Nabla Algebras and Chu Spaces

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Abstract. This paper is a study into some properties and applications of Moss' coalgebraic or 'cover' modality ∇.
First we present two axiomatizations of this operator, and we prove these axiomatizations to be sound and complete with respect to basic modal and positive modal logic, respectively. More precisely, we introduce the notions of a modal ∇-algebra and of a positive modal ∇-algebra. We establish a categorical isomorphism between the category of modal ∇-algebra and that of modal algebras, and similarly for positive modal ∇-algebras and positive modal algebras.
We then turn to a presentation, in terms of relation lifting, of the Vietoris hyperspace in topology. The key ingredient is an F-lifting construction, for an arbitrary set functor F, on the category Chu of two-valued Chu spaces and Chu transforms, based on relation lifting.
As a case study, we show how to realize the Vietoris construction on Stone spaces as a special instance of this Chu construction for the (finite) power set functor. Finally, we establish a tight connection with the axiomatization of the modal ∇-algebras.
Keywords coalgebra, relation lifting, modal algebra, Vietoris hyperspace, Chu space.

1 Introduction

This paper is a study into the algebraic properties of the coalgebraic modal operator ∇, and some of its applications. The connective ∇ takes a finite[1] *set* Φ of formulas and returns a single formula $\nabla\Phi$. Using the standard modal language, ∇ can be seen as a defined operator:

$$\nabla\Phi = \Box(\bigvee \Phi) \wedge \bigwedge \Diamond\Phi, \tag{1}$$

where $\Diamond\Phi$ denotes the set $\{\Diamond\varphi \mid \varphi \in \Phi\}$.

Readers familiar with classical first-order logic will recognize the quantification pattern in (1) from the theory of Ehrenfeucht-Fraïssé games, Scott sentences, and the like, see [9] for an overview. In modal logic, related ideas made

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[1] In this paper we restrict to the finitary version of the operator.

an early appearance in Fine's work on normal forms [8]. As far as we know, however, the first explicit occurrences of the nabla *connective* appeared roughly at the same time, in the work of Barwise & Moss on circularity [4], and that of Janin & Walukiewicz on automata-theoretic approaches towards the modal μ-calculus [10].

The semantics of the nabla modality can be explicitly formulated as follows, for an arbitrary Kripke structure $\mathbb{S}$ with accessibility relation R:

$$\mathbb{S}, s \Vdash \nabla \Phi \text{ if } \begin{array}{l} \text{for all } \varphi \in \Phi \text{ there is a } t \in R[s] \text{ with } \mathbb{S}, t \Vdash \varphi, \text{ and} \\ \text{for all } t \in R[s] \text{ there is a } \varphi \in \Phi \text{ with } \mathbb{S}, t \Vdash \varphi. \end{array} \tag{2}$$

In other words, the semantics of ∇ can be expressed in terms of the *relation lifting* of the satisfaction relation between states and formulas:

$$\mathbb{S}, s \Vdash \nabla \Phi \text{ iff } (R[s], \Phi) \in \overline{\mathsf{P}}(\Vdash). \tag{3}$$

This insight, which is nothing less than a coalgebraic reformulation of modal logic, led Moss [14] to the introduction of *coalgebraic logic*, in which (3) is generalized to an (almost) arbitrary set functor F by introducing a coalgebraic operator ∇_{F}, and interpreting it using the relation lifting $\overline{\mathsf{F}}(\Vdash)$ of the forcing relation.

In this paper we want to look at ∇ as an algebraic operator in its own right. Our motivation for undertaking such a study, besides a natural intellectual curiosity, was twofold: firstly, we hope that such a study might be a first step in the direction of a 'coalgebraic proof theory' (we'll come back to this towards the end of this paper). And second, we belief that a thorough algebraic understanding of the nabla operator might shed light on power lifting constructions, such as the Vietoris hyperspace construction in topology. Let us address these issues in some more detail, and on the way explain what we believe to be the contribution of this paper.

Concerning the algebraic properties, the main issue that we address concerns *axiomatizations*. We were interested in axiomatizing the properties of the nabla operator in terms that only refer to ∇ itself and its interaction with the Boolean connectives, but which does not involve the non-coalgebraic modalities $\Box$ and $\Diamond$. As we will see in the next section, such an 'intrinsic' axiomatization is indeed possible. A remarkable feature of our axiomatization is that it is largely independent of the Boolean negation, so that its natural algebraic setting is that of positive modal algebras [7]. On the other hand, the nabla operator for the power set functor interacts reasonably well with the complementation operator, so that in fact we obtain two sound and complete axiomatizations for ∇, one in the setting of positive modal logic, and one in the setting of classical (i.e., Boolean) modal logic. Both of these results are formulated in terms of an isomorphism between categories of algebras.

The connection between the nabla operator and powering constructions in topology [13] is less obvious — we confine our attention to the Vietoris hyperspace. Formulations of the Vietoris hyperspace construction involving modal logic are well-known [11, 18], and the importance of the Vietoris construction on the interface of coalgebra and modal logic has already been the object of a number of studies [1, 12, 15, 6]. Indeed, one may argue that the coalgebras of the Vietoris endofunctor on Stone spaces provide an adequate semantics for all modal logics since there is an isomorphism between the category of these coalgebras, and the category of the descriptive general frames known in modal logic [12]. Here, however, we take a slightly different angle. Our goal was to somehow define the Vietoris hyperspace construction in a way that would be relevant and useful for coalgebraic applications and that would only refer to category-theoretic properties of the power set functor. Analogous to Moss' coalgebraic approach to modal logic, this might enable one to generalize the Vietoris construction to arbitrary set functors. The key idea in our approach is to formulate the Vietoris construction in terms of the relation lifting $\overline{\in} := \overline{\mathsf{P}}(\in)$ of the *membership* relation between points and (open/closed/clopen) sets.

