On Canonical Modal Logics That Are Not Elementarily Determined

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February 18, 2004

Abstract

There exist modal logics that are validated by their canonical frames but are not sound and complete for any elementary class of frames. Continuum many such bimodal logics are exhibited, including one of each degree of unsolvability, and all with the finite model property. Monomodal examples are also constructed that extend K4 and are related to the proof of non-canonicity of the McKinsey axiom.

1 Introduction

A modal logic L is called *canonical* if it is valid in the canonical frame $\mathcal{F}_{\mathsf{L}}$ whose points are the maximally L-consistent sets of formulas. These special Kripke frames were introduced in the mid-1960's by Lemmon and Scott [24], and independently by Cresswell [2] and Makinson [26], as an extension of the method of completeness proof due to Henkin [17]. Any formula valid in $\mathcal{F}_{\mathsf{L}}$ is an L-theorem, and so if $\mathcal{F}_{\mathsf{L}}$ satisfies some condition on frames for which L is sound, then it follows that L is *determined by* (i.e., sound and complete for) the class of all frames satisfying that condition.

By the early 1970's numerous logics had been shown to be determined by Kripke frames via the technique of using the proof theory of L to establish some *first-order* condition on $\mathcal{F}_{\mathsf{L}}$ for which L is sound. A logic L will be called *elementarily determined* if there is at least one class of frames determining L that is elementary, i.e., is axiomatized by some first-order sentences. Thus these early results gave many proofs of canonicity which at the same time showed that the logic concerned was elementarily determined. Moreover, the only examples of non-canonical logics that were found were ones whose axioms expressed non-first-order properties of frames. The first explicit such example would appear to be that in [4, p. 38], where Fine proves invalidity in the canonical frame for the logic extending S4.3 by the Grzegorczyk axiom

$$\Box(\Box(p \to \Box p) \to p) \to p.$$

Validity of this formula in S4.3-frames is equivalent to the second-order condition that every non-empty subset has a maximal element. Some years earlier Kripke [21] had noted that there are formulas whose validity is not preserved in passing from a modal algebra $\mathcal{A}$ to the algebra of all subsets of the frame whose points are the ultrafilters of $\mathcal{A}$. This can also been seen as a manifestation of non-canonicity. Kripke gave the example of Dummett's Diodorean axiom

$$\Box(\Box(p \to \Box p) \to \Box p) \to (\Diamond\Box p \to \Box p),$$

whose validity expresses *discreteness* of a linear ordering.

The absence of any elementarily determined instances of non-canonicity was soon explained by the following theorem of Fine [5]:

if a modal logic L is determined by some elementary class of frames, then it is valid in the canonical frame $\mathcal{F}_{\mathsf{L}}$.

Fine asked whether the converse was true. If a logic is canonical, must it be elementarily determined? Many affirmative partial solutions have been produced for this question, which we now briefly review. A modal formula is called *r-persistent* if it is validated by a Kripke frame $\mathcal{F}$ whenever it is validated by some general frame based on $\mathcal{F}$ that is *refined* in the sense of Thomason [29]. Every logic with r-persistent axioms is canonical. Lachlan [22] showed that the class of validating frames for an r-persistent formula is definable by a first-order sentence, and hence every r-persistent logic is elementarily determined.[1] Sahlqvist [27] gave a syntactic scheme specifying infinitely many formulas, each of which defines a canonical logic and has its frame-validity equivalent to an explicit first-order condition. Fine [6] proved the elementary determination of any canonical modal logic that is determined by a class of transitive frames that is closed under subframes. Zakharyaschev [34] extended this to logics determined by a class of transitive frames that is closed under *cofinal* subframes. Wolter [33] removed the transitivity restriction in Fine's result, and also proved [32] elementary determination of all canonical normal extensions of linear tense logic. Jónsson [19] gave an algebraic analysis which implies that a modal axiom of the form $\varphi(p \vee q) \leftrightarrow \varphi(p) \vee \varphi(q)$ is canonical whenever $\varphi(p)$ is a positive formula, and Venema [31] showed that logics with such axioms are elementarily determined. In [14] it is shown that the converse of Fine's theorem holds for any logic that is validated by a frame $\mathcal{F}$ whenever it is validated by some general frame based on $\mathcal{F}$ whose propositions include the singleton subsets of $\mathcal{F}$.

Fine's theorem was strengthened to show that if L is determined by some elementary conditions, then it is always determined by elementary conditions that are satisfied by $\mathcal{F}_{\mathsf{L}}$ (see [11]). The result was also expressed algebraically to show [9] that if a variety $\mathcal{V}$ of Boolean algebras with operators is generated by the algebras of subsets of the members of some elementary class of relational structures, then $\mathcal{V}$ is closed under the perfect extension construction of Jónsson and Tarski [20]. The converse of this algebraic formulation has been confirmed for numerous varieties of cylindric, relation, and modal algebras.

In this paper we show that the converse of Fine's theorem fails in general, and fails as badly as it could. We exhibit $2^{\aleph_0}$ different canonical logics that are not determined by any elementary class of frames. These are bimodal logics, with one modality being of S5 type. All of the logics have the finite model property and they include one of each degree of unsolvability. In addition monomodal examples are constructed that are extensions of the logic K4 and are connected to the non-canonicity proof in [10] for the McKinsey axiom $\Box\Diamond p \to \Diamond\Box p$.

Our bimodal examples are related to the modal logic KMT, studied by Hughes [18], whose validating frames are those directed graphs satisfying the non-elementary condition that the children of any node have no finite colouring. KMT has an infinite sequence of axioms whose n-th member rules out colourings that use n colours. But the logic is also *elementarily* determined by the class of graphs whose edge relation R satisfies $\forall x \exists y(xRyRy)$, meaning that every node has a reflexive child. The canonical frame for KMT satisfies this condition.

Here we also use axioms that impose reflexive points on canonical frames. But now a canonical frame is essentially the disjoint union of a family of directed graphs, and it is only the *infinite* members of the family that are required to have a reflexive point to ensure canonicity. This is a non-elementary requirement. To prove that our logics are never elementarily determined we apply a famous piece of graph theory of Erdős [3], who showed that for each integer n there is a finite graph $\mathcal{G}_n$ whose chromatic number and girth are both greater than n, the girth being the length of the shortest cycle in the graph and the chromatic number being the smallest number of colours needed to colour it. The essence of the application is that if a certain logic L were

[1] This result was independently proved also in [5] and [8].

determined by an elementary class $\mathcal{K}$, and infinitely many of the $\mathcal{G}_n$'s validated L, then by a compactness argument it would follow that $\mathcal{K}$ contained an infinite graph that had no cycles of odd length. But such a graph can be coloured using only *two* colours, a property that invalidates one of the axioms defining L. Hence the existence of $\mathcal{K}$ is impossible.

This paper is a companion to the article [16] which takes an algebraic approach to our topic, making use of the duality between frames and modal algebras as well as the theory of discriminator varieties.

2 Colouring Graphs

A *graph* is a structure $\mathcal{G} = (V, E)$ in which E is a symmetric and irreflexive binary relation on a non-empty set V of "vertices". A pair (x, y) in E may be thought of as an *edge* with vertices x and y. $\mathcal{G}$ may also be viewed as a Kripke frame, and in that context symmetry of E is equivalent to validity in $\mathcal{G}$ of the Brouwerian axiom $p \to \Box\Diamond p$. But there is no modal formula whose validity corresponds to irreflexivity, and it is that inability to rule out reflexive points that lies at the heart of our canonicity proofs.

A *colouring* of $\mathcal{G}$ is an assignment of colours to the points in V in such a way that the two vertices of any edge are assigned different colours. An n-colouring is one that uses at most n colours. This can be expressed more set-theoretically by defining a subset W of V to be *independent* if it contains no edge, in the sense that there are no $x, y \in W$ with xEy. An *n-colouring* of graph $\mathcal{G}$ is then a partition of V into at most n independent subsets. The *chromatic number* $\chi(\mathcal{G})$ is the smallest integer n, if it exists, for which $\mathcal{G}$ has an n-colouring, and ∞ if there is no such n. Of course a finite graph has $\chi(\mathcal{G})$ no bigger than the number $|\mathcal{G}|$ of members of V, since we can always give every vertex a different colour. Observe that to obtain an n-colouring it is enough to find n independent sets $W_1, \dots, W_n$ that *cover* V, i.e., $W_1 \cup \cdots \cup W_n = V$, for then this can be refined to a *partition* of V into the independent sets

$$W_1,\ W_2 - W_1, \dots, W_n - (W_1 \cup \cdots \cup W_{n-1}).$$

For $k \geq 3$, a *k-cycle*, or *cycle of length k*, is a a sequence $(x_1, \dots, x_k)$ of distinct nodes of V, such that (x_1, x_2), $\dots$, (x_{k-1}, x_k), (x_k, x_1) are all in E.[2] An *odd cycle* is one of odd length.

Erdős [3] showed that for any integers n, k there is a finite graph $\mathcal{G}$ with $\chi(\mathcal{G}) > n$ such that $\mathcal{G}$ has no cycle of length k or less. He gave an existence proof by a revolutionary probabilistic method whose power was evinced by the fact that it took a decade to find an actual construction of such graphs [25].

