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Representation of Game Algebras

Abstract. We prove that every abstractly defined game algebra can be represented as an algebra of consistent pairs of monotone outcome relations over a game board. As a corollary we obtain Goranko's result that van Benthem's conjectured axiomatization for equivalent game terms is indeed complete.

Keywords: game algebra, game theory, algebraic lattice expansion, representation theory.

1. Introduction

Research into the connections between logic and game theory has become more active in recent years, cf. van Benthem [4]. Our paper fits in a line of research that was initiated by Parikh [2] and has recently been developed further by van Benthem, Pauly, Goranko and others, cf. [3, 1]. These researchers study games from a logical perspective analogous to the theory of processes in computer science. The focus is on an abstract approach in which games or game expressions are analyzed in semantic terms; a crucial role in this perspective is played by so-called outcome relations or effectivity functions.

The basic idea is as follows; assume that we are dealing with 2 players, called 0 and 1, and with a *game board* which for the moment will be just a set B of objects that we call *states* or *positions*. With any game g and each player i we will associate an *outcome relation* R^i_g; that is, a relation between positions and sets of positions. Intuitively, if a position p is in the relation R^i_g with the set T of positions, this means that in position p, player i has a strategy of playing the game g in such a way that after play, the resulting state belongs to the set T. In brief, pR^i_gT holds if in position p, i can force that the outcome of g will be a position in T.

Given these intuitions, there are some restrictions that one should or could impose on such outcome relations. In this paper we confine ourselves to the properties of *monotonicity* and *consistency*:

(monotonicity) if pR^i_gT and $T \subseteq U$ then pR^i_gU,

(consistency) if pR^i_gT then not $pR^{1-i}_g(B - T)$.

Here and in the sequel we use the convention that $1-i$ denotes the adversary of i.

This game board perspective offers a natural notion of equivalence between games, making g and h equivalent on a certain game board if both players have the same power in g as in h; that is, if $R^i_g = R^i_h$ for each player i.

An interesting aspect of the approach by Parikh and others is that various ways to *compose* games are studied from an abstract, algebraic perspective. That is, formal game operations are introduced to construct new games from old. Below we list some of the most natural game operations that one could consider in this context:

(choice) $g \vee_i h$ is the game in which the first move is that player i chooses whether to play g or h;

(dualization) $-g$ is the game g but with the roles of the two players reversed;

(composition) $g \diamond h$ is the game in which a play of g is followed by a play of h. repeatedly

Various other operations have been studied, such as iteration (play g repeatedly until one of the players decides to stop) or idle games; however, in this paper we confine our attention to choice, dualization and composition.

Naturally, the outcome relations of composed games should be based on those of their components according to the following definition:

$$\begin{array}{rll} pR^i_{g\vee_i h}T & \text{iff} & pR^i_g T \text{ or } pR^i_h T, \\ pR^i_{g\vee_{1-i} h}T & \text{iff} & pR^i_g T \text{ and } pR^i_h T, \\ pR^i_{-g}T & \text{iff} & pR^{1-i}_g T, \\ pR^i_{g\diamond h}T & \text{iff} & pR^i_g U \text{ for some set } U \text{ such that } uR^i_h T \text{ for all } u \in U. \end{array} \tag{1}$$

The reader could easily verify that with this definition the conditions of monotonicity and consistency are propagated; for instance, if R^i_g and R^i_h are monotone outcome relations then, given (1), so are $R^i_{g\vee_i h}$, $R^i_{g\vee_j h}$ and $R^i_{g\diamond h}$.

The notion of game equivalence that we introduced earlier can be applied to such composed games as well, and there are certain interesting laws to be discovered here. For instance, the reader can easily verify that the games $(g_1 \vee_0 g_2) \diamond h$ and $(g_1 \diamond h) \vee_0 (g_2 \diamond h)$ will be equivalent on any game board, no matter what the outcome relations of the games g_1, g_2 and h are. Being slightly more formal, we define a *game expression* as a term in the algebraic language over the set of function symbols $\{\vee_0, \vee_1, -, \diamond\}$. A *game board* is a pair $\mathcal{B} = (B, R)$ such that B is a set of positions and R is a map assigning to each *atomic game* or *game variable* x a pair (R^0_x, R^1_x) of outcome relations;

we require the monotonicity and consistency conditions to hold. Inductively, we use (1) to define outcome relations R^i_g for each player i and each game expression g. Now we say that two game expressions are *equivalent on a game board* $\mathcal{B} = (B, R)$, notation: $\mathcal{B} \models g \approx h$, if $R^i_g = R^i_h$ for each player i. We call g and h *equivalent* if they are equivalent on every game board.

An obvious problem is to find a complete axiomatization for this semantic notion of equivalence. A proposal for such an axiomatization was made by van Benthem, cf. our discussion in the next section. It was proved in GORANKO [1] that van Benthem's axioms indeed completely generate the notion of game equivalence. Goranko's proof is based on a syntactic analysis of game expressions and a validity preserving translation of game identities into formulas in the language of basic modal logic.

In this paper we will prove the completeness of van Benthem's axiomatization by purely algebraic means. Our main result, Theorem 1, is a strengthening of Goranko's theorem: we will show that every game algebra (that is, every abstract algebra satisfying van Benthem's axioms) is in fact isomorphic to an algebra of consistent pairs of outcome relations.

We hope that our algebraic approach will lead to more results in the future. In particular, we plan to concentrate on the following questions:

- the precise connection between our approach and that of Goranko,
- axiomatizations of the notion of game equivalence if further constraints are added to the outcome relations,
- axiomatizations of the notion of game equivalence in extended languages; in particular, with iteration and perhaps other fixed point operators,
- connections with the game logic as developed by Parikh.

2. Board algebras and game algebras

The aim of this section is to rephrase the axiomatization problem for equivalent game expressions in purely algebraic terms. We will define two classes of concrete and abstract algebras called *board algebras* and *game algebras*, respectively. We will show that the axiomatization problem can be solved by showing that every game algebra can be represented as a board algebra; in other words, we will prove an result analogous to Stone's Representation Theorem which states that every abstract Boolean algebra is in fact representable as a field of sets.

