Completeness through Flatness in Two-Dimensional Temporal Logic

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Abstract. We introduce a temporal logic *TAL* and prove that it has several nice features. The formalism is a two-dimensional modal system in the sense that formulas of the language are evaluated at pairs of time points. Many known formalisms with a two-dimensional flavor can be expressed in *TAL*, which can be seen as the temporal version of square arrow logic.
We first pin down the expressive power of *TAL* to the three-variable fragment of first-order logic; we prove that this induces an expressive completeness result of 'flat' *TAL* with respect to monadic first order logic (over the class of linear flows of time).
Then we treat axiomatic aspects: our main result is a completeness proof for the set of formulas that are 'flatly' valid in well-ordered flows of time and in the flow of time of the natural numbers.

1 Introduction

Two-dimensional temporal logic In the last twenty years, various disciplines related to logic have seen the idea arising to develop a framework for modal or temporal logic in which the possible worlds are *pairs* of elements of the model instead of the elements themselves. Often the motivation for developing such *two-dimensional* formalisms stems from a dissatisfaction with the expressive power of ordinary one-dimensional modal or temporal logic. Let us mention three examples of two-dimensional modal logics (we refer to Venema [23] for a more substantial overview):

First, in tense logic, there is a research line inspired by linguistic motivations. In the seventies, the development of formal semantics and the strive to give a logical foundation for it, lead people like Gabbay, Kamp and Åqvist (cf. [5] for an overview) to develop two-dimensional modal logics taking care of the linguistic phenomenon that the truth of a proposition may not only change with the time of reference, but also with the time of utterance by the speaker. Second, in artificial intelligence it has been argued that from a philosophical or psychological point of view, it is more natural to consider temporal ontologies where periods of time are the basic entities instead of time points. In some approaches, for instance in Halpern & Shoham [7], modal logics of time intervals have been studied where the possible worlds are intervals; and an interval is identified with the pair consisting of its beginning point and its endpoint. Our last example concerns arrow logic (Venema [26]) which is based on the idea that in its semantics, transitions (arrows) do not link the possible worlds, they *are* the possible worlds. Two-dimensional arrow logic arises if we see transitions as a pair consisting of an input and an output state.

Diverse as all these two-dimensional modal formalisms may be in background and nature, they have many aspects in common, one of which is the close connection with algebraic logic (for an introductory overview of algebraic logic we refer to Németi [17]). For instance arrow logic can serve as a tool to study the theory of Tarski's *relation algebras*. Second, and partly related to the first point, the axiomatics of two-dimensional modal logics is not a trivial matter. In particular, if one is interested in a *full square semantics*, i.e. models where *all* pairs of points are admissible as possible worlds, there are few interesting logics that are decidable or finitely axiomatizable by a standard Hilbert-style derivation system.

There are various ways to get around these negative results: for instance in arrow logic, an interesting approach is to drop the constraint that the universe of a model should be a *full* square. The theory of such *relativized* squares may be both decidable and nicely axiomatizable (cf. Marx et alii [15] for some examples). For the axiomatizability problem, a different solution was found by Gabbay (cf. [3]); by introducing so-called irreflexivity rules, various logics which are not finitely axiomatizable by standard means, do allow a finite derivation system. Obviously, such rules do not affect the undecidability of the logic. In order to deal with the latter issue, one may take up a third idea, viz. to restrict the *interpretation* of the two-dimensional semantics. A very common constraint is to make the truth of an *atomic* formula at a pair dependent on *one* coordinate of the pair only. Such valuations are called *weak* or *flat*. It is not very difficult to see that with this restriction, two-dimensional modal logic corresponds to *monadic* first-order logic instead of to *dyadic* predicate calculus, and hence decidability follows immediately for various classes of models. This approach originates with the literature on two-dimensional tense logic and is also followed in formalisms like process logic, cf. Harel, Kozen & Parikh [8].

The system *TAL* Let us now say a few words about the formalism *TAL* that we investigate in this paper. It is a temporal logic, i.e. its intended models will be flows of time $\mathfrak{T} = (T, <)$, and it is a two-dimensional system: the basic declarative statement of *TAL* will be of the form

$$\mathfrak{T}, V, s, t \Vdash \phi,$$

where ϕ is a formula, V is a (possibly flat) interpretation function for the propositional variables, and (s, t) is a *pair* of time points.

The position of *TAL* within the landscape of two-dimensional modal and temporal logics is best explained by noting that it is the *t̲emporal version of a̲rrow l̲ogic*. To be precise, the language of *TAL* is that of arrow logic, extended with a *constant* λ which refers to the ordering relation, i.e.

$$\mathfrak{M}, s, t \Vdash \lambda \iff s < t.$$

An advantage of the presence of this constant is that it makes it easy to express properties of the ordering relation in the language. Compared to other two-dimensional temporal logics, *TAL* is a quite expressive formalism, being able to express the operators of most of the systems that are known from the literature.

Overview The aim of the paper is to look at the expressive power and the axiomatics of *TAL*.

After giving the necessary formal definitions in the next section, we will see in section 3 that, as a consequence of results known from the theory of relation algebras, *TAL* is expressively equivalent to the three-variable fragment of first-order logic with dyadic predicates; as an interesting corollary we can prove that over the class of linear orders *TAL* is *expressively complete* with respect to first order logic with *monadic* predicates. Concerning axiomatics, in section 4 we first prove some completeness results for derivation systems having an irreflexivity rule. These results are easy consequences of similar results obtained for arrow logic, cf. VENEMA [23].