Here is a summary of the facts from graph theory that we will use. This is essentially standard material, but we give proofs for the sake of completeness.

Theorem 2.1

(1) *A graph has a 2-colouring if, and only if, it has no odd cycles.*

(2) *If $\mathcal{G} = (V, E)$ and $\mathcal{G}' = (V', E')$ are graphs, and $f : V \to V'$ preserves edges, i.e., xEy implies $fxE'fy$, then if $\mathcal{G}$ has an odd cycle of length n or less, so does $\mathcal{G}'$.*

(3) *There exists a computable enumeration $\{\mathcal{G}_n : n \geq 2\}$ of finite graphs $\mathcal{G}_n$ with $\chi(\mathcal{G}_n) > n$ and $\mathcal{G}_n$ having no cycle of length n or less, such that if $2 \leq m < n$, then $|\mathcal{G}_m| < |\mathcal{G}_n|$ and $\chi(\mathcal{G}_m) < \chi(\mathcal{G}_n)$.*

Proof.

[2] In graph theory, $(x_1, \dots, x_k)$, $(x_2, \dots, x_k, x_1)$, and $(x_k, \dots, x_1)$ are regarded as the same cycle; but this is not important here.

(1) If $\mathcal{G}$ has a 2-colouring, and $x_1, \ldots, x_k$ is any cycle in $\mathcal{G}$, then the odd-indexed vertices $x_1, x_3, \ldots$ must have the same colour, while x_k has a different colour to x_1, so k cannot be odd.

For the converse, assume $\mathcal{G}$ has no odd cycles. Define a *walk of length n from x to y* to be a sequence $x = x_0, x_1, \ldots, x_n = y$ of vertices with $x_i E x_{i+1}$ for all $i < n$. The length is the number of edges in the walk, and we allow $n = 0$ here. If $x_0 = x_n$, the walk is *closed.* First we show that $\mathcal{G}$ cannot have any closed walks of odd length. For if it did, we could pick such a walk $x_0, x_1, \ldots, x_n = x_0$ whose odd length n was least possible. Then the minimality of n would ensure that $x_1, \ldots, x_n$ are distinct — for if $1 \leq i < j \leq n$ and $x_i = x_j$ then both $x_i, \ldots, x_j$ and $x_0, \ldots, x_i, x_{j+1}, \ldots, x_n$ would be closed walks of length less than n, and one of them would be of odd length since their combined edges are just the edges of the original odd-length walk. It follows that $x_1, \ldots, x_n$ would be an odd cycle, contrary to hypothesis on $\mathcal{G}$.

Now a graph is *connected* if any two of its vertices have a walk connecting them. Any graph is the disjoint union of connected subgraphs, each of which can be coloured independently. Hence we can assume that $\mathcal{G}$ is connected. To define a 2-colouring, fix a vertex x of $\mathcal{G}$ and then assign any vertex y colour 1 if there is an even-length walk from x to y, and colour 2 otherwise. For any edge $(y, z) \in E$, if y and z got the same colour, then from the definition of the colour assignment and connectivity there would exists walks from x to y and x to z whose lengths had the same parity (both even or both odd). But then taking the walk from x to y, then following the edge (y, z) and finally the reverse of the same-parity walk from x to z would give a closed walk of odd length — which we have just seen does not exist. Thus y and z must get different colours, and the 2-colouring of $\mathcal{G}$ is established.

(2) If $\mathcal{G}$ has an odd cycle $\mathcal{C}$ with $|\mathcal{C}| \leq n$, restrict the edge relation of $\mathcal{G}$ to $\mathcal{C}$ to regard $\mathcal{C}$ as a graph in its own right. Similarly, let the image-set $f(\mathcal{C}) = \{fx : x \in \mathcal{C}\}$ be viewed as a subgraph of $\mathcal{G}'$. Now if $\mathcal{G}'$ had no odd cycle of length $\leq n$, then since $|f(\mathcal{C})| \leq n$, $f(\mathcal{C})$ would have no odd cycle at all, and so by (1) would have a 2-colouring. Then assigning to $x \in \mathcal{C}$ the same colour as fx would give a 2-colouring of $\mathcal{C}$, since edges are preserved. But that contradicts (1), since $\mathcal{C}$ is an odd cycle.

(3) Fix a recursive enumeration of all isomorphism types of finite graphs, in order of their cardinality. If $\mathcal{G}_m$ $(2 \leq m < n)$ have been defined, define $\mathcal{G}_n$ to be the first graph in the enumeration with no cycles of length $\leq n$, chromatic number greater than both n and $\chi(\mathcal{G}_m)$ and $|\mathcal{G}_n| > |\mathcal{G}_m|$ for all m with $2 \leq m < n$.

$\square$

3 Frames and Models

Take a propositional language with two box-type modalities, denoted $\Box$ and A. Their duals will be denoted $\Diamond$ and E. A frame for this language is a structure $\mathcal{F} = (W, R_\Box, R_\mathsf{A})$ with $R_\Box$ and R_A being binary relations on W. For any binary relation R we will use the notation R^x for the set $\{y : xRy\}$ of all *R-alternatives* of a point x. Recall that a *model* $\mathcal{M}$ on frame $\mathcal{F}$ is an assignment to each propositional variable p of a set $\mathcal{M}(p) \subseteq W$, thought of as the set of points at which p is *true*, or *satisfied.* This extends to assign a *truth-set* $\mathcal{M}(\varphi)$ to each formula φ, with the definitions for the modalities given by $\mathcal{M}(\Box\varphi) = \{x : R_\Box^x \subseteq \mathcal{M}(\varphi)\}$ and $\mathcal{M}(\mathsf{A}\varphi) = \{x : R_\mathsf{A}^x \subseteq \mathcal{M}(\varphi)\}$. φ is *valid in* $\mathcal{F}$, $\mathcal{F} \models \varphi$, if $\mathcal{M}(\varphi) = W$ for all models $\mathcal{M}$ on $\mathcal{F}$. φ is *satisfiable in* $\mathcal{F}$ if it is true at some point of some model on $\mathcal{F}$ (i.e., $\mathcal{F} \not\models \neg\varphi$), and is *falsifiable in* $\mathcal{F}$ if it is false at some point of some model on $\mathcal{F}$ (i.e., $\mathcal{F} \not\models \varphi$). For a class $\mathcal{K}$ of frames we write $\mathcal{K} \models \varphi$ to mean that

$\mathcal{F} \models \varphi$ for all $\mathcal{F} \in \mathcal{K}$. For a logic L, an L-*frame* is any frame $\mathcal{F}$ that validates all L-theorems, which we indicate by writing $\mathcal{F} \models \mathsf{L}$.

A frame $\mathcal{F}' = (W', R'_\Box, R'_\mathsf{A})$ is a *subframe* of $\mathcal{F}$ if $W' \subseteq W$ and $R'_\Box$ and R'_A are the restrictions to W' of $R_\Box$ and R_A respectively. If further we have $(R'_\Box)^x \subseteq W'$ and $(R'_\mathsf{A})^x \subseteq W'$ for all $x \in W'$, then $\mathcal{F}'$ is an *inner* subframe of $\mathcal{F}$. In that case, any formula valid in $\mathcal{F}$ is valid in $\mathcal{F}'$. For each point x of $\mathcal{F}$ there is a smallest inner subframe of $\mathcal{F}$ containing x, called the inner subframe *generated by* x.

We will work from now on with *basic* frames, defined as those for which R_A is an equivalence relation, $R_\Box \subseteq R_\mathsf{A}$, and $R_\Box$ is symmetric (n.b.: we do not require $R_\Box$ to be irreflexive.) The R_A-equivalence classes are usually called *clusters*. Each cluster is an inner subframe of $\mathcal{F}$, because $R^x_\Box \subseteq R^x_\mathsf{A}$, and can be viewed as a basic frame in its own right on which R_A is universal. Hence any basic frame on which R_A is universal will simply be called a cluster. If φ is true at some point in a model on a cluster, then $\mathsf{E}\varphi$ is true everywhere in the cluster. Dually, if φ is false at some point, then $\mathsf{A}\varphi$ is false everywhere in the cluster in that model.

Each graph $\mathcal{G} = (V, E)$ will be treated as a basic frame by putting $R_\Box = E$ and $R_\mathsf{A} = V \times V$. Thus any graph is a cluster, and any $R_\Box$-irreflexive cluster is a graph. Observe that in any model $\mathcal{M}$ on $\mathcal{G}$, a truth-set of the form $\mathcal{M}(\varphi \wedge \Box\neg\varphi)$ is an independent set, since it can contain no $R_\Box$-edge.