For notational convenience, and in order to stay close to the algebraic tradition, we will change our notation for the choice operation symbols, writing $\vee$ for $\vee_0$ and $\wedge$ for $\vee_1$.

Board algebras

We first consider the board algebras; these are the concrete game algebras that one can associate with a game board.

DEFINITION 2.1. Given a set B, let $O(B) = \mathcal{P}(B \times \mathcal{P}(B))$ denote the collection of outcome relations on B, and $O_m(B)$, the set of monotone outcome relations. $G(B)$ and $G_m(B)$ denote the set of pairs of outcome relations and the set of pairs of monotone outcome relations, respectively; that is, $G(B) = O(B) \times O(B)$ and $G_m(B) = O_m(B) \times O_m(B)$. Finally, $G_{mc}(B)$ is the set of *consistent* pairs of monotone outcome relations. $\lhd$

Intuitively, any element $(R_0, R_1) \in G_{mc}(B)$ denotes a possible interpretation of a game played on the board B, given our monotonicity and consistency requirements.

It will be convenient for us to rephrase the inductive definition (1) of outcome relations for complex game expressions in terms of operations on $O(B)$ and $G(B)$. Define the binary operation $\circ$ on outcome relations as follows:

$$R \circ S \ := \ \{(p, T) \mid pRU \text{ for some set } U \text{ such that } uST \text{ for all } u \in U\}.$$

Also, note that outcome relations over B, being subsets of the set $\mathcal{P}(B \times \mathcal{P}(B))$, are subject to the standard set-theoretic operations such as taking unions or intersections. belongs Given all of this, we invite the reader to check that (1) can be rephrased as follows:

$$\begin{array}{rcl} R^0_{g \vee h} & = & R^0_g \cup R^0_g, \\ R^0_{g \wedge h} & = & R^0_g \cap R^0_g, \\ R^0_{g \diamond h} & = & R^0_g \circ R^0_g. \end{array}$$

while a similar definition applies to player 1.

Recall that the notion of equivalence between game expressions is defined in terms of the outcome relation for *both* players. For a proper algebraic phrasing, we thus have to interpret the function symbols as operations on the set of *pairs* of outcome relations on a board B.

DEFINITION 2.2. Fix a set B. First, consider the following operations on the set $G(B)$:

$$\begin{aligned}(R_1, R_2) \sqcup (S_1, S_2) &= (R_1 \cup S_1, R_2 \cap S_2),\\ (R_1, R_2) \sqcap (S_1, S_2) &= (R_1 \cap S_1, R_2 \cup S_2),\\ (R_1, R_2)^- &= (R_2, R_1),\\ (R_1, R_2)\Box(S_1, S_2) &= (R_1 \circ S_1, R_2 \circ S_2).\end{aligned}$$

Now define the *full outcome algebra over* B to be the structure $\mathcal{G}(B) = (G(B), \sqcup, \sqcap, ^-, \Box)$; subalgebras of $\mathcal{G}(B)$ are called *outcome algebras over* B. An outcome algebra of the form $(A, \sqcup, \sqcap, ^-, \Box)$ is called *monotone* if $A \subseteq G_m(B)$ and a *board algebra* if $A \subseteq G_{mc}(B)$. The *full monotone outcome algebra over* B and the *full board algebra over* B are defined as the structures $\mathcal{G}_m(B) = (G_m(B), \sqcup, \sqcap, ^-, \Box)$ and $\mathcal{G}_{mc}(B) = (G_{mc}(B), \sqcup, \sqcap, ^-, \Box)$, respectively.

The class of board algebras is denoted as B. $\lhd$

Equivalence of game expressions can now simply be stated as the validity of the corresponding game equation in the class of board algebras.

DEFINITION 2.3. Two game expressions g and h are called *equivalent*, notation: $\mathsf{B} \models g \approx h$, if the equation $g \approx h$ is valid in every board algebra. $\lhd$

The problem of axiomatizing the notion of game equivalence thus reduces to finding an axiomatization for the set of game equations that are valid in the class of board algebras.

Game algebras

The proposal by van Benthem that we mentioned in the introduction, comprises the following axioms.

DEFINITION 2.4. Consider the following (quasi-)equations:

$$x \vee x \approx x \qquad x \wedge x \approx x \tag{G1}$$
$$x \vee y \approx y \vee x \qquad x \wedge y \approx y \wedge x \tag{G2}$$
$$x \vee (y \vee z) \approx (x \vee y) \vee z \qquad x \wedge (y \wedge z) \approx (x \wedge y) \wedge z \tag{G3}$$
$$x \vee (x \wedge y) \approx x \qquad x \wedge (x \vee y) \approx x \tag{G4}$$
$$x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z) \qquad x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z) \tag{G5}$$
$$--x \approx x \tag{G6}$$
$$-(x \vee y) \approx -x \wedge -y \qquad -(x \wedge y) \approx -x \vee -y \tag{G7}$$
$$(x \diamond y) \diamond z \approx x \diamond (y \diamond z) \tag{G8}$$
$$(x \vee y) \diamond z \approx (x \diamond z) \vee (y \diamond z) \qquad (x \wedge y) \diamond z \approx (x \diamond z) \wedge (y \diamond z) \tag{G9}$$
$$-x \diamond -y \approx -(x \diamond y) \tag{G10}$$
$$y \preceq z \rightarrow x \diamond y \preceq x \diamond z \tag{G11}$$

Here $s \preceq t$ is an abbreviation of the equation $s \vee t \approx t$.

A *distributive lattice* is any algebra $\mathcal{D} = (D, \vee, \wedge)$ satisfying the equations G1–5; a *de Morgan lattice* is an algebra $\mathcal{M} = (M, \vee, \wedge, -)$ satisfying the equations G1–7. Finally, a *game algebra* is a structure $\mathcal{G} = (G, \vee, \wedge, -, \diamond)$ satisfying the axioms G1–11. We let G denote the class of game algebras. ◁

In words, a distributive lattice is any algebra $\mathcal{D} = (D, \vee, \wedge)$ in which the *join* $\vee$ and the *meet* $\wedge$ are *idempotent* (G1), *commutative* (G2) and *associative* (G3) operations that satisfy the laws of *absorption* (G4) and *distribution* (G5). Any expansion of a distributive lattice with a unary complementation operation is called a de Morgan lattice if it satisfies the *de Morgan laws* (G6) and (G7). A game algebra is an expansion $\mathcal{G} = (G, \vee, \wedge, -, \diamond)$ of a de Morgan lattice $(G, \vee, \wedge, -)$ with an *associative* (G8) binary operator which satisfies the *left-distributive* laws (G9), the *dualization* axiom (G10), and *right-monotonicity* (G11).

