by a set X is *rigid* w.r.t. X if

$$\bigwedge \Lambda \wedge \bigwedge_{j \in J} \nabla_j Y_j \leq \bot \quad \text{implies} \quad \bigwedge \Lambda \leq \bot \text{ or } \exists j \in J, y \in Y_j \text{ s.t. } y \leq \bot$$

holds in A, where Λ is a finite set of literals, $J \subseteq I$, and, for each $j \in J$, Y_j is a finite possibly empty set of elements of A.

Theorem 21. *The free regular modal $\sharp$-algebra $\mathcal{L}(X)$ is rigid w.r.t. X.*

The proof of the Theorem depends on the following construction. If A is any modal algebra, let us call a pair $\mathfrak{f} = \langle J, \{ Y_j \mid j \in J \} \rangle$ – where $J \subseteq I$ and for each $j \in J$ Y_j is a finite subset of A – a *candidate failure* for rigidness if $y \not\leq \bot$ whenever $j \in J$ and $y \in Y_j$. Given a candidate failure $\mathfrak{f}$, a *repair* for $\mathfrak{f}$ is a collection $\chi = \{ \chi_j^y : A \longrightarrow 2 \mid j \in J, y \in Y_j \}$ of Boolean algebra morphisms such that $\chi_j^y(y) = \top$ for each $j \in J$ and $y \in Y_j$. Observe that, by the prime filter theorem, such a repair always exists. Define $\chi_j(z) = \bigvee_{y \in Y_j} \chi_j^y(z)$ if $j \in J$ and, otherwise, $\chi_j(z) = \bot$.

Definition 22. *The modal algebra $A_{\mathfrak{f},\chi}$ has as Boolean algebra reduct the product Boolean algebra $A \times 2$. For $i \in I$, the modal operators $\Diamond_i$ are defined by:*

$$\Diamond_i^{A_{\mathfrak{f},\chi}}(z, w) = (\Diamond_i^A z, \chi_i(z)) .$$

Observe that $\Diamond_i$ are indeed modal operators, since the functions χ_i preserve finite joins. The point of considering this construction is the following statement. If $\mathcal{K}$ is a category of modal algebras and modal algebra morphisms, such that whenever $\mathfrak{f}$ is a candidate failure in A and χ is some repair of $\mathfrak{f}$, then $A_{\mathfrak{f},\chi}$ (as well as the projection to A) belongs to $\mathcal{K}$, then a modal algebra $\mathcal{F}_{\mathcal{K}}(X)$, which is free within $\mathcal{K}$, is rigid. Thus we prove:

Proposition 23. *If A is a $\sharp$-algebra, then $A_{\mathfrak{f},\chi}$ is also a $\sharp$-algebra and the projection is a $\sharp$-algebra morphism. If moreover A is regular, then $A_{\mathfrak{f},\chi}$ is regular.*

Recall the definition of the cover modalities ∇_i, $i \in I$: $\nabla_i Y = \bigwedge \Diamond_i Y \wedge \Box_i \bigvee Y$, for some set of variables Y. This implies that in order to consider the interpretation of ∇_i in a modal algebra A we need to fix an indexing Y_0. Hence, we shall write $\nabla_{i\,Y_0}^{A} : A^{Y_0} \longrightarrow A$ and observe that, using this notation, it is not the case that, for $Y \subseteq Y_0$, $\nabla_{i\,Y_0}^{A}$ is obtained from $\nabla_{i\,Y}^{A}$ by precomposing with the projections.

Proposition 24. *The Lindenbaum algebra $\mathcal{L}(X)$ is such that, for each finite set $\{\, k_i \in \mathcal{L}(X) \mid i = 1, \ldots, n \,\}$, the collection*

$$\mathcal{F} = \{\, k_i \wedge \bigwedge_{j \in J} \nabla_{j\,Y}^{\mathcal{L}(X)} \mid i = 1, \ldots, n,\ J \subseteq I,\ Y \subseteq Y_0 \,\}$$

is a family of finitary $\mathcal{O}$-adjoints. Moreover the modal operators $\Diamond_i^{\mathcal{L}(X)}$, $i \in I$, are residuated.