As it turned out, *Chu spaces* provide a natural setting for this. A Chu space is a triple $S = \langle X, S, A\rangle$ consisting of two sets X and A, together with a binary relation[2] $S \subseteq X \times A$. In itself, the connection with Chu spaces should not come as a big surprise: as we will show in more detail further on, we may read (3) as saying that the semantics of ∇ itself can be seen as a Chu transform, that is, an arrow in the category Chu. In section 3, we give F-lifting constructions on Chu spaces for arbitrary endofunctors F on Set. The main desiderata of these constructions are functoriality and preservation of the full subcategory of *normal* Chu spaces (see Definition 8 below). Since the latter is not met in general, we also introduce a normalization functor on Chu spaces. We show that if F preserves weak pullbacks, then its associated lifting construction, and also the finite version of it, are functorial on Chu spaces. Then, as a case study in section 4, we show how to realize the Vietoris construction on Stone spaces as a special instance of this Chu construction for the (finite) power set functor (Theorem 4).

Finally, the two parts of the paper come together in Theorem 5, which establishes a tight connection between the Vietoris construction and the axiomatization of the modal ∇-algebras.

2 An axiomatization of ∇

In the introduction we mentioned that the nabla operator enables a coalgebraic reformulation of standard modal logic. The aim of this section is to substantiate this claim.

First of all, while we introduced the nabla operator as an abbreviation in the language of standard modal logic, for a proper use of the word 'reformulation', we need of course *inter*definability of the nabla operator on the one hand, and the standard modal operators on the other. It is in fact an easy exercise to prove that with the semantics of ∇ as given by (2), we have the following semantic

[2] We restrict attention to two-valued Chu spaces in this paper. In fact, these structures are known from the literature under various names, including *topological systems* [18] and *classifications* [5].

equivalences:

$$\begin{array}{l}\Diamond\varphi \equiv \nabla\{\varphi, \top\} \\ \Box\varphi \equiv \nabla\varnothing \vee \nabla\{\varphi\}\end{array} \tag{4}$$

In other words, the standard modalities $\Box$ and $\Diamond$ can be defined in terms of the nabla operator (together with $\vee$ and $\top$).

Taken together, (1) and (4) show that on the *semantic* level of Kripke structures, the language with the nabla operator is indeed a reformulation of standard modal logic. This naturally raises the question whether this equivalence can also be expressed *axiomatically*. That is, we are interested in the question whether we may impose natural conditions which characterize those nablas that behave like the 'real' ones defined using (1). From the semantic interdefinability of ∇ with respect to $\Box$ and $\Diamond$, it follows that a 'roundabout' axiomatization of the nabla operation is possible. However, it is of course much more interesting to try and find a more 'direct' axiomatization, in terms of the intrinsic properties of the nabla operator, and its interaction with the Boolean connectives.

A good starting point for this would be to look for *validities*, i.e., ∇-formulas that are true in every state of every Kripke structure, or, equivalently, for pairs of *equivalent* formulas. As an example of such an equivalence, we give an interesting distributive law; for a concise formulation we need the notion of relation lifting.

Definition 1. *Given a relation $Z \subseteq A \times A'$, define its* power lifting relation $\overline{\mathsf{P}}Z \subseteq \mathsf{P}A \times \mathsf{P}A'$ *as follows:*

$$\begin{aligned}\overline{\mathsf{P}}Z := \{(X, X') \mid \quad & \textit{for all } x \in X \textit{ there is an } x' \in X' \textit{ with } (x, x') \in Z \\ & \& \textit{ for all } x' \in X' \textit{ there is an } x \in X \textit{ with } (x, x') \in Z\}.\end{aligned}$$

We say that $Z \subseteq A \times A'$ is full *on A and A', notation: $Z \in A \bowtie A'$, if $(A, A') \in \overline{\mathsf{P}}Z$. Observe that as a special case, $\varnothing \bowtie A = \varnothing$ if $A \neq \varnothing$, while $\varnothing \bowtie \varnothing = \{\varnothing\}$ (i.e., the empty relation is full on $\varnothing$ and $\varnothing$).*

The distributive law that we mentioned concerns the following equivalence, which holds for arbitrary sets of formulas Φ, Φ':

$$\nabla\Phi \wedge \nabla\Phi' \;\equiv\; \bigvee_{Z \in \Phi \bowtie \Phi'} \nabla\{\varphi \wedge \varphi' \mid (\varphi, \varphi') \in Z\}. \tag{5}$$

For a proof of (5), first suppose that $\mathbb{S}, s \Vdash \nabla\Phi \wedge \nabla\Phi'$. Let $Z_s \subseteq \Phi \times \Phi'$ consist of those pairs (φ, φ') such that the conjunction $\varphi \wedge \varphi'$ is true at some successor $t \in R[s]$. It is then straightforward to derive from (2) that Z_s is full on Φ and Φ', and that $\mathbb{S}, s \Vdash \nabla\{\varphi \wedge \varphi' \mid (\varphi, \varphi') \in Z_s\}$. The converse direction follows fairly directly from the definitions.

We have now arrived at one of the key definitions of the paper, namely that of nabla algebras. Here we provide the desired direct axiomatization of the nabla operator.

Definition 2. *A structure $A = \langle A, \wedge, \vee, \top, \bot, \nabla\rangle$ is a* positive modal ∇-algebra *if its lattice reduct $A_\flat := \langle A, \wedge, \vee, \top, \bot\rangle$ is a distributive*[3] *lattice and $\nabla : \mathsf{P}_\omega(A) \to$*

[3] In this paper with a 'lattice' we shall always mean a *bounded* lattice.

A satisfies the laws $\nabla 1$ – $\nabla 6$ *below. Here Greek lower case letters refer to finite subsets of* A.

$\nabla 1$. *If* $\alpha \overline{\mathsf{P}}(\leq)\beta$, *then* $\nabla\alpha \leq \nabla\beta$,
$\nabla 2$. *If* $\bot \in \alpha$, *then* $\nabla\alpha = \bot$,
$\nabla 3$. $\nabla\alpha \wedge \nabla\beta \leq \bigvee\{\nabla\{a \wedge b \mid (a,b) \in Z\} \mid Z \in \alpha \bowtie \beta\}$,
$\nabla 4$. *If* $\top \in \alpha \cap \beta$, *then* $\nabla\{a \vee b \mid a \in \alpha, b \in \beta\} \leq \nabla\alpha \vee \nabla\beta$,
$\nabla 5$. $\nabla\varnothing \vee \nabla\{\top\} = \top$,
$\nabla 6$. $\nabla\alpha \cup \{a \vee b\} \ \leq\ \nabla(\alpha \cup \{a\}) \vee \nabla(\alpha \cup \{b\}) \vee \nabla(\alpha \cup \{a,b\})$.