Note that although we formulated the right-monotonicity law G11 as a quasi-equation, it can also be phrased equationally as $x \diamond y \preceq x \diamond (y \vee z)$.

Rephrasing the problem algebraically

Van Benthem's conjecture can now be rephrased algebraically as the statement that the classes G and B have the same equational theory. In fact, the main Theorem of this paper states something stronger, namely, that every game algebra can be *represented* as a board algebra.

THEOREM 1. *Every game algebra is isomorphic to a board algebra:*

$$\mathsf{G} = \mathbb{I}\mathsf{B}.$$

To see why this Theorem solves the axiomatization problem, consider two game expressions g and h. By our definitions, g and h are game equivalent expressions if and only if $\mathsf{B} \models g \approx h$. It follows from Birkhoff's Completeness Theorem for equational logic, that $g \approx h$ is derivable from G1–11 if and only if $\mathsf{G} \models g \approx h$. Hence, by Theorem 1 it follows immediately that g and h are equivalent game expressions if and only if the equation $g \approx h$ is derivable from the equations G1–11.

The remainder of the paper is devoted to our proof of Theorem 1. Before we go into the details of the proof, let us briefly sketch the intuitions underlying it. Obviously, the first problem that we encounter when trying to represent a game algebra $\mathcal{G} = (G, \vee, \wedge, -, \diamond)$ as a board algebra is to find a suitable board. Fortunately however, this problem can be tackled easily by concentrating on the lattice reduct $(G, \vee, \wedge)$ of $\mathcal{G}$: we can use the well-known representation theory of distributive lattices and represent $(G, \vee, \wedge)$ as a set lattice over the collection $B_{\mathcal{G}}$ of its prime filters (formal definitions will follow later). Basically, we would like to take this set $B_{\mathcal{G}}$ as the underlying set of the board algebra representing $\mathcal{G}$.

In order to see how this representation should work, take a slightly alternative perspective on the game algebra: associate with each element g of G a map $\Diamond_g : G \to G$ given by

$$\Diamond_g a = g \diamond a.$$

The game algebra can then be seen as forming a structured family of operations on its lattice reduct. Each map $\Diamond_g$ is a monotone operation on the lattice $(G, \vee, \wedge)$ and thus naturally corresponds to a monotone outcome relation Q_g on the set $B_{\mathcal{G}}$. Putting these observations together we will show that there is a natural homomorphism from any game algebra to the full monotone outcome algebra over the set of prime filters of the lattice reduct of the game algebra.

However, there are still two problems that need solving before the above considerations yield a proof of Theorem 1:

separability In general, we cannot guarantee that the 'natural homomorphism' mentioned above is in fact an *embedding*. For instance, it could very well be the case that g and h are distinct elements of $\mathcal{G}$, while $\Diamond_g a = \Diamond_h a$ for all elements $a \in G$; but in such a situation, g and h

would be represented as identical outcome relations. In order to solve this problem we will first *add* elements to $\mathcal{G}$, one of which will *separate* g from h.

consistency The above procedure shows that an arbitrary game algebra is isomorphic to a monotone outcome algebra, but in order to prove Theorem 1 we need to represent game algebras as board algebras, that is, *consistent* monotone outcome algebras. Fortunately, this problem will be quite easy to solve, since we can show that any monotone outcome algebra can be embedded in a board algebra.

Game modules

It will be quite useful to formalize the perspective of a game algebra as a structured family of monotone operations on its lattice reduct. First however, we should note that for technical reasons, when trying to represent a game algebra $\mathcal{G} = (G, \vee, \wedge, -, \diamond)$ we will use its *de Morgan reduct* $(G, \vee, \wedge, -)$ rather than its lattice reduct $(G, \vee, \wedge)$ as the algebra on which the operations $\Diamond_g$ act.

DEFINITION 2.5. Let $\mathcal{G} = (G, \vee, \wedge, -, \diamond)$ be a game algebra. A *module over* $\mathcal{G}$ is an algebra $\mathcal{M} = (M, \vee, \wedge, -, \Diamond_g)_{g \in G}$ such that $(M, \vee, \wedge, -)$ is a de Morgan algebra, and $(\Diamond_g)_{g \in G}$ is a family of unary monotone operations on M satisfying the following equations:

$(M1)$ $\Diamond_{g \vee h} x \approx \Diamond_g x \vee \Diamond_h x$

$(M2)$ $\Diamond_{g \wedge h} x \approx \Diamond_g x \wedge \Diamond_h x$

$(M3)$ $\Diamond_{g \diamond h} x \approx \Diamond_g \Diamond_h x$

$(M4)$ $\Diamond_{-g} x \approx -\Diamond_g - x$

A game module is *separable* if for all distinct elements g and h of G there is an $x \in M$ such that $\Diamond_g x \neq \Diamond_h x$. ◁

Note that with this definition, we may indeed see a given game algebra $\mathcal{G} = (G, \vee, \wedge, -, \diamond)$ as a module over its de Morgan reduct if we put $\Diamond_g a = g \diamond a$. We will not introduce any notation to distinguish these two perspectives on game algebras.

Proof of Theorem 1

Our proof of the representation theorem for game algebras involves the following three steps:

1. In Proposition 5.1 we will prove that every game algebra, seen as a module over itself, can be embedded in a *separable* module over itself. From this it follows that over every game algebra there is a separable module.

2. Proposition 4.2 states that if $\mathcal{M}$ is a separable module over $\mathcal{G}$, then $\mathcal{G}$ is isomorphic to some monotone outcome algebra over $\mathcal{M}$.

3. Finally, we will prove that any monotone outcome algebra can be embedded in a board algebra, cf. Proposition 6.1.

The proof of Theorem 1 is immediate from these results.