Now fix two disjoint infinite lists $p_1, p_2, \dots$ and $q_1, q_2, \dots$ of propositional variables. For $m \geq 1$, let E_m be the formula

$$\mathsf{E}p_1 \wedge \mathsf{E}(p_2 \wedge \neg p_1) \wedge \cdots \wedge \mathsf{E}(p_m \wedge \neg p_1 \wedge \cdots \wedge \neg p_{m-1}).$$

For $n \geq 1$, let χ_n be the formula

$$\mathsf{E}\big((q_1 \to \Diamond q_1) \wedge \cdots \wedge (q_n \to \Diamond q_n)\big).$$

If E_m is true at a point x in a model on a basic frame, then the cluster R^x_A contains *distinct* points $x_1, \dots, x_m$ with p_i true at x_i for all $i \leq m$. Conversely, if a cluster has at least m points then we can define a model on it that satisfies E_m.

The formula χ_n is a variant of the axiom MT_n of [18]. Note that if $n \geq m$, then the formula $\chi_n \to \chi_m$ is valid in all frames. If a cluster contains a *reflexive* point x, i.e., $xR_\Box x$, then no formula of the form $\varphi \to \Diamond\varphi$ can ever be falsified at x, and so the cluster validates χ_n for all $n \geq 1$. In the case of a graph $\mathcal{G}$, if χ_n is falsifiable in some model $\mathcal{M}$ on $\mathcal{G}$, then for each point x there must be some $i \leq n$ with q_i true and $\Diamond q_i$ false at x. Hence the n independent sets $\mathcal{M}(q_i \wedge \Box\neg q_i)$ cover the graph and so can be refined to an n-colouring. Conversely, given an n-colouring of $\mathcal{G}$ we can associate a variable q_i with each colour (independent set) to obtain a falsifying model on $\mathcal{G}$ for χ_n.

For $m, n \geq 1$, let $\chi[m, n]$ be the formula $\mathsf{E}_m \to \chi_n$.

Lemma 3.1

(1) *E_m is satisfiable in a cluster $\mathcal{F}$ iff $\mathcal{F}$ has at least m elements.*

(2) *χ_n is falsifiable in a graph $\mathcal{G}$ iff $\chi(\mathcal{G}) \leq n$. Equivalently, $\mathcal{G} \models \chi_n$ iff $\chi(\mathcal{G}) > n$.*

(3) *For any $m, n \geq 2$, $\chi[\,|\mathcal{G}_m|, m]$ is valid in $\mathcal{G}_n$.*

Proof. Parts (1) and (2) summarize the above observations. For (3), if the antecedent $\mathsf{E}_{|\mathcal{G}_m|}$ of $\chi[\,|\mathcal{G}_m|, m]$ is true at some point in a model on $\mathcal{G}_n$, then by (1), $\mathcal{G}_n$ has at least $|\mathcal{G}_m|$ elements, so $n \geq m$. Since $\chi(\mathcal{G}_n) > n$, we then have $\chi(\mathcal{G}_n) > m$ so by (2), $\mathcal{G}_n$ validates the consequent χ_m of $\chi[\,|\mathcal{G}_m|, m]$. $\Box$

4 Canonical Logics With the FMP

If our propositional language is generated by an infinite set of variables of size κ, then canonical frames built from this language will typically be of size 2^κ. The canonicity results of this paper hold with κ any infinite cardinal here: all that is required is that there be at least a countably infinite set of variables.

By a *basic logic* we will mean any normal propositional bimodal logic L, in the language of $\Box$ and A, that obeys the rule of uniform substitution of formulas for variables and includes the following axioms:

$$\begin{array}{ll} \mathrm{S5}_{\mathsf{A}}: & \mathsf{A}p \to p,\ \mathsf{E}p \to \mathsf{AE}p \\ \mathrm{Sub}: & \mathsf{A}p \to \Box p \\ \mathrm{B}_{\Box}: & p \to \Box\Diamond p. \end{array}$$

A frame validates these axioms if, and only if, it is a basic frame.

Recall that the canonical frame $\mathcal{F}_{\mathsf{L}} = (W_{\mathsf{L}}, R_{\Box}, R_{\mathsf{A}})$ for a normal logic L has W_{L} as the set of all maximally L-consistent sets of formulas, with

$$xR_{\Box}y \quad \text{iff} \quad \{\varphi : \Box\varphi \in x\} \subseteq y \quad \text{iff} \quad \{\Diamond\varphi : \varphi \in y\} \subseteq x,$$

and likewise $xR_{\mathsf{A}}y$ iff $\{\varphi : \mathsf{A}\varphi \in x\} \subseteq y$ iff $\{\mathsf{E}\varphi : \varphi \in y\} \subseteq x$.

For each formula φ, let $\|\varphi\|_{\mathsf{L}} = \{x \in W_{\mathsf{L}} : \varphi \in x\}$. The canonical model $\mathcal{M}_{\mathsf{L}}$ has $\mathcal{M}_{\mathsf{L}}(\varphi) = \|\varphi\|_{\mathsf{L}}$. In general, a formula is an L-theorem iff it belongs to every maximally L-consistent set, so if $\mathcal{F}_{\mathsf{L}} \models \varphi$, then $\mathcal{M}_{\mathsf{L}} \models \varphi$ so $\|\varphi\|_{\mathsf{L}} = \mathcal{M}_{\mathsf{L}}(\varphi) = W_{\mathsf{L}}$, and thus $\mathsf{L} \vdash \varphi$.

Lemma 4.1 *For any finite sequence $x_1, \dots, x_m$ of distinct points of W_{L} there exist formulas $\varphi_1, \dots, \varphi_m$ such that $\varphi_i \in x_j$ iff $i = j$. Hence if S is any finite subset of W_{L}, then for each set $X \subseteq S$ there is a formula φ_X such that $X = \|\varphi_X\|_{\mathsf{L}} \cap S$.*

Proof. If $i \neq j$, there exists $\varphi_{ij} \in x_i$ with $\varphi_{ij} \notin x_j$. Put $\varphi_i = \bigwedge_{i \neq j} \varphi_{ij}$. Then if $S = \{x_1, \dots, x_m\}$ and $X = \{x_{i_1}, \dots, x_{i_k}\}$, put $\varphi_X = \varphi_{i_1} \vee \cdots \vee \varphi_{i_k}$. $\Box$

Lemma 4.2 *Let L be a normal logic obeying the rule of uniform substitution. If $\mathcal{F}$ is a finite inner subframe of $\mathcal{F}_{\mathsf{L}}$, then $\mathcal{F} \models \mathsf{L}$.*

Proof. This is standard: a finite inner subframe of any canonical frame for any normal logic validates that logic. To see why, let $\mathcal{M}$ be a model on $\mathcal{F}$, and suppose $\mathcal{F}$ has underlying set S. For each variable p there is, by Lemma 4.1, a formula φ_p such that $\mathcal{M}(p) = \|\varphi_p\|_{\mathsf{L}} \cap S$. Then an induction on formation of formulas shows that for any formula ψ, $\mathcal{M}(\psi) = \|\psi^*\|_{\mathsf{L}} \cap S$, where ψ^* is the result of uniformly replacing each variable p of ψ by φ_p. But if $\mathsf{L} \vdash \psi$, then $\mathsf{L} \vdash \psi^*$, so $\|\psi^*\|_{\mathsf{L}} \cap S = S$. Hence any L-theorem is true at every point of every model on $\mathcal{F}$. $\Box$

Now let L be a basic logic. The axioms $\mathrm{S5}_{\mathsf{A}}$ ensure that R_{A} is an equivalence relation on W_{L}, axiom Sub enforces $R_{\Box} \subseteq R_{\mathsf{A}}$, and $\mathrm{B}_{\Box}$ makes $R_{\Box}$ symmetric, so $\mathcal{F}_{\mathsf{L}}$ is a basic frame.

Lemma 4.3 *Let L be any basic logic with the property that there are infinitely many n for which there exists an m such that $\mathsf{L} \vdash \chi[m, n]$. If $\mathcal{F}$ is any infinite cluster of $\mathcal{F}_{\mathsf{L}}$, then $\mathcal{F}$ contains an $R_{\Box}$-reflexive point.*

Proof. Take any point x in $\mathcal{F}$, and let

$$y_0 = \{\varphi : \mathsf{A}\varphi \in x\} \cup \{\psi \to \Diamond\psi : \psi \text{ is a formula}\}.$$

If y_0 is L-consistent, then it extends to a set $y \in W_{\mathsf{L}}$. Then $xR_{\mathsf{A}}y$, so y belongs to the cluster $\mathcal{F}$, and $\{\Diamond\psi : \psi \in y\} \subseteq y$, so $yR_{\Box}y$ as desired.