A structure $A = \langle A, \wedge, \vee, \top, \bot, \neg, \nabla\rangle$ *is a* modal ∇-algebra *if* $\langle A, \wedge, \vee, \top, \bot, \neg\rangle$ *is a Boolean algebra and the structure satisfies, in addition to the axioms* $\nabla 1$ – $\nabla 6$ *above, the following*

$\nabla 7$. $\neg\nabla\alpha = \nabla\{\bigwedge\alpha, \top\} \vee \nabla\varnothing \vee \bigvee\{\nabla\{a\} \mid a \in \alpha\}$.

The category of (positive) modal ∇*-algebras with homomorphisms is denoted as* $(\mathsf{P})\mathsf{MA}_\nabla$.

Remark 1. It is not hard to see that the following formulas can be derived from the axioms $\nabla 1$ – $\nabla 6$:

$\nabla 3'$. If $\alpha \neq \varnothing$, then $\nabla\varnothing \wedge \nabla\alpha = \bot$,
$\nabla 3^n$. $\bigwedge_{i=1}^n \nabla\alpha_i \leq \bigvee\{\nabla Z^\wedge \mid Z \in \bigodot_i \alpha_i\}$,
where, for a finite collection α_i of finite subsets of A, $\bigodot_i \alpha_i := \{Z \subseteq \Pi_i\alpha_i \mid \pi_i[Z] = \alpha_i$ for every $i\}$, and, for $Z \in \bigodot_i \alpha_i$, $Z^\wedge := \{\bigwedge_i a_i : (a_i)_{i\in I} \in Z\}$,
$\nabla 6'$. $\nabla\alpha = \nabla\alpha \cup \{\bigvee\alpha\}$,

For instance, $\nabla 3'$ follows by instantiating β with the empty set in $\nabla 3$, and $\nabla 3^n$ is just the n-ary version of $\nabla 3$ and can be shown by induction on n. Recall from Definition 1 that $\varnothing \bowtie \alpha = \varnothing$ in this case, and $\bigvee\varnothing = \bot$.

Definition 3. *A structure* $A = \langle A, \wedge, \vee, \top, \bot, \Diamond, \Box\rangle$ *is a* positive modal algebra *if the* lattice reduct $A_\flat := \langle A, \wedge, \vee, \top, \bot\rangle$ *is a distributive lattice, and* $\Box, \Diamond$ *are unary operations on* A *that satisfy the following axioms:*

$\Diamond(a \vee b) = \Diamond a \vee \Diamond b$ $\quad$ $\Diamond\bot = \bot$
$\Box(a \wedge b) = \Box a \wedge \Box b$ $\quad$ $\Box\top = \top$
$\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$
$\Box(a \vee b) \leq \Box a \vee \Diamond b$

A modal algebra *is an algebra* $A = \langle A, \wedge, \vee, \top, \bot, \neg, \Diamond, \Box\rangle$ *such that* $A_\flat := \langle A, \wedge, \vee, \top, \bot, \neg\rangle$ *is a Boolean algebra and the operations* $\Box$ *and* $\Diamond$ *satisfy, in addition to the axioms above:*

$\neg\Diamond a = \Box\neg a$.

We let MA *and* PMA *denote the categories of modal algebras (positive modal algebras, respectively) as objects, and algebraic homomorphisms as arrows.*

Definition 4. *Let* A *be a positive modal algebra (modal algebra, respectively). Then we let* A^∇ *denote the structure* $\langle A_\flat, \nabla)\rangle$, *where* ∇ *is defined using (1).*

Conversely, if B is a positive modal ∇-algebra (modal ∇-algebra, respectively), we let $B^{\Diamond}$ denote the structure $\langle B_\flat, \Diamond, \Box\rangle$, where $\Diamond$ and $\Box$ are defined using (4).

We let both $(\cdot)^{\nabla}$ and $(\cdot)^{\Diamond}$ operate as the identity on maps, i.e., $f^{\nabla} := f$ and $f^{\Diamond} := f$ whenever applicable.

Theorem 1. *The functors $(\cdot)^{\nabla}$ and $(\cdot)^{\Diamond}$ establish a categorical isomorphism between the categories* PMA *and* PMA_∇*, and between the categories* MA *and* MA_∇*.*

Proof. We restrict ourselves to a proof of the following two claims, for an arbitrary positive modal ∇-algebra A:

1. $A^{\Diamond}$ is a positive modal algebra;
2. $(A^{\Diamond})^{\nabla} \cong A$.

1. $\nabla 2$ implies that $\Diamond\bot = \bot$.
$\nabla 3$ instantiated with $\alpha = \{a\}$ and $\beta = \{b\}$ yields $\Box a \wedge \Box b = \Box(a \wedge b)$.
$\nabla 4$ instantiated with $\alpha = \{a, \top\}$ and $\beta = \{b, \top\}$ yields $\Diamond a \vee \Diamond b = \Diamond(a \vee b)$.
$\nabla 5$ says that $\Box\top = \top$.
$\nabla 6$ instantiated with $\alpha = \emptyset$ yields that $\nabla\{a \vee b\} \leq \nabla\{a\} \vee \nabla\{b\} \vee \nabla\{a, b\}$. Since $\{b\}\overline{\mathsf{P}}(\leq)\{b,\top\}$ and $\{a,b\}\overline{\mathsf{P}}(\leq)\{b,\top\}$, then by $\nabla 1$, $\nabla\{b\}\vee\nabla\{a,b\} \leq \nabla\{b,\top\}$. Hence $\nabla\{a\vee b\} \leq \nabla\{a\}\vee\nabla\{b,\top\}$, from which we get $\Box(a\vee b) \leq \Box a\vee\Diamond b$. In order to show that $\Box a\wedge\Diamond b \leq \Diamond(a\wedge b)$ we need to show that $(\nabla\{a\}\vee\nabla\emptyset)\wedge\nabla\{b,\top\} \leq \nabla\{a\wedge b,\top\}$.

$$\begin{aligned}(\nabla\{a\} \vee \nabla\emptyset) \wedge \nabla\{b,\top\} &= [\nabla\{a\} \wedge \nabla\{b,\top\}] \vee [\nabla\emptyset \wedge \nabla\{b,\top\}] \\ &= [\nabla\{a\} \wedge \nabla\{b,\top\}] \vee \bot && (\nabla 3') \\ &= \nabla\{a \wedge b, a \wedge \top\} && (\nabla 3) \\ &\leq \nabla\{a \wedge b, \top\} && (\nabla 1)\end{aligned}$$

The last inequality holds since $\{a\wedge b, a\}\overline{\mathsf{P}}(\leq)\{a\wedge b,\top\}$. This completes the proof that $A^{\Diamond}$ is a positive modal algebra.

