A Crash Course in Arrow Logic

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to appear in : M. Marx, L. Pólós & M. Masuch (eds.),
Arrow Logic

Overview This contribution gives a short introduction to arrow logic. We start by explaining the basic idea underlying arrow logic and the motivation for studying it (sections 1 and 2). We discuss some elementary duality theory between arrow logic and the algebraic theory of binary relations (section 3). In the sections 4 and 5 we give a brief survey of the theory that has been developed on the semantics (definability), axiomatics and decidability of various systems of arrow logic. We briefly describe some closely related formalisms and some extensions and reducts of arrow logic in section 6. We end with mentioning some promising research lines and open problems, in section 8.

Acknowledgements I would like to thank the following people for reading and commenting on earlier versions of the manuscript: Hajnal Andréka, Patrick Blackburn, Jan van Eijck, Maarten Marx, Istvan Németi, Szabolcs Mikulás, Ildikó Sain and Jerry Seligman. Special thanks are due to Johan van Benthem.

1 The basic idea

Summarizing in six words what arrow logic is about, one could say that

arrow logic is the basic modal logic of arrows,

a slogan calling for a few more words to discuss its key words: 'basic', 'modal' and 'arrow'.

To start with the latter, the language of arrow logic is designed to talk about all such objects as may be represented in a picture by arrows. As a concrete representation of an arrow the mathematically inclined reader might think of a vector, a function or a morphism in some category; the computer scientist of a program; the linguist of the dynamic meaning of a grammatically well-formed piece of text or discourse; the philosopher of some agent's (cognitive) action; etc. Note that in the last three examples (which will be discussed in some more detail later on) arrows are *transitions* related to some space of (information) states.

The essential characteristic of arrow logic is that the entities at which the truth of a formula is evaluated are arrows, i.e. that the basic statement made in arrow logic is of the form

$$\mathfrak{M}, a \models \phi,$$

Here $\mathfrak{M}$ an arrow model, i.e. a structure of which the universe consists of arrows and arrows only, and a is an *arrow*. This is not to say that arrows need always be the primitive entities of the model. For instance, in the semantics of arrow logic an important rôle is played by two-dimensional models. Here an arrow a is seen as a *pair* (a_0, a_1) of which a_0 may be thought of as the starting point of a and a_1 as the endpoint of a.

Having defined the intended models of arrow logic as consisting of objects that are graphically representable as arrows, we should say something about the *structure* imposed on these arrows. Let us consider the question what the *basic* relations between arrows are. The obvious first candidate is *composition*: vector spaces have an additive structure, functions can be composed, language fragments can be concatenated, etc. Therefore, the central relation of arrows will be a ternary *composition relation* C, $Cabc$ denoting the fact that a can be seen as an outcome of the composition of b and c, or conversely, that a can be decomposed into b and c. Note that in many concrete examples, C is actually a (partial) function; for instance, in the two-dimensional framework we have

$$Cabc \text{ iff } a_0 = b_0,\ a_1 = c_1 \ \&\ b_1 = c_0. \tag{1}$$

Second, in all the examples listed, the composition function has a neutral element; think for instance of the identity function or the `SKIP`-program. So, arrow models will contain degenerate arrows, transitions that do not lead to a different state. Formally, there will be a designated subset I of *identity arrows*; in the pair-representation, I will be (a subset of) the diagonal:

$$Ia \text{ iff } a_0 = a_1. \tag{2}$$

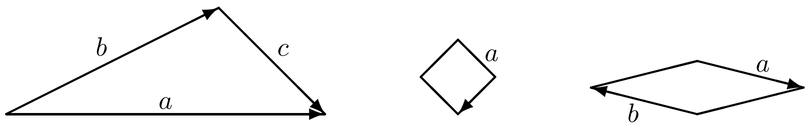
Slightly more debatable is the presence among the basic arrow relations of the third candidate: converse. For instance, in linguistic applications it seems to be difficult to imagine what the meaning of a reversed arrow could be. On the other hand, in some theories of cognitive science, reverse arrows appear with the clear meaning of 'undo-ing' an action; and in many fields of mathematics, arrow-like objects have converses (vectors) or inverses (bijective functions). Therefore, a generous viewpoint prevails: an arrow structure will have a binary *reverse relation* R. Again, in many cases this relation will be a function; for instance in the two-dimensional picture, the function f given by

$$fa = (a_1, a_0) \qquad \text{(if } fa \text{ is defined).} \tag{3}$$

Now there is a question as to whether this triple of arrow relations really forms the basic set. A natural further candidate would be some manifestation of *iteration*, if only because of its fundamental importance in computer science. However, it is less obvious how to give a mathematically simple and transparent definition of a primitive 'iteration relation' between arrows than it is for the other relations; besides that, the notion of iteration can be captured nicely without extending the signature of arrow frames (cf. section 7). As there are no other plausible candidates[1], we can now give a formal definition of an arrow frame:

Definition 1 *An arrow frame is a quadruple* $\mathfrak{F} = (W, C, R, I)$ *such that* $C \subseteq W \times W \times W$, $R \subseteq W \times W$ *and* $I \subseteq W$.

A nice aspect of arrow logic is that one can draw quite perspicuous pictures, clarifying the meaning of the relations:



$Cabc$ $\qquad$ Ia $\qquad$ Rab

Now that we have explained two of the three key words of our slogan, we should discuss the *modal* nature of arrow logic. Let us first consider briefly what we understand with the notion 'modal logic'. The last decade has seen a development in modal logic towards a more abstract and technical approach. In this perspective of what we will call *abstract modal logic*, arbitrary relational structures can be seen as models for an (extended) modal language: any relation is a potential accessibility relation of some suitably defined modal operator. The essentially *modal* aspects of the framework include the following. First, in contrast to what happens in the semantics of first order logic, the quantification over the model is restricted to an accessible part of the structure. And second, modal logicians focus on both the structure of the model itself (like in first order logic) and the power set algebra of the structure (like in algebraic logic).

What this boils down to for arrow logic is that we define a modal operator-language such that the truth of its formulas can be evaluated at arrows. If one wants to use traditional terminology from modal logic, this means that the transitions are *not* links between the possible worlds of the model; we treat the transitions *themselves* as the possible worlds. The three modal operators of the

[1]This is not entirely true: for instance, one could study relations like R_{ll}: $R_{ll}ab$ holds between two arrows a and b iff a and b start at the same point (cf. the discussion on Bulgarian-style arrow logic in section 7). However, in many classes of arrow frames, these relations can be *defined* using our 'basic' signature, for instance $R_{ll}ab \equiv \exists x\ Cabx$.

language are taken such that the composition, converse and identity relation are their accessibility relations. For instance, the language has a binary operator $\circ$ for C; intuitively speaking, the truth definition of $\circ$ states that $\phi \circ \psi$ is true at an arrow iff it can be decomposed into two arrows at which ϕ and ψ hold, respectively.

Now we are ready to give a formal definition of the syntax and semantics of arrow logic:

Definition 2 *The alphabet of arrow logic consists of an infinite set of propositional variables, the boolean connectives $\neg$ and $\vee$, and the modal operators $\circ$ (composition), $\otimes$ (converse) and δ (identity).*[2] *Its set of formulas is defined as follows:*

(i) *every propositional variable is a formula, and so is δ,*
(ii) *if ϕ and ψ are formulas, then so are $\neg\phi$, $\phi \vee \psi$, $\phi \circ \psi$ and $\otimes\psi$,*
(iii) *and nothing else is a formula.*

We will freely use the standard boolean abbreviations $\wedge$, $\rightarrow$ and $\leftrightarrow$, and also $\underline{\circ}$ and $\underline{\otimes}$ as duals of $\circ$ and $\otimes$, i.e. $\phi \,\underline{\circ}\, \psi = \neg(\neg\phi \circ \neg\psi)$ and $\underline{\otimes}\phi = \neg\otimes\neg\phi$.

Definition 3 *An arrow model is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ such that $\mathfrak{F} = (W, C, R, I)$ is an arrow frame and V is a valuation, i.e. a function mapping propositional variables to subsets of W. Truth of a formula ϕ at an arrow a of $\mathfrak{M}$, notation: $\mathfrak{M}, a \Vdash \phi$, is inductively defined as follows:*

$$\begin{array}{lll}
\mathfrak{M}, a \Vdash p & \textit{if} & a \in V(p) \\
\mathfrak{M}, a \Vdash \delta & \textit{if} & Ia \\
\mathfrak{M}, a \Vdash \neg\phi & \textit{if} & \mathfrak{M}, a \nVdash \phi \\
\mathfrak{M}, a \Vdash \phi \vee \psi & \textit{if} & \mathfrak{M}, a \Vdash \phi \textit{ or } \mathfrak{M}, a \Vdash \psi \\
\mathfrak{M}, a \Vdash \phi \circ \psi & \textit{if} & \textit{there are } b, c \textit{ with } Cabc,\ \mathfrak{M}, b \Vdash \phi \textit{ and } \mathfrak{M}, c \Vdash \psi \\
\mathfrak{M}, a \Vdash \otimes\phi & \textit{if} & \textit{there is a } b \textit{ with } Rab \textit{ and } \mathfrak{M}, b \Vdash \phi.
\end{array}$$

A formula ϕ is valid in a class of frames K, notation $\mathsf{K} \models \phi$, if for every frame $\mathfrak{F}$ in K, every valuation V on $\mathfrak{F}$ and every arrow a in $\mathfrak{F}$, we have $\mathfrak{F}, V, a \Vdash \phi$. A formula ϕ is valid in a frame $\mathfrak{F}$ if it is valid in $\{\mathfrak{F}\}$.

