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Lax extensions of coalgebra functors and their logic

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ABSTRACT

We discuss the use of relation lifting in the theory of set-based coalgebra and coalgebraic logic. On the one hand we prove that the neighborhood functor does not extend to a relation lifting of which the associated notion of bisimilarity coincides with behavioral equivalence. On the other hand we argue that relation liftings may be of use for many other functors that do not preserve weak pullbacks, such as the monotone neighborhood functor. We prove that for any relation lifting L that is a lax extension extending the coalgebra functor T and preserving diagonal relations, L-bisimilarity captures behavioral equivalence. We also show that a finitary T admits such an extension iff it has a separating set of finitary monotone predicate liftings. Finally, we present the coalgebraic logic, based on a cover modality, for an arbitrary lax extension.

1. Introduction

There are at least two reasons why the notion of relation lifting plays an important role in the theory of (set-based) coalgebras: to characterize bisimulations, and to define the semantics of Moss-type coalgebraic logics. In both cases, coalgebraists generally have the Barr extension $\overline{T}$ in mind, which, for a functor T and a relation $R \subseteq X \times Y$, is the relation given by

$$\overline{T}R := \big\{ \big(T\pi_X(\rho), T\pi_Y(\rho)\big) \in TX \times TY \;\big|\; \rho \in TR \big\},$$

where $\pi_X : R \to X$ and $\pi_Y : R \to Y$ are the two projections. This relation lifting characterizes a bisimulation between two coalgebras $\xi : X \to TX$ and $\upsilon : Y \to TY$ as a relation $R \subseteq X \times Y$ such that $(\xi(x), \upsilon(y)) \in \overline{T}R$ whenever $(x, y) \in R$. It is well known, however, that these applications only work properly in case the functor T satisfies the category-theoretic property of preserving weak pullbacks. The key observation here is that $\overline{T}$ distributes over relation composition iff T preserves weak pullbacks. As an example, the above characterization of bisimilarity only coincides with that of behavioral equivalence (that is the relation of identifiability of two states by morphisms sharing their codomain) if T has this property. For this reason relation liftings are often thought to be of interest only in a setting of coalgebras for a weak pullback preserving functor.

On the other hand, the monotone neighborhood functor $\mathcal{M}$ is an important example of a coalgebra functor which does not preserve weak pullbacks, but which has a relation lifting $\widetilde{\mathcal{M}}$ that is essentially different from the Barr extension $\overline{\mathcal{M}}$ and whose notion of bisimilarity exactly captures behavioral equivalence [4]. And recently it has been shown that this notion of relation lifting can also be used to define the semantics of a Moss-style coalgebraic modality [15].

For this reason we study the notions of relation lifting that can be associated with a set functor T from a more general perspective. Here we take a relation lifting for a set functor T to be a collection of relations LR for every relation R, such

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that $LR \subseteq TX \times TY$ if $R \subseteq X \times Y$ (in the sequel we will give a more precise definition). Such studies have already been undertaken in the past. In [18] Thijs introduced a class of relation liftings, which he calls 'relators', to generalize different notions of coalgebraic simulation. Later, Baltag used Thijs' framework in [3] to give a semantics for the coalgebraic cover modality nabla. In [6] Hughes and Jacobs defined a generalization of the Barr extension for functors that carry an order. Very recently, Levy investigated the relation between the concept of similarity given by a relation lifting and final coalgebras [10].

In this paper we focus mainly on the question when such a relation lifting captures behavioral equivalence, in the sense that L-bisimilarity (defined in the obvious way) coincides with behavioral equivalence for any pair of T-coalgebras. Our work is concerned with similar notions as Levy's paper [10]. The difference is that, whereas Levy looks for endofunctors in some suitable order-category such that the notion of behavioral equivalence for its coalgebras (in his case: identification in the final coalgebra) coincides with similarity for a fixed relation lifting, we go the other way round and try to find relation liftings, whose notion of bisimilarity captures behavioral equivalence for a fixed functor.

Our main results can be summarized as follows. On the negative side, we prove that there is no way to capture behavioral equivalence between coalgebras for the (arbitrary) neighborhood functor $\mathcal{N}$ by means of relation lifting (Theorem 12). On the positive side, an important notion studied here is that of a lax extension of a functor T [17]. We will see that if such a lax extension preserves diagonals, then it indeed captures behavioral equivalence (Theorem 11) – this takes care of all cases known to us. Furthermore, we will provide some additional evidence that this combination of properties (lax extension preserving diagonals) is a natural one: in Theorem 14 we will prove that any finitary functor T has such an extension iff it admits a separating set of finitary monotone predicate liftings. The notion of a predicate lifting is familiar from coalgebraic modal logic [13]. Our theorem helps to clarify the relation between coalgebraic modal logic using a cover modality, and coalgebraic modal logic for a separating set of predicate liftings.

This paper is an extension of the earlier paper [12], which itself contained some of the results from the MSc thesis [11], authored by the first author and supervised by the second. We added a part on the logic that results when one uses lax extensions that preserve diagonals to give a semantics for the coalgebraic cover modality in the style of [3]. Also in this context the properties of lax extensions are exactly what is needed to have well-behaved logic. To compare the cover modality to more standard coalgebraic logics we also investigate translations between different coalgebraic logics. In this we follow [8], but we work on the more concrete level of formulas rather than equivalence classes.

2. Preliminaries

This paper presupposes knowledge of the theory of coalgebras [14]. In this section we recall some of the central definitions in this section, mainly to fix the notation.

2.1. Relations

In the following we consider relations to be arrows in the category of sets and relations. That is, we think of a relation $R : X \nrightarrow Y$ between sets X and Y as not just a subset of $X \times Y$ but as also specifying its codomain X and domain Y. Nevertheless, we often write $R = R'$, $R \subseteq R'$, $R \cup R'$, $R \cap R' : X \nrightarrow Y$ or $(x, y) \in R$ as if the relations $R, R' : X \nrightarrow Y$ were sets. We use $R^{gr} \subseteq X \times Y$ if we want to make explicit that we mean the set of pairs, considered as an object in the category of sets and functions, that stand in a relation $R : X \nrightarrow Y$.

We write $R \,;\, S : X \nrightarrow Z$ for the composition of two relations $R : X \nrightarrow Y$, $S : Y \nrightarrow Z$, and $R^\circ : Y \nrightarrow X$ for the converse of $R : X \nrightarrow Y$ with $(y, x) \in R^\circ$ iff $(x, y) \in R$. The graph of any function $f : X \to Y$ is a relation $f : X \nrightarrow Y$ between X and Y for which we also use the symbol f. It will be clear from the context in which a symbol f occurs whether it is meant as an arrow in the category of sets and functions or as an arrow in the category of relations. The composition of relations is written the other way round than the composition of functions. So we have for functions $f : X \to Y$ and $g : Y \to Z$ that $g \circ f = f \,;\, g$.