The proof, crucially involving Theorem 21, is along the same lines as in [10], see Propositions 5.1, 6.7, and 7.2. We are ready to state and prove the main goal of this section.

Proposition 25. *The Lindenbaum algebra $\mathcal{L}(X)$ is constructive.*

Proof. Let us remark first that, for each fixed vector $\boldsymbol{k} \in \mathcal{L}(X)^P$, the least prexifed point of $\mathcal{P}_+(T_\sharp)_{\boldsymbol{k}}^{\mathcal{L}(X)} : \mathcal{L}(X)^Y \longrightarrow \mathcal{L}(X)^Y$ exists by the definition of a regular modal $\sharp$-algebra. Let us verify that such a least prefixed point is constructive. For each $S \in \mathcal{P}_+(Z)$, $\pi_{y_S} \circ \mathcal{P}_+(T_\sharp)_{\boldsymbol{k}}^{\mathcal{L}(X)}$ is of the form

$$\bigvee_{j \in J_S} k_j \wedge \bigwedge_{i \in I_j} \nabla_{i\,W_i}^{\mathcal{L}(X)} \boldsymbol{f}^i$$

with k_j constant and, for each $w \in W_i$, $\boldsymbol{f}^i_w$ is a join of elements in Y. Since families of finitary $\mathcal{O}$-adjoints can be closed under joins and substitution, it follows from Proposition 24 that $\{\, \pi_{y_S} \circ \mathcal{P}_+(T_\sharp)_{\boldsymbol{k}}^{\mathcal{L}(X)} \mid S \subseteq \mathcal{P}_+(Z) \,\}$ is a family of finitary $\mathcal{O}$-adjoints. By [10, Proposition 6.3.4], $\mathcal{P}_+(T_\sharp)_{\boldsymbol{k}}^{\mathcal{L}(X)}$ is itself a finitary $\mathcal{O}$-adjoint and its least prefixed point, which exists, is constructive. By [10, Lemma 7.4], it follows that the least prefixed point of $(T_\sharp^{\mathcal{L}(X)})_{\boldsymbol{k}}$ is constructive. Finally, it follows from Lemma 26 below that the least prefixed point of $\gamma_{\boldsymbol{k}}^{\mathcal{L}(X)}$ is itself constructive. $\Box$

We end this section stating the mentioned Lemma, which is an analogous of Lemma 14 for continuous functions and constructive fixed points.

Lemma 26. *Let us suppose that the operations of the $\sharp$-algebra A are continuous. Let $\boldsymbol{v} \in A^P$ be arbitrary. The least prefixed point of $(T_\sharp^A)_{\boldsymbol{v}}$ exists and is constructive if and only if the least prefixed point of $\gamma_{\boldsymbol{v}}^A$ exists and is constructive.*

6 A representation theorem

In this section we shall prove Theorem 20. Let us fix a modal $\sharp$-algebra A as in the statement of the Theorem. For simplicity we restrict attention to a language with a single diamond $\Diamond$, and a single fixpoint connective $\sharp$. We let $\gamma(x, \boldsymbol{p})$ denote the associated formula of $\sharp$, where $\boldsymbol{p} = (p_1, \ldots, p_n)$. The main lemma in the proof of Theorem 20 is the following.

Lemma 27. *For each $a \in A$ there is a Kripke frame S_a and a modal $\sharp$-homomorphism $\rho_a : A \to S_a^\sharp$ such that $\rho_a(a) > \bot$.*

We shall prove Lemma 27 by a step-by-step approximation process involving the notion of a *network* [2]. Let ω^* denote the set of finite sequences of natural numbers. We denote concatenation of such sequences by juxtaposition, and write ϵ for the empty sequence. If $s = tk$ for some $k \in \omega$ we say that s is the parent of t and write either $s = t^-$ or $s \lhd t$. A *tree* is a subset T of ω^* which is both downward and leftward closed; that is, if $t \neq \epsilon$ belongs to T, then so does t^-, and if $sm \in T$ then $sk \in T$ for all $k < m$. Obviously, a tree T, together with the relation $\lhd$, forms a Kripke frame; this frame will simply be denoted as T, and its complex $\sharp$-algebra, as $T^\sharp$.