The reader will have noticed that while we fixed the syntax of arrow logic, we are more liberal concerning its semantics. For instance, any *group* $\mathfrak{G} = (G, \cdot, (\cdot)^{-1}, e)$ counts as an arrow frame, if we put

$$\begin{array}{lll}
C & = & \{(x, y, z) \mid x = y \cdot z\}, \\
R & = & \{(x, y) \mid y = x^{-1}\}, \\
I & = & \{e\}.
\end{array} \tag{4}$$

We agree that such examples may stretch one's intuitions concerning 'arrows' to the limit[3]. Nevertheless, the generality of our semantics reflects the opinion that there is no such thing as *the* proper semantics for arrow logic: different applications invoke different kinds of model. It is the task of the logician to clarify the choices that are to be made, to investigate the relations between the various options and to search for logical patterns in the landscape.

[2]There are other notations in the literature as well, e.g. $\bullet$ or ; for $\circ$, $\phi^{\smile}$ for $\otimes\phi$ and *id* for δ.

[3]Note that groups exist of which the elements do have a natural representation as arrows, consider for instance the bijective functions on a given set.

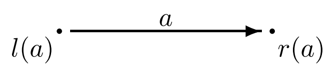
This is not to say that discussions concerning the ontology of arrows are not interesting or important. Consider for instance the problem whether arrows should be primitive entities or constructed from more basic material (for instance, as pairs of states). These matters touch upon the philosophical question whether dynamic or static aspects of knowledge and action should have primacy. Technically speaking however, the two-dimensional frames form just another special subclass of the class of arrow frames.

As a last remark, let us mention some other dimensions along which the semantics of arrow logic may be varied. As an example, one might consider arrow models in which the valuations are subjected to certain restrictions. For instance, in the two-dimensional case one might think of the so-called flat models, in which the truth of atomic propositions is not dependent on both coordinates, but perhaps only on the first one. Or, having some informational interpretation in mind, we may need a many-valued semantics to formalize reasoning with incomplete information. As these options have not yet been explored, we will not go into detail here.

2 Some intuitive examples

Let us now consider those examples of arrow frames that have been studied most intensively in the literature: squares, graphs (or relativized squares) and multigraphs.

To start with the latter, a *multigraph* is defined as a quadruple (E, P, l, r) of which E is a set of directed edges, P is a set of points and l and r are total functions mapping edges to points Intuitively we understand an edge a as an arrow leading from its left point $l(a)$ to its right point $r(a)$:

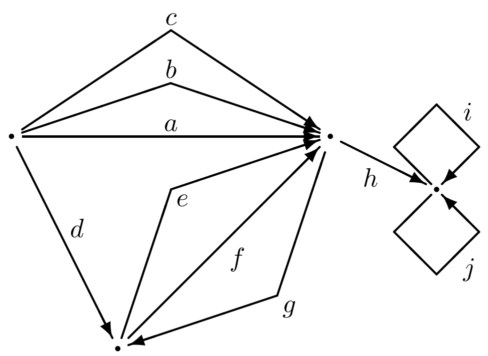


In a most natural way, we can define the following relations on the set E of edges of a multigraph:

$$\begin{array}{lll} Cabc & \text{iff} & a \text{ leads from } l(b) \text{ to } r(c) \text{ and } r(b) = l(c), \\ Rab & \text{iff} & a \text{ leads from } r(b) \text{ to } l(b), \\ Ia & \text{iff} & l(a) = r(a). \end{array}$$

thus creating a multigraph arrow frame with universe E[4] Now an arrow *model* arises if we *label* the edges of the multigraph with sets of propositions, consider the example below:

[4] Although we will not treat *category* frames in detail, it is easy to imagine how one may change the definition of a multigraph frame into that of a category frame. We may see a (small) category as a multigraph endowed with a binary *composition function* on arrows; this composition function will give the ternary accessibility relation C of the arrow frame. Note that we cannot define a reverse relation R on category frames, hence we can only interpret the $\otimes$-free arrow formulas in category frames. This is different for *allegories* (cf. Freyd & Scedrov [21]), where arrows do have reverses; the reader is invited to investigate the connections between arrow logic and allegories.



If we label the edges of this multigraph by the following valuation:

$$\begin{array}{rcl} V(p) & = & \{a, b, e, i\} \\ V(q) & = & \{i, g\} \\ V(r) & = & \{h\}, \end{array}$$

we obtain that

$$\begin{array}{l} d \Vdash p \circ q \\ g \Vdash \otimes p \wedge \otimes \neg p \\ h \Vdash r \circ (\delta \wedge q) \\ \text{etc.} \end{array}$$

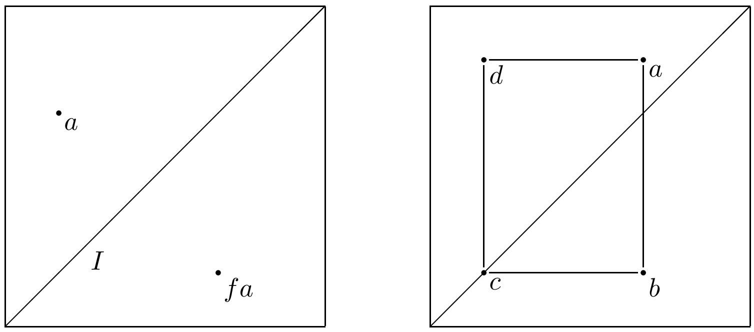
Note that in a multigraph frame, there may be *various* arrows leading from one point to another, as in the picture above. This implies that neither C nor R needs to be functional in a multigraph frame.

This is different in *graph* frames: a graph is a multigraph with *at most one* edge between each pair of points. So we arrive at the two-dimensional semantics mentioned earlier on, as we may identify an arrow a with the *pair* $(l(a), r(a))$. From this perspective, a graph frame is a structure $\mathfrak{F} = (E, C, R, I)$ such that E is a subset of a cartesian square $P \times P$; C and I are defined as in (1) resp. (2); and R is the function f satisfying (3). This viewpoint also explains the other names in use for graph frames: *pair* frames, *two-dimensional* frames and *relativized squares* (in this paper, we will mainly use the latter name).

Note that in a graph model, the modal clauses of the truth definition boil down to the following:

$$\begin{array}{lll} \mathfrak{M}, (x, y) \Vdash \phi \circ \psi & \text{iff} & \text{there is a } z \text{ such that } \mathfrak{M}, (x, z) \Vdash \phi \text{ and } \mathfrak{M}, (z, y) \Vdash \psi, \\ \mathfrak{M}, (x, y) \Vdash \otimes \phi & \text{iff} & \mathfrak{M}, (y, x) \Vdash \phi, \\ \mathfrak{M}, (x, y) \Vdash \delta & \text{iff} & x = y. \end{array}$$

Relativized squares can be depicted as graphs, but we can also draw them in a way that reflects their two-dimensional aspects more clearly:



The identity relation I consists of the *diagonal* elements of the universe; f is the operation of *mirroring* in the diagonal. C consists of all triples (a, b, d) with the property that we can draw a *rectangle abcd* such that: b lies on the vertical line through a; d lies on the horizontal line through a; and c (which does not have to belong to the universe of the model) lies on the diagonal.

We can narrow down the class of graphs further by demanding that the universe E of 'available' arrows satisfies certain properties, like *reflexivity* (if (x, y) is available, then so are its starting arrow (x, x) and its ending arrow (y, y)) or *symmetry* (if (x, y) is available, then so is its converse (y, x)). A particularly interesting condition is that *all* pairs of points are present as arrows. In the setting of multigraphs, this means that there is *precisely one* arrow running between every pair of points. With this constraint we arrive at the class of *square* frames or *full* graphs having a full cartesian square as their universe.

Definition 4 *An arrow frame is called a two-dimensional frame, a pair frame, a relativized square or a graph frame, if for some base set U, W is a subset of $U \times U$, C and I are given as in (1) and (2) and R is a partial function f satisfying (3). If W is a reflexive and symmetric binary relation, we call the arrow frame locally square; if W is equal to the full cartesian product over U, i.e. $W = U \times U$, the frame is called a square. The classes of squares, relativized squares and locally square arrow frames are denoted by SQ, SQ^{r} and $\mathsf{SQ}^{\mathsf{rs}}$, respectively.*

3 The motivation

As we have already hinted in the introduction, arrow logic is a widely applicable system: many different notions from various disciplines like mathematics, computer science, linguistics and cognitive science can be formalized in it. Here we will briefly discuss the possible applications of arrow logic in some more detail, first focusing on the notion of *dynamic semantics*, which unifies insights from computer science, linguistics and philosophy, and then describing the position of arrow logic with respect to other parts of (mathematical) logic.

So let us start with a brief explanation of the dynamic perspective on semantics; for more detailed information we refer the reader to van Benthem [6]. Traditionally, the central notion of semantics in logic is a static one: a proposition is interpreted as a declarative statement over a model. In recent years however, alternative viewpoints have been proposed in which not only the information content, but also the use of this information forms an object of study. In particular, the basic idea underlying dynamic semantics is that part of the meaning of a proposition is formed by its potential to *change* the information that a given agent has about the model. Let us give a few examples.

Consider for instance the meaning of anaphora in natural language. Comparing the texts *a man walks in the park; he whistles* with *he whistles; a man walks in the park*, the reader will notice that a purely truth-conditional approach to semantics cannot explain the difference in meaning. Clearly, the meaning of the pronoun *he* has an active component, in that it passes information concerning the anaphoric linkage. In Groenendijk & Stokhof [27], the system of Dynamic Predicate Logic is introduced, with the intention of formalizing this dynamic view on the semantics of anaphora; an essential characteristic of this formalism is that the formulas of ordinary first order logic are interpreted as sets of transitions over a state space. To be more precise: the objects at which the evaluation of formulas takes place, are *pairs* consisting of an input- and an output-state.

Related to the anaphora problem in natural language is the philosophical notion of presupposition; the intuitive idea to see a presupposition as a condition to be fulfilled in order to enable the agent to process some piece of information, turns presupposition into an undeniably dynamic notion (cf. van Eijck [20], Visser [74]).