Identity elements in the category of sets and relations are the diagonal relations $\Delta_X : X \nrightarrow X$ with $(x, x') \in \Delta_X$ iff $x = x'$. Note that $\Delta_X = \mathsf{id}_X$, if we consider the identity function $\mathsf{id}_X : X \to X$ as a relation. Given sets $X' \subseteq X$ and $Y' \subseteq Y$ we define the restriction $R\restriction_{X' \times Y'} : X' \nrightarrow Y'$ of the relation $R : X \nrightarrow Y$ as $R\restriction_{X' \times Y'} = R \cap (X' \times Y')$.

2.2. Set functors

In the following we assume, if not explicitly stated otherwise, that functors are covariant endofunctors in the category of sets and functions.

We first introduce some of the functors that concern us in this paper. The *powerset functor* $\mathcal{P}$ maps a set X to the set of all its subsets $\mathcal{P}X$. A function $f : X \to Y$ is sent to $\mathcal{P}f : \mathcal{P}X \to \mathcal{P}Y, U \mapsto f[U]$. The *contravariant powerset functor* $\breve{\mathcal{P}}$ also maps a set X to $\breve{\mathcal{P}}X = \mathcal{P}X$. On functions $\breve{\mathcal{P}}$ is the inverse image map, that is for an $f : X \to Y$ we have $\breve{\mathcal{P}}f : \breve{\mathcal{P}}Y \to \breve{\mathcal{P}}X$, $V \mapsto f^{-1}[V]$.

The *neighborhood functor* or double contravariant powerset functor $\mathcal{N} = \breve{\mathcal{P}}\breve{\mathcal{P}}$ maps a set X to $\mathcal{N}X = \breve{\mathcal{P}}\breve{\mathcal{P}}X$ and a function $f : X \to Y$ to $\mathcal{N}f = \breve{\mathcal{P}}\breve{\mathcal{P}}f : \mathcal{N}X \to \mathcal{N}Y$ or more concretely for all $\xi \in \mathcal{N}X = \breve{\mathcal{P}}\breve{\mathcal{P}}X$ we have

$$\mathcal{N}f(\xi) = \left\{ V \subseteq Y \;\middle|\; f^{-1}[V] \in \xi \right\}.$$

For any cardinal α there is an α-ary variant ${}^{\alpha}\mathcal{N}$ of $\mathcal{N}$ that maps a set X to ${}^{\alpha}\mathcal{N}X = \breve{\mathcal{P}}((\breve{\mathcal{P}}X)^{\alpha})$. This means that the elements $\xi \in {}^{\alpha}\mathcal{N}X$ are sets of α-tuples of subsets of X. For an object $U \in (\breve{\mathcal{P}}X)^{\alpha}$ and a $\beta \in \alpha$ we write U_{β} for $U(\beta)$ that is the β-th component of U. So if α is a finite number, that is $\alpha = n \in \omega$, then we have that $U = (U_0, U_1, \ldots, U_{n-1})$. A function $f : X \to Y$ is mapped by ${}^{\alpha}\mathcal{N}$ to ${}^{\alpha}\mathcal{N}f : {}^{\alpha}\mathcal{N}X \to {}^{\alpha}\mathcal{N}Y$ such that for all $\xi \in {}^{\alpha}\mathcal{N}X = \breve{\mathcal{P}}((\breve{\mathcal{P}}X)^{\alpha})$

$${}^{\alpha}\mathcal{N}f(\xi) = \big\{ V \in (\breve{\mathcal{P}}Y)^{\alpha} \;\big|\; \big(f^{-1}[V_{\beta}]\big)_{\beta \in \alpha} \in \xi \big\}.$$

A restriction of the neighborhood functor $\mathcal{N}$ is the *monotone neighborhood functor* $\mathcal{M}$. It maps a set X to the collection $\mathcal{M}X$ of objects ξ in $\mathcal{N}X$ that are *upsets*, meaning that for all $U, U' \subseteq X$, if $U' \subseteq U$ and $U' \in \xi$ then also $U \in \xi$. On functions $\mathcal{M}$ is the same as $\mathcal{N}$. So we have for $f : X \to Y$ that

$$\mathcal{M}f : \mathcal{M}X \to \mathcal{M}Y,$$
$$\xi \mapsto \big\{ V \subseteq Y \;\big|\; f^{-1}[V] \in \xi \big\}.$$

It is straightforward to check that this is well-defined. There is also an α-ary version ${}^{\alpha}\mathcal{M}$ of $\mathcal{M}$ that is defined analogously to ${}^{\alpha}\mathcal{N}$ where the monotonicity requirement becomes that if $U'_{\beta} \subseteq U_{\beta}$ for all $\beta \in \alpha$ and $U' \in \xi$ then also $U \in \xi$.

The next two functors F^3_2 and $\mathcal{P}_n$ are interesting examples for us, because they, like the monotone neighborhood functor, do not preserve weak pullbacks but still allow for a relation lifting that captures behavioral equivalence.

The functor F^3_2 maps a set X to

$$F^3_2 X = \big\{ (x_0, x_1, x_2) \in X^3 \;\big|\; \big|\{x_0, x_1, x_2\}\big| \le 2 \big\}$$

the set of all triples over X that consist of at most two distinct elements. On functions the functor F^3_2 is defined exactly as $(-)^3$, that is a function $f : X \to Y$ is mapped by F^3_2 such that $F^3_2 f(x_0, x_1, x_2) = (f(x_0), f(x_1), f(x_2))$.

The restricted powerset functor $\mathcal{P}_n$ for an $n \in \omega$ maps a set X to the set $\mathcal{P}_n X = \{U \subseteq X \mid |U| < n\}$ of all its subsets of cardinality smaller than n. On functions it has the same definitions as $\mathcal{P}$, that is $\mathcal{P}_n f(U) = f[U]$.

In the context of coalgebraic logic one pays special attention to functors that preserve finite sets and are finitary. A functor T *preserves finite sets* if TX is finite whenever X is. All the functors mentioned above restrict to finite sets.