3. Monotone operations on distributive lattices

In this section we briefly sketch the required background knowledge on the representation theory of distributive lattices and their monotone expansions. None of the results in this section are originally ours.

Representing distributive lattices

The prime examples of distributive lattices are given by the *lattices of sets*; these are algebras of the form $(A, \cup, \cap)$ with A being some collection of sets which is closed under taking unions and intersections. In fact, it is well-known that every distributive lattice can be represented as such a set algebra. We briefly recall the basic notions that are needed for stating the result that we need.

DEFINITION 3.1. Let $\mathcal{D} = (D, \vee, \wedge)$ be a distributive lattice. A *filter* is a subset F of D which is upward closed (if $a \in F$ and $a \leq b$ then $b \in F$) and closed under meets (if $a, b \in F$ then $a \wedge b \in F$). A filter F is *prime* if $a \vee b \in F$ implies that at least one of a and b belongs to F. Let $B_{\mathcal{D}}$ denote the set of prime filters of $\mathcal{D}$.

Given an element $a \in D$, define

$$\widehat{a} = \{p \in B_{\mathcal{D}} \mid a \in p\},$$

that is: $\widehat{a}$ denotes the set of prime filters to which a belongs. ◁

Fact 3.2. *For any distributive lattice $\mathcal{D}$, the map $\widehat{(\cdot)}$ is an embedding of $\mathcal{D}$ in $(\mathcal{P}(B_{\mathcal{D}}), \cup, \cap)$.*

It will also be useful to introduce the notion of a *closed* set of prime filters. There is of course an entire topological theory lurking behind the corner here, but we only need the following tip of this iceberg.

Definition 3.3. Let $\mathcal{D} = (D, \vee, \wedge)$ be a distributive lattice. Given a set T of prime filters, let F_T denote the set of elements a of D such that $a \in p$ for every $p \in T$, or, equivalently,

$$F_T = \{a \in D \mid T \subseteq \widehat{a}\}.$$

A set C of prime filters is *closed* if it is the intersection of sets of the form $\widehat{a}$; or, equivalently, if $C = \bigcap\{\widehat{a} \mid a \in F_C\}$. Given a set T of prime filters, let $\overline{T}$ be the smallest closed superset of T; it is not hard to see that $\overline{T} = \bigcap_{a \in F_T} \widehat{a}$. $\lhd$

Monotone lattice operations

Definition 3.4. Let $\mathcal{D} = (D, \vee, \wedge)$ be a distributive lattice. A map $\Diamond : D \to D$ is *monotone* if $\Diamond a \leq \Diamond b$ whenever $a \leq b$.

A *monotone lattice expansion* is an algebra $(D, \vee, \wedge, \Diamond)$ such that $\Diamond$ is a monotone operation on the distributive lattice $(D, \vee, \wedge)$. All definitions concerning distributive lattices apply to monotone lattice expansions as well. $\lhd$

The prime example of monotone lattice operations stems from monotone outcome relations. Let R be an outcome relation on B, and define the operation $m_R : \mathcal{P}(B) \to \mathcal{P}(B)$ by

$$m_R(T) = \{p \in B \mid pRT\}.$$

It is easy to verify that R is monotone if and only if m_R is a monotone relation on the power set lattice of B.

In fact, one can show that every monotone lattice operation can be represented as an operation of the form m_R for some monotone outcome relation on the set of prime filters of the lattice.

Definition 3.5. Given a monotone lattice expansion $\mathcal{D} = (D, \vee, \wedge, \Diamond)$, let $Q_\Diamond$ be the outcome relation on the board of prime filters of $\mathcal{D}$ given by

$$pQ_\Diamond T \text{ iff } \Diamond a \in p \text{ for all } a \in F_T.$$

$\lhd$

Note that for any set T of prime filters we have $pQ_\diamond T$ if and only if $pQ_\diamond \overline{T}$.

Proposition 3.6. *For any monotone lattice expansion $\mathcal{D} = (D, \vee, \wedge, \diamond)$, the map $\widehat{(\cdot)}$ is an embedding of $\mathcal{D}$ in $(\mathcal{P}(B_\mathcal{D}), \cup, \cap, m_{Q_\diamond})$.*

Proof. By the earlier fact it suffices to prove that

$$\widehat{\diamond a} = m_{Q_\diamond} \widehat{a}$$

for any element $a \in D$. This follows immediately from the observation that for any prime filter $p \in B_\mathcal{D}$ and any $a \in D$:

$$pQ_\diamond \widehat{a} \text{ iff } \diamond a \in p. \tag{2}$$

In order to prove (2), first assume that $pQ_\diamond \widehat{a}$. Since $a \in F_{\widehat{a}}$ it follows that $\diamond a \in p$ by definition of $Q_\diamond$. For the other direction, suppose that $\diamond a \in p$ and let b be an arbitrary element of $F_{\widehat{a}}$. By definition of $F_{\widehat{a}}$ this means that $\widehat{a} \subseteq \widehat{b}$, so by Fact 3.2 we obtain that $a \leq b$. Monotonicity of $\diamond$ gives that $\diamond a \leq \diamond b$, so we find $\diamond b \in p$ since p is a prime filter. Because b was arbitrary this means that $pQ_\diamond \widehat{a}$, as required. ∎

It is obvious that if $\diamond$ is a monotone operation on the lattice $\mathcal{D}$, then $Q_\diamond$ is a monotone outcome relation on $B_\mathcal{D}$.

We will also need the following rather technical lemma which in essence stems from L. Esakia. Recall that a set A of elements in a lattice $\mathcal{D}$ is *downward directed* if for any finite subset $A_0 \subseteq A$ there is an element $a \in A$ such that $a \leq a'$ for every $a' \in A_0$.

Lemma 3.7. *Let $A \subseteq D$ be a downward directed set in the monotone lattice expansion $\mathcal{D} = (D, \vee, \wedge, \diamond)$, and let p be some prime filter of $\mathcal{D}$. Then*

$$pQ_\diamond \bigcap_{a \in A} \widehat{a} \ \text{ iff } \ \diamond a \in p \text{ for all } a \in A.$$

Proof. Since the direction from left to right follows immediately from the monotonicity of $Q_\diamond$, we concentrate on the other direction.