Similar proposals have been made in the literature on Categorial Grammar, cf. van Benthem [9]. The paradigm that parsing a sentence is performing a logical deduction, has been combined with the observation that these processes require an activity from the agent. A surprising insight resulting from this connection is that Lambek's Syntactic Calculus, in fact the main Categorial Grammar in use, allows a dynamic, procedural interpretation, and as such can be taken as a generalized reduct of arrow logic (we will give more details in section 7).

These ideas have also been investigated in the wider context of cognitive science, cf. for instance Gärdenfors [25] or Veltman [68]. Here the old philosophical idea of cognition as activity is molded into a formal framework, in which information processing is modeled by epistemic operations for changing information states, e.g. by updating or revising them.

The central idea emerging from the literature is that propositions are interpreted as sets of *transitions*. Because of this, dynamic semantics forms a bridge from formalisms developed in linguistics and philosophy towards approaches followed in computer science, such as process algebra (cf. Milner [48], Bergstra & Klop [14]) or dynamic logic (cf. Harel [28]). Note that in the denotational semantics of programming languages, the standard meaning of a program α is a set of input-output pairs.

It seems as if a research field of modal transition logics is arising, i.e. modal formalisms designed to talk about transition structures, and as we saw in the previous section, arrow logic occupies a central position in this landscape.

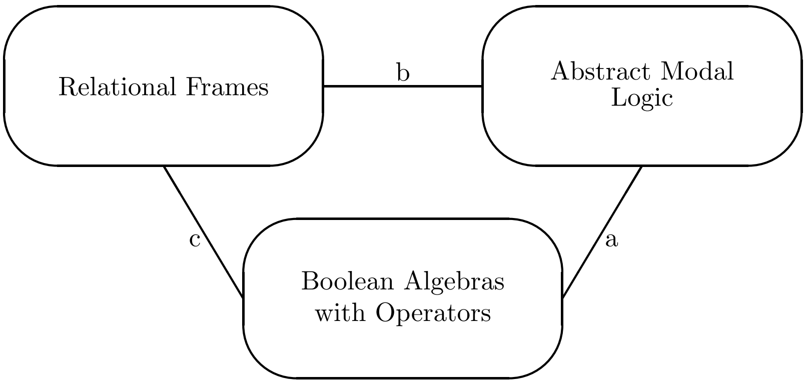
Finally, within the field of mathematics and logic arrow logic is of interest because of its intimate and important ties with algebraic logic and with the predicate calculus.

The program of algebraic logic (cf. Németi [53] for an overview) is to study logical formalisms with algebraic tools. A familiar example is the use of Boolean Algebras to study classical propositional logic. Several kinds of algebras of relations have been studied in the literature of algebraic logic, one of the most important ones is formed by the Relation Algebras defined in Tarski [64]. One of the motivations for developing arrow logics was to investigate the theory of Relation Algebras from an abstract modal point of view. In section 4 we will see how Relation Algebras emerge as the *modal* or *complex* algebras of arrow logic.

With respect to the predicate calculus, arrow logic is interesting from the viewpoint of correspondence theory (cf. van Benthem [7]), the field of modal logic in which the expressive power of modal logic is compared to that of first order logic and other formalisms. One of the motivations for breaking out of the traditional modal framework was to increase the expressive power of modal languages. An interesting aspect of arrow logic is that over the class of square models, arrow logic is equally expressive as the three-variable fragment of the predicate calculus: everything that can be said about binary relations using three variables (and first order quantifiers) only, is expressible in arrow logic as well, and vice versa. For more details we refer to Venema [70]. yde4.tex

4 The algebras

As we already mentioned in the previous section, arrow logic has intimate connections with algebraic logic; the insights and tools obtained in the literature on Relation Algebras have played an important rôle in the development of arrow logic. Therefore, we feel it might be useful to provide the reader with a brief introduction to the algebraic theory of binary relations and its connections with arrow logic. As a starting point, consider the following picture, due to C. Brink (cf. [16]):



It has manifestations for a manifold of logical formalisms; for instance, in the case of classical propositional logic (which we may see as a degenerate kind of modal logic) the relational frames are just unstructured sets, and the algebras are boolean algebras without extra operators.

Let us now mention some key words to explain the connections in the picture. The relation (a) between abstract modal logic and boolean algebras with operators[5] (BAOs) is very tight; for instance, BAOs appear as the Lindenbaum-Tarski algebras of extended modal logics. As we saw in the introductory section, (b) relational frames are the central structures in the semantics of modal logic, and this is so since the work of Kripke. Note however that, years before the terminology of

[5]A boolean algebra with operators is a structure $\mathfrak{A} = (A, +, -, f_i)_{i \in I}$ where $(A, +, -)$ is a boolean algebra and every operator f_i is *normal* ($f_i(a_0, \ldots, 0, \ldots, a_{n-1}) = 0$) and *additive* ($f_i(a_0, \ldots, a_i + a'_i, \ldots, a_{n-1}) = f_i(a_0, \ldots, a_i, \ldots, a_{n-1}) + f_i(a_0, \ldots, a'_i, \ldots, a_{n-1})$).

'possible worlds' connected by 'accessibility relations' was introduced, Jónsson and Tarski (cf. [32]) had already investigated the relation (c) between BAOs and relational frames — in particular, they had already studied *arrow frames*. For a recent overview of the *duality theory* which was started by the Jónsson-Tarski paper we refer to GOLDBLATT [26]. From a mathematical perspective, the development of arrow logic can be seen as *filling in* the modal part of the picture above, where the structural and algebraic sides already existed.

So it seems to be time to say something about the algebras involved. The basic idea is that one starts with algebras consisting of a set of binary relations, together with some natural operations on them. The next step is to abstract away from this class of concrete algebras to the class of all boolean algebras with operators of the appropriate similarity type. One aim of the algebraic theory is then to find necessary and sufficient conditions for the representability of an abstract algebra as a concrete algebra of relations.

Definition 5 *Let U be a set; consider the following operations on $\mathcal{P}(U \times U)$:*

$$\begin{array}{lcl} R \mid S & = & \{(u,v) \in U \times U \mid \exists w((u,w) \in R \;\&\; (w,v) \in S)\} \\ R^{-1} & = & \{(u,v) \in U \times U \mid (v,u) \in R\} \\ Id & = & \{(u,v) \in U \times U \mid u = v\}. \end{array}$$

Any algebra of the form $\mathfrak{A} = (\mathcal{P}(U \times U), \cup, -, |, (\cdot)^{-1}, Id)$ is called a full relational set algebra; the class of such algebras is denoted by FRA*. The class* RRA *of representable relation algebras is defined as the class* **ISP**FRA*, i.e., isomorphic copies of subalgebras of direct products of full relational set algebras.*

A relational type algebra is a boolean algebra augmented with the following operators: a binary ;, a unary $\breve{}$ and a constant 1'.

A full relational set algebra contains *all* binary relation on a given set U. RRA consists of all 'real' relation algebras: algebras in which the elements can be seen as binary relations, and the operations behave in the expected manner. In order to enumerate the equational theory of the class RRA, which happens to be a variety, Tarski proposed the following:

Definition 6 *A relation algebra is a relation type algebra $\mathfrak{A} = (A, +, -, ;, \breve{}, 1')$ satisfying the following axioms:*

($RA0$) *Axioms stating that $(A, +, -)$ is a Boolean Algebra*
($RA1$) $(x + y);z = x;z + y;z$
($RA2$) $(x + y)\breve{} = x\breve{} + y\breve{}$
($RA4$) $(x;y);z = x;(y;z)$
($RA5$) $x;1' = x$
($RA6$) $(x\breve{})\breve{} = x$
($RA7$) $(x;y)\breve{} = y\breve{};x\breve{}$
($RA8$) $x\breve{};-(x;y) \leq -y$.

The class of relation algebras is denoted by RA.

For an introduction to the theory of relation algebras we refer to Jónsson [30, 31] or MADDUX [41]. It soon turned out that the RA-axioms do not exhaustively generate all valid principles governing binary relations, in other words: RA is only an *approximation* of RRA. In fact, RRA is not axiomatizable by a finite (purely equational) axiomatization, as was shown by Monk (cf. [50]). This negative result has been considerably strengthened by various authors, cf. ANDRÉKA [1] for the most recent version. Infinite axiomatizations of RRA are known, cf. for instance LYNDON [38].

One way to overcome the negative results is to allow derivation systems that are not in equational form, cf. WADGE [75], MADDUX [40] or ORŁOWSKA [55] where Gentzen-type axiomatizations are given in which variables referring to elements of the base set U occur, or VENEMA [70]. Another possibility is to widen the class of representable relation algebras, for instance by starting with power set algebras where the top set is not necessarily a *full* cartesian square, but a *subset* of it, and to *relativize* the operations to this subset:

Definition 7 *Let W be a subset of the cartesian square $U \times U$, for some set U. The operations $-^W$, $\cup^W$, $|^W$, $(\cdot)^{-1:W}$ and Id^W are defined as the relativizations to W of $-$, $\cup$, $|$, $(\cdot)^{-1}$ and Id respectively, i.e.,*

$$\begin{array}{rcl} -^W R & = & -R \cap W \\ R \,|^W\, S & = & (R \mid S) \cap W \end{array}$$

Any algebra of the form $\mathfrak{A} = (\mathcal{P}(W), -^W, \cup^W, |^W, (\cdot)^{-1:W}, Id^W)$ with $\mathcal{P}(W)$ and the operations as defined above, is called a relativized relational power set algebra; the class of such algebras is denoted by $\mathsf{FRA}^{\mathsf{r}}$*;* $\mathsf{FRA}^{\mathsf{rs}}$ *is the subclass of* $\mathsf{FRA}^{\mathsf{r}}$ *in which the algebras have a reflexive, symmetric relation W as their top element. The class* $\mathsf{RRA}^{\mathsf{r}}$ *of relativized representable relation algebras is defined as* $\mathbf{IS}\mathsf{FRA}^{\mathsf{r}}$ *(isomorphic copies of subalgebras of relativized relational power set algebras),* $\mathsf{RRA}^{\mathsf{rs}}$ *as* $\mathbf{IS}\mathsf{FRA}^{\mathsf{rs}}$.