For the definition of finitary functors we use $\iota_{X',X} : X' \to X, x \mapsto x$ for the inclusion of a subset $X' \subseteq X$ into X. A functor T is *finitary* if all sets X

$$TX = \bigcup \big\{ T\iota_{X',X}\big[TX'\big] \subseteq TX \;\big|\; X' \subseteq X, X' \text{ is finite} \big\}.$$

The idea behind this definition is that finitary functors have the property that in order to describe an element $\xi \in TX$ one has to use only a finite amount of information from the possibly infinite set X. From the functors introduced above only F^3_2 and $\mathcal{P}_n$ for $n \in \omega$ are finitary. However one can define for every set functor T its finitary version T_{ω} that maps a set X to

$$T_{\omega}X = \bigcup \big\{ T\iota_{X',X}\big[TX'\big] \subseteq TX \;\big|\; X' \subseteq X, X' \text{ is finite} \big\}.$$

A function $f : X \to Y$ is mapped by T_{ω} to the function

$$T_{\omega}f : T_{\omega}X \to T_{\omega}Y,$$
$$\xi \mapsto T\iota_{f[X'],Y} \circ Tf_{X'}\big(\xi'\big),$$

where $\xi' \in TX'$ is such that $\xi = T\iota_{X',X}(\xi')$ for a finite $X' \subseteq X$ and $f_{X'}$ is the function $f_{X'} : X' \to f[X'], x' \mapsto f(x')$. This is well-defined, that means independent of the choice of X', because the following diagram commutes for all $X', X'' \subseteq X$

$$\begin{tikzcd}
X' \arrow[r, "\iota_{X',X}"] \arrow[d, "f_{X'}"] & X \arrow[d, "f"] & X'' \arrow[l, "\iota_{X'',X}"'] \arrow[d, "f_{X''}"] \\
f[X'] \arrow[r, "\iota_{f[X'],Y}"] & Y & f[X''] \arrow[l, "\iota_{f[X''],Y}"']
\end{tikzcd}$$

It is immediate from the definition that there is an inclusion $\tau_X : T_{\omega}X \subseteq TX$ for all sets X and that this actually defines a natural transformation $\tau : T_{\omega} \Rightarrow T$. If the functor T is already finitary then T identical is to T_{ω} and τ is the identity. Therefore we will often write T_{ω} when we work with an arbitrary functor that we assume to be finitary.

An example of a finitary version of a functor that we make use of is $\mathcal{P}_{\omega}$. One can see by instantiating the above definition that this functor maps a set X to the set of all its finite subsets.

A last property of functors that is important in coalgebraic logic is preservation of inclusions. A functor T *preserves inclusions* if $T\iota_{X',X} = \iota_{TX',TX}$ for any inclusion $\iota_{X',X} : X' \to X$ of some subset $X' \subseteq X$. In [2, Chapter III, p. 132] it is proved that for every set functor T there is a functor T' that preserves inclusions and that is naturally isomorphic to it with the only possible exception of the empty set. In fact a stronger result is showed and one can check the proof that if one only wants T' to preserve inclusions then T' can in fact be constructed to be isomorphic to T, even on the empty set.

In the last two sections on coalgebraic logic we will presuppose that we are working with a functor that preserves inclusions. In all other parts of the paper we explicitly mention if preservation of inclusions is used as an assumption for some result.

2.3. Coalgebras

A *T-coalgebra* for a covariant functor T on a set X is a function $\xi : X \to TX$. The elements of X are called the *states* of ξ and the function ξ is called the *transition structure*. A *T-coalgebra morphism* from a T-coalgebra $\xi : X \to TX$ to a T-coalgebra $\zeta : Z \to TZ$ is a function $f : X \to Z$ such that $\zeta \circ f = Tf \circ \xi$.

The T-coalgebras together with the T-coalgebra morphisms are a category where the identity arrows, and composition of arrows is the same as for the underlying set functions. This category is cocomplete and all colimits are computed as for the underlying sets.

The central notion of equivalence between states in coalgebra is behavioral equivalence. Two states, x_0 in a T-coalgebra $\xi : X \to TX$ and y_0 in T-coalgebra $\upsilon : Y \to TY$, are *behaviorally equivalent* if there exists a T-coalgebra ζ and coalgebra morphisms f from ξ to ζ and g from υ to ζ such that $f(x_0) = g(y_0)$.

2.4. Predicate liftings

A notion from coalgebraic modal logic that we are using later are predicate liftings. Predicate liftings for a functor T were originally introduced in [13], but see also [16], to define a modal logic for T-coalgebras that resembles the standard modal logic with boxes and diamonds on Kripke frames.

An n-ary *predicate lifting* for T is a natural transformation $\lambda : \breve{\mathcal{P}}^n \Rightarrow \breve{\mathcal{P}}T$. We write $\mathsf{ar}(\lambda)$ for the arity of a predicate lifting $\lambda : \breve{\mathcal{P}}^n \Rightarrow \breve{\mathcal{P}}T$. The *transposite* $\lambda^\flat : T \Rightarrow {}^n\mathcal{N} = \breve{\mathcal{P}}\breve{\mathcal{P}}^n$ of predicate lifting λ for a functor T is a natural transformation that is defined at a set X as

$$\begin{aligned}
&\lambda^\flat_X : TX \to {}^n\mathcal{N}X = \breve{\mathcal{P}}(\breve{\mathcal{P}}X)^n, \\
&\xi \mapsto \{U \in (\breve{\mathcal{P}}X)^n \mid \xi \in \lambda_X(U)\}.
\end{aligned}$$

An n-ary predicate lifting $\lambda : \breve{\mathcal{P}}^n \Rightarrow \breve{\mathcal{P}}T$ is *monotone* if $U_i \subseteq U'_i$ for all $i \in n$ implies that $\lambda(U) \subseteq \lambda(U')$ for any $U, U' \in (\breve{\mathcal{P}}X)^n$. The following observation about monotone predicate liftings is crucial for the proof of Theorem 14. The routine proof is left to the reader.

Proposition 1. *If $\lambda : \breve{\mathcal{P}}^n \Rightarrow \breve{\mathcal{P}}T$ is a monotone n-ary predicate lifting for T then the codomain of its transposite $\lambda^\flat : T \Rightarrow {}^n\mathcal{N}$ can be restricted to ${}^n\mathcal{M}$. That means $\lambda^\flat : T \Rightarrow {}^n\mathcal{M}$ defined as above is well-defined.*

When dealing with multiple predicate liftings of possibly different finite arity it can be useful to compose them with the injective natural transformation $e^n : {}^n\mathcal{N} \Rightarrow {}^\omega\mathcal{N}$ defined by

$$\begin{aligned}
&e^n_X : {}^n\mathcal{N}X \to {}^\omega\mathcal{N}X, \\
&\xi \mapsto \{U \in (\breve{\mathcal{P}}X)^\omega \mid (U_0, U_1, \ldots, U_{n-1}) \in \xi\}.
\end{aligned}$$

This natural transformation restricts to the monotone neighborhood functor to $E^n : {}^n\mathcal{M} \Rightarrow {}^\omega\mathcal{M}$ which we use if the predicate liftings are monotone.

A last concept that is relevant when working with predicate lifting is the one of separating sets. It comes in many interdependent versions of which we here introduce the ones that we are going to need later. A family of sets $F \subseteq \mathcal{P}X$ is *separating* if for all $x, x' \in X$ with $x \neq x'$ there is an $U \in F$ such that either $x \in U$ and $x' \notin U$ or $x \notin U$ and $x' \in U$. A tuple $S = (S_0, S_1, \ldots, S_{n-1}) \in (\breve{\mathcal{P}}X)^n$ is separating if for all $x, x' \in X$ with $x \neq x'$ there is an $i < n$ such that either $x \in S_i$ and $x' \notin S_i$ or $x \notin S_i$ and $x' \in S_i$.