In section 5 we pay special attention to the well-ordered flows of time and in particular, to the flow of time ω of the natural numbers. There are two reasons to do so: first of all, for these structures we can prove a completeness result for *flat* validity of a system *without any non-orthodox derivation rules*. An interesting aspect of the proof is that it essentially uses the expressive completeness of *TAL* over the class of linear orderings; this completeness-by-completeness argument was first used in GABBAY & HODKINSON [6] for Kamp's one-dimensional functional complete logic with S ('Since') and U ('Until'). Second, well-ordered models for *TAL* have close connections with relation algebras defined in MADDUX [13], but for limitations of space we cannot go into detail here.

To motivate of the special attention for ω, let us note that the set of *intervals* $\{(s,t) \in N \times N \mid s \leq t\}$ over ω can be seen as to represent *finite computation* paths.[3] Hence our results may have applications in the theory of program verification.

2 Definitions

Definition 1. *TAL* is the similarity type having, besides the boolean connectives, the following set of modal operators: a dyadic operator $\circ$, a monadic $\otimes$ and two constants δ and λ. The set of *TAL*-formulas are defined as usual, i.e. given a set *VAR* of propositional variables, a *TAL*-formula is either atomic (i.e. in the set $VAR \cup \{\delta, \lambda\}$), or it has the form $\neg\phi$, $\phi \vee \psi$, $\otimes\phi$ or $\phi \circ \psi$, where ϕ and ψ are formulas.

As abbreviations we will use the usual boolean connectives and constants, and the following *compass diamonds*

[3] Infinite computation paths could come into the picture if we would consider the ordering of the successor ordinal of ω; for this flow of time $(\omega + 1, <)$ results can be obtained that are similar to the ones reported on in this paper.

$$
\begin{array}{llll}
\Diamond_N\phi = \phi \circ \otimes\lambda & & \Diamond_W\phi = \otimes\lambda \circ \phi \\
\Diamond_S\phi = \phi \circ \lambda & & \Diamond_E\phi = \lambda \circ \phi \\
\Diamond_V\phi = \Diamond_N\phi \vee \phi \vee \Diamond_S\phi & & \Diamond_H\phi = \Diamond_W\phi \vee \phi \vee \Diamond_E\phi \\
\Diamond\phi \;\; = \Diamond_H\Diamond_V\phi & &
\end{array}
$$

together with their obvious duals $\Box_N$, $\Box_S$, $\Box_V$, etc. Besides these, we define the following diamond D:

$$D\phi \equiv \neg\delta \circ (\phi \circ \top) \;\vee\; (\top \circ \phi) \circ \neg\delta.$$

Definition 2. A *frame* for *TAL* is a pair $\mathfrak{T} = (T, <)$ with $<$ a binary relation on T. A *flow of time* is a frame where $<$ is a transitive, irreflexive relation on T. A *model* is a triple $\mathfrak{M} = (T, <, V)$ such that $(T, <)$ is a frame and V is a valuation, i.e. a function mapping propositional variables to subsets of $T \times T$.

Truth of a formula ϕ at a pair (s, t) in a model $\mathfrak{M}$, notation: $\mathfrak{M}, s, t \Vdash \phi$, is defined as follows:

$$
\begin{array}{ll}
\mathfrak{M}, s, t \Vdash p & \text{if } (s,t) \in V(p), \\
\mathfrak{M}, s, t \Vdash \delta & \text{if } s = t, \\
\mathfrak{M}, s, t \Vdash \lambda & \text{if } s < t, \\
\mathfrak{M}, s, t \Vdash \neg\phi & \text{if } \mathfrak{M}, s, t \nVdash \phi, \\
\mathfrak{M}, s, t \Vdash \phi \vee \psi & \text{if } \mathfrak{M}, s, t \Vdash \phi \text{ or } \mathfrak{M}, s, t \Vdash \psi, \\
\mathfrak{M}, s, t \Vdash \otimes\phi & \text{if } \mathfrak{M}, t, s \Vdash \phi, \\
\mathfrak{M}, s, t \Vdash \phi \circ \psi & \text{if there is a } u \text{ such that } \mathfrak{M}, s, u \Vdash \phi \text{ and } \mathfrak{M}, u, t \Vdash \psi.
\end{array}
$$

Validity is defined and denoted as usual. For instance, a formula is valid in a class K of frames, notation: $\mathsf{K} \models \phi$, if for every frame $\mathfrak{T}$ in K, every valuation V on $\mathfrak{T}$, and every pair s, t of points in $\mathfrak{T}$ we have $\mathfrak{T}, V, s, t \Vdash \phi$. A formula ϕ is *satisfiable* in a model if $\neg\phi$ is not valid in the model.

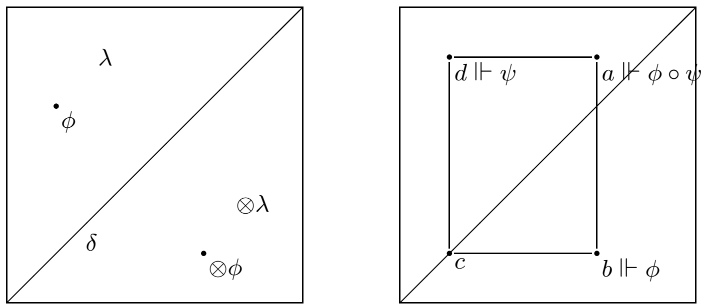
A valuation is called *flat* if for for every propositional variable p and every s, t and u in T, we have $(s, t) \in V(p)$ iff $(s, u) \in V(p)$. *Flat validity*, notation: $\models_\flat$, is defined like ordinary validity, but with the restriction to flat valuations.