2. We need to show that ∇ coincides with the operator $\widetilde{\nabla}$ associated with the ∇-induced modal operators. For every finite subset α of A,

$$\begin{aligned}\widetilde{\nabla}\alpha &= [\nabla\{\textstyle\bigvee\alpha\} \vee \nabla\emptyset] \wedge \textstyle\bigwedge\{\nabla\{a,\top\} : a \in \alpha\} \\ &= \textstyle\bigwedge\{[\nabla\{\bigvee\alpha\} \vee \nabla\emptyset] \wedge \nabla\{a,\top\} : a \in \alpha\} \\ &= \textstyle\bigwedge\{[\nabla\{\bigvee\alpha\} \wedge \nabla\{a,\top\}] \vee [\nabla\emptyset \wedge \nabla\{a,\top\}] : a \in \alpha\} \\ &= \textstyle\bigwedge\{[\nabla\{\bigvee\alpha\} \wedge \nabla\{a,\top\}] \vee \bot : a \in \alpha\} && (\nabla 3') \\ &= \textstyle\bigwedge\{[\nabla\{\bigvee\alpha, a\}] : a \in \alpha\} && (\nabla 3) \\ &= \textstyle\bigvee\{\nabla Z^{\wedge} : Z \in \bigodot_{a\in\alpha}\{\bigvee\alpha, a\}\} && (\nabla 3^n) \\ &= \nabla\alpha \cup \{\textstyle\bigvee\alpha\} && (*) \\ &= \nabla\alpha && (\nabla 6')\end{aligned}$$

Let us show the $(*)$-marked equality: let $\alpha = \{a_i, i = 1\ldots,n\}$ for every i, let $\beta_i = \{\bigvee\alpha, a_i\}$. Then consider the following relation:

$$Z = \{(b_i)_i \in \Pi_i\beta_i : \text{for at most one } i, b_i \neq \textstyle\bigvee\alpha\}.$$

Then $Z^{\wedge} = \{\bigvee \alpha, a_1, \ldots, a_n\}$, and moreover it is not difficult to see that for every $W \in \bigodot_i \beta_i$, the pair $(W^{\wedge}, Z^{\wedge})$ belongs to the relation $\overline{\mathsf{P}}(\leq)$, so the statement follows by $\nabla 1$. □

Remark 2. As an easy corollary of Theorem 1, we can obtain a completeness result for modal logic formulated in terms of the nabla operator.

Finally, in section 4 we will need the construction which can be seen as a kind of *power set lifting* of a Boolean algebra A. Following terminology and notation of [18], in the definition below we present a Boolean algebra by generators and relations.

Definition 5. *Let A be a Boolean algebra. Then*

$$\mathsf{BA}\langle\ \{\nabla\alpha \mid \alpha \in \mathsf{P}_\omega A\} : \nabla 1 - \nabla 7\ \rangle$$

presents a Boolean algebra that we shall denote as A^{P}.

In words, A^{P} is the Boolean algebra we obtain as follows: first freely generate a Boolean algebra by taking the set $\{\nabla\alpha \mid \alpha \in \mathsf{P}_\omega A\}$ as *generators*, and then take a quotient of this algebra, by identify those elements that can be proven equal on the basis of the *relations* (axioms) $\nabla 1$–$\nabla 7$. In section 4 we will see a different characterization of this algebra: Theorem 5 states that A^{P} is in fact isomorphic to the algebra of clopens of the Vietoris hyperspace of the Stone space which is dual to A.

3 Chu spaces and their liftings

Chu spaces [16] unify a wide range of mathematical structures, including relational, algebraic and topological ones. Surprisingly this degree of generality can be achieved with a remarkably simple form of structure. As we mentioned already, in this paper we will only enter a small part of Chu territory since we restrict attention to two-valued Chu spaces. These can be defined as follows.

Definition 6. *A (two-valued) Chu space is a triple $\mathbb{S} = \langle X, S, A\rangle$ consisting of two sets X and A, together with a binary relation $S \subseteq X \times A$. Elements of X are called* objects *or* points*, and elements of A,* attributes*; the relation S is the* matrix *of the space. Given two Chu spaces $\mathbb{S}' = \langle X', S', A'\rangle$ and $\mathbb{S} = \langle X, S, A\rangle$, a* Chu transform *from $\mathbb{S}'$ to $\mathbb{S}$ is a pair (f, f') of functions $f : X' \to X$, $f' : A \to A'$ that satisfy the* (generalized) adjointness condition

$$f(x')Sa \iff xS'f'(a). \tag{6}$$

for all $x' \in X'$ and $a \in A$. We let Chu *denote the category with Chu spaces as objects and Chu transforms as arrows.*

As a motivating example of a Chu transform, consider once more the semantics of ∇. One may read (3) above as saying that the pair $(R[\cdot] : S \to \mathsf{P}(S), \nabla : \mathsf{P}_\omega(Fma) \to Fma)$ is a *Chu transform* from the Chu space $(S, \Vdash, Fma)$ to its power set lifting $(\mathsf{P}S, \overline{\mathsf{P}}(\Vdash), \mathsf{P}_\omega(Fma))$. In a slogan: the semantics of ∇ is an arrow in the category Chu of Chu spaces and Chu transforms.

Clearly, the generalized adjointness condition specializes to adjointness in the right context, for example if partial orders $\langle P, \leq, \rangle$ are represented as the Chu spaces $\langle P, \leq, P \rangle$, then the Chu transforms between two such structures are exactly tuples of residuated maps.