A set Λ of predicate liftings for a functor T is separating if in every component the union of their images is separating. That means that for all sets X and all $\xi, \xi' \in TX$ with $\xi \neq \xi'$ there is a $\lambda \in \Lambda$ and an $U \in (\breve{\mathcal{P}}X)^{\mathsf{ar}(\lambda)}$ such that either $\xi \in \lambda_X(U)$ and $\xi' \notin \lambda_X(U)$ or $\xi \notin \lambda_X(U)$ and $\xi' \in \lambda_X(U)$. Intuitively, a set of natural transformations for a functor T is separating if it is expressive enough to recognize every difference between elements in TX.

A family $\mathcal{F}$ of functions from X to Y is *jointly injective* if given any $x, x' \in X$ we have that $f(x) = f(x')$ for all $f \in \mathcal{F}$ implies that $x = x'$. It can be checked that a set of predicate liftings is separating if and only if the family of functions $\{e \circ \lambda^\flat_X : TX \to {}^\omega\mathcal{N}X\}_{\lambda \in \Lambda}$ is jointly injective at every set X.

A set of predicate liftings is *finitely separating* if it is separating in the above sense but only for finite sets X. That means for instance that for all finite sets X the set $\{\lambda_X(U) \subseteq TX \mid \lambda \in \Lambda, U \in (\breve{\mathcal{P}}X)^{\mathsf{ar}(\lambda)}\}$ is separating.

The reader can check that if Λ is a finitely separating set of predicate liftings for a functor T then $\{\breve{\mathcal{P}}\tau \circ \lambda \mid \lambda \in \Lambda\}$, where $\tau : T_\omega \Rightarrow T$ is the inclusion, is a separating set of predicate liftings for T_ω. For the case where T is finitary we have that τ

is the identity and obtain the following:

Proposition 2. *Every finitely separating set of predicate liftings Λ for a finitary functor T_ω is separating.*

2.5. Relation liftings and bisimilarity

Fix a covariant set functor T. A *relation lifting* L for T associates with every relation $R : X \nrightarrow Y$ a relation $LR : TX \nrightarrow TY$. Throughout this paper we shall require relation liftings to preserve converses, this means that $L(R^\circ) = (LR)^\circ$ for all relations R. This restriction simplifies the presentation and is not essential for our results because behavioral equivalence, the notion we want to capture with relation liftings, is symmetrical.

Given a relation lifting L for a set functor T and two T-coalgebras $\xi : X \to TX$ and $\upsilon : Y \to TY$, an *L-bisimulation between* ξ and υ is a relation $R : X \nrightarrow Y$ such that $(\xi(x), \upsilon(y)) \in LR$ for all $(x, y) \in R$. The relation $\underline{\leftrightarrow}^L_{\xi,\upsilon} : X \nrightarrow Y$ of L-bisimilarity between ξ and υ is defined as the union of all L bisimulations between ξ and υ. We sometimes omit the subscripts and just write $x \,\underline{\leftrightarrow}^L\, y$ if the coalgebras to which x and y are clear from the context. We also write $\underline{\leftrightarrow}^L_\xi = \underline{\leftrightarrow}^L_{\xi,\xi} : X \nrightarrow X$ for bisimilarity on one single coalgebra $\xi : X \to TX$.

A relation lifting L for T *captures behavioral equivalence* if for any pair of states x and y in T-coalgebras $x \,\underline{\leftrightarrow}^L\, y$ holds iff x and y are behaviorally equivalent.

3. Lax extensions

In this section we introduce lax extensions. These are relation liftings satisfying certain conditions that make them well-behaved in the context of coalgebra. We summarize some general properties of lax extensions and show that they capture behavioral equivalence if they preserve diagonals. For some additional discussion of lax extensions, although in a different context, we refer to [17].

Definition 3. A relation lifting L for a functor T is a *lax extension* of T if it satisfies the following conditions for all relations $R, R' : X \nrightarrow Z$ and $S : Z \nrightarrow Y$, and all functions $f : X \to Z$:

(L1) $R' \subseteq R$ implies $LR' \subseteq LR$,
(L2) $LR \,;\, LS \subseteq L(R \,;\, S)$,
(L3) $Tf \subseteq Lf$.

A lax extension L *preserves diagonals* if it additionally satisfies:

(L4) $L\Delta_X \subseteq \Delta_{TX}$.

Condition (L3) in [17] additionally requires that $(Tf)^\circ \subseteq L(f^\circ)$. For us this follows automatically from the preservation of converses.

Only one inclusion is needed in (L4) for a lax extension to preserve diagonals. This is enough because, as shown in Proposition 5 below, together with condition (L3) condition (L4) implies that $L\Delta_X = \Delta_{TX}$.

Remark 4. In [6] a generalization of the Barr extension is defined with the name 'lax relation lifting'. This lax relation lifting is in general not a lax extension in our sense, even if we would not require preservation of converses, because it does not satisfy (L2). The lax relation lifting of [6] always satisfies $LR \,;\, LS \supseteq L(R \,;\, S)$ which is exactly the condition that distinguishes lax extension that preserve diagonals from the Barr extension and makes them useful for functors that do not preserve weak pullbacks.

Lax extensions have already been studied in the context of coalgebra under the name 'monotone relator' in [18, Section 2.1] and very recently in [10, Definition 6], where they are just called 'relators'. In [18] it is additionally required that composition of relation is preserved, that means $=$ instead of $\subseteq$ in our condition (L2) of Definition 3, but it is noted that the $\supseteq$-inclusion can be omitted for most of the proofs. Both [18] and [10] use a different set of conditions in their definitions, but it can be checked that they are equivalent to our Definition 3. Instead of (L3) [18] requires that

(R3) $\Delta_{TX} \subseteq L\Delta_X$,
(R4) $Tf \,;\, LR \,;\, (Tg)^\circ \subseteq L(f \,;\, R \,;\, g^\circ)$.

In [10] condition (R4) has $=$ instead of just $\subseteq$. This is redundant, because we can show that (R3) and (R4) imply (L3). Hence every relator is a lax extension and the equality in (R4) follows from Proposition 5(ii) below. To see that (R3) and (R4) imply (L3) consider for any function $f : X \to Z$

$$\begin{aligned} Tf = Tf \,;\, \Delta_{TZ} \,;\, (T\mathsf{id}_Z)^\circ &\subseteq Tf \,;\, L\Delta_Z \,;\, (T\mathsf{id}_Z)^\circ && \text{(R3)} \\ &\subseteq L\big(f \,;\, \Delta_Z \,;\, \mathsf{id}_Z^\circ\big) = Lf. && \text{(R4)} \end{aligned}$$

That every lax extension is a monotone relator, that is every lax extension satisfies (R3) and (R4) follows from our next proposition that summarizes some basic properties of lax extensions.