Note that informally, a valuation is flat if the truth of a propositional variable at a pair (s, t) only depends on the first coordinate s.

In the sequel, we will consider *linear* flows of time mainly, i.e. flows of time $(T, <)$ where $<$ is a total relation. Such frames allow a nice, two-dimensional representation, cf. the pictures below. The set of pairs where δ holds consists of the *diagonal* elements of the universe; λ is true precisely in the 'north-western' halfplane. The operator $\otimes$ corresponds to *mirroring* in the diagonal. A formula $\phi \circ \psi$ holds at a pair a, if we can draw a *rectangle abcd* such that: b lies on the vertical line through a and ϕ holds at b, d lies on the horizontal line through a and ψ holds at d; and c lies on the diagonal.

The subscripts N, S, W, E, H and V are mnemonics for respectively north, south, west, east, horizontal and vertical. Note that according to the truth definition given above, these compass operators receive their natural interpretation,

Fig. 1. *TAL*'s operators in linear frames.



e.g.

$$\mathfrak{M}, s, t \Vdash \Diamond_S \phi \iff \text{there is a point } (s, u) \textit{ south} \text{ of } (s, t) \text{ with } s, u \Vdash \phi,$$
$$\mathfrak{M}, s, t \Vdash \Diamond_V \phi \iff \text{there is a point } (s, u) \text{ with } s, u \Vdash \phi,$$

The diamond D is a so-called *difference operator*, i.e. its 'accessibility relation' is the inequality relation:

$$\mathfrak{M}, (s, t) \Vdash D\phi \text{ iff there are } s', t' \text{ with } (s, t) \neq (s', t') \text{ and } \mathfrak{M}, (s', t') \Vdash \phi. \quad (1)$$

Finally, $\Diamond$ is a *universal* operator, i.e.

$$\mathfrak{M}, (s, t) \Vdash \Diamond\phi \text{ iff there are } s', t' \text{ with } \mathfrak{M}, (s', t') \Vdash \phi. \quad (2)$$

We leave it to the reader to verify (1) and (2).

A central role in our paper is played by the class of well-orderings and the flow of time of the natural numbers.

Definition 3. Let ω denote the flow of time of the natural numbers, i.e. ω is the frame $(N, <)$ where N is the set of natural numbers and $<$ is the usual ordering on N. WO denotes the class of well-ordered flows of time, i.e. linear frames such that every non-empty set of time-points has a *smallest* element.

3 Expressiveness

In this section we investigate the expressive power of *TAL*. First we will see how some properties of temporal frames can be expressed in the language; then we turn to the level of models, where we compare the expressive power of *TAL* to that of first-order logic. It turns out that on the model level, *TAL* is as expressive as the three variable fragment of first-order logic; as a consequence, we obtain an expressive completeness result for flat *TAL* over the class of linear orders.

So let us start with some correspondence theory on the frame level. Note that as a nice consequence of having an explicit referent to the (ordering) relation in the object language, it becomes very easy to *characterize* properties of $<$:

Definition 4. Consider the following *TAL*-formulas:

(TR)	$\lambda \circ \lambda \to \lambda$	(transitivity)
(IR)	$\lambda \to \neg\delta$	(irreflexivity)
(TO)	$\lambda \vee \delta \vee \otimes\lambda$	(totality)
(LN)	$TR \wedge IR \wedge TO$	(linearity)
(DI)	$\lambda \circ \lambda \to \lambda \circ (\lambda \wedge \neg(\lambda \circ \lambda)) \wedge (\lambda \wedge \neg(\lambda \circ \lambda)) \circ \lambda$	(discreteness)
(DE)	$\lambda \to \lambda \circ \lambda$	(denseness)
(W)	$\Diamond p \to \Diamond(p \wedge \Box_S \neg p \wedge \Box_W \Box_V \neg p)$	(well-orderings)
(UL)	$\Diamond_W \top$	(left-serial)
(UR)	$\Diamond_N \top$	(right-serial)

Proposition 5. *Let $\mathfrak{T} = (T, <)$ be a frame. Then*

(i) $\mathfrak{T} \models TR \iff <$ *is transitive,*
(ii) $\mathfrak{T} \models IR \iff <$ *is irreflexive,*
(iii) $\mathfrak{T} \models TO \iff <$ *is total,*
(iv) $\mathfrak{T} \models LN \iff <$ *is linear,*

Now suppose $<$ is linear. Then

(v) $\mathfrak{T} \models DI \iff <$ *is discrete,*
(vi) $\mathfrak{T} \models DE \iff <$ *is dense,*
(vii) $\mathfrak{T} \models W \iff <$ *is well-ordered,*
(viii) $\mathfrak{T} \models UL \iff T$ *is left-serial,*
(ix) $\mathfrak{T} \models UR \iff T$ *is right-serial,*
(x) $\mathfrak{T} \models DI \wedge W \wedge UR \iff T \cong \omega.$

Proof. As an example, we prove **(v)**, one direction of **(vii)**, and **(x)**. Let $\mathfrak{T}$ be linear.