But if y_0 were not consistent, then since the set $\{\varphi : \mathsf{A}\varphi \in x\}$ is closed under finite conjunctions it would follow that there are formulas $\mathsf{A}\varphi \in x$, and $\psi_1, \dots, \psi_k$ for some $k \geq 1$, such that $\mathsf{L} \vdash \varphi \to \neg\big((\psi_1 \to \Diamond\psi_1) \wedge \cdots \wedge (\psi_k \to \Diamond\psi_k)\big)$. By the given property of L there must be an $n \geq k$ and an m such that $\mathsf{L} \vdash \chi[m,n]$. For $k < j \leq n$ put $\psi_j = \psi_k$, so then

$$\mathsf{L} \vdash \varphi \to \neg\big((\psi_1 \to \Diamond\psi_1) \wedge \cdots \wedge (\psi_n \to \Diamond\psi_n)\big).$$

Since L is normal this implies that

$$\mathsf{L} \vdash \mathsf{A}\varphi \to \mathsf{A}\neg\big((\psi_1 \to \Diamond\psi_1) \wedge \cdots \wedge (\psi_n \to \Diamond\psi_n)\big),$$

and hence as $\mathsf{A}\varphi \in x$ we get

$$\mathsf{E}\big((\psi_1 \to \Diamond\psi_1) \wedge \cdots \wedge (\psi_n \to \Diamond\psi_n)\big) \notin x. \tag{1}$$

But $\mathcal{F}$ is infinite, so we can choose m distinct points $x_1, \dots, x_m$ in $\mathcal{F}$. Let $\varphi_1, \dots, \varphi_m$ be the formulas given by Lemma 4.1. Then $(\varphi_i \wedge \bigwedge_{1 \leq j < i} \neg\varphi_j) \in x_i$, and so as R_{A} is universal on $\mathcal{F}$, $\mathsf{E}(\varphi_i \wedge \bigwedge_{1 \leq j < i} \neg\varphi_j) \in x$, for all $i \leq m$. Hence

$$\bigwedge_{1 \leq i \leq m} \mathsf{E}(\varphi_i \wedge \textstyle\bigwedge_{1 \leq j < i} \neg\varphi_j) \in x. \tag{2}$$

However (1) and (2) contradict the fact that every substitution instance of $\chi[m,n]$ belongs to x. It follows that y_0 is L-consistent. $\Box$

Theorem 4.4 *Let L be a basic logic defined by additional axioms of the form χ_n or $\chi[m,n]$. If there are infinitely many n for which there exists an m such that $\mathsf{L} \vdash \chi[m,n]$, then L is a canonical logic that has the finite model property and is determined by a class of finite clusters.*

Proof. $\mathcal{F}_{\mathsf{L}}$ is the disjoint union of its clusters. Let $\mathcal{F}$ be any cluster. If $\mathcal{F}$ is finite, then $\mathcal{F} \models \mathsf{L}$ by Lemma 4.2. If $\mathcal{F}$ is infinite, then it contains an $R_{\Box}$-reflexive point by Lemma 4.3. Hence $\mathcal{F} \models \chi_n$ for every $n \geq 1$, and so $\mathcal{F}$ validates every $\chi[m,n]$. Thus $\mathcal{F}$ is a basic frame validating every additional axiom of L, so again $\mathcal{F} \models \mathsf{L}$. Altogether $\mathcal{F}_{\mathsf{L}}$ is the disjoint union of a set of frames that each validate L, so $\mathcal{F}_{\mathsf{L}} \models \mathsf{L}$, i.e., L is canonical.

For the finite model property, suppose that $\mathsf{L} \nvdash \varphi$. We have to show that φ is falsifiable in a model on a *finite* L-frame. Now there is some $x \in W_{\mathsf{L}}$ with $\varphi \notin x$, so φ is false at x in the canonical model $\mathcal{M}_{\mathsf{L}}$. Let $\mathcal{F} = (S, R_{\Box}, R_{\mathsf{A}})$ be the cluster of x in $\mathcal{F}_{\mathsf{L}}$, and $\mathcal{M}$ the restriction of $\mathcal{M}_{\mathsf{L}}$ to $\mathcal{F}$, i.e., $\mathcal{M}(p) = \mathcal{M}_{\mathsf{L}}(p) \cap S$. Then φ is false at x in $\mathcal{M}$. If S is finite, then $\mathcal{F} \models \mathsf{L}$ and we are done.

If however S is infinite, then it contains an $R_{\Box}$-reflexive point. We then carry out a standard filtration process through the finite set Γ of all subformulas of φ, to get a falsifying model for φ on a finite basic frame that also has a reflexive point and so validates L. This is done by defining an equivalence relation $\sim$ on S by putting $y \sim z$ iff $y \cap \Gamma = z \cap \Gamma$. Let S' be the quotient set $S/{\sim}$, and $f : S \to S'$ the natural map. Put $\mathcal{F}' = (S', R'_{\Box}, R'_{\mathsf{A}})$, where $fyR'_{\Box}fz$ iff $y'R_{\Box}z'$ for some $y' \sim y$ and some $z' \sim z$, and similarly for R'_{A}. Putting $\mathcal{M}'(p) = f(\mathcal{M}(p))$ for all $p \in \Gamma$ gives a model on $\mathcal{F}'$ such that $\mathcal{M}'(\psi) = f(\mathcal{M}(\psi))$ for all $\psi \in \Gamma$. It follows that $fx \notin \mathcal{M}'(\varphi)$, so φ is false at fx in $\mathcal{M}'$.

$\mathcal{M}'$ is what is known as the *least filtration of $\mathcal{M}$ through* Γ. Its underlying set S' is finite, with at most $2^{|\Gamma|}$ elements. The symmetry of $R_{\Box}$ and the universality of R_{A} on S transfer to

$R'_\Box$ and R'_A on S', respectively, so $\mathcal{F}'$ is a finite cluster. But there is some $y \in S$ with $yR_\Box y$, and hence $fyR'_\Box fy$, so $\mathcal{F}'$ has a reflexive point, which is enough to force $\mathcal{F}' \models \mathsf{L}$ as explained above.

To sum up, we have seen that every non-theorem of L is falsifiable on a finite L-frame that is a cluster. $\Box$

As a first example of a logic fulfilling this Theorem, let EG be the basic logic[3] with additional axioms $\{\chi_2\} \cup \{\chi[\,|\mathcal{G}_n|, n] : n > 2\}$. (The role of χ_2 will be explained in the next section.)

Theorem 4.5 EG *is a decidable logic.*

Proof. From the computable enumeration $\{\mathcal{G}_n : n \geq 2\}$ (Theorem 2.1(3)) we obtain a computable enumeration of the formulas $\{\chi[\,|\mathcal{G}_n|, n] : n > 2\}$. Since there are only finitely many other axioms of EG, it follows that the set of all axioms is computably enumerable, and therefore so is the set of all EG-theorems.

Now a finite cluster $\mathcal{F}$ validates EG iff it either contains an $R_\Box$-reflexive point, or else is a graph with $\chi(\mathcal{F}) > 2$ and also $\chi(\mathcal{F}) > n$ for all $n > 2$ such that $|\mathcal{G}_n| \leq |\mathcal{F}|$ (see Lemma 3.1). There are finitely many such n, so it is decidable whether $\mathcal{F} \models \mathsf{EG}$. Hence we can computably enumerate the (isomorphism types of) finite EG-clusters. By simultaneously enumerating all formulas and checking whether they are valid in finite EG-clusters, we can obtain a computable enumeration of the set of all formulas that are falsifiable in some finite cluster that validates EG. By Theorem 4.4, this is an enumeration of the set of all non-theorems of EG.

Since every formula appears in just one of these two enumerations, either that of the theorems or that of the non-theorems, the set of EG-theorems is decidable. $\Box$

We will now construct continuum many logics fulfilling Theorem 4.4. We write $[2)$ for the set $\{n \in \omega : n \geq 2\}$. For each subset J of $[2)$ let EG_J be the basic logic with additional axioms

$$\{\chi_2\} \cup \{\chi[\,|\mathcal{G}_n|, \chi(\mathcal{G}_n)] : n \in J\}.$$

The formula $\chi[\,|\mathcal{G}_n|, \chi(\mathcal{G}_n)]$ can be interpreted as asserting of a graph that if it has at least as many vertices as $\mathcal{G}_n$ then its chromatic number is greater than the chromatic number $\chi(\mathcal{G}_n)$ of $\mathcal{G}_n$. If J is infinite, then by Theorem 4.4 EG_J is canonical and has the finite model property.

Lemma 4.6 *Let* $m, n \in [2)$ *and* $J \subseteq [2)$.

(1) $\mathcal{G}_n \models \chi[\,|\mathcal{G}_m|, \chi(\mathcal{G}_m)]$ *iff* $m \neq n$.

(2) *If* $n \notin J$, *then* $\mathcal{G}_n \models \mathsf{EG}_J$.

(3) $n \in J$ *if, and only if,* $\mathsf{EG}_J \vdash \chi[\,|\mathcal{G}_n|, \chi(\mathcal{G}_n)]$.

Proof. Note that $\mathcal{G}_n \models \chi_2$ by Lemma 3.1(2), since $\chi(\mathcal{G}_n) \geq \chi(\mathcal{G}_2) > 2$. Let $c_m = \chi(\mathcal{G}_m)$.

(1) Suppose $\mathcal{G}_n \models \chi[\,|\mathcal{G}_m|, c_m]$. Then if $m = n$, the antecedent of $\chi[\,|\mathcal{G}_m|, c_m]$ is satisfied in $\mathcal{G}_n$, so $\mathcal{G}_n \models \chi_{c_n}$, which by Lemma 3.1(2) gives the absurdity $\chi(\mathcal{G}_n) > c_n$. Hence $m \neq n$.