Proposition 5. *If L is a lax extension of T then for all functions $f : X \to Z$, $g : Y \to Z$ and relations $R : X \nrightarrow Z$, $S : Z \nrightarrow Y$:*

(i) $\Delta_{TX} \subseteq L\Delta_X$,
(ii) $Tf \,;\, LS = L(f \,;\, S)$ *and* $LR \,;\, (Tg)^\circ = L(R \,;\, g^\circ)$,

and if L preserves diagonals then

(iii) $\Delta_{TX} = L\Delta_X$ *and* $Tf = Lf$,
(iv) $Tf \,;\, (Tg)^\circ = L(f \,;\, g^\circ)$.

Proof. For (i) recall that we identify a function with the relation of its graph. So we have that $\Delta_X = \mathsf{id}_X$ and we can calculate

$$\begin{aligned} \Delta_{TX} &= \mathsf{id}_{TX} = T\mathsf{id}_X && T \text{ functor} \\ &\subseteq L\mathsf{id}_X = L\Delta_X. && \text{(L3)} \end{aligned}$$

The $\subseteq$-inclusion of $Tf\,;LS = L(f\,;S)$ in (ii) holds because $Tf\,;LS \subseteq Lf\,;LS \subseteq L(f\,;S)$ where the first inclusion is condition (L3) and the second inclusion is (L2). For the $\supseteq$-inclusion consider

$$\begin{aligned} L(f \,;\, S) &\subseteq Tf \,;\, (Tf)^\circ \,;\, L(f \,;\, S) && \Delta_{TX} \subseteq Tf \,;\, (Tf)^\circ \\ &\subseteq Tf \,;\, (Lf)^\circ \,;\, L(f \,;\, S) && \text{(L3)} \\ &\subseteq Tf \,;\, Lf^\circ \,;\, L(f \,;\, S) && \text{preservation of converses} \\ &\subseteq Tf \,;\, L\big(f^\circ \,;\, f \,;\, S\big) && \text{(L2)} \\ &\subseteq Tf \,;\, LS. && f^\circ \,;\, f \subseteq \Delta_Y \text{ and (L1)} \end{aligned}$$

The other claim $LR\,;(Tg)^\circ = L(R\,;g^\circ)$ follows from $Tf\,;LS = L(f\,;S)$ because L preserves converses.

For (iv) and (iii) first notice that if L preserves diagonals then $\Delta_{TX} = L\Delta_X$ because of (L4) and (i).

The equation $Tf = Lf$ from (iii) holds because of

$$\begin{aligned} Tf &= Tf\,; L\Delta_X && \Delta_{TX} = L\Delta_X \\ &= L(f\,;\,\Delta_X) = Lf. && \text{(ii)} \end{aligned}$$

For claim (iv) consider

$$\begin{aligned} Tf \,;\, (Tg)^\circ &= Tf \,;\, L\Delta_X \,;\, (Tg)^\circ && \Delta_{TX} = L\Delta_X \\ &= L\big(f \,;\, \Delta_X \,;\, g^\circ\big) = L\big(f \,;\, g^\circ\big). && \text{(ii) twice} \qquad \square \end{aligned}$$

The following proposition states that lax extensions of inclusion preserving functors commute with restrictions of relations. This property is useful in coalgebraic logic in Section 6.

Proposition 6. *Let L be a lax extension of a functor T that preserves inclusions. Then for all relations $R : X \nrightarrow Y$ and subsets $X' \subseteq X$ and $Y' \subseteq Y$ it holds that*

$$L(R\restriction_{X'\times Y'}) = (LR)\restriction_{TX'\times TY'}.$$

Proof. We can write the restriction of a relation $R : X \nrightarrow Y$ to the sets $X' \subseteq X$ and $Y' \subseteq Y$ as $R\restriction_{X'\times Y'} = \iota_{X',X}\,; R\,; \iota^\circ_{Y',Y}$ where $\iota_{X',X} : X' \to X$ and $\iota_{Y',Y} : Y' \to Y$ are inclusions. Then we compute that

$$\begin{aligned} L(R\restriction_{X'\times Y'}) &= L\big(\iota_{X',X} \,;\, R \,;\, \iota^\circ_{Y',Y}\big) \\ &= T\iota_{X',X} \,;\, LR \,;\, (T\iota_{Y',Y})^\circ && \text{Proposition 5 (ii)} \\ &= \iota_{TX',TX} \,;\, LR \,;\, \iota^\circ_{TY'TY} && T \text{ preserves inclusions} \\ &= (LR)\restriction_{TX'\times TY'}. && \square \end{aligned}$$

Example 7.

(i) For any functor T there is a trivial lax extension C that maps any relation $R : X \nrightarrow Y$ to the maximal relation $CR = TX \times TY : TX \nrightarrow TY$. For most functors this lax extension does not preserve diagonals.

(ii) The *Egli–Milner lifting* $\overline{\mathcal{P}}$ is a lax extension of the covariant powerset functor $\mathcal{P}$ that preserves diagonals. It is defined such that $\overline{\mathcal{P}}R : \mathcal{P}X \nrightarrow \mathcal{P}Y$ for any $R : X \nrightarrow Y$ and $(U, V) \in \overline{\mathcal{P}}R$ iff
- for all $u \in U$ there is a $v \in V$ such that $(u, v) \in R$ (forth condition), and
- for all $v \in V$ there is a $u \in U$ such that $(u, v) \in R$ (back condition).

More concisely we can write $\overline{\mathcal{P}}R = \overrightarrow{\mathcal{P}}R \cap \overleftarrow{\mathcal{P}}R$ where we use the abbreviations

$$\overrightarrow{\mathcal{P}}R = \{(U, V) \in \mathcal{P}X \times \mathcal{P}Y \mid \forall u \in U.\exists v \in V.(u, v) \in R\},$$
$$\overleftarrow{\mathcal{P}}R = \{(U, V) \in \mathcal{P}X \times \mathcal{P}Y \mid \forall v \in V.\exists u \in U.(u, v) \in R\}.$$

(iii) The Egli–Milner lifting from item (ii) is an instances of a relation lifting that is definable for arbitrary functors T. The *Barr extension* $\overline{T}$ of a functor T on a relation $R : X \nrightarrow Y$ with projections $\pi_X : R \to X$ and $\pi_Y : R \to Y$ is

$$\overline{T}R = \{(T\pi_X(\rho), T\pi_Y(\rho)) \mid \rho \in TR^{gr}\}.$$

It is easy to see that the Barr extension $\overline{T}$ of a functor T satisfies (L1). One can also show that $\overline{T}f = Tf$ for all function $f : X \to Y$. This means that $\overline{T}$ satisfies (L3) and (L4). For proofs of this basic properties of the Barr extension consult for instance [7].