Assume that $\diamond a \in p$ for all $a \in A$, and suppose for contradiction that $pQ_\diamond \bigcap_{a \in A} \widehat{a}$ does *not* hold. Then by definition of $Q_\diamond$ and the fact that $\bigcap_{a \in A} \widehat{a}$ is closed, there is a $b \in D$ such that $\diamond b \notin p$ and $\bigcap_{a \in A} \widehat{a} \subseteq \widehat{b}$. We claim that there there are finitely many elements $a_0, \ldots, a_n$ in A satisfying $a_0 \wedge \cdots \wedge a_n \leq b$.

To see why this must be the case, consider the filter F generated by A; that is, F is the set of elements d in D for which there are $a_0, \ldots, a_n \in A$ such that $a_0 \wedge \cdots \wedge a_n \leq d$. If b would not belong to F then by the Prime Filter Theorem there would be a prime filter q with $F \subseteq q$ and $b \notin q$. This q would then be such that $q \in \bigcap_{a \in A} \widehat{a}$ while $q \notin \widehat{b}$, which clearly cannot be the case. Hence, b does belong to F which proves our claim.

Since A is downward directed, there is an element $a \in A$ such that $a \leq a_0 \wedge \cdots \wedge a_n$. But then we also have that $a \leq b$, so by monotonicity of $\Diamond$ and the fact that $\Diamond b \notin p$ we find that $\Diamond a \notin p$ which contradicts our assumption that $\Diamond a \in p$ for all $a \in A$. ■

4. Representing game algebras

In this section we will show how we can represent a game algebra $\mathcal{G}$ as a monotone outcome algebra once we know that there is some separable module over $\mathcal{G}$.

The basic idea is as follows. Assume that $\mathcal{M} = (M, \vee, \wedge, -, \Diamond_g)_{g \in G}$ is a module over the game algebra $\mathcal{G}$. By Definition 3.5, with every operation $\Diamond_g$ of $\mathcal{M}$ we may associate a monotone outcome relation Q_g on $B_{\mathcal{M}}$ (for brevity, we will write Q_g rather than $Q_{\Diamond_g}$). The representation map embedding the game algebra $\mathcal{G}$ into the full monotone outcome algebra over $B_{\mathcal{M}}$ will map an element g of G to the pair (Q_g, Q_{-g}) of outcome relations on $B_{\mathcal{M}}$. The injectivity of this map will follow from the separability of the module; in order to prove that it is a homomorphism we need the following lemma which is one of the main technical results of the paper.

Proposition 4.1. *Let $\mathcal{M} = (M, \vee, \wedge, -, \Diamond_g)_{g \in G}$ be a module over the game algebra $\mathcal{G} = (G, \vee, \wedge, -, \diamond)$, and let g and h be arbitrary elements of G. Then we have*

1. $Q_{g \vee h} = Q_g \cup Q_h$,
2. $Q_{g \wedge h} = Q_g \cap Q_h$,
3. $Q_{g \diamond h} = Q_g \circ Q_h$,
4. *if $\Diamond_g a \neq \Diamond_h a$ for some $a \in M$ then $Q_g \neq Q_h$.*

Proof. Let p be an arbitrary prime filter and T an arbitary set of prime filters of $\mathcal{M}$.

For part 1, we distinguish cases depending on the nature of T. We first assume that T is of the form $\widehat{a}$ for some $a \in M$. In this case we have the following chain of equivalent statements:

$$\begin{array}{rlll} pQ_{g\vee h}\widehat{a} & \iff & (2) & \Diamond_{g\vee h}a \in p \\ & \iff & (\text{axiom M1}) & \Diamond_g a \vee \Diamond_h a \in p \\ & \iff & (p \text{ is prime}) & \Diamond_g a \in p \text{ or } \Diamond_h a \in p \\ & \iff & (2) & pQ_g\widehat{a} \text{ or } pQ_g\widehat{a} \\ & \iff & & p(Q_g \cup Q_h)\widehat{a}. \end{array}$$

Now allow T to be an arbitrary set. If $pQ_{g\vee h}T$ then $\Diamond_{g\vee h}a \in p$ for all $a \in F_T$, so by the proof of the first case for all $a \in F_T$ we have $\Diamond_g a \in p$ or $\Diamond_h a \in p$. We claim that pQ_gT or pQ_hT, for suppose otherwise. Then there are elements $b_g, b_h \in F_T$ such that $\Diamond_g b_g \notin p$ and $\Diamond_h b_h \notin p$. Define $b := b_g \wedge b_h$; it is straightforward to check that $b \in F_T$. But note that since $b \leq b_g$ and $\Diamond_g b_g \notin p$, we have $\Diamond_g b \notin p$ by monotonicity of $\Diamond_g$ and upward closure of prime filters. Likewise, we can prove that $\Diamond_h b \notin p$. But then we have that $\Diamond_{g\vee h}b \notin p$, since by M1 it holds that $\Diamond_{g\vee h}b = \Diamond_g b \vee \Diamond_h b$, and p, being a prime filter, cannot contain the join $\Diamond_g b \vee \Diamond_h b$ without containing $\Diamond_g b$ or $\Diamond_h b$.

For the other direction, suppose that $p(Q_g \cup Q_h)T$; that is, we have that pQ_gT or pQ_hT. Without loss of generality, suppose the first; now take any $a \in F_T$; from pQ_gT it follows that $\Diamond_g a \in p$, whence also $\Diamond_{g\vee h}a \in p$ since $\Diamond_{g\vee h}a \geq \Diamond_g a \in p$. Since a was an arbitrary element of F_T, this shows that $pQ_{g\vee h}T$.

Part 2 of the proposition can be proved along similar lines — we omit the details.

For part 3, first suppose that $p(Q_g \circ Q_h)T$. That is, we have some set $U \subseteq B_{\mathcal{M}}$ such that pQ_gU and every $u \in U$ satisfies uQ_hT. Now consider an arbitrary element a such that $T \subseteq \widehat{a}$; by definition of Q_h it holds that $\Diamond_h a \in u$ for all $u \in U$, whence $U \subseteq \widehat{\Diamond_h a}$. is a So by the assumption that pQ_gU we find that $\Diamond_g\Diamond_h a \in p$. But by $M3$ we know that $\Diamond_g\Diamond_h a = \Diamond_{g\diamond h}a$. So, $\Diamond_{g\diamond h}a \in p$, and since a was arbitrary, this shows that $pQ_{g\diamond h}T$.