Conversely, suppose $m \neq n$. If $m > n$, then the formula $\chi[\,|\mathcal{G}_m|, c_m]$ is valid in $\mathcal{G}_n$ because its antecedent $\mathsf{E}_{|\mathcal{G}_m|}$ cannot be satisfied in $\mathcal{G}_n$. But if $m < n$, then the formula is again valid in $\mathcal{G}_n$ because $\chi(\mathcal{G}_n) > c_m$ and therefore the consequent is valid in $\mathcal{G}_n$.

(2) Suppose $n \notin J$. Then if $m \in J$, we get $\mathcal{G}_n \models \chi[\,|\mathcal{G}_m|, \chi(\mathcal{G}_m)]$ from part (1).

(3) If $n \in J$, then $\chi[\,|\mathcal{G}_n|, \chi(\mathcal{G}_n)]$ is an axiom of EG_J. But if $n \notin J$, then we have $\mathcal{G}_n \models \mathsf{EG}_J$ by part (2) and $\mathcal{G}_n \not\models \chi[\,|\mathcal{G}_n|, \chi(\mathcal{G}_n)]$ by part (1), so $\mathsf{EG}_J \nvdash \chi[\,|\mathcal{G}_n|, \chi(\mathcal{G}_n)]$ by soundness.

[3] EG stands for "Erdős graphs".

$\square$

Part (3) of this lemma immediately gives

Corollary 4.7 *If $J \neq J'$, then $\mathsf{EG}_J \neq \mathsf{EG}_{J'}$.* $\square$

Theorem 4.8 *If J is an infinite subset of $[2)$, then J and EG_J have the same degree of unsolvability.*

Proof. We show that each of the properties "$n \in J$" and "$\mathsf{EG}_J \vdash \varphi$" is decidable relative to an oracle for deciding the other property.

For any $n \geq 2$ we can effectively find the formula $\chi[\,|\mathcal{G}_n|, \chi(\mathcal{G}_n)]$, so by Lemma 4.6(3), relative to an oracle that decides provability in EG_J we can decide membership of J.

The converse is similar to the proof of Theorem 4.5. From the computable enumeration $\{\mathcal{G}_n : n \geq 2\}$ we obtain, relative to an oracle for deciding membership of J, a computable enumeration of $\{\chi[\,|\mathcal{G}_n|, \chi(\mathcal{G}_n)] : n \in J\}$. Hence the set of all axioms of EG_J is computably enumerable relative to this oracle for J, and therefore so is the set of all EG_J-theorems.

But a finite cluster $\mathcal{F}$ validates EG_J iff it either contains an $R_\Box$-reflexive point, or else is a graph with $\chi(\mathcal{F}) > 2$ and also $\chi(\mathcal{F}) > \chi(\mathcal{G}_n)$ for all $n \in J$ such that $|\mathcal{G}_n| \leq |\mathcal{F}|$. Hence relative to J it is decidable whether $\mathcal{F} \models \mathsf{EG}_J$. This implies, similarly to 4.5, that there is a computable enumeration of the set of all formulas that are falsifiable in some finite cluster that validates EG_J. Since J is infinite this is an enumeration of the set of all non-theorems of EG_J, by Theorem 4.4.

Altogether then, given an oracle for J we can computably enumerate both the set of theorems and the set of non-theorems of EG_J, and so we can decide theoremhood in EG_J. $\square$

5 Failure of Elementary Determination

We are going to show that certain of the logics EG_J are not sound and complete with respect to any elementary class of frames. For this purpose we use the notion of a *homomorphism* $f : \mathcal{F} \to \mathcal{F}'$ between two clusters, defined as a function that preserves the $R_\Box$ relations, i.e., $xR_\Box y$ implies $fxR'_\Box fy$.

Lemma 5.1 *Let $\mathcal{K}$ be any class of basic frames, and $\mathsf{L}_\mathcal{K} = \{\varphi : \mathcal{K} \models \varphi\}$ the logic it determines. Then for any $n \geq 2$ such that $\mathcal{G}_n \models \mathsf{L}_\mathcal{K}$ there exists a frame $\mathcal{F}_n \in \mathcal{K}$ and a cluster $\mathcal{C}_n$ of $\mathcal{F}_n$ for which there is a homomorphism $f_n : \mathcal{C}_n \to \mathcal{G}_n$.*

Proof. Suppose the elements of $\mathcal{G}_n$ are $x_1, \ldots, x_k$. Take variables $p_1, \ldots, p_k$, and let Δ_n be the *finite* set consisting of the following formulas.

$$\begin{array}{ll} \mathsf{A}(p_1 \vee \cdots \vee p_k) & \\ \mathsf{A}\neg(p_i \wedge p_j) & \text{for } 1 \leq i < j \leq k \\ \mathsf{A}(p_i \to \Box\neg p_j) & \text{for all } i, j \text{ such that not } x_i R_\Box x_j. \end{array}$$

Take a model $\mathcal{M}_n$ on the cluster $\mathcal{G}_n$ with $\mathcal{M}_n(p_i) = \{x_i\}$ for all $i \leq k$. Then every member of Δ_n is true at all points in the model, hence so is the conjunction δ_n of the members of Δ_n. Since $\mathcal{G}_n \models \mathsf{L}_\mathcal{K}$, it follows that $\mathsf{L}_\mathcal{K} \not\vdash \neg\delta_n$, so there must be a model $\mathcal{M}$ based on a frame in $\mathcal{K}$ having a point t such that δ_n is true at t in $\mathcal{M}$. Hence all members of Δ_n are true at t.

Let $\mathcal{C}_n$ be the cluster of t in $\mathcal{M}$. Then for each point x in $\mathcal{C}_n$ there is exactly one i such that p_i is true at x in $\mathcal{M}$. Put $f_n(x) = x_i \in \mathcal{G}_n$, to define $f_n : \mathcal{C}_n \to \mathcal{G}_n$. The formulas $\mathsf{A}(p_i \to \Box\neg p_j)$ ensure that f_n is a homomorphism. $\square$

Theorem 5.2 *Let $\mathcal{K}$ be any class of basic frames that validate χ_2. If there are infinitely many n such that $\mathcal{G}_n \models \mathsf{L}_{\mathcal{K}}$, then $\mathcal{K}$ is not an elementary class.*

Proof. Suppose, for the sake of contradiction, that $\mathcal{K}$ is elementary. Then there is a set Σ of sentences in the first-order language of frames $\mathcal{F} = (W, R_\Box, R_\mathsf{A})$ such that $\mathcal{F} \models \Sigma$ iff $\mathcal{F} \in \mathcal{K}$.

Add to the first-order language a unary relation symbol C, and let δ be the sentence

$$\exists x Cx \wedge \forall x(Cx \to \neg\, xR_\Box x) \wedge \forall x\big(Cx \to \forall y(Cy \leftrightarrow xR_\mathsf{A} y)\big)$$

asserting that the interpretation of C in $\mathcal{F}$ is an irreflexive cluster, and hence is a graph.

For each k, let γ_k be the sentence

$$\exists x_1 \cdots \exists x_k \Big(\textstyle\bigwedge_{1 \le i \ne j \le k} (Cx_i \wedge x_i \ne x_j) \wedge \bigwedge_{1 \le i < k} x_i R_\Box x_{i+1} \wedge x_k R_\Box x_1 \Big)$$

asserting that the interpretation of C contains a k-cycle. Put

$$\Delta = \Sigma \cup \{\delta\} \cup \{\neg\, \gamma_k : k \text{ is odd}\}.$$

We will show that Δ has a model. This model must be of the form $(\mathcal{F}, \mathcal{C})$, where $\mathcal{F} \models \Sigma$ and $\mathcal{C}$ is an irreflexive cluster of $\mathcal{F}$ that contains no odd cycle. Then $\mathcal{F} \in \mathcal{K}$ and so by hypothesis $\mathcal{F} \models \chi_2$, hence $\mathcal{C} \models \chi_2$ as $\mathcal{C}$ is an inner subframe of $\mathcal{F}$. But as $\mathcal{C}$ has no odd cycle it is 2-colourable, and therefore $\mathcal{C} \not\models \chi_2$ by Lemma 3.1(2). This contradiction shows that $\mathcal{K}$ cannot be an elementary class after all.