It is an easy and instructive exercise to prove that $\mathsf{RRA} \subseteq \mathsf{RRA}^{\mathsf{r}}$ by showing that any representable relation algebra is (isomorphic to) a subalgebra of a relativized relational power set algebra where the top set W is an *equivalence* relation.

The variety $\mathsf{RRA}^{\mathsf{r}}$ *does* allow a finite axiomatization, cf. KRAMER [35], or MARX [42] for a simpler system. A nice result by Maddux (cf. [39]) states that $\mathsf{RRA}^{\mathsf{rs}}$ is precisely the class WA of *weakly associative* algebras which arises if we replace the associativity axiom ($RA4$) in the definition of RA by its weaker variant

(WA) $\quad$ $(1';x);1;1 = (1';x);1.$

Now one of the most important connections with arrow logic lies in the fact that relation type algebras are the *complex algebras* of arrow frames; complex algebras form one of the fundamental structural operations in the duality theory of relational frames and BAOs. To explain what a complex algebra is, consider a relational frame $\mathfrak{F} = (W, R_i)_{i \in I}$. With each $n+1$-ary *relation* R_i we associate an n-ary *operation* f_{R_i} on the power set $\mathcal{P}(W)$:

$$f_{R_i}(X_1, \ldots, X_n) = \{x_0 \mid \exists x_1 \ldots x_n (R_i x_0 x_1 \ldots x_n \;\&\; \bigwedge_{1 \le i \le n} x_i \in X_i)\}.$$

The complex algebra $\mathfrak{Cm}\mathfrak{F}$ of $\mathfrak{F}$ is given as

$$\mathfrak{Cm}\mathfrak{F} = (\mathcal{P}(W), \cup, -, f_{R_i})_{i \in I};$$

for a class K of relational structures, we let $\mathbf{Cm}\mathsf{K}$ denote the class of its complex algebras. In the case of arrow logic we find that the full relation algebras are the complex algebras of the square arrow frames, and a similar result applies to the relativized squares:

Proposition 8

$$\begin{array}{rcl} \mathsf{FRA} & = & \mathbf{Cm}\mathsf{SQ} \\ \mathsf{FRA}^{\mathsf{r}} & = & \mathbf{Cm}\mathsf{SQ}^{\mathsf{r}} \\ \mathsf{FRA}^{\mathsf{rs}} & = & \mathbf{Cm}\mathsf{SQ}^{\mathsf{rs}} \end{array}$$

Proof.
Straightforward; note for instance that

$$\begin{array}{rcl} R \mid S & = & \{(u,w) \in U \times U \mid \exists v[(u,v) \in R \;\&\; (v,w) \in S]\} \\ & = & \{(u,w) \in U \times U \mid \exists v[C(u,w)(u,v)(v,w) \;\&\; (u,v) \in R \;\&\; (v,w) \in S]\} \\ & = & \{(u,w) \in U \times U \mid \exists rs[C(u,w)rs \;\&\; r \in R \;\&\; s \in S]\} \\ & = & f_C(R,S). \end{array}$$

□

These observations have important consequences for the axiomatics of arrow logic. First, observe that, modulo a trivial translation τ (with clauses like $\tau(\phi \vee \psi) = \tau\phi + \tau\psi$, $\tau(\phi \circ \psi) = \tau\phi ; \tau\psi$, etc.), the *formulas* of arrow logic are the *terms* of the algebraic language for relation type algebras. Now the following proposition is almost immediate:

Proposition 9

$$\begin{array}{lcl} \mathsf{SQ} \models \phi & \iff & \mathsf{RRA} \models \tau\phi = 1 \\ \mathsf{SQ}^{\mathsf{r}} \models \phi & \iff & \mathsf{RRA}^{\mathsf{r}} \models \tau\phi = 1 \\ \mathsf{SQ}^{\mathsf{rs}} \models \phi & \iff & \mathsf{RRA}^{\mathsf{rs}} \models \tau\phi = 1. \end{array}$$

These propositions form a basic reason for the applicability of techniques from algebraic logic in arrow logic (and vice versa!): axiomatizing the modal theory of SQ and the equational theory of RRA are two sides of the same coin. However, note that axiomatics is not the only area in which duality theory proves it use; in fact, almost all properties of logics have an algebraic counterpart.

5 Characterization results

In this section we discuss some basic model theory of arrow logic. First we work on the level of arrow *models*, for which we define arrow bisimulations as the fundamental notion of similarity between two arrow models. In the second part of the section we move to the level of frames, considering the question, how classes of arrow frames resp. properties of arrow frames can be defined in the language of arrow models.

So let us start with considering the notion of a bisimulation between structures. Bisimulations are the key tools to compare (labeled) transition systems, cf. van Benthem & Bergstra [11], van Benthem, van Eijck & Stebletsova [12], and thus of fundamental importance to computer science, cf. Park [56] for a first reference. It is interesting to note that in the latter area, a bisimulation is usually defined as a relation between *states*, whereas in arrow logic, we need to compare *transitions*:

Definition 10 *Let $\mathfrak{M} = (W, C, R, I, V)$ and $\mathfrak{M}' = (W', C', R', I', V')$ be two arrow models. A relation $Z \subseteq W \times W'$ is an arrow bisimulation if it satisfies, for any propositional variable p:*

$$\text{if } aZa', \text{ then } a \in V(p) \text{ iff } a' \in V'(p),$$

the following 'forth'-conditions ($a, b, c \in W$ and $a' \in W'$):

$(ZZ)^{\rightarrow}_{C}$	*$Cabc$ and aZa'*	*only if there are $b', c' \in W'$ such that bZb', cZc' and $C'a'b'c'$,*
$(ZZ)^{\rightarrow}_{R}$	*Rab and aZa'*	*only if there is a $b' \in W'$ such that bZb' and $Ra'b'$,*
$(ZZ)^{\rightarrow}_{I}$	*Ia and aZa'*	*only if $I'a'$*

and the converse 'back'-conditions $(ZZ)^{\leftarrow}_{C}$, $(ZZ)^{\leftarrow}_{R}$ and $(ZZ)^{\leftarrow}_{I}$ (with the obvious definition).

We write $\mathfrak{M}, a \underline{\leftrightarrow} \mathfrak{M}', a'$ if there is an arrow bisimulation Z between $\mathfrak{M}$ and $\mathfrak{M}'$ such that aZa'.

In de Rijke [59], the crucial rôle of bisimulations in the model theory of modal logic is investigated, the slogan being that bisimulations are to modal logic what partial isomorphisms are to first order logic. For instance, a nice result of de Rijke (in terms of arrow logic) states that two *rooted*[6] arrow models have the same arrow theory iff they have bisimilar ultraproducts. In particular, the existence of a bisimulation between two arrow models is an indication of the fact that the two models are very much alike. The following theorem states that the modal language of arrow logic cannot distinguish two bisimilar arrows:

Theorem 11 *Assume that $\mathfrak{M}, a \underline{\leftrightarrow} \mathfrak{M}', a'$. Then for every arrow formula ϕ:*

$$\mathfrak{M}, a \Vdash \phi \iff \mathfrak{M}', a' \Vdash \phi.$$

This theorem has an easy proof by induction on the complexity of arrow formulas. Now another manifestation of the fundamental importance of bisimulations lies in the fact, that a (restricted) kind of converse of Theorem 11 holds as well: a property of arrows that is first-order definable, can be expressed by a modal formula if and only if it is invariant under bisimulations. To be a bit more precise:

Theorem 12 *Let $\alpha(x)$ be a formula in the first order language over arrow models*[7]*. Assume that truth of $\alpha(x)$ is invariant under arrow bisimulations, i.e. $\mathfrak{M}_0, a_0 \underline{\leftrightarrow} \mathfrak{M}_1, a_1$ implies that*

$$\mathfrak{M}_0 \models \alpha[x \mapsto a_0] \iff \mathfrak{M}_1 \models \alpha[x \mapsto a_1].$$

Then $\alpha(x)$ is equivalent to an arrow formula, i.e. there is a formula ϕ in the modal language of arrow logic such that for every arrow model $\mathfrak{M}$ and every arrow a in $\mathfrak{M}$

$$\mathfrak{M} \models \alpha[x \mapsto a] \iff \mathfrak{M}, a \Vdash \phi.$$

---

[6] A rooted arrow model is an arrow model with a distinguished arrow, as in the definition of $\mathfrak{M}, a \underline{\leftrightarrow} \mathfrak{M}', a'$; the theory of a rooted arrow frame is the set of formulas holding at its root.

[7] This is to say that $\alpha(x)$ may use the following fixed predicate symbols: a ternary C, a binary R and a unary I, and further arbitrary unary predicate symbols $P_0, P_1, \ldots$.

A proof of Theorem 12 (for the ordinary modal case with one diamond $\diamond$) can be found in van Benthem [7].