For the other direction, suppose that $pQ_{g\diamond h}T$. First assume that T is some closed set C; recall that F_C is the set of elements b of $\mathcal{M}$ such that $C \subseteq \widehat{b}$. Define

$$U = \bigcap\{\widehat{\Diamond_h b} \mid b \in F_C\}.$$

From this definition it is immediate that for every element $u \in U$, and each $b \in F_C$ it holds that $\Diamond_h b \in u$; this shows that uQ_hC for every $u \in U$. We also claim that

$$pQ_gU. \tag{3}$$

First observe that the set $A = \{\Diamond_h b \mid b \in F_C\}$ is a downward directed subset of M. Hence, by Lemma 3.7, in order to prove that pQ_gU it suffices to show that $\Diamond_g a \in p$ for every $a \in A$. Hence, take an arbitrary element $a \in A$; by definition of A, a is of the form $\Diamond_h b$ for some b in F_C. It then follows from the assumption $pQ_{g\diamond h}C$ that $\Diamond_{g\diamond h} b \in p$, so from $\Diamond_{g\diamond h} b = \Diamond_g \Diamond_h b = \Diamond_g a$ we have established that indeed $\Diamond_g a \in p$. This proves (3) and hence, shows that $p(Q_g \circ Q_h)C$.

In case T is an arbitrary, not necessarily closed, set of prime filters, we may use the above proof and infer from $pQ_{g\diamond h}\overline{T}$ that there is some set U such that pQ_gU and $uQ_h\overline{T}$ for all $u \in U$. Since this implies that uQ_hT for all $u \in U$, we have indeed established that $p(Q_g \circ Q_h)T$.

Finally, for part 4 of the proposition, suppose that $\Diamond_g a \neq \Diamond_h a$ for some element of $\mathcal{M}$. It follows from the Prime Filter Theorem that there is a prime filter p that contains exactly one of these two elements, say, $\Diamond_g a \in p$, $\Diamond_h a \notin p$. It is then immediate that $pQ_g\widehat{a}$ but not $pQ_h\widehat{a}$ whence $Q_g \neq Q_h$. ■

From Proposition 4.1 the following is virtually immediate.

Proposition 4.2. *Let $\mathcal{M} = (M, \vee, \wedge, -, \Diamond_g)_{g \in G}$ be a separable module over the game algebra $\mathcal{G} = (G, \vee, \wedge, -, \diamond)$. Then the map* rep : $G \to G(B_{\mathcal{M}})$ *given by*

$$\text{rep}(g) \;=\; (Q_g, Q_{-g})$$

is an embedding of $\mathcal{G}$ into $\mathcal{G}_m(B_{\mathcal{M}})$.

Proof. We first prove that rep is a homomorphism, that is, preserves the game operations $\vee$, $\wedge$, $-$ and $\diamond$. For the join operator this follows from:

$$\begin{array}{rcl} \text{rep}(g \vee h) & = & (Q_{g\vee h}, Q_{-(g\vee h)}) \\ & = & (Q_{g\vee h}, Q_{-g\wedge -h}) \\ & = & (Q_g \cup Q_h, Q_{-g} \cap Q_{-h}) \\ & = & (Q_g, Q_{-g}) \sqcup (Q_h, Q_{-h}) \\ & = & \text{rep}(g) \sqcup \text{rep}(h); \end{array}$$

each step in this series of identities is an obvious consequence of the definitions, of the fact that $\mathcal{G}$ is a game algebra, or of Proposition 4.1. We omit the proof for the meet operator which is similar. Concerning dualization, we have

$$\text{rep}(-g) = (Q_{-g}, Q_{--g}) = (Q_{-g}, Q_g) = (Q_g, Q_{-g})^- = (\text{rep}(g))^-.$$

As our last game operation we treat composition:

$$\begin{aligned}
\mathrm{rep}(g \diamond h) &= (Q_{g\diamond h}, Q_{-(g\diamond h)}) \\
&= (Q_{g\diamond h}, Q_{-g\diamond -h}) \\
&= (Q_g \circ Q_h, Q_{-g} \circ Q_{-h}) \\
&= (Q_g, Q_{-g}) \Box (Q_h, Q_{-h}) \\
&= \mathrm{rep}(g) \Box \mathrm{rep}(h).
\end{aligned}$$

This proves that rep is indeed a homomorphism: $\mathcal{G} \to (G_m(B), \sqcup, \sqcap, ^-, \Box)$

Finally, the injectivity of the representation map follows from the assumed separability of the module and part 4 of Proposition 4.1. ■

5. Separability

In this section we will prove that every game algebra can be seen as a *separable* game module.

PROPOSITION 5.1. *Let $\mathcal{G} = (G, \vee, \wedge, -, \diamond)$ be a game algebra. Then $\mathcal{G}$, seen as a game module over itself, can be embedded in a separable game module $\mathcal{G}'$ over $\mathcal{G}$.*

PROOF. In this proof we will fix a game algebra $\mathcal{G} = (G, \vee, \wedge, -, \diamond)$. Recall that the module perspective on $\mathcal{G}$ means that we identify $\mathcal{G}$ with the structure $(G, \vee, \wedge, -, \Diamond_g)_{g\in G}$. We will show that G can be embedded in a $\mathcal{G}$-module $(G', \vee', \wedge', -', \Diamond'_g)_{g\in G}$. To do so, we will first concentrate on extending the de Morgan reduct $(G, \vee, \wedge, -)$ of $\mathcal{G}$, and then show how to extend the operations $\Diamond'_g$ to the extended de Morgan lattice. The basic intuition underlying our approach is that we aim at adding a single separating element to M; that is, an object s that will satisfy $\Diamond_g s = g$ for every $g \in G$.

The first part of the construction can be applied to arbitrary de Morgan lattices; fix such an algebra $\mathcal{M} = (M, \vee, \wedge, -)$. In a number of steps we will define an extension $\mathcal{M}'$ of $\mathcal{M}$ which satisfies certain nice properties.