For **(v)** first assume that $\mathfrak{T}$ is discrete, and that $\mathfrak{M}$ is a model on $\mathfrak{T}$ with $\mathfrak{M}, s, t \Vdash \lambda \circ \lambda$. Clearly then t is a successor of s, but not the immediate one. So let u be the immediate successor of s. By linearity of $<$ we have $s < u < t$, and as u is the immediate successor of s: $s, u \Vdash \lambda \wedge \neg(\lambda \circ \lambda)$. So $s, t \Vdash (\lambda \wedge \neg(\lambda \circ \lambda)) \circ \lambda$. The other conjunct in the consequent of DI is treated likewise.

For the other direction, assume that $\mathfrak{T} \models DI$ and let $s < t$. We have to find an immediate successor for s. If t is the immediate successor of s, we are finished. Otherwise, $s, t \Vdash \lambda \circ \lambda$ (in every model on $\mathfrak{T}$), so $s, t \Vdash (\lambda \wedge \neg(\lambda \circ \lambda)) \circ \lambda$ by assumption. By the truth definition, there is a u with $s, u \Vdash \lambda \wedge \neg(\lambda \circ \lambda)$ and $u, t \Vdash \lambda$. It is then straightforward to verify that this u is the immediate successor of s.

Now we will show the direction $\Leftarrow$ of **(vii)**. Assume that $<$ is well-ordering of T, and suppose that V is a valuation on $\mathfrak{T}$ such that for some $s, t \in T$, we have $\mathfrak{T}, s, t \Vdash \Diamond p$. This implies that $V(p) \neq \varnothing$, so we obtain that the set X, defined by $X = \{s \in T \mid \exists t\ (s, t) \in V(p)\}$, is not empty. As $<$ is a well-ordering, X has a *smallest element* x. Now let Y be the set $\{t \in T \mid (x, y) \in T\}$, then $Y \neq \varnothing$ by definition of X. So also Y has a *smallest* element y. It is then straightforward to verify that $\mathfrak{T}, V, x, y \Vdash (p \wedge \Box_S \neg p \wedge \Box_W \Box_V \neg p)$, whence $\mathfrak{T}, V, s, t \Vdash \Diamond(p \wedge \Box_S \neg p \wedge \Box_W \Box_V \neg p)$.

For **(x)**, note that ω is the only well-ordering which is discrete and right-serial. Hence, (x) follows from (v), (vii) and (ix). □

Compared to the existing two-dimensional tense logics, we feel that *TAL* has the advantage of being both quite expressive and perspicuous. In fact, concerning the first point, all of the systems known to us can be seen as subsystems of *TAL*. For example, the system studied by Åqvist in [27] uses a set of operators all of which can be defined in *TAL*:

$$\{bf = \lambda, id = \delta, af = \otimes\lambda,$$
$$\langle P\rangle\phi = \Diamond_W\phi, \langle F\rangle\phi = \Diamond_E\phi, \langle O\rangle\phi = \Diamond_H(\delta \wedge \phi), \langle X\rangle\phi = \otimes\phi\}.$$

As a second example, one of the systems discussed by Gabbay in [5] has two modal operators, F and P, with F having the following semantics:

$$\mathfrak{M}, s, t \Vdash F\phi \iff \begin{array}{ll} \text{either} & s = t \text{ and for some } t' > t,\ \mathfrak{M}, s, t' \Vdash \phi \\ \text{or} & s < t \text{ and } \mathfrak{M}, t, t \Vdash \phi \\ \text{or} & s > t \text{ and for some } s < u < t,\ \mathfrak{M}, u, u \Vdash \phi. \end{array}$$

It is a straightforward exercise to show that $F\phi$ can be defined in *TAL* as

$$(\delta \to \Diamond_N\phi) \wedge (\lambda \to \Box_H(\delta \to \phi)) \wedge (\otimes\lambda \to \Diamond_W\Diamond_N(\delta \wedge \phi)).$$

Of course, for practical purposes such operators may be necessary: Gabbay's motivation for the introduction of F is to capture the future perfect tense in English. However, we feel that it is better to use a formalism where the *primitive* operators have a more perspicuous semantics, provided that this clarity does not stand in the way of the system's expressive power.

In the second part of this section we compare the expressive power of *TAL* to that of first-order logic on the level of models. To start with, let us define the first-order language used to describe our models.

Definition 6. Let $L^{2<}$ be a signature of first-order logic with a designated dyadic predicate symbol R and the following set of *dyadic* predicates $\{P_i \mid p_i \in VAR\}$. $L^{1<}$ is defined as $L^{2<}$, but now all predicates P_i are *monadic*.

Let N be a set of L-formulas, k a natural number and X a set of variables in L. We define
$N(X) = \{\phi \in N \mid$ all *free* variables of ϕ are in $X\}$,
$N_k \quad = \{\phi \in N \mid$ *all* variables of ϕ are among $x_0, \ldots, x_{k-1}\}$.

Now we can view the models of our modal formalism as structures for $L^{2<}$ as follows. Let $\mathfrak{M} = (T, <, V)$ be a model for *TAL*, then we can define an interpretation I for $L^{2<}$ on T by putting $I(R) = <$ and $I(P_i) = V(p_i)$. If $\mathfrak{M}$ is *flat*, its induced $L^{1<}$-structure will be the pair (T, I) where $I(R) = <$ and

$$I(P_i) = \{t \in T \mid (t, u) \in V(p_i) \text{ for some } u \in T\}.$$

In the sequel, we will identify *TAL*-models with their induced L-structures.