Definition 7. *Any Chu space $S = \langle X, S, A \rangle$ gives rise to an order on X, the* specialization order $\sqsubseteq_S$*, defined as follows: $x \sqsubseteq_S y$ iff for every $a \in A$ ($xSa \Rightarrow ySa$). The specialization order then induces the following equivalence relation $\equiv_S$ on X: $x \equiv_S y$ iff $x \sqsubseteq_S y$ and $y \sqsubseteq_S x$, i.e. iff for every $a \in A$ ($xSa \Leftrightarrow ySa$).*

Normal Chu spaces A prominent role within Chu, from the point of view of logic, is played by the so-called *normal* Chu spaces. Normal Chu spaces provide a general and uniform setting for algebraic, set-based and topological semantics of propositional logics.

Definition 8. *A Chu space $S = \langle X, S, A \rangle$ is* normal *if $A \subseteq \mathsf{P}(X)$ and S is the membership relation restricted to A, that is, xSa iff $x \in a$.* NChu *denotes the full subcategory of* Chu *based on these normal spaces.*

To mention an important example, any Stone space $X = \langle X, \tau \rangle$ can be represented as $S_X = \langle X, \in, C \rangle$, C being the Boolean algebra of the clopen subsets in τ. Then a map f between Stone spaces is continuous exactly when (f, f^{-1}) is a Chu transform between their associated Chu spaces. In fact, any Chu transform from one normal Chu space to another is of the form (f, f^{-1}).

Since not all our constructions on Chu spaces preserve normality, we shall need a *normalization* operation on Chu spaces.

Definition 9. *Given a Chu space $S = \langle X, S, A \rangle$ define the map $E_S : A \to \mathsf{P}(X)$ by putting $E_S(a) := S^{-1}[a]$, that is:*

$$E_S(a) := \{x \in X \mid xSa\}.$$

Then the normalization *of S is given as the structure $\mathsf{N}(S) = \langle X, \in, E_S[A] \rangle$. Extending this definition to transforms, we define the normalization $\mathsf{N}(f, f')$ of a Chu transform $(f, f') : S' \to S$ as the pair (f, f^{-1}).*

Proposition 1. *The normalization construction* N *is a functor from* Chu *to* NChu.

Proof. We confine ourselves to checking that the normalization of a Chu transform is again a Chu transform. Suppose that $(f, f') : S' \to S$ is a Chu transform from $S' = \langle X', S', A' \rangle$ to $S = \langle X, S, A \rangle$. It is obvious that any elements $x' \in X'$

and $Y \in E_S[A]$ satisfy the adjointness condition (6) with respect to f and f^{-1}. The point is to prove that f^{-1} is a well-defined map from $E_S[A]$ to $E_{S'}[A']$. For this purpose, take an arbitrary element $E_S(a) = S^{-1}[a] \in E_S[A]$. Then

$$\begin{aligned} f^{-1}(S^{-1}[a]) &= \{x' \in X' \mid f(x')Sa\} \\ &= \{x' \in X' \mid x'S'f'(a)\} \\ &= (S')^{-1}[f'(a)] \\ &= E_{S'}(a), \end{aligned}$$

which shows that, indeed, $f^{-1}(S^{-1}[a])$ belongs to $E_{S'}[A']$. □

Strongly normal Chu spaces In the next section we will be interested in Chu spaces that satisfy a strong form of normality that we will describe now. Normal Chu spaces are extensional in that every attribute is completely determined by the set of objects that it is related to, but they do not necessarily satisfy the dual property of *separation*.

Definition 10. *A Chu space $S = \langle X, S, A\rangle$ is* separated *if for every distinct pair of points x and y in X there is an attribute $a \in A$ separating x from y, in the sense that it is either related to x and not to y, or related to y and not to x.*

The following, slightly technical definition will be of use in section 4, when we will understand the Vietoris construction as a special power lifting construction:

Definition 11. *Let $S = \langle X, S, A\rangle$ be a Chu space. A subset $Y \subseteq X$ is called a* representative subset of X*, if Y contains exactly one representant of every $\equiv_S$-cell of S (where $\equiv_S$ is as defined in Definition 7). For any such Y, the* strong normalization $\mathsf{N}^Y(S)$ *of S based on Y is the Chu space $\langle Y, \in, E^Y_S[A]\rangle$, where $E^Y_S : A \to \mathsf{P}(Y)$ is the map given by $E^Y_S(a) := \{y \in Y \mid ySa\}$.*

It is not difficult to prove that for any representative subset Y of X the Chu space $\mathsf{N}^Y(S)$ is strongly normal, and that the pair (ι_{YX}, E^Y_S) (with ι_{XY} the inclusion) is a Chu transform from $\mathsf{N}^Y(S)$ to S.

Remark 3. One of the referees pointed out that Proposition 1 can be expanded to state that NChu is a coreflective subcategory of Chu, and that separated Chu spaces form a reflective subcategory of Chu, cf. [2].

Lifting Chu spaces Many category-theoretic operations can be defined on Chu spaces, for instance orthogonality, tensor product, transposition (see [16] for an overview). Here our focus will be on lifting constructions; our aim is to define, for an arbitrary set functor F and for an arbitrary Chu space $S = \langle X, \in, A\rangle$, a Chu space $\widetilde{\mathsf{F}}(S)$ which is based on the set $\mathsf{F}(X)$. Although we are mainly interested in a lifting construction for normal Chu spaces, we take a little detour to first define a functorial power lifting construction on the full category Chu. For that purpose we need the notion of relation lifting for an arbitrary set functor. Recall that in Definition 1 we gave the power set lifting of a binary relation.