Let us now move on to the level of arrow *frames*. As a corollary of Theorem 11 we find that the validity of arrow formulas is preserved under certain operations on frames. As an example, we consider zigzag morphisms:

Definition 13 *Let $\mathfrak{F}_0 = (W_0, C_0, R_0, I_0)$ and $\mathfrak{F}_1 = (W_1, C_1, R_1, I_1)$ be two arrow frames. A function $z : W_0 \mapsto W_1$ is a zigzag morphism from $\mathfrak{F}_0$ to $\mathfrak{F}_1$ if its graph*[8] *satisfies the back & forth conditions of Definition 10. We call $\mathfrak{F}_1$ a zigzagmorphic image of $\mathfrak{F}_0$, if there is a total, surjective zigzag morphism from $\mathfrak{F}_0$ to $\mathfrak{F}_1$.*

Theorem 14 *Let $\mathfrak{F}_1$ be a zigzagmorphic image of $\mathfrak{F}_0$. Then for every arrow formula ϕ:*

$$\mathfrak{F}_0 \models \phi \ \Rightarrow\ \mathfrak{F}_1 \models \phi.$$

As an example of a zigzag morphism, consider the square frame $\mathfrak{F}$ over the set ζ of integers, and the arrow frame $\mathfrak{Z}$ of the additive group over ζ, cf. (4). We invite the reader to check that the map $q : \zeta \times \zeta \mapsto \zeta$ defined by

$$q(y, z) = z - y,$$

is a zigzag morphism from $\mathfrak{F}$ onto $\mathfrak{Z}$ [9].

This example takes us to the second part of the section, where we consider *characterization results* in arrow logic. We say that a set Γ of arrow formulas *defines* or *characterizes* a class K of arrow frames if for any arrow frame $\mathfrak{F}$, validity of Γ in $\mathfrak{F}$ is equivalent to membership of $\mathfrak{F}$ in K. Then it follows from Theorem 14 that a class of frames is *not* modally definable if it is not closed under taking zigzagmorphic images. So, the previous example shows that the class of (relativized) squares cannot be characterized in the language of arrow logic. We will come back to this point later on.

Let us first consider some positive results: it turns out that many natural *properties* of arrow frames do allow a modal characterization. We say that a modal formula ϕ *defines* or *characterizes* a property P of arrow frames if it defines the class of frames having this property. Now consider the following list of formulas:[10]

$(A1)$	$\neg\otimes p$	$\rightarrow$	$\otimes\neg p$
$(A2)$	$\otimes\neg p$	$\rightarrow$	$\neg\otimes p$
$(A3)$	$\otimes\otimes p$	$\rightarrow$	p
$(A4)$	$\otimes(p \circ q)$	$\rightarrow$	$\otimes q \circ \otimes p$
$(A5)$	$p \circ \neg(\otimes p \circ q)$	$\rightarrow$	$\neg q$
$(A6)$	δ	$\rightarrow$	$\otimes\delta$
$(A7)$	$\delta \circ p$	$\rightarrow$	p
$(A8)$	p	$\rightarrow$	$\delta \circ p$
$(A9)$	$p \circ (q \circ r)$	$\leftrightarrow$	$(p \circ q) \circ r$

[8]The graph of a function $f : W_0 \mapsto W_1$ is the relation $\{(x_0, x_1) \in W_0 \times W_1 \mid x_1 = f(x_0)\}$.

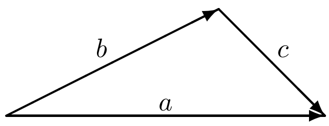
[9]In fact, the Cayley representation of an arbitrary group induces a zigzag morphism from the square over the carrier of the group to the group itself.

[10]Note that all of these formulas are in so-called *Sahlqvist* form; thus it is immediate that they correspond to first-order properties of frames (cf. Sahlqvist [62]).

$A1$ defines the arrow frames $\mathfrak{F}$ in which the reverse relation R is *serial*, i.e. $\mathfrak{F} \models \forall x \exists y Rxy$. For, let R be serial and suppose that $\mathfrak{M}, x \Vdash \neg \otimes p$ for some model $\mathfrak{M}$ based on $\mathfrak{F}$. By assumption, x has an R-successor y, and as p cannot be true at y, we have $y \Vdash \neg p$. So $\mathfrak{M}, x \Vdash \otimes \neg p$. Conversely, suppose that R is not serial; then there is an x in $\mathfrak{F}$ without an R-successor. Consider the valuation V with $V(p) = \emptyset$, then $x \Vdash \neg \otimes p$, but not $x \Vdash \otimes \neg p$.

Likewise, we can show that $A2$ characterizes the arrow frames in which R is *functional*, i.e. $\mathfrak{F} \models \forall xyz((Rxy \wedge Rxz) \to y = z)$, and that among these functional frames, $A3$ defines the ones with *idempotent* functions, i.e. $ffx = x$. Let us call a frame $\mathfrak{F}$ an f-frame if R is serial, functional and idempotent. In the sequel it will be convenient to represent such frames as $\mathfrak{F} = (W, C, f, I)$, i.e., in the signature we replace the relation R by the function f.

Now consider a C-triple (a, b, c) in an f-frame $\mathfrak{F}$:



It seems to be a very natural property that given this C-triangle, *all* of the triples of the following set $CYC_{a,b,c}$ should be in C:

$$CYC_{a,b,c} = \{(a, b, c),\ (fa, fc, fb),\ (b, a, fc),\ (fb, c, fa),\ (c, fb, a),\ (fc, fa, b)\}. \quad (5)$$

This property is well known from the theory of relation algebras; in LYNDON [38] triples (a, b, c) satisfying $CYC_{a,b,c} \subseteq C$ are called *cycles*. The property "$CYC_{a,b,c} \subseteq C$ for all arrows a, b and c with $Cabc$" is taken care of by $A4$ and $A5$:

$$\begin{array}{lcl} \mathfrak{F} \models A4 & \iff & \mathfrak{F} \models \forall xyz\ (Cfxyz \to Cxfzfy) \\ \mathfrak{F} \models A5 & \iff & \mathfrak{F} \models \forall xyz\ (Cxyz \to Czfyx) \end{array}$$

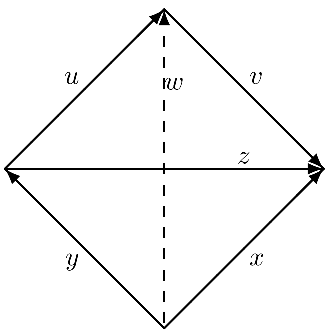
To show the equivalence for $A5$ first assume that $\mathfrak{F} \models \forall xyz\ (Cxyz \to Czfyx)$, and that in some model over $\mathfrak{F}$ there is an arrow x at which $p \circ \neg(\otimes p \circ q)$ is true. We will derive a contradiction from the assumption that $x \Vdash q$. Note that there must be y and z such that $Cxyz$, $y \Vdash p$ and $z \Vdash \neg(\otimes p \circ q)$. So $fy \Vdash \otimes p$, whence by $Czfyx$ and our assumption we find $z \Vdash \otimes p \circ q$: the desired contradiction. Conversely, if $\mathfrak{F} \not\models \forall xyz\ (Cxyz \to Czfyx)$, then there must be x, y and z such that $Cxyz$ and not $Czfyx$. Now consider a valuation V with $V(p) = \{y\}$ and $V(q) = \{x\}$; the crucial observation is that now fy is the *only* arrow at which $\otimes p$ is true, and x the *only* one with $x \Vdash q$. So $\otimes p \circ q$ must be *false* at z, whence $x \Vdash p \circ \neg(\otimes p \circ q)$. As we also have $x \Vdash q$, this gives $\mathfrak{F} \not\models A5$.

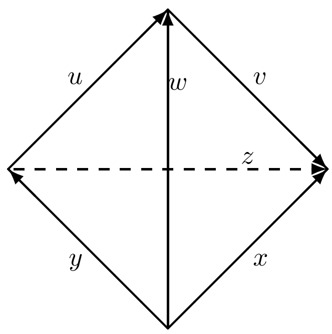
Of the formulas involving the identity, and in the class of f-frames, $A6$ characterizes the arrow frames in which reverses of identity arrows are identity arrows ($\forall x(Ix \to Ifx)$), $A8$ defines those frames in which every arrow has a starting arrow ($\forall x \exists y(Ix \wedge Cxyx)$), and $A7$ defines the following (perhaps most essential) property of an identity arrow y: $\forall xyz\ ((Cxyz \wedge Iy) \to x = z)$.

Now we turn to the *associativity* axiom $A9$, which plays a very interesting rôle in arrow logic, as we will see in the next section. It has a first order correspondent on the class of arrow frames as well:

$$\mathfrak{F} \models A9 \iff \mathfrak{F} \models \forall xyuv\ (\ \exists z(Cxyz \wedge Czuv)\ \leftrightarrow\ \exists w(Cxwv \wedge Cwyu)\ ),$$

viz. the picture below:





The formulas $A1 \ldots A9$ have been chosen for a special reason; taken together, they constitute the arrow-counterpart of Tarski's relation algebras:

Proposition 15 *Let $\mathfrak{F}$ be an arrow frame. Then*

$$\mathfrak{F} \models A1 \ldots A9 \iff \mathfrak{Cm}\mathfrak{F} \textit{ in } \mathsf{RA}.$$

Now we return to the question of what to do with classes of frames which are not modally definable. Even when in modal logic *characterizations* of frame classes do not lead automatically to complete *axiomatizations*, this issue is of importance. We will only treat the cases of the squares and the relativized squares.