The basic result concerning the expressive power of our system is that *TAL* has the same expressive power as $L^2_3(x_0, x_1)$, a fragment of L which we will call 'the three variable fragment of first-order logic', by a slight abus de langue. We hasten to remark that this claim is an immediate consequence of a well-known result in algebraic logic (cf. for instance TARSKI & GIVANT [21]).

Proposition 7. **(i)** *There is an effective translation* τ *from* TAL- *formulas to* $L^2(x_0, x_1)$*-formulas such that for every model* $\mathfrak{M}$ *for* TAL*, every* TAL*-formula* ϕ *and every pair* $(t_0, t_1) \in T \times T$

$$\mathfrak{M}, t_0, t_1 \Vdash \phi \iff \mathfrak{M} \models \tau(\phi)[x_0 \mapsto t_0, x_1 \mapsto t_1].$$

(ii) *There is an effective translation* μ *from* $L^2(x_0, x_1)$*-formulas to* TAL*-formulas such that for every* L^2*-structure* $\mathfrak{M}$ *every* $L^2(x_0, x_1)$*-formula* ϕ*, and every pair* $(t_0, t_1) \in T \times T$

$$\mathfrak{M} \models \phi[x_0 \mapsto t_0, x_1 \mapsto t_1] \iff \mathfrak{M}, t_0, t_1 \Vdash \mu(\phi).$$

Omitting the proof of this theorem, we just mention the definition of τ:

$$\begin{array}{ll} \tau p_i & = P_i x_0 x_1 \\ \tau\delta & = x_0 = x_1 \\ \tau\lambda & = R x_0 x_1 \\ \tau\neg\phi & = \neg\tau\phi \\ \tau(\phi \vee \psi) & = \tau\phi \vee \tau\psi \\ \tau\otimes\phi & = \tau\phi[x_0/x_1, x_1/x_0] \\ \tau(\phi \circ \psi) & = \exists x_2 \, [\tau\phi[x_2/x_1] \wedge \tau\psi[x_2/x_0], \end{array} \tag{3}$$

where we assume that we have a suitable devise to perform the variable substitutions ($[x_0/x_1, x_1/x_0]$, $[x_2/x_1]$ and $[x_0/x_1]$) *within* the three variable fragment of $L^{2<}$.

As a straightforward consequence of this proposition, we see that over the class of *flat* models, every *TAL*-formula ϕ has an equivalent $\tau_\flat\phi$ in $L^{1<}_3(x_0, x_1)$ and vice versa. The crucial (and only) difference in the translation $\tau_\flat$ lies in the atomic clause:

$$\tau_\flat p_i = P_i x_0 \wedge x_1 = x_1. \tag{4}$$

A fortiori, every *TAL*-formula has an equivalent in $L^{1<}(x_0, x_1)$. We will now show the converse of this fact to hold as well, and thus establish an *expressive completeness* theorem, in the style of Kamp's famous result concerning the operators S ('Since') and U ('Until') (cf. KAMP [10])[4].

Theorem 8 (Flat expressive completeness over linear orders). *Over the class of flat models based on a linear frame, every* $L^{1<}(x_0, x_1)$*-formula has an equivalent in* TAL*, and vice versa.*

[4] Kamp's theorem states that that over the class of Dedekind-complete linear orderings, every formula in $L^{1<}(x_0)$ has an equivalent in the one-dimensional formalism with operators S and U; we refer to GABBAY E.A. [5] for a more accessible proof.

Proof. Let ϕ be a formula in $L^{1<}(x_0, x_1)$. By results in Gabbay [4], resp. Immerman & Kozen [9], $L^{1<}$ has *Henkin-dimension* three, resp. the *three-variable property* over the class of linear orderings, both implying that ϕ has an equivalent in $L_3^{1<}(x_0, x_1)$. Then by proposition 7, ϕ has a *TAL*-equivalent over the class of flat linear models. □

Note that the restriction to *flat TAL/monadic* predicates is essential here, as it is shown in Venema [22] that no *finite* system of two-dimensional temporal operators can be as expressive as $L_3^{2<}(x_0, x_1)$. Finally, we should mention that *TAL* is not the only two-dimensional expressively complete system, and that our notion of two-dimensional expressive completeness is not the only one possible. We refer to Gabbay e.a. [5] for more details.

4 Axiomatics: the general case

In this section we develop the axiomatics of *TAL*. First we will define a derivation system which is sound and complete with respect to the class of all frames; then we will define complete axiom systems for various classes of flows of time, like the linear, dense and discrete ones. At the end of the section we treat axiom systems for flat validity.