For the definition of relation lifting with respect to a general set functor F, consider a binary relation $Z \subseteq S \times S'$, with associated projections π, π':

$$S \xleftarrow{\pi} Z \xrightarrow{\pi'} S'$$

Applying F to this diagram we obtain

$$\mathsf{F}S \xleftarrow{\mathsf{F}\pi} \mathsf{F}Z \xrightarrow{\mathsf{F}\pi'} \mathsf{F}S'$$

so that by the properties of the product $\mathsf{F}S \times \mathsf{F}S'$, we may consider the product map $(\mathsf{F}\pi, \mathsf{F}\pi') : \mathsf{F}Z \to \mathsf{F}S \times \mathsf{F}S'$. This map need not be an inclusion (or even an injection), and $\mathsf{F}Z$ need not be a binary relation between $\mathsf{F}S$ and $\mathsf{F}S'$. However, we may consider the *range* $\overline{\mathsf{F}}(Z)$ of the map $(\mathsf{F}\pi, \mathsf{F}\pi')$ which is of the right shape.

Definition 12. *Let F be a set functor. Given two sets S and S', and a binary relation Z between $S \times S'$, we define the* lifted relation $\overline{\mathsf{F}}(Z) \subseteq \mathsf{F}S \times \mathsf{F}S'$ *as follows:*

$$\overline{\mathsf{F}}(Z) := \{((\mathsf{F}\pi)(\varphi), (\mathsf{F}\pi')(\varphi)) \mid \varphi \in \mathsf{F}Z\},$$

where $\pi : Z \to S$ and $\pi' : Z \to S'$ are the projection functions given by $\pi(s, s') = s$ and $\pi'(s, s') = s'$.

Definition 13. *Let F be a set functor, and let $\mathbb{S} = \langle X, S, A\rangle$ be a Chu space. Then we define the* F-lifting of $\mathbb{S}$ *to be the Chu space*

$$\widetilde{\mathsf{F}}\mathbb{S} := \langle \mathsf{F}(X), \overline{\mathsf{F}}(S), \mathsf{F}(A)\rangle.$$

Given a Chu transform (f, f') from $\mathbb{S}' = \langle X', S', A'\rangle$ to $\mathbb{S} = \langle X, S, A\rangle$, we define $\widetilde{\mathsf{F}}(f, f')$ as the pair $(\mathsf{F}f, \mathsf{F}f')$ of maps.

We need some of the properties of relation lifting. Given a function $f : A \to B$, we let $Gr(f)$ denote the graph of f, i.e., $Gr(f) := \{(a, b) \in A \times B \mid b = f(a)\}$.

Fact 2 *Let F be a set functor. Then the relation lifting $\overline{\mathsf{F}}$ satisfies the following properties, for all functions $f : S \to S'$, all relations $R, Q \subseteq S \times S'$, and all subsets $T \subseteq S$, $T' \subseteq S'$:*
(1) $\overline{\mathsf{F}}$ extends F: $\overline{\mathsf{F}}(Gr(f)) = Gr(\mathsf{F}f)$;
(2) $\overline{\mathsf{F}}$ preserves the diagonal: $\overline{\mathsf{F}}(\mathsf{Id}_S) = \mathsf{Id}_{\mathsf{F}S}$;
(3) $\overline{\mathsf{F}}$ commutes with relation converse: $\overline{\mathsf{F}}(R^{\smile}) = (\overline{\mathsf{F}}R)^{\smile}$;
(4) $\overline{\mathsf{F}}$ is monotone: if $R \subseteq Q$ then $\overline{\mathsf{F}}(R) \subseteq \overline{\mathsf{F}}(Q)$;
(5) $\overline{\mathsf{F}}$ distributes over composition: $\overline{\mathsf{F}}(R \circ Q) = \overline{\mathsf{F}}(R) \circ \overline{\mathsf{F}}(Q)$, if F preserves weak pullbacks.

For proofs we refer to [14, 3], and references therein. The proof that Fact 2(5) depends on the property of weak pullback preservation goes back to Trnková [17].

Theorem 3. *If F preserves weak pullbacks, then $\widetilde{\mathsf{F}}$ is an endofunctor on Chu.*

Proof. We restrict our proof to showing that $\widetilde{\mathsf{F}}$ turns Chu transforms into Chu transforms. Let $S' = \langle X', S', A' \rangle$ and $S = \langle X, S, A \rangle$ be two Chu spaces, and let $f : X' \to X$ and $f' : A \to A'$ be two maps. It is easily verified that (f, f') is a Chu transform iff

$$Gr(f) \circ S' = (Gr(f') \circ S)^{\smile}.$$

But then it follows from the properties of relation lifting for weak pullback preserving functors that

$$\begin{aligned} Gr(\mathsf{F}f)) \circ \overline{\mathsf{F}}(S') &= \overline{\mathsf{F}}(Gr(f) \circ S') \\ &= \overline{\mathsf{F}}((Gr(f') \circ S)^{\smile}) \\ &= (Gr(\mathsf{F}f') \circ \overline{\mathsf{F}}(S))^{\smile}. \end{aligned}$$

In other words, $(\mathsf{F}f, \mathsf{F}f')$ is a Chu transform as well. □

Unfortunately, normality of a Chu space is not preserved under taking liftings of Chu spaces. But clearly, we can combine lifting with normalization.

Definition 14. *Assume that* F *preserves weak pullbacks. Then* $\widehat{\mathsf{F}}$ *denotes the endofunctor on* NChu *defined by* $\widehat{\mathsf{F}} := \mathsf{N} \circ \widetilde{\mathsf{F}}$.

Remark 4. It will be useful in the next section to have a more concrete definition of the normalization operation for Chu spaces of the form $\widetilde{\mathsf{F}}S$, where S is normal. Suppose that $S = \langle X, \in, A \rangle$, then $\widetilde{\mathsf{F}}S = \langle \mathsf{F}(X), \overline{\in}, \mathsf{F}(A) \rangle$, where we write $\overline{\in}$ for the lifted membership relation $\overline{\mathsf{F}}(\in)$. Now the normalization map $E_{\overline{\in}}$ is given by

$$E_{\overline{\in}}[\alpha] = \{\varphi \in \mathsf{F}(X) \mid \varphi \overline{\in} \alpha\}.$$

4 Stone spaces

As a case study, let us show how the Vietoris construction on Stone spaces naturally arises in the framework that we have developed in the previous two sections.

As we mentioned earlier, any Stone space $\langle X, \tau \rangle$ can be represented as a Chu space $S = \langle X, \in, A \rangle$. The Boolean algebra A of the clopen subsets of X is a base for the topology τ, so for instance, for every $Y \subseteq X$, the τ-closure of Y is $Y^{\bullet} = \bigcap\{a \in A \mid Y \subseteq a\}$.