Condition (L2) is more difficult. It is the case that $\overline{T}R ; \overline{T}S = \overline{T}(R ; S)$ for all relations $R : X \nrightarrow Z$ and $S : Z \nrightarrow Y$ iff T preserves weak pullbacks [7, Fact 3.6]. So we have that the Barr extension $\overline{T}$ of a weak pullback preserving functor T is a lax extension that preserves diagonals.

Also note that the condition $\overline{T}R ; \overline{T}S = \overline{T}(R ; S)$ for all relations $R : X \nrightarrow Z$ and $S : Z \nrightarrow Y$ is very strong. Together with $\overline{T}f = Tf$ for all function $f : X \to Y$ it means that $\overline{T}$ is a functor from Rel to Rel that extends T. Such an extension of a functor T is unique if it exists because for every relation $R : X \nrightarrow Y$ with projections $\pi_X : R^{gr} \to X$ and $\pi_Y : R^{gr} \to Y$ we have that

$$\begin{aligned} \overline{T}R &= \overline{T}(\pi_X^\circ ; \pi_Y) && R = \pi_X^\circ ; \pi_Y \\ &= \overline{T}\pi_X^\circ ; \overline{T}\pi_Y && \overline{T}R ; \overline{T}S = \overline{T}(R ; S) \\ &= T(\pi_X^\circ) ; T\pi_Y. && \overline{T}f = Tf \end{aligned}$$

(iv) Even though one can show that the Barr extension $\overline{\mathcal{M}}$ of the monotone neighborhood functor does not satisfy (L2), there is a lax extension $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ that preserves diagonals. For this definition recall the notation $\overrightarrow{\mathcal{P}}R$ and $\overleftarrow{\mathcal{P}}R$ from item (ii). The lax extension $\widetilde{\mathcal{M}}$ is defined on a relation $R : X \nrightarrow Y$ as

$$\widetilde{\mathcal{M}}R : \mathcal{M}X \nrightarrow \mathcal{M}Y$$
$$\widetilde{\mathcal{M}}R = \overrightarrow{\mathcal{P}}\overleftarrow{\mathcal{P}}R \cap \overleftarrow{\mathcal{P}}\overrightarrow{\mathcal{P}}R.$$

One can also define the α-ary version of $\widetilde{\mathcal{M}}$ that maps an $R : X \nrightarrow Y$ to

$$\widetilde{{}^\alpha\mathcal{M}}R : {}^\alpha\mathcal{M}X \nrightarrow {}^\alpha\mathcal{M}Y$$
$$\widetilde{{}^\alpha\mathcal{M}}R = \{(\xi, \upsilon) \mid \forall U \in \xi.\exists V \in \upsilon.\forall \beta \in \alpha.(U_\beta, V_\beta) \in \overleftarrow{\mathcal{P}}R\}$$
$$\cap \{(\xi, \upsilon) \mid \forall V \in \upsilon.\exists U \in \xi.\forall \beta \in \alpha.(U_\beta, V_\beta) \in \overrightarrow{\mathcal{P}}R\}.$$

It is easy to check the conditions (L1) and (L2) for $\widetilde{\mathcal{M}}$. To check (L3) we show that $(\xi, \mathcal{M}f(\xi)) \in \widetilde{\mathcal{M}}f$ for all functions $f : X \to Y$ and $\xi \in \mathcal{M}X$. For $(\xi, \mathcal{M}f(\xi)) \in \overrightarrow{\mathcal{P}}\overleftarrow{\mathcal{P}}f$ observe that $(U, f[U]) \in \overleftarrow{\mathcal{P}}f$ and $f[U] \in \mathcal{M}f(\xi)$ for any $U \in \xi$ because ξ is an upset. To get $(\xi, \mathcal{M}f(\xi)) \in \overleftarrow{\mathcal{P}}\overrightarrow{\mathcal{P}}f$ take any $V \in \mathcal{M}f(\xi)$. By the definition of $\mathcal{M}$ on morphisms this means that $f^{-1}[V] \in \xi$ and for this we have $(f^{-1}[V], V) \in \overrightarrow{\mathcal{P}}f$. To check condition (L4) we prove that $\xi \subseteq \xi'$ for any $(\xi, \xi') \in \widetilde{\mathcal{M}}\Delta_X$. A similar argument shows $\xi \supseteq \xi'$ and hence $(\xi, \xi') \in \Delta_{\mathcal{M}X}$. So let $(\xi, \xi') \in \widetilde{\mathcal{M}}\Delta_X$ and take any $U \in \xi$. It follows that there exists a $U' \in \xi'$ such that $(U, U') \in \overleftarrow{\mathcal{P}}\Delta_X$. This means that $U \supseteq U'$ and because ξ' is an upset, we get that $U \in \xi'$. Completely analogously one can verify that $\widetilde{{}^\alpha\mathcal{M}}$ is a lax extension of ${}^\alpha\mathcal{M}$ that preserves diagonals.

(v) The F_2^3 functor has a lax extension L_2^3 that preserves diagonals. L_2^3 is defined componentwise for any relation $R : X \nrightarrow Y$:

$$L_2^3R : F_2^3X \nrightarrow F_2^3Y,$$
$$L_2^3R = \{((x_0, x_1, x_2), (y_0, y_1, y_2)) \mid (x_0, y_0), (x_1, y_1), (x_2, y_2) \in R\}.$$

There is an easy counterexample to (L2) for the Barr extension $\overline{F_2^3}$ of F_2^3.

(vi) There is a lax extension $\widetilde{\mathcal{P}_n}$ of the restricted powerset functor $\mathcal{P}_n$ that preserves diagonals. It is defined in the same way as the Egli–Milner lifting $\overline{\mathcal{P}}$ of $\mathcal{P}$, that is $\widetilde{\mathcal{P}_n}R = \overrightarrow{\mathcal{P}}R \cap \overleftarrow{\mathcal{P}}R$ for any relation $R : X \nrightarrow Y$. Nevertheless, $\widetilde{\mathcal{P}_n}$ is distinct from the Barr extension $\overline{\mathcal{P}_n}$ of $\mathcal{P}_n$. As for $\overline{F_2^3}$ one can given a counterexample to (L2) for $\overline{\mathcal{P}_n}$ provided that $n > 3$.

Remark 8. In Example 7(iii) we noticed that the properties $LR\,;LS = L(R\,;S)$ and $Tf = Lf$ together uniquely determine a relation lifting. It seems likely that this is no longer the case if we loosen the latter condition and consider lax extensions that preserve diagonals. But we do not know of an example of a functor that has two distinct lax extensions that both preserve diagonals.