It remains to prove that Δ has a model. Now take any n with $\mathcal{G}_n \models \mathsf{L}_{\mathcal{K}}$. By Lemma 5.1 there is an $\mathcal{F}_n \in \mathcal{K}$ and a cluster $\mathcal{C}_n$ of $\mathcal{F}_n$ with a homomorphism $f_n : \mathcal{C}_n \to \mathcal{G}_n$. Since reflexive points are preserved by homomorphisms, and $\mathcal{G}_n$ has no reflexive points, it follows that $\mathcal{C}_n$ is irreflexive. Also if $\mathcal{C}_n$ had an odd cycle of length $\le n$, then by Theorem 2.1(2) so too would $\mathcal{G}_n$, which is false. Hence

$$(\mathcal{F}_n, \mathcal{C}_n) \models \Sigma \cup \{\delta\} \cup \{\neg\, \gamma_k : k \text{ is odd and } k \le n\}.$$

Since by assumption there are arbitrarily large n for which $\mathcal{G}_n \models \mathsf{L}_{\mathcal{K}}$, this suffices to show that every finite subset of Δ has a model, and hence by Compactness that Δ itself has one too. □

Theorem 5.3 *There are exactly $2^{\aleph_0}$ distinct basic logics that are canonical and have the finite model property but are not sound and complete for any elementary class of frames. They include the decidable logic EG and logics having every possible degree of unsolvability, as well as undecidable logics that have decidable axiomatizations.*

Proof. Let J be any subset of [2) that is infinite and *coinfinite*, i.e., the complement $[2) - J$ is also infinite. By Theorem 4.4, EG_J is canonical and has the finite model property. Suppose that EG_J is determined by a class of frames $\mathcal{K}$, i.e., $\mathsf{EG}_J = \mathsf{L}_{\mathcal{K}}$. Since EG_J is a basic logic and has χ_2 as an axiom, each member of $\mathcal{K}$ is a basic frame validating χ_2. But by Lemma 4.6(2), the set $\{n : \mathcal{G}_n \models \mathsf{L}_{\mathcal{K}}\}$ includes $[2) - J$ and so is infinite. In that case $\mathcal{K}$ is not elementary by Theorem 5.2.

Now there are $2^{\aleph_0}$ distinct subsets J of [2) that are infinite and coinfinite, and each defines a distinct logic EG_J by Corollary 4.7. Since there are $2^{\aleph_0}$ logics altogether, this proves the first statement of the Theorem. The case of EG is similar, since $\{n : \mathcal{G}_n \models \mathsf{EG}\} = [2)$. The case of the different possible degrees of unsolvability follows from Theorem 4.8, since every degree is the degree of some infinite coinfinite subset of [2).

Finally, let J be any computably enumerable but undecidable subset of [2). Then J is infinite and coinfinite. Since J is undecidable, so too is EG_J by Lemma 4.6(3). From the enumerability of J we obtain a computable enumeration $\varphi_1, \varphi_2, \varphi_3, \dots$ of the axioms of EG_J, and then by Craig's trick we get $\{\varphi_1, \varphi_1 \wedge \varphi_2, \varphi_1 \wedge \varphi_2 \wedge \varphi_3, \dots\}$ as a decidable set of axioms for EG_J. $\square$

The inclusion of the Brouwerian axiom $p \to \Box\Diamond p$ in basic logics ensured that their frames have the symmetry property enjoyed by the edge relations of graphs, and this led to a proof of Theorem 5.2 by a simple compactness argument together with some accessible graph theory. The analysis could be carried through without the symmetry assumption, but that would require a more involved proof of the appropriate version of 5.2. What really lies in the background here is the following fact, which provides a criterion for failure of elementary determination that applies to logics in any kind of modal language.

Theorem 5.4 *Let L be a normal modal logic for which there exists a set $\{\mathcal{F}_i : i \in I\}$ of finite L-frames and an ultrafilter D on I such that the ultraproduct $\prod_D \mathcal{F}_i$ is not an L-frame. Then L is not determined by any elementary class of frames.*

Proof. (Sketch.) This is shown in [15] for a monomodal logic, but the result holds in general. One approach to it is to use the fact, which follows from [13, 4.12] or [12, 11.4.2], that if L is determined by *some* elementary class of frames, then it is determined by an elementary class $\mathcal{K}$ that is closed under inner subframes and has $\mathcal{F}_{\mathsf{L}} \in \mathcal{K}$. But every finite L-frame is isomorphic to an inner subframe of $\mathcal{F}_{\mathsf{L}}$, so this implies $\{\mathcal{F}_i : i \in I\} \subseteq \mathcal{K}$. Then $\prod_D \mathcal{F}_i \in \mathcal{K}$ as elementary classes are closed under ultraproducts. But that contradicts the fact that $\prod_D \mathcal{F}_i \not\models \mathsf{L}$ and L is sound for $\mathcal{K}$.

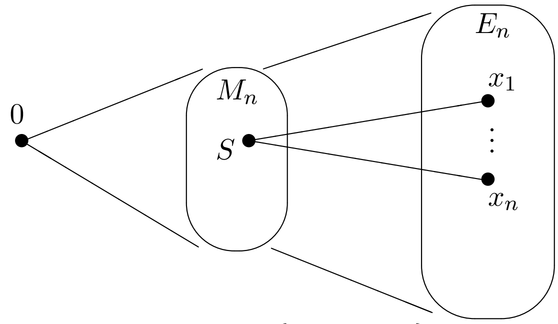
To see why the assertion about finite L-frames holds, let $\mathcal{F}$ be any such frame, and take a model $\mathcal{M}$ on $\mathcal{F}$ such that for each element x of $\mathcal{F}$ there is a variable p with $\mathcal{M}(p) = \{x\}$. Put $fx = \{\varphi : x \in \mathcal{M}(\varphi)\}$. Since $\mathcal{F} \models \mathsf{L}$, fx is maximally L-consistent, so this gives a map $f : \mathcal{F} \to \mathcal{F}_{\mathsf{L}}$. It is straightforward to check that f is an isomorphism between $\mathcal{F}$ and a subframe of $\mathcal{F}_{\mathsf{L}}$ that is inner (the latter requires the finiteness of $\mathcal{F}$). $\square$

We will use the criterion of this Theorem in the next sections to demonstrate further failures of elementary determination.

6 Monomodal Examples

By applying a translation of bimodal logic into monomodal logic due to Thomason [30] it would be possible to convert the logics EG_J into single-modality logics that are canonical but not elementarily determined. Here instead we give more natural examples by adapting some ideas that were used in [10] to prove the non-canonicity of the McKinsey axiom $\Box\Diamond p \to \Diamond\Box p$, or equivalently $\Diamond(\Diamond p \to \Box p)$.

We define a sequence $\{\mathcal{H}_n : n \geq 1\}$ of finite monomodal frames $\mathcal{H}_n = (W_n, R_n)$, depicted as



$S = \{x_1, \dots, x_n\}$

Here $E_n = \{k \in \omega : 1 \leq k \leq n2^n\}$ (as will become apparent, any set with $n2^n$ elements would do for E_n). Then $M_n = \{S \subseteq E_n : |S| = n\}$ is the set of all n-element subsets of E_n, and $W_n = \{0\} \cup M_n \cup E_n$. Thus $|\mathcal{H}_n| = 1 + \binom{n2^n}{n} + n2^n$. The binary relation R_n is specified by

$$\begin{aligned} R_n^0 &= M_n \cup E_n \\ R_n^S &= S \quad \text{for } S \in M_n \\ R_n^e &= \emptyset \quad \text{for } e \in E_n. \end{aligned}$$

The frame $\mathcal{H}_n$ is generated by the point 0 whose R_n-alternatives are all points of W_n except itself. The members of M_n are the *middle-points of* $\mathcal{H}_n$. The alternatives of each $S \in M_n$ are just the members of S. The members of E_n are *end-points* of $\mathcal{H}_n$ and have no R_n-alternatives. Note that

$$\textit{every middle-point of } \mathcal{H}_n \textit{ has exactly } n \textit{ alternatives,} \tag{3}$$

a fact that will be crucial below (in the proof of Lemma 6.7). It is readily checked that R_n is transitive. Hence $\mathcal{H}_n$ validates the logic K4, which is the normal modal logic with axiom 4: $\Box p \to \Box\Box p$, valid in precisely the transitive frames.

We use a single diamond modality $\Diamond$ with dual $\Box$. For $m \geq 1$, let $\exists_m$ be the formula

$$\Diamond p_1 \wedge \Diamond(p_2 \wedge \neg p_1) \wedge \cdots \wedge \Diamond(p_m \wedge \neg p_1 \wedge \cdots \wedge \neg p_{m-1}).$$

$\exists_m$ is satisfiable at x in a frame (W, R) iff $|R^x| \geq m$.

For $n \geq 1$, let μ_n be the formula

$$\Diamond^2\top \to \Diamond\big(\Diamond\top \wedge (\Diamond q_1 \to \Box q_1) \wedge \cdots \wedge (\Diamond q_n \to \Box q_n)\big),$$

where $\Diamond^2\top$ abbreviates $\Diamond\Diamond\top$.

Lemma 6.1 (1) *$\mathcal{H}_n \models \mu_m$ for all $m \leq n$.*

(2) *For $n \geq 2$, $\mathcal{H}_n \models \mu_m$ iff $m \leq n$.*

Proof.

(1) It is enough to show that $\mathcal{H}_n \models \mu_n$, since $\mu_n \to \mu_m$ is valid whenever $m \leq n$. The antecedent $\Diamond^2\top$ of μ_n is satisfiable just at the generator 0 in $\mathcal{H}_n$, so the only issue is whether the consequent of μ_n is also satisfied at 0. That depends on the truth-values of variables at end-points.