The non-definability of SQ^{r} leaves open the possibility that one can find a set of formulas characterizing the class $\mathbf{P_f}\mathsf{SQ}^{\mathsf{r}}$ of *zigzagmorphic images* of relativized squares. And indeed, this aim can be achieved, as a result by Marx shows:

Theorem 16 *Let $\mathfrak{F}$ be an arrow frame. Then $\mathfrak{F}$ is a zigzagmorphic image of a relativized square iff the following formulas are valid in $\mathfrak{F}$:*

(A^r1)	$\otimes\neg p$	$\rightarrow$	$\neg\otimes p$
(A^r2)	$\otimes\otimes p$	$\rightarrow$	p
(A^r3)	$p \wedge \delta$	$\rightarrow$	$\otimes p$
(A^r4)	δ	$\rightarrow$	$\delta \circ \delta$
(A^r5)	$\delta \circ p$	$\rightarrow$	p
(A^r5')	$p \circ \delta$	$\rightarrow$	p
(A^r6)	$p \circ \neg\otimes p$	$\rightarrow$	$\neg\delta$
(A^r7)	$\otimes p \circ \neg(p \circ q)$	$\rightarrow$	$\neg q$
(A^r7')	$\neg(p \circ q) \circ \otimes q$	$\rightarrow$	$\neg p$
(A^r8')	$((p \wedge \delta) \circ q) \circ r$	$\leftrightarrow$	$(p \wedge \delta) \circ (q \circ r)$
(A^r8'')	$(p \circ (q \wedge \delta)) \circ r$	$\leftrightarrow$	$p \circ ((q \wedge \delta) \circ r)$
(A^r8''')	$(p \circ q) \circ (r \wedge \delta)$	$\leftrightarrow$	$p \circ (q \circ (r \wedge \delta))$

Note that the formulas listed above *all* have first-order correspondents, as each of them is in Sahlqvist form. We refer the reader to Marx e.a. [46] for the precise form of these first-order correspondents as well as for the proof of the theorem.

The situation for the squares is rather different, because the modal theory of SQ is not finitely axiomatizable by standard means (cf. the results by Monk and Andréka in the previous section).

Therefore, we have to look for a different *kind* of characterization. First, define the operator D as the following abbreviation:

$$D\phi \equiv \neg\delta \circ (\phi \circ \top) \ \vee \ (\top \circ \phi) \circ \neg\delta. \tag{6}$$

For an arbitrary frame $\mathfrak{F}$, we can define an accessibility relation R_D for D by setting

$$R_D = \{(a,b) \mid \mathfrak{F} \models \exists x_1 x_2 y \ \big(\ (Cax_1x_2 \wedge \neg Ix_1 \wedge Cx_2by) \ \vee \ (Cax_1x_2 \wedge \neg Ix_2 \wedge Cx_1yb) \ \big), \}$$

i.e. we have in every arrow model $\mathfrak{M}$:

$$\mathfrak{M}, a \Vdash D\phi \text{ iff there is a } b \text{ with } R_Dab \text{ and } \mathfrak{M}, b \Vdash \phi.$$

The reader is invited to check that in a *square* model we have:

$$\mathfrak{M}, a \Vdash D\phi \text{ iff there is a } b \text{ with } a \neq b \text{ and } \mathfrak{M}, b \Vdash \phi.$$

In other words, over the class of squares we have defined the *difference operator*. Now our characterization theorem below states that this property (viz. that the defined difference operator has the inequality relation as its accessibility relation), precisely characterizes the squares among the RA-frames (a proof could be distilled from Venema [70]):

Theorem 17 SQ *consists precisely of the arrow frames $\mathfrak{F}$ satisfying*
(i) $\mathfrak{F} \models A1 \ldots A9$,
(ii) $\mathfrak{F} \models \forall xy(R_Dxy \leftrightarrow x \neq y)$.

6 Completeness and Decidability

The axiomatics and decidability aspects of arrow logics have been investigated intensively; in this section we will try to give a sketchy overview of what is known.

Completeness

To start with axiomatics, let us first define the kinds of derivation systems that have been considered in the literature.

Definition 18 *A derivation system for arrow logic is a pair (A, R) with A a set of axioms and R a set of derivation rules. A derivation system is called normal if A contains the following axioms:*
(CT) *all classical tautologies*
(DB) $(p \to p') \,\underline{\circ}\, q \to (p \,\underline{\circ}\, q \to p' \,\underline{\circ}\, q)$
$p \,\underline{\circ}\, (q \to q') \to (p \,\underline{\circ}\, q \to p \,\underline{\circ}\, q')$
$\underline{\otimes}(p \to q) \to (\underline{\otimes}p \to \underline{\otimes}q)$
and R contains the rules of Modus Ponens, Universal Generalization and Substitution:

(MP)	$\phi, \phi \to \psi \;/\; \psi$
(UG)	$\phi \;/\; \phi \,\underline{\circ}\, \psi, \psi \,\underline{\circ}\, \phi$
	$\phi \;/\; \underline{\otimes}\phi$
(SUB)	$\phi \;/\; \sigma\phi$

where σ is a map uniformly substituting formulas for propositional variables in formulas.

A normal derivation system is called orthodox if (MP), (UG) and (SUB) are the only derivation rules of the system. For a set Σ of formulas, we let $\Omega(\Sigma)$ denote the orthodox logic generated by Σ, i.e. Ω is the orthodox derivation system having as axioms Σ, (CT) and (DB). A formula is a theorem of the derivation system $\Delta = (A, R)$, notation: $\Delta \vdash \phi$, if ϕ is the last item ϕ_n of a sequence $\phi_0, \ldots, \phi_n$ of formulas such that each ϕ_i is either an axiom or the result of applying a rule to formulas of $\{\phi_0, \ldots, \phi_{i-1}\}$. A derivation system Δ is sound with respect to a class K of frames if every theorem of Δ is valid in K, complete if every K-valid formula is a theorem of Δ.

The notion of a normal arrow logic is the straightforward generalization to the similarity type $\{\circ, \otimes, \delta\}$ of the notion of a normal modal logic in the similarity type with one diamond $\Diamond$. Orthodox modal logics are the direct modal correspondents of standard algebraic equational axiomatizations for varieties of boolean algebras with operators. For instance, if the orthodox modal logic generated by the set of axioms Σ is sound and complete with respect to a class K of frames, then the set $\{\tau\sigma = 1 \mid \sigma \in \Sigma\}$ [11] completely axiomatizes the class $\mathbf{Cm}\mathsf{K}$ (and, equivalently, the variety generated by it).

A nice aspect of arrow logic is that almost all interesting modal formulas are in Sahlqvist form — for instance, all formulas that we encountered in section 5. Therefore, many easy completeness results are obtainable:

Proposition 19 *Let Σ be a subset of the formulas occurring in section 5. Then $\Omega(\Sigma)$ is sound and complete with respect to the class of frames that is characterized by (the first order equivalents of the formulas in) Σ.*

The proof of Proposition 19 follows from the *completeness* part of Sahlqvists theorem, cf. Sahlqvist [62].

For the two-dimensional semantics the situation is more complicated. For instance, the squares are not finitely axiomatizable by an orthodox system (this follows from the non-finite axiomatizability result for Representable Relation Algebras mentioned in section 4). Fortunately, many classes of *relativized squares* allow finite orthodox axiomatizations. For instance, the following theorem is an immediate consequence of Proposition 19 and the result by Maddux that $\mathsf{WA} = \mathsf{RRA}^{\mathsf{rs}}$ (cf. section 3):

Theorem 20 *Let Σ be the set of axioms $A1 \ldots A8$ and*
$(A10)$ $\quad (\delta \circ p) \circ \top \circ \top \;\leftrightarrow\; (\delta \circ p) \circ \top$;
The $\Omega(\Sigma)$ is sound and complete with respect to the class $\mathsf{SQ}^{\mathsf{rs}}$ of locally square arrow frames.

In a similar way, the next theorem (due to M. Marx, cf. [42]), follows from Proposition 19 and Theorem 16:

[11] τ is the trivial translation from arrow logic *formulas* to relation algebraic *terms*, cf. the paragraph preceding Proposition 9.

Theorem 21 *Let Σ be the set of axioms $A^r1 \ldots A^r8'''$ from Theorem 16. Then $\Omega(\Sigma)$ is sound and complete with respect to* SQ^r.

If one allows unorthodox derivation rules, full square validity can be axiomatized by a finite system too. The proof of the following result can be found in VENEMA [70].

Theorem 22 *Let $\mho$ be the derivation system having as its axioms the formulas $A1 \ldots A9$ of section 4, and as its rules the orthodox derivation rules and*

$$\begin{array}{l}(p \wedge \neg Dp) \rightarrow \phi \;\; / \;\; \phi \\ \text{provided } p \text{ does not occur in } \phi.\end{array} \tag{7}$$

where D is the operator defined in (6). Then $\mho$ is sound and complete with respect to SQ.

The use of rather odd looking derivation rules like (7) to axiomatize theories lacking a finite orthodox axiomatization originates with GABBAY [23], and is discussed in detail in VENEMA [72].

Decidability

With one exception, we will be rather sketchy concerning decidability and related matters, as most of the known results can be found in other contributions to this volume. The watershed in the hierarchy of arrow logics seems to be the associativity axiom[12]:

$$p \circ (q \circ r) \leftrightarrow (p \circ q) \circ r. \tag{A9}$$

From results in ANDRÉKA E.A. [3] and KURUCZ E.A. [37] we can distil the following theorem:

Theorem 23 *Let Λ be any non-trivial[13] arrow logic containing $A9$. Then it is undecidable whether a given formula is derivable in Λ.*

As the proof of Theorem 23 contains some nice arguments of which the central ideas are not too difficult to follow, we will give a proof sketch here:

Proof.
Assume that Λ is the orthodox arrow logic axiomatized by $(A9)$. Recall that a semigroup is an algebra $\mathfrak{G} = (G, \cdot)$ such that $\cdot$ is an associative operation on G, and that a quasi-equation is an expression of the form

$$s_1 = t_1 \wedge \ldots \wedge s_n = t_n \;\rightarrow\; s_0 = t_0, \tag{8}$$

[12] Recent results have shown that undecidability already occurs under weaker conditions than associativity of $\circ$. We refer the reader to XXXX.

[13] We prefer to keep this notion informal, referring to ANDRÉKA E.A. [3] for a precise formulation; as an *example* of a sufficient condition for the theorem to hold, one might demand that every theorem of Λ is valid on squares of arbitrary size.

where the s_i's and t_i's are terms. We will show that Λ is undecidable by reducing the (undecidable!) quasi-equational theory of semigroups to it[14].