Definition 15. *Given a Stone space* $S = \langle X, \in, A \rangle$*, we let* $K(S)$ *denote the collection of closed sets of* S*. We define the operations* $\langle \ni \rangle, [\ni] : \mathsf{P}(X) \to \mathsf{P}(K(S))$ *by*

$$\begin{aligned} [\ni]a &:= \{F \in K(S) \mid F \subseteq a\}\,, \\ \langle \ni \rangle a &:= \{F \in K(S) \mid F \cap a \neq \varnothing\}\,. \end{aligned}$$

We let $V(A)$ *denote the Boolean subalgebra of* $\mathsf{P}(K(S))$ *generated by the set* $\{\langle \ni \rangle a, [\ni]a \mid a \in A\}$*.* $V(A)$ *is the Boolean algebra of clopen subsets of the* Vietoris topology *on* $K(S)$*.*

Modal logicians will recognize the above notation as indicating that $[\ni]$ and $\langle\ni\rangle$ are the 'box' and the 'diamond' associated with the converse membership relation $\ni\, \subseteq K(S) \times X$.

It is well-known that the Vietoris hyperspace of a Stone space $S = \langle X, \in, A\rangle$ is a Stone space, so $V(A)$ is a base for the Vietoris topology. Then $V(S) = \langle K(S), \in, V(A)\rangle$ is the Chu-representation of the Vietoris hyperspace of S.

For the remainder of this section fix a Stone space $S = \langle X, \in, A\rangle$. Here is a summary of our approach:

1. First, as a minor variation on Chu power set lifting, consider the Chu space $\widetilde{\mathsf{P}}_\omega(S) := \langle \mathsf{P}(X), \overline{\in}, \mathsf{P}_\omega(A)\rangle$ where $\overline{\in}$ denotes the relation lifting $\overline{\mathsf{P}}(\in)$, restricted to $\mathsf{P}(X) \times \mathsf{P}_\omega(A)$. Thus the variation consists in taking the *finite* power set $\mathsf{P}_\omega(A)$ rather than the full power set $\mathsf{P}(A)$.
2. We then show that every equivalence class of the relation $\equiv_{\overline{\in}}$ contains exactly one closed element, so that we may take the collection $K(S)$ as the 'canonical representants' in order to define a strong normalization $\widehat{\mathsf{P}}_\omega := \langle K(S), \in, Q\rangle$ of $\langle \mathsf{P}(X), \overline{\in}, \mathsf{P}_\omega(A)\rangle$ (see Definition 11).
3. We then prove that the Boolean algebra generated by Q is *identical* to the Vietoris algebra $V(A)$.
4. Finally we prove that the Vietoris algebra is isomorphic to the algebra A^{P} (defined in section 2 as the Boolean algebra generated by the set $\{\nabla\alpha \mid \alpha \in \mathsf{P}_\omega(A)\}$ modulo the ∇ axioms).

Definition 16. *Given a Stone space $S = \langle X, \in, A\rangle$, let $\overline{\in}$ denote the relation lifting $\overline{\mathsf{P}}(\in)$, restricted to $\mathsf{P}(X) \times \mathsf{P}_\omega(A)$. Define the Chu space $\widetilde{\mathsf{P}}_\omega(S)$ as the structure $\langle \mathsf{P}(X), \overline{\in}, \mathsf{P}_\omega(A)\rangle$.*

For the following proposition, recall that the *closure* of a set $Y \subseteq X$ is denoted by $Y^\bullet$, and that for any Chu space $T = \langle P, T, B\rangle$, the specialization order $\sqsubseteq_T$ on X induced by T is given by $p \sqsubseteq_T q$ iff $(pTb \Rightarrow qTb)$ for all $b \in B$.

Proposition 2. *Let $\sqsubseteq$ and $\equiv$ be the specialization order and the equivalence relation associated with the Chu space $\widetilde{\mathsf{P}}_\omega(S)$, respectively. Then, for every set $Y \in \mathsf{P}(X)$, its closure $Y^\bullet$ is the maximum element of the $\equiv$-cell $Y/\!\equiv$. In particular, $Y^\bullet$ is the unique closed set in $Y/\!\equiv$.*

Proof. Clearly it suffices to prove that

$$Y \sqsubseteq Z \Rightarrow Y^\bullet \subseteq Z^\bullet \tag{7}$$

and

$$Y \equiv Y^\bullet. \tag{8}$$

For (7), suppose that $Y^\bullet \not\subseteq Z^\bullet = \bigcap\{a \in A \mid Z \subseteq a\}$. Then, since every clopen is closed and $Y^\bullet$ is the smallest closed set that contains Y, there must be some $a \in A$ such that $Z \subseteq a$ and $Y \not\subseteq a$. Let $\alpha = \{\neg a, X\} \in \mathsf{P}_\omega(A)$: it holds that $Y \overline{\in} \alpha$ but $Z \overline{\notin} \alpha$, hence $Y \not\sqsubseteq Z$.

For (8), if $\alpha \in \mathsf{P}_\omega(A)$ and $Y \overline{\in} \alpha$, then $a \cap Y \neq \varnothing$ for every $a \in \alpha$ and $Y \subseteq \bigcup \alpha$. Then, as $Y \subseteq Y^\bullet$, we get $a \cap Y^\bullet \neq \varnothing$ for every $a \in \alpha$. Also, as $\bigcup \alpha \in A$ is in particular closed, from $Y \subseteq \bigcup \alpha$ we get $Y^\bullet \subseteq \bigcup \alpha$, which proves that $Y^\bullet \overline{\in} \alpha$. This shows that $Y \sqsubseteq Y^\bullet$. Conversely, if $Y^\bullet \overline{\in} \alpha$ then $Y \subseteq Y^\bullet \subseteq \bigcup \alpha$. In addition, for every $a \in A$, $Y^\bullet \cap a \neq \varnothing$ implies $Y \cap a \neq \varnothing$, for if not, then $Y \subseteq \neg a \in K(S)$, which would imply $Y^\bullet \subseteq \neg a$, contradiction. □

The proposition above says that $K(S)$ is a representative subset of $\mathsf{P}(X)$ (see Definition 11). So we can consider the strong normalization of $\widetilde{\mathsf{P}}_\omega(S)$:

Definition 17. *Given a Stone space $S = \langle X, \in, A \rangle$, define $\widehat{\mathsf{P}}_\omega(S)$ as the strong normalization of $\widetilde{\mathsf{P}}_\omega(S)$ w.r.t. $K(S)$, i.e. $\widehat{\mathsf{P}}_\omega(S)$ is the normal and separated Chu space*

$$\langle K(S), \in, Q \rangle,$$

where $Q = E[\mathsf{P}_\omega(A)]$ and $E : \mathsf{P}_\omega(A) \to \mathsf{P}(K(S))$ is the map given by $E(\alpha) := \{F \in K(S) \mid F \overline{\in} \alpha\}$.