There are 2^n possible truth-value assignments to the list of variables $q_1, \ldots, q_n$. Given a model $\mathcal{M}$ on $\mathcal{H}_n$, by labelling each end-point by the valuation it gives to these variables we get a partition of E_n into at most 2^n subsets, with all the members of any one partition-set assigning the same values to $q_1, \ldots, q_n$. One of these $\leq 2^n$ subsets must have at least n elements, or else there could be at most $(n-1)2^n$ end-points altogether, contradicting the fact that $|E_n| = n2^n$. Hence there exists an n-element set S, i.e., a middle-point of $\mathcal{H}_n$, such that all members of S assign the same values to $q_1, \ldots, q_n$. Thus each q_i has a constant truth-value on R^S, and therefore $(\Diamond q_i \to \Box q_i)$ is true at S for all $i \leq n$. But $\Diamond\top$ is also true at S, by (3), so S makes the consequent of μ_n true at the generator 0 in the arbitrary model $\mathcal{M}$ on $\mathcal{H}_n$, as desired.

(2) For $n \geq 2$ it holds that $(n-1)2^{n+1} \geq n2^n$, so there exists a partition of E_n into at most 2^{n+1} sets each of size at most $n-1$. Associate with each partition-set X a distinct truth-valuation of $q_1, \ldots, q_{n+1}$, and let each member of X assign this valuation to these variables. The result is a model in which there is no n-element set of end-points whose members all

give the same valuation to these variables. Indeed if S is any middle-point of $\mathcal{H}_n$, then S is larger than any partition set, so there must exist at least two elements of S that belong to different partition-sets, so assign a different truth-value to q_i for some $i \leq n+1$. For that i, $\Diamond q_i \to \Box q_i$ is false at S, therefore so is

$$\Diamond\top \wedge (\Diamond q_1 \to \Box q_1) \wedge \cdots \wedge (\Diamond q_n \to \Box q_{n+1}).$$

But this last formula is also false at all points of E_n, since those points falsify $\Diamond\top$, and therefore altogether the formula is false throughout R_n^0. This shows that the consequent of μ_{n+1} is false at the generator 0, and so $\mathcal{H}_n \not\models \mu_{n+1}$. It follows that $\mathcal{H}_n \not\models \mu_m$ for all $m > n$. $\Box$

For $m, n \geq 1$, let $\mu[m,n]$ be the formula $\exists_m \to \mu_n$.

Lemma 6.2 *For any $m, n \geq 1$, $\mu[\,|\mathcal{H}_m|, m]$ is valid in $\mathcal{H}_n$.*

Proof. If the antecedent of $\mu[\,|\mathcal{H}_m|, m]$ is true at some point in a model on $\mathcal{H}_n$, then $\mathcal{H}_n$ has at least $|\mathcal{H}_m|$ elements, so $n \geq m$. Hence $\mathcal{H}_n \models \mu_m$ by Lemma 6.1(1). $\Box$

Let $\mu_R(x)$ be the first-order formula

$$\exists y\big(xRy \wedge \exists z \forall w(yRw \leftrightarrow w = z)\big)$$

asserting that there exists $y \in R^x$ with $|R^y| = 1$. It is evident that if $\mu_R(x)$ holds of a point x in a frame, then for all $n \geq 1$, the consequent of μ_n will be true at x in any model on that frame. Thus the elementary condition

$$\forall x(\exists z(xR^2z) \to \mu_R(x))$$

(where xR^2z iff $\exists y(xRyRz)$) is sufficient for validity of μ_n. It is not in general necessary, as $\mu_R(0)$ fails in $\mathcal{H}_n$ for $n > 1$.

Lemma 6.3 *Let L be any normal extension of K4 such that there are infinitely many n for which there exists an m such that $\mathsf{L} \vdash \mu[m,n]$. Let x be a point in the canonical frame $\mathcal{F}_\mathsf{L}$ that generates an infinite inner subframe and has $\Diamond^2\top \in x$. Then $\mu_R(x)$ holds in $\mathcal{F}_\mathsf{L}$.*

Proof. (This is analogous to Lemma 4.3.) Let $\mathcal{F}_\mathsf{L} = (W_\mathsf{L}, R)$. Put

$$y_0 = \{\varphi : \Box\varphi \in x\} \cup \{\Diamond\top\} \cup \{\Diamond\psi \to \Box\psi : \psi \text{ is a formula}\}.$$

If y_0 is L-consistent, it is included in some $y \in W_\mathsf{L}$. Then xRy; $\Diamond\top \in y$ and so $|R^y| \geq 1$; and $(\Diamond\psi \to \Box\psi) \in y$ for all formulas ψ, which ensures that $|R^y| \leq 1$ and establishes $\mu_R(x)$ as required.

But if y_0 were not L-consistent, there would be some φ with $\Box\varphi \in x$, and some formulas $\psi_1, \ldots, \psi_k$, for some $k \geq 1$, such that

$$\mathsf{L} \vdash \varphi \to \neg\big(\Diamond\top \wedge (\Diamond\psi_1 \to \Box\psi_1) \wedge \cdots \wedge (\Diamond\psi_k \to \Box\psi_k)\big).$$

By assumption there is some $n \geq k$ and some m such that $\mathsf{L} \vdash \mu[m,n]$. Putting $\psi_j = \psi_k$ for $k < j \leq n$ and applying normality of L then leads to

$$\mathsf{L} \vdash \Box\varphi \to \Box\neg\big(\Diamond\top \wedge (\Diamond\psi_1 \to \Box\psi_1) \wedge \cdots \wedge (\Diamond\psi_n \to \Box\psi_n)\big),$$

and so as $\Box\varphi \in x$,

$$\Diamond\big(\Diamond\top \wedge (\Diamond\psi_1 \to \Box\psi_1) \wedge \cdots \wedge (\Diamond\psi_n \to \Box\psi_n)\big) \notin x. \tag{4}$$

Since $\mathsf{L} \vdash \Box p \to \Box\Box p$, the relation R in $\mathcal{F}_{\mathsf{L}}$ is transitive (and hence $\mathcal{F}_{\mathsf{L}} \models \Box p \to \Box\Box p$). Transitivity ensures that the inner subframe generated by x is based on the set $\{x\} \cup R^x$. Since this subframe is infinite, R^x must be infinite, so we can choose m distinct points $x_1, \ldots, x_m$ in R^x. Then there are formulas $\varphi_1, \ldots, \varphi_m$ with $\varphi_i \in x_j$ iff $i = j$. Then $(\varphi_i \wedge \bigwedge_{1 \le j < i} \neg\varphi_j) \in x_i$ for all $i \le m$, and so

$$\bigwedge_{1 \le i \le m} \Diamond(\varphi_i \wedge \bigwedge_{1 \le j < i} \neg\varphi_j) \in x.$$

Together with (4) and the fact that $\Diamond^2\top \in x$, this contradicts the fact that every instance of $\mu[m,n]$ belongs to x. So y_0 is L-consistent, and the proof is complete. □

Theorem 6.4 *Let L be any normal extension of K4 that is defined by additional axioms of the form μ_n or $\mu[m,n]$. If there are infinitely many n for which there exists an m such that $\mathsf{L} \vdash \mu[m,n]$, then L is canonical.*

Proof. Let $\mathcal{F}$ be the canonical frame of L. The K4 axiom $\Box p \to \Box\Box p$ is canonical so is valid in $\mathcal{F}$ (as explained in the proof of Lemma 6.3).

Now let $x \in \mathcal{F}$. If the inner subframe $\mathcal{F}^x$ generated by x is finite, then it validates L (by the proof of Lemma 4.2), so L cannot be falsified at x in any model on $\mathcal{F}$. Alternatively, if $\Diamond^2\top \notin x$, then in any model on $\mathcal{F}$ the antecedent $\Diamond^2\top$ of every μ_n is false at x, so every μ_n is true at x, and hence so is every $\mu[m,n]$.

This leaves the case that $\mathcal{F}^x$ is infinite and $\Diamond^2\top \in x$. But then by Lemma 6.3 the condition $\mu_R(x)$ holds at x in $\mathcal{F}$, which ensures that every μ_n, and hence every $\mu[m,n]$, is true at x in all models on $\mathcal{F}$.

Thus $\mathcal{F}$ validates the axioms of L. □

Now let D be a non-principal ultrafilter on $\{n : n \ge 1\}$, and $\prod_D \mathcal{H}_n$ the associated ultraproduct of the frames $\mathcal{H}_n = (W_n, R_n)$. Recall that $\prod_D \mathcal{H}_n = (\prod_D W_n, R)$ where $\prod_D W_n$ is the quotient set of the direct product $\prod_{n \ge 1} W_n$ by the equivalence relation $\equiv$ defined by: $f \equiv g$ iff $\{n : f(n) = g(n)\} \in D$. We write f_D for the $\equiv$-equivalence class of any $f \in \prod_{n \ge 1} W_n$. Then $f_D R g_D$ in $\prod_D \mathcal{H}_n$ iff $\{n : f(n) R_n g(n)\} \in D$.

We are going to show that μ_1 is falsified in some model on the ultraproduct $\prod_D \mathcal{H}_n$. To do this we use the following criterion, adapted from [10, Theorem 1].