Now let Q be a quasi-equation in the language of semigroups; we will first show that

$$\mathsf{SG} \models Q \iff \mathsf{BA}^{\circ}_{A9} \models Q, \tag{9}$$

where SG is the variety of semigroups and BA°_{A9} is the class of boolean algebras with an associative, normal and additive operator $\circ$. Note that by duality, the *equational* theory of BA°_{A9} is decidable iff Λ is decidable.

To prove (9), the direction $\Rightarrow$ is trivial, as the $\circ$-reduct of any algebra in BA°_{A9} *is* a semigroup. For the other direction, let $\mathfrak{G}$ be a semigroup such that $\mathfrak{G} \not\models Q$. Without loss of generality we may assume that $\mathfrak{G}$ is a monoid, i.e. that $\mathfrak{G}$ has a designated element e such that $\mathfrak{G} \models ex = xe = x$. (For, if $\mathfrak{G}$ is not a monoid, we can *embed* it in a monoid $\mathfrak{G}'$; if $\mathfrak{G} \not\models Q$, then Q will fail to hold in $\mathfrak{G}'$ as well). Now we will embed $\mathfrak{G}$ in a complex algebra $\mathfrak{A} = (\mathcal{P}(W), \circ)$ in BA°_{A9}. $\mathfrak{A}$ is defined by

$$\begin{array}{lcl} W & = & \{(u, uv) \mid u, v \in G\}, \\ X \circ Y & = & X \mid^{W} Y, \end{array}$$

i.e. $\mathfrak{A}$ is the $\{+, -, ;\}$-reduct of a power set algebra of a relativized square over G. Note that $|^{W}$ is identical to the relational composition $|$, as W is a transitive relation over G. So $\circ$ is associative, and hence $\mathfrak{A}$ is in BA°_{A9}.

Now to embed $\mathfrak{G}$ in $\mathfrak{A}$, we define a map h reminiscent of the Cayley representation of groups:

$$h(u) = \{(v, vu) \mid v \in G\}.$$

To show that h is a homomorphism, first assume $(x, y) \in h(uu')$. Then $y = x(uu') = (xu)u'$, so by $(x, xu) \in h(u)$ and $(xu, (xu)u') \in h(u')$ we obtain $(x, y) \in h(u) \mid h(u')$. Thus we find $h(uu') \subseteq h(u) \mid h(u')$. For the other direction, assume that there is a z with $(x, z) \in h(u)$ and $(z, y) \in h(u')$. By definition of h, $z = xu$ and $y = zu'$, so $y = (xu)u' = x(uu')$, implying that $(x, y) \in h(uu')$. Therefore, $h(u) \mid h(u') \subseteq h(uu')$. So to show that h is an embedding it suffices to prove injectivity. Assume $h(u) = h(u')$; then $(e, u) \in h(u')$, and as u' is the *only* element $x \in G$ such that $(e, x) \in h(u')$, this gives $u = u'$. *This proves (9).*

The second part of the proof consists in showing that over BA°_{A9}, the validity problem of quasi-equations is reducible to that of equations, by defining a (recursive!) translation E from quasi-equations to equations such that

$$\mathsf{BA}^{\circ}_{A9} \models Q \iff \mathsf{BA}^{\circ}_{A9} \models E(Q). \tag{10}$$

It would go too far to define the translation which works for BA°_{A9} itself; to capture the essential idea, let us look at classes $\mathsf{K} \subseteq \mathsf{BA}^{\circ}_{A9}$ for which there is a term $c(x)$ satisfying, for all $\mathfrak{A}$ in K[15]:

$$\mathfrak{A} \models \begin{cases} c(a) = 0 & \text{if } a = 0, \\ c(a) = 1 & \text{if } a \neq 0, \end{cases}$$

[14]Note that this is a very strong simplification of the proof method used in the cited papers. The 'core' of the undecidability of arrow logic with associativity is not (just) the undecidability of the quasi-equational theory of semigroups itself, but (also) the fact that various classes of (semi-)groups have *recursively inseparable* quasi-equational theories.

[15]As an example of a term satisfying this condition, take the term $c(x) = 1; x; 1$, then (10) holds for every FRA.

In other words, we are in a *discriminator* variety of boolean algebras with operators. Now for such a class K, we can define the equation $E(Q)$ as

$$c(s_0 \oplus t_0) \ \leq \ c(s_1 \oplus t_1) \cdot \ldots \cdot c(s_n \oplus t_n)$$

where Q is as in (8) and $\oplus$ is the the symmetric difference operation given by $x \oplus y = x \cdot -y \, + \, -x \cdot y$.

We leave it to the reader to prove (10) by showing that for all $\mathfrak{A}$ in K, $\mathfrak{A} \models Q$ iff $\mathfrak{A} \models E(Q)$; the crucial observation is that in any $\mathfrak{A}$ in K, $c(s \oplus t)$ is either 0 or 1, according to whether $s \neq t$ or $s = t$. □

Many logics not containing $A9$ are decidable, cf. [46], as are many theories in simpler languages, cf. the literature mentioned in section 7. Two quite interesting examples of decidable logics are formed by the Dynamic Arrow Logic of van Benthem [10] and the modal theory of relativized squares, cf. Marx [42]. The most common method to prove that a logic Λ is decidable is to show that it has the finite model property (fmp), i.e. to prove that every non-theorem of the logic is falsifiable in a model based on a *finite* frame on which Λ is valid (and to show that it is decidable whether this condition is met by the frame). For instance, this method is used to prove the two results mentioned above.

It is interesting to note that for pair arrow logics, one can define an even stronger property that we will call the *finite base property* (fbp), which applies to a logic Λ if each of its non-theorems is falsifiable in a two-dimensional Λ-frame based on a finite *base set*[16]. This property is strictly stronger than the fmp, as a relatively simple example involving a dense linear ordering will show.

Interpolation

Let us finish this section by noting that with respect to other logical properties, the landscape of arrow logic still seems to be terra incognita. An exception concerns interpolation properties; in Sain [61] and Venema [70] negative results can be found for logics *with* the associativity axiom. For many classes of relativized frames, positive results are proved by Sain, Németi and Marx; an overview of known results is given in [45].

7 The family

In this section we briefly discuss, or just mention a number of systems that are closely related to the basic arrow logic, and/or fragments or extensions of it.

Dynamic Arrow Logic

In many of the application areas of arrow logic, e.g. linguistics, computer science or the algebraic theory of binary relations, *iteration* plays an essential rôle; therefore the Kleene star would be the first candidate to add to the language of arrow logic. Van Benthem introduces the system of dynamic arrow logic, as a proposal for the 'computational core calculus' of dynamic logics (cf. van

[16]Recently, it was shown by H. Andréka that non-square arrow logics have this fbp.

Benthem [10]). Here the focus is on abstract arrow semantics; the truth definition of $*$ is given by

$$\mathfrak{M}, a \Vdash *\phi \iff a \text{ can be } C\text{-decomposed into some finite sequence of arrows, each satisfying } \phi \text{ in } \mathfrak{M},$$

where C-decomposition means that there is at least one way of successive composition of the arrows in the sequence so as to obtain a. As axioms van Benthem proposes, besides $A1 \ldots A8$ of section 5, the following:

$$\begin{array}{rcl} \phi & \rightarrow & *\phi \\ *\phi \circ *\phi & \rightarrow & *\phi \end{array}$$

and the induction principle can be found in the rule

$$\phi \rightarrow \alpha,\ \alpha \circ \alpha \rightarrow \alpha\ \ /\ \ *\phi \rightarrow \alpha.$$

Both a completeness result and a decidability result are proved by van Benthem.

Arrow Logic, Bulgarian style

In Vakarelov [65] systems are studied that are closely related to the arrow logics discussed in this introduction, and that are also called arrow logics. Here the universe of a model consists of arrows as well, but the signature of these arrow models is different, and so is the similarity type of the modal language. The basic system has frames of the form $\mathfrak{F} = (W, R_{ij})_{i,j\in\{l,r\}}$ with every R_{ij} a binary relation on the set W of arrows. In the intended frames, W is the set of edges in a multigraph (cf. section 2) and the accessibility relations are defined by

$$R_{ij}ab \text{ iff } i(a) = j(b),$$

for instance $R_{lr}ab$ holds if the left point of a is the right point of b. The modal language has a diamond $\Diamond_{ij}$ for every relation R_{ij}. Note that in the full square semantics, these operators are definable in the language of ordinary arrow logic, for instance $\Diamond_{lr}\phi$ as $\otimes\phi\circ\top$. Vakarelov introduces an interesting construction called *copying*, and proves several completeness and decidability results (cf. also Venema [71] for a completeness result for the full square semantics). In Vakarelov & Arsov [67] the connection between the two approaches to arrow logic are investigated; one of the main results of the paper is a completeness result with respect to the class of multigraphs, for a language combining the operators of the two approaches.

Modal Transition Logics

Arrow logic occupies an interesting position in the landscape of what we will call modal transition logics. With a modal transition logic we understand (informally) a modal formalism (in the sense of abstract modal logic, cf. the discussion in the first section) with an intended semantics in which at least part of the well-formed expressions are interpreted as sets of transitions. To classify such systems, the main criteria seem to be (i) the relation of states versus transitions in the intended semantics, and (ii) the set of modal operators governing transitions, states and their connection. The extremity of arrow logic in this landscape lies in the fact that its models consists of transitions only. This goes too for Dynamic Implication Logic (Blackburn & Venema [15]) which considers a relatively poor fragment of arrow logic, and for Action Logic (PRATT [57]0, which deals with the connectives $*$, $\vee$, $\circ$ and the residuals / and \ of $\circ$ (to be discussed below).