The basic systems in our axiomatics are the following:

Definition 9. The axiom system AR is given by the following sets of axioms (where $\phi \,\underline{\circ}\, \psi$ and $\underline{\otimes}\phi$ abbreviate $\neg(\neg\phi \circ \neg\psi)$ and $\neg\otimes\neg\phi$, respectively):

(CT)	all classical tautologies	
(DB)	$(p \to p') \,\underline{\circ}\, q$	$\to (p \,\underline{\circ}\, q \to p' \,\underline{\circ}\, q)$
	$p \,\underline{\circ}\, (q \to q')$	$\to (p \,\underline{\circ}\, q \to p \,\underline{\circ}\, q')$
	$\underline{\otimes}(p \to q)$	$\to (\underline{\otimes}p \to \underline{\otimes}q)$
$(A1)$	$\neg\otimes p$	$\leftrightarrow \otimes\neg p$
$(A2)$	$\otimes\otimes p$	$\to p$
$(A3)$	$\otimes(p \circ q)$	$\leftrightarrow \otimes q \circ \otimes p$
$(A4)$	$\delta \circ p$	$\leftrightarrow p$
$(A5)$	$p \circ \neg(\otimes p \circ q)$	$\to \neg q$
$(A6)$	$p \circ (q \circ r)$	$\leftrightarrow (p \circ q) \circ r$

and the following set of derivation rules: Modus Ponens, Universal Generalization and Substitution:

(MP) $\phi, \phi \to \psi \;/\; \psi$

(UG) $\phi \;/\; \phi \,\underline{\circ}\, \psi, \psi \,\underline{\circ}\, \phi$

$\phi \;/\; \underline{\otimes}\phi$

(SUB) $\phi \;/\; \sigma\phi$

where σ is a map uniformly substituting formulas for propositional variables in formulas.

The derivation system AR^+ is the extension of AR with the Irreflexivity Rule for D:

(IR_D) $\quad (p \wedge \neg Dp) \to \phi \;/\; \phi,$ provided p does not occur in ϕ.

Definition 10. Let $\Lambda = (A, R)$ be a derivation system with A being the set of axioms and R the set of derivation rules. A *derivation* is a sequence $\phi_0, \ldots, \phi_n$ such that every ϕ_i is either an axiom or the result of applying a rule to formulas of $\{\phi_0, \ldots \phi_{i-1}\}$. A formula ϕ is a *theorem* of Λ, notation: $\Lambda \vdash \phi$, if ϕ appears as the last item of a derivation in Λ. A formula ϕ is a *Λ-consistent* if $\neg\phi$ is not a theorem. A derivation system Λ is *sound* with respect to a class K of frames if every Λ-theorem is valid in K and *complete* if every K-valid formula is a theorem of Λ.

We refer the reader to VENEMA [26] for more information on the axioms. The meaning of the Irreflexivity Rule for D is perhaps best understood by reading it as follows: 'if ψ is consistent and p does not occur in ϕ, then $(p \wedge \neg Dp) \wedge \psi$ is consistent'. Note that the formula $p \wedge \neg Dp$ ('p is true here, and nowhere else') can be seen as a *name* for a world (s, t). So (IR_D) states that every consistent formula remains consistent if we take its conjunction with a name. For more information concerning rules like IR_D, which originate with GABBAY [3], we refer to VENEMA [25].

Theorem 11. AR$^+$ *is sound and complete with respect to the class of all frames.*

Note that Theorem 11 is the *TAL*-version of a square completeness theorem for arrow logic. For a *proof* we refer to VENEMA [23].

Definition 12. Let $AL^{(+)}$ be the derivation system $AR^{(+)}$ extended with the set $\{TR, IR, TO\}$ as axioms (cf. Definition 4). For $X \subseteq \{DI, DE, UL, UR, W\}$, we define $AL^{(+)}X$ as the derivation system $AL^{(+)}$ extended with the set X as axioms.

Theorem 13. *If* $X \subseteq \{DI, DE, UL, UR\}$*, then* $AL^{+}X$ *is sound and complete with respect to the class* K_X *of frames where* X *is valid.*

Proof. An inspection of the proof of Theorem 11 reveals that it can be easily modified to prove Theorem 13. The basic observation is that the theorem only mentions sets X of *closed* axioms, i.e. formulas without any propositional variables. The proof method of Theorem 11 yields, for an arbitrary consistent formula ϕ, a model $\mathfrak{M}$ such that (i) ϕ is satisfiable in $\mathfrak{M}$ and (ii), X is true in every world of $\mathfrak{M}$. It is then a straightforward consequence of Proposition 5 that the underlying frame of $\mathfrak{M}$ belongs to the intended class. □

As instances of Theorem 13, we obtain that AL^+ is sound and complete with respect to the class of all linear flows of time, and $AL^+\{DE, UL, UR\}$ with respect to the class of unbounded, dense flows of time, and thus with respect to the flow of time of the rational numbers.

Finally, we turn to the matter of flat validity. Note that the set of formulas which are flatly valid in a class of frames cannot be a logic in the ordinary sense of the word, since it will not be closed under substitution. For, the formula $p \rightarrow \Box_V p$ will be flatly valid, but for instance the formula $\Diamond_H \otimes q \rightarrow \Box_V \Diamond_H \otimes q$ can easily be falsified in a flat model. Therefore, we have to use a trick.

Definition 14. Let Ω be one of the axiom systems defined above, and ϕ a *TAL*-formula. Let VAR_ϕ be the set of propositional variables that occur in ϕ. We say that ϕ is *flatly* derivable in Ω, notation: $\Omega_\flat \vdash \phi$, if

$$\Omega \vdash (\bigwedge_{p \in VAR_\phi} \Box(p \leftrightarrow \Box_V p)) \rightarrow \phi.$$

Corollary 15. *If* $X \subseteq \{DI, DE, UL, UR\}$*, then* $\mathrm{AL}^+X_\flat$ *is 'flatly sound and complete' with respect to the class* K_X*, i.e. for any formula* ϕ*:*

$$\Omega_\flat \vdash \phi \iff \mathsf{K}_X \models_\flat \phi.$$

Proof. For a change, we only prove soundness. Suppose that $\Omega_\flat \vdash \phi$, and let $\mathfrak{M}$ be a flat model $(T, <, V)$; we have to show that $\mathfrak{M} \models \phi$.