Lemma 6.5 *Let $\mathcal{F}$ be a frame containing a point r such that the set*

$$[R^r] = \{y \in R^r : R^y \ne \emptyset\}$$

is non-empty and for any $y \in [R^r]$, R^y is an infinite set that is at least as large in cardinality as $[R^r]$. Then μ_1 is falsifiable at r in some model on $\mathcal{F}$.

Proof. Let κ be the cardinality of $[R^r]$, and let $\{y_\lambda : \lambda < \kappa\}$ be an indexing of the members of $[R^r]$ by the ordinals λ less than κ. For each λ, distinct points $y_{\lambda 0}, y_{\lambda 1} \in R^{y_\lambda}$ will then be defined in such a way that $\{y_{\lambda 0}, y_{\lambda 1}\} \cap \{y_{\mu 0}, y_{\mu 1}\} = \emptyset$ whenever $\lambda \ne \mu < \kappa$. Then declaring q_1 to be true just at the points in $\{y_{\lambda 1} : \lambda < \kappa\}$ defines a model on $\mathcal{F}$ in which q_1 is false at $y_{\lambda 0}$, and true at $y_{\lambda 1}$, making $\Diamond q_1 \to \Box q_1$ false at y_λ. Since this is the case for every member y_λ of $[R^r]$, while $\Diamond\top$ is false at every member of $R^r - [R^r]$, it follows that $\Diamond(\Diamond\top \wedge (\Diamond q_1 \to \Box q_1))$ is false at r in this model. On the other hand, $\Diamond^2\top$ is true at r, since $[R^r] \ne \emptyset$. Hence μ_1 is false at r.

It remains then to show that the $y_{\lambda i}$ can be defined as claimed. Fix $\lambda < \kappa$, and suppose inductively that $y_{\mu i}$ has been defined for all $\mu < \lambda$ and $i \in \{0,1\}$, such that $y_{\mu i} \ne y_{\nu j}$ whenever $\mu \ne \nu < \lambda$ and $j \in \{0,1\}$. Let

$$Y_\lambda = \{y_{\mu 0}, y_{\mu 1} : \mu < \lambda\}.$$

Then if λ is a finite ordinal, Y_λ is a finite set, so as R^{y_λ} is infinite by hypothesis, distinct points $y_{\lambda 0}, y_{\lambda 1}$ can be selected from $R^{y_\lambda} - Y_\lambda$. If however λ is infinite, then the cardinality of Y_λ is at most that of λ, and hence is less than κ. But R^{y_λ} has cardinality at least κ, so again the selection of $y_{\lambda 0}, y_{\lambda 1} \in R^{y_\lambda}$ can be made to ensure that $y_{\mu i} \neq y_{\nu j}$ for all $\mu \neq \nu \leq \lambda$. Hence the construction extends to λ, and so goes through by induction.

□

We also need the following fact about cardinalities of ultraproducts that is due to [7, Theorem 1.28] (the proof can also be found in [1, Theorem 6.3.12]):

Lemma 6.6 *If $\{X_n : n \geq 1\}$ is a collection of finite sets, and $\{n : |X_n| = k\} \notin D$ for all $k \in \omega$, then $|\prod_D X_n| = 2^{\aleph_0}$.* □

Lemma 6.7 *μ_1 is falsifiable in $\prod_D \mathcal{H}_n$.*

Proof. Let $r = \langle r_n : n \geq 1 \rangle_D$ in $\prod_D \mathcal{H}_n$, where $r_n = 0 \in W_n$. Notice that in $\mathcal{H}_n$, the set $[R_n^0] = \{x \in R_n^0 : R_n^x \neq \emptyset\}$ is just the set M_n of middle points. Hence $[R_n^0] \neq \emptyset$ for all $n \geq 1$, and so $[R^r] \neq \emptyset$ in $\prod_D \mathcal{H}_n$. Indeed if $g \in \prod_{n \geq 1} M_n$, then $g_D \in [R^r]$. Moreover, each $y \in [R^r]$ is equal to g_D for some $g \in \prod_{n \geq 1} M_n$.

Since D is non-principal, it contains only infinite sets. Thus as $|W_n|$ is a strictly increasing function of n, it is not constant on any set in D, so by Lemma 6.6 it follows that $\prod_D \mathcal{H}_n$ has $|\prod_D W_n| = 2^{\aleph_0}$ elements. But if $y \in [R^r]$ in $\prod_D \mathcal{H}_n$, we can similarly show that R^y is of size $2^{\aleph_0}$, so that r satisfies Lemma 6.5, giving the desired result that μ_1 is falsifiable at r. To see this, let $y = g_D$ for some $g \in \prod_{n \geq 1} M_n$. Then for each $h \in \prod_{n \geq 1} R_n^{g(n)}$ we have $g(n) R_n h(n)$ for all $n \geq 1$, so yRh_D in $\prod_D \mathcal{H}_n$, i.e., $h_D \in R^y$. This shows that the natural injection $\prod_D R_n^{g(n)} \rightarrowtail \prod_D W_n$ has its range included in R^y. But for each $n \geq 1$, $g(n)$ is a middle-point of $\mathcal{H}_n$, so $|R_n^{g(n)}| = n$ by (3). So $|R_n^{g(n)}|$ is a strictly increasing function of n, hence cannot be constant on any set in D, which ensures that $|\prod_D R_n^{g(n)}| = 2^{\aleph_0}$, hence $|R^y| = 2^{\aleph_0}$. □

Now let H be the normal extension of K4 with axioms μ_1 and $\mu[|\mathcal{H}_n|, n]$ for all $n \geq 2$.

Theorem 6.8 *H is canonical but not determined by any elementary class of frames.*

Proof. Canonicity follows from Theorem 6.4. The frame $\mathcal{H}_n$ validates the logic H for all $n \geq 1$, but the ultraproduct $\prod_D \mathcal{H}_n$ does not, since it invalidates the axiom μ_1. Failure of elementary determination then follows from Theorem 5.4. □

These constructions can be adapted to show that there are $2^{\aleph_0}$ distinct normal extensions of K4 that are canonical but not elementarily determined. For each set J of positive integers let H_J be the normal extension of K4 with axioms μ_1 and $\mu[|\mathcal{H}_n| - 1, n+1]$ for all $n \in J$. Using Lemma 6.1 it can be shown that if $n \notin J$ then $\mathcal{H}_n \models \mathsf{H}_J$, and that

$$J = \{n : \mathsf{H}_J \vdash \mu[|\mathcal{H}_n| - 1, n+1]\}.$$

If J is infinite, then H_J is canonical by Theorem 6.4. If J is also coinfinite then we can take a non-principal ultrafilter on $\{n : n \notin J\}$, and then the resulting ultraproduct of the family $\{\mathcal{H}_n : n \notin J\}$ of H_J-frames will falsify μ_1, showing that H_J is not elementarily determined. It is left to the reader to verify the details of these claims.

Now let H^+ be the normal extension of H by the additional axiom $\Box^3\bot$ whose validating frames are defined by the condition $\forall x \forall y \neg(xR^3y)$. An H^+-frame has *depth* at most two, where the depth of a frame is the length of its longest cycle-free R-path. If $\mathcal{V}$ is any finite set of propositional variables, then up to provable equivalence in H^+, there are only finitely many non-equivalent formulas whose variables come from $\mathcal{V}$. This property was established in [28, Theorem II.6.5] for any frame-complete normal logic L extending K4 that has a fixed finite upper bound on the depth of L-frames. It implies that the Lindenbaum algebra of L generated by $\mathcal{V}$ is finite, and hence, since Lindenbaum algebras are free, that every finitely generated L-algebra is finite. Then any L-algebra is *locally finite*, meaning that all of its finitely generated subalgebras are finite.[4]

The interest in H^+ is that it shows that even the strong hypothesis of local finiteness of algebraic models does not imply the converse of Fine's theorem. H^+ is canonical, but is not elementarily determined because all of the frames $\mathcal{H}_n$ validate H^+. Note that local finiteness readily implies that H^+ has the finite model property.

We end now with some questions for further investigation. First, is there a canonical but not elementarily determined logic that is *Halldén complete*? Recall that Halldén completeness of L means that for any two formulas φ and ψ that have no variables in common, if $\mathsf{L} \vdash \varphi \vee \psi$ then either $\mathsf{L} \vdash \varphi$ or $\mathsf{L} \vdash \psi$. The axioms of the form $\chi[m,n]$ and $\mu[m,n]$ show that none of the logics discussed in this paper are Halldén complete.

Second, is every canonical normal extension of K4.3 elementarily determined? As already mentioned, it was shown in [32] that every canonical normal extension of linear tense logic is elementarily determined, so the question asks: does this hold for the future fragment of tense logic? A counter example including S4.3 would be of particular interest, since all extensions of S4.3 are finitely axiomatizable.

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[4] See [13, Theorem 6.3] for a direct proof that if $\mathcal{F}$ is transitive and has finite depth, then the modal algebra of all subsets of $\mathcal{F}$ is locally finite.

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