Note that the 'classical' dynamic logic, Propositional Dynamic Logic (PDL, cf. HAREL [28]) is a modal transition logic as well: although the formulas of PDL are evaluated at states, the programs are evaluated at transitions. (It is interesting to note that the decidability of PDL depends on the impossibility to express the equivalence of two programs, cf. NÉMETI [52].) The system of Dynamic Modal Logic (DML, cf. DE RIJKE [58]) is similar to PDL in that it has an algebra of diamonds as well; the main difference being that the algebra of diamonds is not dynamic but relational.

Finally, there are hybrid systems: languages with two sorts of formulas, referring to states and transitions respectively, and a rich set of operators including modalities that relate the two sorts. We refer the reader to VAN BENTHEM [10] for an abstract approach, MARX [44] for results on concrete one- and two-dimensional interpretation of sorted transition systems, and DE RIJKE [59] for a very rich language, and completeness results for the full square case.

Residuals and Conjugates

In section 3 we already mentioned the dynamic interpretation of the Lambek calculus due to van Benthem (cf. [9]); in this semantics, the connectives of categorial logic (\, / and $\circ$) are interpreted in two-dimensional models (with a transitive and irreflexive universe $W \subseteq U \times U$) as follows:

$$\begin{array}{lll}
\mathfrak{M},(x,y) \Vdash \phi \circ \psi & \text{if} & \text{there are } (x,z),(z,y) \in W \text{ with } \mathfrak{M},(x,z) \Vdash \phi \text{ and } \mathfrak{M},(z,y) \Vdash \psi, \\
\mathfrak{M},(x,y) \Vdash \phi / \psi & \text{if} & \text{for all } (y,z) \in W\text{: } \mathfrak{M},(y,z) \Vdash \psi \text{ implies } \mathfrak{M},(x,z) \Vdash \phi, \\
\mathfrak{M},(x,y) \Vdash \phi \backslash \psi & \text{if} & \text{for all } (z,x) \in W\text{: } \mathfrak{M},(z,x) \Vdash \phi \text{ implies } \mathfrak{M},(z,y) \Vdash \psi.
\end{array}$$

In other words, the slashes are the *residuals* of $\circ$ — the following derivation rules form the basis of the Lambek calculus:

$$\phi \to \chi/\psi \iff \phi \circ \psi \to \chi \iff \psi \to \phi \backslash \chi$$

Note that the residuals can be defined in the language of arrow logic, e.g. / by $\phi/\psi \equiv \neg(\neg\phi \circ \otimes \psi)$.

Van Benthem observed that with the interpretation defined above, the Lambek calculus is sound with respect to the class of relativized squares where the universe is a transitive binary relation. In ANDRÉKA & MIKULÁS [4] completeness is proved, via an algebraic representation theorem (cf. also KURTONINA [36] for a derivation system related to the Gentzen calculus for RRA discussed in WADGE [75]).

In his contribution [47] to this volume, Mikulás studies the composition operator $\circ$ together with its *conjugates*, i.e. connectives $\circ_1$ and $\circ_2$ with the following truth definition on arrow models:

$$\begin{array}{lll}
\mathfrak{M}, b \Vdash \phi \circ_1 \psi & \text{if} & \text{there are } a, c \text{ with } Cabc,\ \mathfrak{M}, a \Vdash \phi \text{ and } \mathfrak{M}, c \Vdash \psi, \\
\mathfrak{M}, c \Vdash \phi \circ_2 \psi & \text{if} & \text{there are } a, b \text{ with } Cabc,\ \mathfrak{M}, a \Vdash \phi \text{ and } \mathfrak{M}, b \Vdash \psi.
\end{array}$$

Multi-Dimensional Modal Logic

Let us call an abstract modal system multi-dimensional if it has an intended semantics in which the 'possible worlds' are tuples over some base set, cf. VENEMA [70] for an overview. Then arrow logic is an example of a two-dimensional logic, as part of its intended semantics is formed by the (relativized) squares. Therefore it is interesting to see which phenomena of arrow logic are shared by other multi-dimensional modal logics. We give a few examples: to start with, in VENEMA [73] a formalism of cylindric modal logic is developed which can be seen as the modal version of a restricted kind of first order logic. The main result of that paper, an axiomatization of the class of cubes (higher-dimensional analogues of squares) is closely related to Theorem 22. Our second example is the paper NÉMETI [54]; the author shows how the undecidability of first-order logic vanishes if we generalize the class of models in the same way that the relativized squares form a generalization of the class of full squares. In ANDRÉKA, VAN BENTHEM & NÉMETI [2] the model theory of this 'restricted first-order logic' is developed. Finally, in VAKARELOV [66] a higher-dimensional version of the Bulgarian-style arrow logic is studied.

8 Questions

We finish this introduction with indicating some lines for further research, and listing a few technical open problems. There are still a lot of areas to be explored in the landscape of arrow logic.

Recall that in the introduction, we mentioned a list of possible entities that arrows might represent. One direction of research could be to extend this list with other application areas for arrow logic, and to develop the arrow-logical theory of various applications; for instance, what can be said about the arrow logic of various kinds of *categories*? Note that one need not only think here of questions concerning axiomatics or decidability; it would also be very interesting to see some (more) general results on other logical properties like interpolation, Beth definability, functional completeness, etc. In particular, we mentioned in section 1, that notions like *partiality* or *constraints* on the valuations might be needed for applications of arrow logic in for instance dynamic semantics. These are areas which have not yet been exploited.

These question can also be approached from a more theoretical perspective: the general question would be to study the *lattice of arrow logics*. In particular, it would be nice to have more results along the lines of Theorem 23 — it is interesting to know exactly where in the lattice of arrow logics the right combinations of complexity and expressiveness are situated.

Another area in which quite general questions can be asked, is given by the connections between arrow logics and *substructural logics*. In section 7 we mentioned the connections between arrow logic and the Lambek Calculus. Now the Lambek Calculus is only one particular spot in the landscape of *substructural logics* (cf. DOŠEN [17]), i.e. Gentzen-style calculi where the ordinary structural rules are absent. It would be interesting to establish more general connections between the hierarchy of substructural logics and formalisms of arrow logics.

Mentioning Gentzen-style derivation systems takes us to an undeveloped area of arrow logic — its *proof theory*. It would be nice to have complete sequent calculi for logics which allow finite Hilbert-style axiomatizations (like the relativized squares or the locally square arrow frames), in particular calculi with nice properties like cut-elimination and decidability. For some known derivation systems in the Gentzen-style, the reader is referred to WADGE [75], MADDUX [40] or ORŁOWSKA [55]; of these, the second paper reveals some nice connections between various varieties of relation (type) algebras and the number of variables needed in a proof.

A huge research field opens up if we shift some of the parameters underlying the *similarity type* of arrow logic. For instance, we may study languages for arrow frames having modal operators different from $\circ$, $\otimes$ and δ. We already encountered the difference operator D, the residuals / and \, and the conjugates $\circ_1$ and $\circ_2$, but there is of course an infinite hierarchy of modal languages that can be interpreted in arrow frames. Another parameter shift would be to study other (almost) basic arrow relations besides C, R and I. For instance, the *parallel* composition operator of processes in process algebra might be generalized into a ternary relation of arrows, for which a binary operator could be added to the language.

An even further generalization of the arrow logic ideology would be the following. Recall from section 3 that over the class of square models, arrow logic is equivalent to the three variable fragment of the predicate calculus of binary relations. The general arrow semantics can then be seen as an abstraction of the usual semantics for first order logic: dyadic predicates are interpreted as sets of arrows which may but *need not* be identified with pairs. This abstraction from concrete pairs to abstract arrows can just as well be made in the higher-dimensional case: n-ary relations may be interpreted as sets of n-dimensional arrows instead of as sets of n-tuples over some domain. Some first exercises in this area have been made[17], but as yet, the questions outnumber the answers by far.

Finally, there are some *technical* nuts to crack:

1. *Axiomatize the square logic of* $\{\circ, \delta, \wedge, \bot\}$*!*
For various fragments of the language of arrow logics, axiomatizations have been found, or non-finite axiomatizability results obtained (cf. [46]). For the fragment listed above, the question is still open whether there is a finite axiomatization of the set of formulas that are valid in the class SQ. This problem is interesting because the fragment plays an important rôle in *situation theory*, cf. MOSS & SELIGMAN [51].

2. *Does the Lambek Calculus have the finite base property?*
Recall from section 6 that the fbp of a logic states that any non-theorem of the logic can be falsified in a two-dimensional model based on a finite base set. For the associative Lambek Calculus the problem is the following: given two formulas ϕ and ψ, built up using only /, \ and $\circ$ and such that $\phi \to \psi$ is not a theorem of the Lambek Calculus, is there a finite two-dimensional model (with a transitive universe) in which $\phi \to \psi$ can be falsified? For more information, cf. ANDRÉKA & MIKULÁS [4].

3. *Is the class* RF *of representable arrow frames elementary?*
We call an arrow frame $\mathfrak{F}$ representable if its complex algebra $\mathfrak{Cm}\mathfrak{F}$ is representable, i.e. if $\mathfrak{Cm}\mathfrak{F}$ is in $\mathbf{SP}(\mathsf{FRA})$. Note that if an arrow frame $\mathfrak{F}$ is *not* representable, then this fact is witnessed by a formula ϕ from the modal theory of squares, which is not valid in $\mathfrak{F}$. So the class of non-representable arrow frames is closed under taking ultrapowers. Therefore, to solve this open problem, it suffices to answer the question whether RF itself is closed under taking ultrapowers. A positive solution would be given by finding a (necessarily infinite) orthodox axiomatization of the class SQ using only Sahlqvist axioms. For more information, cf. VENEMA [70].

[17]Cf. the part on multi-dimensional modal logics in the previous section; note however that most of these authors concentrate on non-standard, but still multi-dimensional models.

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