The following theorem states that the Vietoris construction of a Boolean space can indeed be seen as an instance of power lifting.

Theorem 4. *Let $S = \langle X, \in, A \rangle$ be a Stone space. Then $V(A)$ is the Boolean algebra generated by the set $Q \in \mathsf{P}(K(S))$, where Q is the set of attributes of $\widehat{\mathsf{P}}_\omega(S)$.*

Proof. For every $F \in K(S)$ and every $\alpha \in \mathsf{P}_\omega(A)$,

$$F \in E(\alpha) \text{ iff } F \overline{\in} \alpha \text{ iff } F \in [\ni](\bigcup \alpha) \cap \bigcap_{a \in \alpha} \langle \ni \rangle a,$$

which means that E is the nabla operator defined from $[\ni]$ and $\langle \ni \rangle$. Hence $Q \subseteq V(A)$, and moreover for every $a \in A$, $[\ni]a = E(\{a\}) \cup E(\varnothing)$ and $\langle \ni \rangle a = E(\{a, \top\})$, which makes Q a set of generators for $V(A)$. □

To see where does the material of the second section come in: The ∇-axioms are an important ingredient for the following representation theorem for the lifted Boolean algebra A^{P}:

Theorem 5. *Let $S = \langle X, \in, A \rangle$ be a Stone space. Then $V(A)$ is isomorphic to the power lifting A^{P} of A.*

Proof. It is not difficult to see, given the axioms $\nabla 1$–$\nabla 7$, that an arbitrary element of A^{P} can be represented as a finite *join* of generators. Now define the map $\rho : A^{\mathsf{P}} \to \mathsf{P}(K(S))$ by putting

$$\rho(\nabla \alpha_1 \vee \ldots \vee \nabla \alpha_n) := E(\alpha_1) \cup \ldots \cup E(\alpha_n),$$

where E is the strong normalization map of $\widehat{\mathsf{P}}_\omega(S)$, given by

$$E(\alpha) = \{F \in K(S) \mid F \overline{\in} \alpha\}.$$

In order to establish the theorem, it suffices to prove our claim that

$$\rho \text{ is an isomorphism.}$$

We omit the argument why ρ is a homomorphism and only sketch the proof that it is an injection. Given a homomorphism between two Boolean algebras, in order to prove injectivity, it suffices to show that the homomorphism maps nonzero elements to nonzero elements. So let $\nabla\alpha_1 \vee \ldots \vee \nabla\alpha_n$ be an arbitrary nonzero element of A^{P}, then at least one of the $\nabla\alpha_i$, say $\nabla\alpha$, is nonzero. Then it follows from axiom $\nabla 2$ that $\bot \notin \alpha$.

Now consider the set $Y = \bigcup\alpha$. Y is a finite union of clopens and hence, certainly closed. Since $\bot \notin \alpha$, it is also straightforward to verify that $Y \overline{\in} \alpha$. In other words, we have found that $Y \in E(\alpha) = \rho(\nabla\alpha) \subseteq \rho(\nabla\alpha_1 \vee \ldots \vee \nabla\alpha_n)$. Hence we have proved indeed that ρ maps an arbitrary nonzero element of A^{P} to a nonempty set of closed elements, i.e., a nonzero element of the algebra $V(A)$. □

The point of carrying out the Vietoris construction in terms of the nabla operator rather than the standard modalities $[\ni]$ and $\langle\ni\rangle$ is that the former is coalgebraic in nature, and the latter are not. This will be advantageous when it comes to generalizing the Vietoris construction to other functors (and other categories).

5 Conclusions

We presented an algebraic study of the coalgebraic modal operator ∇, and we related this to a presentation of the Vietoris power construction on Stone spaces. We believe the main contribution of the paper to be threefold. First, on the algebraic side, we gave an axiomatization for ∇ that characterizes the class of ∇-algebras that is category-theoretically isomorphic (see Theorem 1) to the (positive) modal algebras. Second, using the concept of relation lifting, we showed how an arbitrary set functor F naturally gives rise to various lifting constructions on the category Chu of two-valued Chu spaces. These constructions are functorial in case F preserves weak pullbacks (Theorem 3). And finally, we showed how to realize the Vietoris construction on Stone spaces as a special instance of this Chu construction for the (finite) power set functor (Theorem 4), and linked this approach to the axiomatization of the modal ∇-algebras (Theorem 5).

In the future we hope to expand the work presented here in various directions. Because of space limitations we have to be brief.

1. First of all, there is no strong reason to confine ourselves to a finitary setting. The first natural generalization of this work is to move to a 'localic' setting and study the case of logical languages with infinitary conjunctions and/or disjunctions, and an infinitary version of the nabla operator.
2. In such a generalized setting, it would make sense to look at power constructions for other topologies than just Stone spaces, and to formalize these constructions not in terms of clopens but in terms of closed or open sets.

3. We think it is very interesting to try and generalize the results in this paper to other functors than P. This is the reason why we have taken care to formulate all our results as generally as possible, see for instance our remark following the proof of Theorem 5.
4. We already mentioned in the introduction that our first motivation was to pave the way for a ‘coalgebraic proof theory’, by which we mean to try and give an algebraic and syntactic account of nabla operators associated with weak pullback-preserving endofunctors. As a first step in this direction, we are currently working on a Gentzen-style derivation system for the modal nabla operator.

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