An *A-network* is a pair $N = \langle T, L \rangle$ such that T is a tree, and $L : T \to \mathcal{P}(A)$ is some labelling. Such a network N induces a map $r_N : A \to \mathcal{P}(T)$, given by

$$r_N(a) := \{t \in T \mid a \in L(t)\}. \tag{7}$$

The aim of the proof will be to construct, for an arbitrary nonzero $a \in A$, a network $N = \langle T, L \rangle$, with $a \in L(\epsilon)$, and such that r_N is a modal $\sharp$-homomorphism from A to $T^\sharp$. We need some definitions.

A network $N = \langle T, L \rangle$ is called *locally coherent* if $\bigwedge X > \bot$, whenever X is a finite subset of $L(t)$ for some $t \in T$; *modally coherent* if $\bigwedge X \wedge \Diamond \bigwedge Y > \bot$, for all $s, t \in T$ such that $s \lhd t$ and all finite subsets X and Y of respectively $L(s)$ and $L(t)$; and *coherent* if it satisfies both coherence conditions. N is *prophetic* if for every $s \in T$, and for every $\Diamond a \in L(s)$, there is a *witness* $t \rhd s$ such that $a \in L(t)$; *decisive* if either $a \in L(t)$ or $-a \in L(t)$, for every $t \in T$ and $a \in A$; and $\sharp$-*constructive* if, for every $t \in T$, and every sequence $\boldsymbol{a}$ in A such that $\sharp\boldsymbol{a} \in L(t)$, there is a natural number n such that $(\gamma^A)^n_{\boldsymbol{a}}(\bot) \in L(t)$. A network is *perfect* if it has all of the above properties.

Lemma 28. *If N is a perfect A-network, then r_N is a modal $\sharp$-homomorphism.*

Clearly, we shall have that $r_N(a) \neq \varnothing$ for all $a \in A$ for which there is a $t \in T$ with $a \in L(t)$. From the above proposition it follows that in order to prove Lemma 27 it suffices to construct, for an arbitrary nonzero $a \in A$, a perfect network with $a \in L(\epsilon)$. Our construction will be carried out in a step-by-step process, where at each stage we are dealing with a finite *approximation* of the final network. Since these approximations are not perfect themselves, they will suffer from certain *defects*. We will only be interested in those defects that can be *repaired* in the sense that the network can be extended to a bigger version that is lacking the defect.

Formally we define a *defect* of a network $N = \langle T, L\rangle$ to be an object d of one of the following three kinds:

1. $d = (t, a, -)$, with $t \in T$ and $a \in A$ such that neither a nor $-a$ belongs to $L(t)$,
2. $d = (t, a, \Diamond)$, with $t \in T$ and $a \in A$ such that $\Diamond a \in L(t)$, but there is no witness $s \rhd t$ such that $a \in L(s)$,
3. $d = (t, \boldsymbol{a}, \sharp)$, with $t \in T$ and $\boldsymbol{a} \in A^n$ such that $\sharp\boldsymbol{a} \in L(t)$, but there is no $n \in \omega$ such that $(\gamma^A)^n_{\boldsymbol{a}}(\bot) \in L(t)$.

While in principle we could construct a perfect network as a limit of coherent networks, the networks that we will actually use will in fact satisfy a much stronger version of coherency. In order to define this notion, we first need to extend the *local* labelling function L of the network to a global one. Recall that the operator of the algebra A is *residuated*, and hence, *conjugated*. That is, there is an operation $\blacklozenge : A \to A$ such that

$$a \wedge \Diamond b > \bot \text{ iff } \blacklozenge a \wedge b > \bot, \tag{8}$$

for all $a, b \in A$. Using this operation $\blacklozenge$, we can in fact define the global labelling map $\widetilde{L}$ as follows:

$$\begin{aligned}
\Delta_{\downarrow}(t) &:= \bigwedge L(t) \wedge \bigwedge_{t \lhd s} \Diamond \Delta_{\downarrow}(s)\,, \qquad \Delta_{\downarrow,-u}(t) := \bigwedge L(t) \wedge \bigwedge_{t \lhd s, s \neq u} \Diamond \Delta_{\downarrow}(s)\,,\\
\Delta_{\uparrow}(t) &:= \begin{cases} \top\,, & \text{if } t = \epsilon\,,\\ \blacklozenge(\,\Delta_{\uparrow}(t^-) \wedge \Delta_{\downarrow,-t}(t^-)\,)\,, & \text{otherwise}\,,\end{cases}\\
\widetilde{L}(t) &:= \Delta_{\downarrow}(t) \wedge \Delta_{\uparrow}(t)\,.
\end{aligned}$$

The following observation is a consequence of the conjugacy relation (8) and of the fact that the tree is connected.

Lemma 29. *If N is a finite network and $s, t \in N$, then $\widetilde{L}(s) > \bot$ iff $\widetilde{L}(t) > \bot$.*

Call a finite network $N = \langle T, L\rangle$ *globally coherent* if $\widetilde{L}(t) > \bot$ for all $t \in T$. We can now prove our *repair lemma*. We say that N' *extends* N, notation: $N \leq N'$, if $T \subseteq T'$ and $L(t) \subseteq L'(t)$ for every $t \in T$.

Lemma 30 (Repair Lemma). *Let $N = \langle T, L\rangle$ be a globally coherent A-network. Then for any defect d of N there is a globally coherent extension N^d of N which lacks the defect d.*

Proof. We have to take action depending on the type of the defect d. In each case we will make heavily use of the global extension $\widetilde{L}$ of L. For instance, suppose $d = (t, \boldsymbol{a}, \sharp)$ is a defect of the third kind. By strong coherency, $\widetilde{L}^N(t) > \bot$. Suppose for contradiction that $\widetilde{L}^N(t) \wedge (\gamma^A)^n_{\boldsymbol{a}}(\bot) = \bot$ for all numbers n. Then for all n we have $(\gamma^A)^n_{\boldsymbol{a}}(\bot) \leq -\widetilde{L}^N(t)$, and so by constructiveness of $\sharp$ on A it follows that $\sharp^A \boldsymbol{a} \leq -\widetilde{L}^N(t)$. But this contradicts the fact that N is coherent.

It follows that $\widetilde{L}^N(t) \wedge (\gamma^A)^n_{\boldsymbol{a}}(\bot) > \bot$ for some natural number n. Now define $N' := \langle T, L' \rangle$, where $L'(s) := L(s)$ for $s \neq t$, while $L'(t) := L(t) \cup \{(\gamma^A)^n_{\boldsymbol{a}}(\bot)\}$. It is not difficult to check that N' satisfies all the requirements stated in the Lemma. □

Lemma 31. *Every globally coherent A-network can be extended to a perfect network.*

Proof. On the basis of successive applications of Lemma 30, properly scheduled, one may define a sequence of networks $N = N_0 \leq N_1 \leq N_2 \leq \ldots$ such that for each $i \in \omega$ and each defect d of N_i there is a $j > i$ such that d is not a defect of N_j. Then define $N' := \langle T', L' \rangle$, with $T' := \bigcup_{i<\omega} T_i$ and for each $t \in T'$, $L'(t) := \bigcup_{i<\omega} L_i(t)$. It is then straightforward to verify that N' is a perfect extension of N. □

Proof of Lemma 27. Consider an arbitrary nonzero element $a \in A$, and let N_a be the network $\langle \{\epsilon\}, L_a \rangle$, L_a given by $L_a(\epsilon) := \{a\}$. It is obvious that N_a is globally coherent, so Lemma 27 follows by a direct application of the Lemmas 31 and 28.
Proof of Theorem 20. Let S be the disjoint union of the family $\{S_a \mid \bot \neq a \in A\}$, where the S_as are given by Lemma 27. It is straightforward to verify that A can be embedded into the product $\prod_{a \neq \bot} S_a^\sharp$, and that this latter product is isomorphic to $S^\sharp$, the complex $\sharp$-algebra of S.

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