To start with, recall that a distributive lattice (and hence, a de Morgan algebra) is *bounded* if it contains a smallest element $\bot$ and a largest element $\top$. An equivalent, equational definition would be to require that the identities $x \vee \top \approx \top$ and $x \wedge \bot \approx \bot$ hold (plus, in the case of a de Morgan algebra, $-\top \approx \bot$). It is easy to see that any distributive lattice $\mathcal{D}$ can be embedded in a bounded such lattice $\mathcal{D}^b$ with carrier $D^b := D \uplus \{\top, \bot\}$ — the analogous statement holds for de Morgan algebras but we will not have direct need of this. Note that in our definition of the set D^b we *always* add new elements to D (even if the algebra $\mathcal{D}$ itself already has a top and bottom element).

Given our de Morgan lattice $\mathcal{M} = (M, \vee, \wedge, -)$, let M'' be the set $(M \cup \{\top, \bot\})^2$ (that is, M'' is the cartesian square of the set M^b), and let $\vee'$ and $\wedge'$ be the coordinatewise join and meet operations on M''. In other words, the algebra $(M'', \vee', \wedge')$ is the distributive lattice product $(M, \vee, \wedge)^b \times (M, \vee, \wedge)^b$. The operation $-' : M' \to M'$ is defined by

$$-'(x, y) = (-y, -x),$$

and we define $\mathcal{M}''$ as the structure $(M'', \vee', \wedge', -')$. It should be stressed that $\mathcal{M}''$ is in general *not* isomorphic to the product $\mathcal{M}^b \times \mathcal{M}^b$.

CLAIM 1. If $\mathcal{M}$ is a de Morgan algebra, then $\mathcal{M}''$ is a bounded de Morgan algebra. Furthermore, the diagonal map $\Delta : M \to M''$ given by $\Delta(a) = (a, a)$ is an embedding of $\mathcal{M}$ in $\mathcal{M}''$.

PROOF OF CLAIM We leave this proof as an exercise to the reader. ◀

The separating element s of $\mathcal{M}'$ will be the pair $(\top, \bot)$; note that this pair is its own de Morgan dual: $-(\top, \bot) = (\top, \bot)$. Our target algebra $\mathcal{M}'$ will be in a sense the minimal *subalgebra* of the algebra $\mathcal{M}''$ which (i) contains the image of M under the embedding Δ together with the separating element s and (ii) can be made into a module over $\mathcal{G}$ later on. For a more precise definition, consider the following subset M' of M'':

$$M' \;=\; (M \cup \{\top\}) \times (M \cup \{\bot\}) \;\cup\; \{(\top, \top), (\bot, \bot)\}.$$

M' of the We will use the same notation for the operations $\vee'$, $\wedge'$ and $-'$ and their restrictions to to M', and we leave it to the reader to verify that the set M' is closed under these operations. Let $\mathcal{M}'$ be the algebra $(M', \vee', \wedge', -')$.

CLAIM 2. If $\mathcal{M}$ is a de Morgan algebra, then $\mathcal{M}'$ is a bounded de Morgan algebra. Moreover, the diagonal map $\Delta : M \to M'$ is an embedding of $\mathcal{M}$ in $\mathcal{M}'$.

The proof of this Claim is straightforward by our earlier observations.

Finally, we turn to the definition of the module $\mathcal{G}'$. Let $(G', \vee', \wedge', -')$ be defined as above. It is left to define an operation $\diamond'_g : G' \to G'$, for every element $g \in G$. Before we give the formal definition, let us first mention some of the requirements that guided our intuitions:

1. Since we want the diagonal map $\Delta : G \to G'$ to be an embedding of $\mathcal{G}$ in $\mathcal{G}'$, we need that $\diamond'_g(a, a) = (\diamond_g a, \diamond_g a)$ whenever $a \in G$. This makes it natural to put $\diamond'_g(a, b) = (\diamond_g a, \diamond_g b)$ for arbitrary $a, b \in G$.

2. The element $(\top, \bot)$ will be the separating element of $\mathcal{G}'$; this means we have to put $\Diamond'_g(\top, \bot) = (g, g)$ for every $g \in G$.

3. The above considerations leave open the problem of what to do with the other elements of M'. Note that since the pair $(\bot, \top)$ is not an element of M', these other elements will be either of the form $(\top, a)$ with $a \neq \bot$, or else $(a, \bot)$ with $a \neq \top$. Since these elements are bigger (smaller, respectively) than any of the elements we encountered in the first two items above, we can and will simply define $\Diamond'_g(\top, a)$ to be the top element $(\top, \top)$ of the algebra, and $\Diamond'_g(a, \bot)$ to be the bottom element $(\bot, \bot)$. (This is in fact the only reason why we need these top and bottom elements in the algebra $\mathcal{G}'$.)

Formally, we define, for an arbitrary element $g \in G$, the map $\Diamond'_g : G' \to G'$ by putting

$$\Diamond'_g(a, b) \quad = \begin{cases} (\Diamond_g a, \Diamond_g b) & \text{if } a, b \in G, \\ (g, g) & \text{if } a = \top \text{ and } b = \bot, \\ (\top, \top) & \text{if } a = \top \text{ and } b \neq \bot, \\ (\bot, \bot) & \text{if } a \neq \top \text{ and } b = \bot. \end{cases}$$

Given a game algebra $\mathcal{G}$, let $\mathcal{G}'$ be the module $(G', \vee', \wedge', -', \Diamond'_g)_{g \in G}$.

Claim 3. If $\mathcal{G}$ is a game algebra, then $\mathcal{G}$ is a separable game module.

Proof of Claim We have already seen that the structure $(G', \vee', \wedge', -')$ is a de Morgan algebra, and it is easy to see that the operations $\Diamond'_g$ are all monotone. The conditions M1–4 can be checked via straightforward case distinctions.

For instance, take M4 and consider an arbitrary element $g \in G$ and an arbitrary element $(a, b) \in G'$. We will show that

$$\Diamond'_{-g} -'(a, b) = -'\Diamond'_g(a, b).$$

by a case distinction on the nature of a and b.