There is a large class of relations whose lifting by a lax extensions that preserve diagonals is unique: Because of Proposition 5(iv) lax extensions that preserve diagonals are uniquely determined on all relations $R : X \nrightarrow Y$ that arise as a pullback in the category of sets, that is they can be written as $R = f\,;g^\circ$ for functions $f : X \to Z$ and $g : Y \to Z$.

The conditions (L1), (L2) and (L3) of a lax extension L directly entail useful properties of L-bisimulations. The condition (L1) ensures that the union of L-bisimulations is again an L-bisimulation, (L2) yields that the composition of L-bisimulations is an L-bisimulation and because of (L3) coalgebra morphisms are L-bisimulations. Note also that our requirement that relation liftings preserve converses immediately implies that the converse of a bisimulation is a bisimulation. This facts are summarized in the following proposition whose easy proof is left to the reader.

Proposition 9. *For a lax extension L of T and T-coalgebras $\xi : X \to TX$, $\upsilon : Y \to TY$ and $\zeta : Z \to TZ$ it holds that*

(i) *The graph of every coalgebra morphism f from ξ to υ is an L-bisimulation between ξ and υ.*
(ii) *If $R : X \nrightarrow Z$ respectively $S : Z \nrightarrow Y$ are L-bisimulations between ξ and ζ respectively ζ and υ then their composition $R\,;S : X \nrightarrow Y$ is an L-bisimulation between ξ and υ.*
(iii) *Every union of L-bisimulations between ξ and υ is again an L-bisimulation between ξ and υ.*

Corollary 10. *Let L be a lax extension of T and $\xi : X \to TX$ and $\upsilon : Y \to TY$ be two T-coalgebras. The relation of L-bisimilarity $\leftrightarroweq^L_{\xi,\upsilon}$ between ξ and υ is itself an L-bisimulation between ξ and υ. Moreover L-bisimilarity $\leftrightarroweq^L_\xi : X \nrightarrow X$ on one single coalgebra ξ is an equivalence relation.*

We are now ready to prove that lax extensions that preserve diagonals capture behavioral equivalence. Note that in the proof the preservation of diagonals is only used for the application of Proposition 5(iv) at the end of the direction from bisimilarity to behavioral equivalence.

Theorem 11. *If L is a lax extension of T that preserves diagonals then L captures behavioral equivalence.*

Proof. We have to show that a state x_0 in a T-coalgebra $\xi : X \to TX$ and a state y_0 in a T-coalgebra $\upsilon : Y \to TY$ are behaviorally equivalent iff they are L-bisimilar.

For the direction from left to right assume that x_0 and y_0 are behaviorally equivalent. That means that there are a T-coalgebra $\zeta : Z \to TZ$ and coalgebra morphisms f from ξ to ζ and g from υ to ζ such that $f(x_0) = g(y_0)$. To see that x_0 and y_0 are L-bisimilar observe that by Proposition 9(i) and (ii) the relation $f\,;g^\circ : X \nrightarrow Y$ is an L-bisimulation between ξ and υ because it is the composition of graphs of coalgebra morphisms. This implies that x_0 and y_0 are L-bisimilar because $(x_0, y_0) \in f\,;g^\circ$.

In the other direction we have to show that for any pair $(x, y) \in R$, for an L-bisimulation $R : X \nrightarrow Y$ between ξ and υ, the states x and y are behaviorally equivalent. Without loss of generality we can consider the case of two states z and z' in one single coalgebra $\zeta : Z \to TZ$ with an L-bisimulation $S : Z \nrightarrow Z$ on ζ such that $(z, z') \in S$. This is because otherwise we let ζ be the coproduct of ξ and υ with injections i_X and i_Y and then consider the relation $S = i_X^\circ\,;R\,;i_Y$ which, using Proposition 9, can be shown to be an L-bisimulation on ζ.

Now consider the relation $\leftrightarroweq^L_\zeta : Z \nrightarrow Z$ of L-bisimilarity on ζ which by Corollary 10 is both an equivalence relation and an L-bisimulation. Our goal is to put a transition structure $\delta : Z/{\leftrightarroweq^L_\zeta} \to T(Z/{\leftrightarroweq^L_\zeta})$ on the quotient $Z/{\leftrightarroweq^L_\zeta}$ such that the projection $p : Z \to Z/{\leftrightarroweq^L_\zeta}, z \mapsto [z]$ becomes a coalgebra morphism from ζ to δ. Since we assume that $z \leftrightarroweq^L_\zeta z'$ it then follows that $p(z) = p(z')$ which witnesses that z and z' are behaviorally equivalent.

We intend to define the transition function δ on $Z/{\leftrightarroweq^L_\zeta}$ such that

$$\delta\big([z]\big) = Tp \circ \zeta(z).$$

This definition clearly satisfies $\delta \circ p = Tp \circ \zeta$ which means that p is a coalgebra morphism from ζ to δ as required. But we have to show that δ is well-defined. To prove this we need that $Tp \circ \zeta(z) = Tp \circ \zeta(z')$ for arbitrary $z, z' \in Z$ with $z \leftrightarroweq^L_\zeta z'$. Because $\leftrightarroweq^L_\zeta$ is an L-bisimulation it follows that $(\zeta(z), \zeta(z')) \in L{\leftrightarroweq^L_\zeta}$ and moreover

$$\begin{aligned} L{\leftrightarroweq^L_\zeta} &= L\big(p\,;p^\circ\big) && {\leftrightarroweq^L_\zeta} = p\,;p^\circ \\ &= Tp\,;(Tp)^\circ. && \text{Proposition 5(iv)} \end{aligned}$$

Hence $(\zeta(z), \zeta(z')) \in Tp\,;(Tp)^\circ$ and so $Tp \circ \zeta(z) = Tp \circ \zeta(z')$, as required. $\square$

4. (No) bisimulations for neighborhood frames

Already [5] examines relation liftings for the neighborhood functor $\mathcal{N}$, and the notions of bisimilarity they give rise to. It is found that none of the proposed relation liftings captures behavioral equivalence. In this section we show that actually no relation lifting for the neighborhood functor captures behavioral equivalence. Nevertheless, it should be mentioned that, for the simpler case of behavioral equivalence on one single coalgebra, already the Barr extension $\overline{\mathcal{N}}$ of the neighborhood functor captures behavioral equivalence [5, Proposition 3.20].