To start with, by definition of flat derivability we have that

$$\Omega \vdash (\bigwedge_{p \in VAR_\phi} \Box(p \leftrightarrow \Box_V p)) \rightarrow \phi.$$

So by our soundness assumption, for *any* model $\mathfrak{N}$:

$$\mathfrak{N} \models (\bigwedge_{p \in VAR_\phi} \Box(p \leftrightarrow \Box_V p)) \rightarrow \phi.$$

As it is straightforward to verify that the formula $\Box(p \leftrightarrow \Box_V p)$ is valid in any flat model, validity of ϕ in $\mathfrak{M}$ follows immediately. $\Box$

5 Completeness for well-orderings.

In this section we prove our main result, viz. soundness and completeness of $ALW_\flat$ with respect to flat validity in the class of well-ordered frames. In some sense, this result is the best we can get for WO, for ordinary validity in this class does not allow a recursive axiomatization[5]. Neither do we have *strong* completeness for flat validity in WO, as an easy compactness argument shows.

On the other hand, what makes the results in this section interesting is that the complete axiom systems are orthodox in the sense that they do not use an irreflexivity rule. This is interesting from a theoretical point of view and may also have applicational virtues: note that derivations involving the irreflexivity rule use material (the proposition letter p which does not occur in ϕ) in a very 'resource-unconscious' way.

Theorem 16 (Flat soundness and completeness for well-orderings). *For any* TAL*-formula* ϕ

$$\mathrm{ALW}_\flat \vdash \phi \iff \mathsf{WO} \models_\flat \phi.$$

[5] In Halpern & Shoham [7] the authors show how to code the behavior of Turing machines in a subsystem *HS* of *TAL*. It easily follows from their results that the *TAL*-theory of WO is not recursively enumerable.

Proof. We leave it to the reader to establish soundness. For completeness, let ϕ be an $ALW_\flat$-consistent formula. We will find a flat well-ordered model $\mathfrak{M}$ in which ϕ is satisfiable. Define

$$\phi' \equiv (\bigwedge_{p \in VAR_\phi} \Box(p \leftrightarrow \Box_V p)) \ \rightarrow \ \phi,$$

then by definition of $ALW_\flat$, ϕ' is ALW-consistent. Now we need the following lemma:

Lemma 17. *For every formula ψ:*

$$\mathrm{ALW} \vdash \psi \iff \mathrm{AL}^+ W \vdash \psi.$$

Proof. It suffices to show that the irreflexivity rule for D is conservative over ALW. So, let us assume that

$$ALW \vdash (p \wedge \neg Dp) \rightarrow \phi. \tag{5}$$

Then we have to prove that $ALW \vdash \phi$. Abbreviate $ALW \vdash \chi$ by $\vdash \chi$ and let $\texttt{first}(\chi)$ denote the formula

$$\texttt{first}(\chi) \equiv (\chi \wedge \Box_S \neg\chi \wedge \Box_W \Box_V \neg\chi),$$

then the well-ordering axiom W reads: $\Diamond p \rightarrow \Diamond\texttt{first}(p)$. From (5) it follows by the rule of substitution that

$$\vdash (\texttt{first}(\neg\phi) \wedge \neg D\texttt{first}(\neg\phi)) \rightarrow \phi. \tag{6}$$

We leave it to the reader to verify that for every q

$$\vdash \texttt{first}(q) \rightarrow \neg D\texttt{first}(q). \tag{7}$$

From (6) and (7) it follows that

$$\vdash \texttt{first}(\neg\phi) \rightarrow \phi.$$

On the other hand, by definition of $\texttt{first}$, we have

$$\vdash \texttt{first}(\neg\phi) \rightarrow \neg\phi,$$

so we find that

$$\vdash \neg\texttt{first}(\neg\phi),$$

whence an application of NEC gives

$$\vdash \neg\Diamond\texttt{first}(\neg\phi).$$

But then by the instantiation $W(\neg\phi)$ of the well-ordering axiom W we find the desired

$$\vdash \neg\phi \rightarrow \bot.$$

$\Box$

It follows from the lemma that $AL^+ \not\vdash \neg\phi'$, so by Theorem 13[6] ϕ has a *linear* model $\mathfrak{M} = (T, <, V)$ such that V is *flat* and $\mathfrak{M} \models AL^+W$, i.e. every theorem of AL^+W is valid in $\mathfrak{M}$. Unfortunately $\mathfrak{M}$ need not be well-ordered, as not *every* subset of T needs to have a smallest element. Fortunately however, $\mathfrak{M}$ is definably well-ordered: call a first-order structure $\mathfrak{N}$ for $L^{1<}$ *definably well-ordered* if every first-order definable subset of the domain has a smallest element, or to be more precise, if $\mathfrak{N}$ satisfies the condition, that for every first-order formula $\psi(x_0) \in L^{1<}$, the set

$$S_\psi = \{t \in T \mid \mathfrak{M} \models \psi(x_0)[x_0 \mapsto t]\}$$

has a smallest element (provided that $S \neq \varnothing$).