If both a and b belong to G, then $\Diamond'_{-g} -'(a, b) = \Diamond'_{-g}(-b, -a) = (\Diamond_{-g}(-b), \Diamond_{-g}(-a)) = (-\Diamond_g b, -\Diamond_g a) = -'(\Diamond_g a, \Diamond_g b) = -'\Diamond'_g(a, b)$. The case that $a = \top$ and $b = \bot$ gives $\Diamond'_{-g} -'(a, b) = \Diamond'_{-g} -'(\top, \bot) = \Diamond'_{-g}(\bot, \top) = (-g, -g) = -'(g, g) = -'\Diamond'_g(\top, \bot) = -'\Diamond'_g(a, b)$. If $a = \top$ and b is distinct from $\bot$, we find $\Diamond'_{-g} -'(a, b) = \Diamond'_{-g} -'(\top, b) = \Diamond'_{-g}(-b, \bot) = (\bot, \bot) = -'(\top, \top) = -'\Diamond'_g(\top, b) = -'\Diamond'_g(a, b)$. Finally, the case that $a \neq \top$ while $b = \bot$ is similar to the previous one. Since this distinction covers all possible cases this proves that M4 holds of $\mathcal{G}'$. We omit the proofs concerning the other axioms since they are in fact simpler than the one given here for M4.

Note that separability of $\mathcal{G}'$ is immediate by the definition: if g and h are distinct elements of G then $\Diamond'_g(\top,\bot) = (g,g) \neq (h,h) = \Diamond'_h(\top,\bot)$. ◀

Since the diagonal map $\Delta : a \mapsto (a,a)$ is obviously an embedding of the $\mathcal{G}$-module $\mathcal{G}$ in the $\mathcal{G}$-module $\mathcal{G}'$, this proves the proposition. ■

6. Consistency

In this section we will prove that every monotone outcome algebra is isomorphic to a board algebra. This means that the consistency requirement does not give any extra valid equations.

PROPOSITION 6.1. *Let $\mathcal{A} = (A, \sqcup, \sqcap, ^-, \Box)$ be a monotone outcome algebra over the set B. Then $\mathcal{A}$ is isomorphic to a board algebra over the set $B' = B \cup \{\infty\}$ (where $\infty \notin B$).*

PROOF. Suppose that $\mathcal{A} = (A, \sqcup, \sqcap, ^-, \Box)$ is a monotone outcome algebra over the set B, and let ∞ be an object not in B.

Given a monotone outcome relation R over B, define R' as the following outcome relation over the set $B' = B \cup \{\infty\}$:

$$R' \; = \; \{(\infty, T) \mid \infty \in T\} \cup \{(p,T) \mid p \in B, \; \infty \in T \text{ and } (p, T^-) \in R\},$$

where T^- denotes the set $T \setminus \{\infty\}$. It is obvious that any pair of relations (R', S') is consistent since for any $p \in B'$ we have $(p,T) \in R'$ only if $\infty \in T$ and likewise for S'. Thus $(p,T) \in R'$ implies that $(p, B' \setminus T) \notin S'$.

We claim that the function mapping a pair of outcome relations (R,S) to the pair (R', S') is an embedding of $\mathcal{A}$ in the board algebra over B. This follows immediately from the observation that the operation $(\cdot)'$ distributes over unions, intersections and compositions of relations.

We first consider union and prove that

$$(R \cup S)' = R' \cup S'. \tag{4}$$

In order to prove (4), first suppose that $(p,T) \in (R \cup S)'$. In case $p = \infty$ it is easy to see that (p,T) belongs to *both* R' and S', so we certainly have that $(p,T) \in R' \cup S'$. Now assume that $p \in B$. In this case, $(p,T) \in (R \cup S)'$ gives that $\infty \in T$ and $(p, T^-) \in R \cup S$. Now, if $(p,T^-) \in R$ then $(p,T) \in R'$ and if $(p, T^-) \in S$ then $(p,T) \in S'$. In both cases we find that $(p,T) \in R' \cup S'$.

This shows that $(R \cup S)' \subseteq R' \cup S'$. We omit the proof for the other inclusion which is equally straightforward.

This proves (4) and thus shows that $(\cdot)'$ distributes over unions; the case for intersections is similar and left to the reader. Concerning composition, we will prove that

$$(R \circ S)' = R' \circ S'. \tag{5}$$

First suppose that $(p, T) \in (R \circ S)'$. In case $p = \infty$ we have $\infty \in T$. Then we also have $\infty S' T$, so by $\infty R' \{\infty\}$ and the definition of composition on outcome relations we find $(\infty, T) \in R' \circ S'$, with $\{\infty\}$ as the 'middle set'. If on the other hand p belongs to B we have $\infty \in T$ and $(p, T^-) \in R \circ S$. That is, for some set $U \subseteq B$ we have $(p, U) \in R$ and $(u, T) \in S$ for all $u \in U$. Defining $U^+ = U \cup \{\infty\}$, we see that $(p, U^+) \in R'$. Also, for an arbitrary element $u \in U$ we have $(u, T) \in S'$, and since $\infty \in T$ we also find $(\infty, T) \in S'$. Thus we obtain $(p, T) \in R' \circ S'$, with U^+ as the 'middle' set.

For the other direction, assume that $(p, T) \in R' \circ S'$. This means that for some set $U \subseteq B'$ we have $(p, U) \in R'$ and $(u, T) \in S'$ for all $u \in U$. In case $p = \infty$ we find $\infty \in U$ by $(\infty, U) \in R'$; then by $(\infty, T) \in S'$ we obtain that $\infty \in T$; thus by definition of $(R \circ S)'$ it follows that $(\infty, T) \in (R \circ S)'$. In case $p \in B$, we obtain $(p, U^-) \in R$ by definition of R'. Likewise, for all $u \in U^-$ it follows that $(u, T^-) \in S$. Thus $(p, U^-) \in R \circ S$. Then, from $(p, U) \in R'$ it follows that $\infty \in U$, so from $(\infty, T) \in S'$ we find that $\infty \in T$. By definition of $(R \circ S)'$ this means that $(p, T) \in (R \circ S)'$.

This proves (5) and hence finishes the proof of the proposition. ■

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