Lemma 18. *$\mathfrak{M}$ is definably well-ordered.*

Proof. Assume that ψ is such that $S_\psi \neq \varnothing$. First note that by our expressive completeness result Theorem 8, the formula $x_1 = x_1 \wedge \psi(x_0)$ has an equivalent ψ' in TAL, whence we have

$$\mathfrak{M}, s, t \Vdash \psi' \iff s \in S_\psi.$$

Second, as $\mathfrak{M}$ is an AL^+W-model, the ψ'-instantiation of the well-orderedness axiom W is valid in $\mathfrak{M}$, so

$$\mathfrak{M} \models \Diamond\psi' \to \Diamond(\psi' \wedge \Box_W \Box_V \neg\psi').$$

and as $S_\psi \neq \varnothing$, this immediately gives a smallest element for $S = V(\psi)$. □

We now consider the first-order equivalent $\tau_\flat(\phi) \in L_3^{1<}(x_0, x_1)$ of ϕ', cf. (3) and (4). Note that, translated into first-order logic, our problem is that $\tau_\flat(\phi)$ is satisfiable in a definably well-ordered model, while we need to satisfy it in a truely well-ordered model. The solution of this problem is given by the lemma below, for which we need some terminology.

Let $\equiv_n$ denote the following relation between two structures of first-order logic: $\mathfrak{M} \equiv_n \mathfrak{M}'$ iff $\mathfrak{M}$ and $\mathfrak{M}'$ satisfy the same first-order sentences of quantifier depth $\leq n$.

Lemma 19 (Doets). *Let $\mathfrak{N}$ be a definably well-ordered model for $L^{1<}$ and n a natural number. Then $\mathfrak{N}$ has a well-ordered n-equivalent, i.e. there is a well-ordered structure $\mathfrak{N}'$ such that $\mathfrak{N}$ is well-ordered and $\mathfrak{N} \equiv_n \mathfrak{N}'$.*

For a *proof* of this lemma we refer the reader to Doets [2], Corollary 4.4.

Now let n be the quantifier depth of the formula $\tau_\flat(\phi)$; lemma 19 supplies us with a well-ordered structure $\mathfrak{M}'$ such that $\mathfrak{M}' \equiv_{n+2} \mathfrak{M}$. It is then immediate to verify that $\mathfrak{M}' \models \exists x_0 x_1 \tau_\flat(\phi)(x_0, x_1)$.

This implies that $\mathfrak{M}'$ (now seen as a $TAL_\flat$-model) is a well-ordered model for ϕ. □

[6] Actually, we need a strong completeness theorem (which for lack of space we could not state or prove here).

Corollary 20 (Flat soundness and completeness for ω). *Let* ALω *be the axiom system* AL *extended with the axioms* W, DI *and* UR. *Then for any* TAL-*formula* ϕ

$$\mathrm{AL}\omega_\flat \vdash \phi \iff \omega \models_\flat \phi$$

Proof. By a suitable adaptation of the proof of Theorem 16, one can satisfy any $AL\omega_\flat$-consistent formula in a well-ordering which is discrete and unbounded to the right. The underlying frame of the model must then be isomorphic to the ordering of the natural numbers (cf. Proposition 5(x)). □

6 Questions

We finish the paper with mentioning two open problems:

1. In the introduction we already mentioned the fact that the proof method applied in section 5 stems from GABBAY & HODKINSON [6] for the uni-dimensional case. In the cited paper the authors prove a completeness result for the flow of time of the real numbers. However, their derivation system crucially uses an *irreflexivity rule*. In VENEMA [24] we applied the method to an *orthodox* system, to obtain a complete axiomatization for the S, U-logic of the class of well-orders and the flow of time of the natural numbers. Reynolds [19] solved the more difficult problem to find an orthodox complete axiomatization for the S, U-logic of the reals.
   Concerning two-dimensional temporal logics, it is an intrigueing question whether there is a derivation system Λ such that (i) Λ gives a complete enumeration of the flat[7] *TAL*-logic of the real number flow of time. Similar questions may be asked for other flows of times and other classes of (linear) frames. Note that the method applied in this paper cannot be extended to other linear flows of time in a straightforward manner, as our proof of Lemma 17 crucially depends on the presence of the well-orderedness axiom W in the logic.
2. One may read the completeness results of Theorem 16 (and Theorem 20) as follows: $L_3^{1<}(x_0, x_1)$ is sufficiently expressive to contain a Hilbert-style derivation system which is complete for the class WO (resp. for ω). The question is whether it is also expressive enough to define complete Gentzen-style calculi in it (not necessarily only for well-orderings)[8]. In particular, it would be interesting to have cut-free Gentzen-style calculi with nice properties like the subformula property or decidability.

[7] The related question for ordinary validity is solved in the negative, as the *TAL*-theory of the reals is not recursively enumerable, cf. footnote 5 and HALPERN & SHOHAM [7].

[8] Again, the restriction to *monadic* first-order logic is essential here: it follows from results in algebraic logic that there is no upper bound on the number of variables needed to prove theorems of first-order logic with *dyadic* predicates. There are calculi known that are complete for $L_3^{2<}(x_0, x_1)$, cf. MADDUX [12] or ORŁOWSKA [18], but these calculi essentially use formulae of $L_n^{2<}(x_0, x_1)$, n arbitrary large.

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