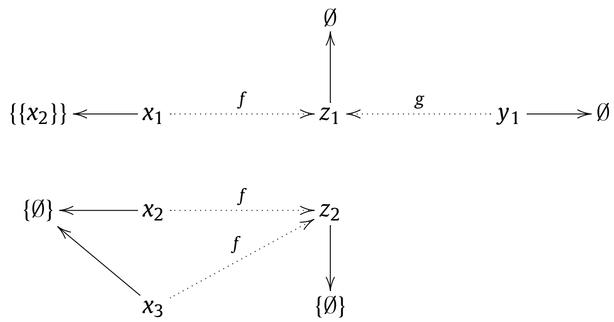
Theorem 12. *There is no relation lifting for the neighborhood functor $\mathcal{N}$ that captures behavioral equivalence.*

Proof. For the proof we need the fact that for any two functions $f : X \to Z$ and $g : Y \to Z$ we have that $\mathcal{N}f(\{\emptyset\}) \neq \mathcal{N}g(\emptyset)$. This holds because otherwise we would get by unfolding the definition of $\mathcal{N}$ on functions that

$$\begin{array}{rlr}
\emptyset & \in \big\{W \subseteq Z \,\big|\, f^{-1}[W] \in \{\emptyset\}\big\} & f^{-1}[\emptyset] = \emptyset \\
& = \mathcal{N}f\big(\{\emptyset\}\big) & \text{definition of } \mathcal{N} \\
& = \mathcal{N}g(\emptyset) & \text{assumption} \\
& = \big\{W \subseteq Z \,\big|\, g^{-1}[W] \in \emptyset\big\} & \text{definition of } \mathcal{N} \\
& = \emptyset, & V \notin \emptyset \text{ for all } V
\end{array}$$

which is clearly impossible.

Now suppose for a contradiction that there is a relation lifting L for $\mathcal{N}$ that captures behavioral equivalence. Consider an example with the coalgebras $\xi : X \to \mathcal{N}X$, where $X = \{x_1, x_2, x_3\}$ with $x_1 \mapsto \{\{x_2\}\}, x_2, x_3 \mapsto \{\emptyset\}$, $\upsilon : Y \to \mathcal{N}Y$ where $Y = \{y_1\}$ with $y_1 \mapsto \emptyset$, and $\zeta : Z \to \mathcal{N}Z$ with $Z = \{z_1, z_2\}$ with $z_1 \mapsto \emptyset, z_2 \mapsto \{\emptyset\}$. For these coalgebras, one can verify, that the functions $f : X \to Z, x_1 \mapsto z_1, x_2, x_3 \mapsto z_2$ and $g : Y \to Z, y_1 \mapsto z_1$ are coalgebra morphisms from ξ to ζ and from υ to ζ. Because $f(x_1) = g(y_1)$ this shows that x_1 and y_1 are behaviorally equivalent. The situation is depicted in the figure:



It follows from the assumption that L captures behavioral equivalence that there is an L-bisimulation $R : X \nrightarrow Y$ between ξ and υ such that $(x_1, y_1) \in R$. Moreover we can show that $(x_2, y_1), (x_3, y_1) \notin R$. We do this only for (x_2, y_1) since the argument for (x_3, y_1) is similar. Suppose for a contradiction that x_2 and y_1 are L-bisimilar. Because L captures behavioral equivalence, it follows that there is a coalgebra $\zeta'' : Z'' \to \mathcal{N}Z''$ and coalgebra morphisms j from ξ' to ζ'' and l from υ to ζ'' such that $j(x_2) = l(y_1)$. Using that j and l are coalgebra morphisms we get following contradiction to what we showed above:

$$\mathcal{N}j\big(\{\emptyset\}\big) = \mathcal{N}j \circ \xi(x_2) = \zeta'' \circ j(x_2) = \zeta'' \circ l(y_1) = \mathcal{N}l \circ \upsilon(y_1) = \mathcal{N}l(\emptyset).$$

So it follows that $R = \{(x_1, y_1)\}$ and because R is an L-bisimulation we find that $(\{\{x_2\}\}, \emptyset) = (\xi(x_1), \upsilon(y_1)) \in LR$.

Next we replace ξ with the coalgebra $\xi' : X \to \mathcal{N}X, x_1 \mapsto \{\{x_2\}\}, x_2 \mapsto \{\emptyset\}, x_3 \mapsto \emptyset$. We still have that $(\xi'(x_1), \upsilon(y_1)) = (\{\{x_2\}\}, \emptyset) \in LR$ which entails that $R = \{(x_1, y_1)\}$ is an L-bisimulation linking x_1 in ξ' and y_1 in υ. Because L captures behavioral equivalence it follows that there is a coalgebra $\zeta' : Z' \to \mathcal{N}Z'$ and there are coalgebra morphisms h from ξ to ζ' and k from υ to ζ' such that $h(x_1) = k(y_1)$. Because h and k are coalgebra morphism this implies that

$$\mathcal{N}h\big(\big\{\{x_2\}\big\}\big) = \mathcal{N}h \circ \xi'(x_1) = \zeta' \circ h(x_1) = \zeta' \circ k(y_1) = \mathcal{N}k \circ \upsilon(y_1) = \mathcal{N}k(\emptyset).$$

By writing out the definition of $\mathcal{N}$ one can see that this means

$$h^{-1}[C] \in \big\{\{x_2\}\big\} \quad \text{iff} \quad k^{-1}[C] \in \emptyset, \quad \text{for all } C \subseteq Z'.$$

Because the right hand side is never true it follows that $h^{-1}[C] \neq \{x_2\}$ for all $C \subseteq Z'$. In the special case $C = \{h(x_2)\}$ this means $h^{-1}[\{h(x_2)\}] \neq \{x_2\}$. Certainly $x_2 \in h^{-1}[\{h(x_2)\}]$ so it must be that $x_1 \in h^{-1}[\{h(x_2)\}]$ or $x_3 \in h^{-1}[\{h(x_2)\}]$. Thus

$h(x_2) = h(x_1)$ or $h(x_2) = h(x_3)$. Using that h and k are coalgebra morphisms we can calculate in the former case that

$$\begin{aligned}\mathcal{N}h\big(\{\emptyset\}\big) &= \mathcal{N}h \circ \xi'(x_2) = \zeta' \circ h(x_2) = \zeta' \circ h(x_1) = \zeta' \circ k(y_1) = \mathcal{N}k \circ \upsilon(y_1)\\ &= \mathcal{N}k(\emptyset)\end{aligned}$$

and in the latter case that

$$\mathcal{N}h\big(\{\emptyset\}\big) = \mathcal{N}h \circ \xi'(x_2) = \zeta' \circ h(x_2) = \zeta' \circ h(x_3) = \mathcal{N}h \circ \xi'(x_3) = \mathcal{N}h(\emptyset).$$

Both cases lead to the situation $\mathcal{N}f(\{\emptyset\}) = \mathcal{N}g(\emptyset)$ which, we argued above, is a contradiction. □

As a corollary we obtain that the neighborhood functor has no lax extension that preserves diagonals, since we know from Theorem 11 that such a relation lifting would capture behavioral equivalence.

Corollary 13. *There is no lax extension that preserves diagonals for the neighborhood functor $\mathcal{N}$.*

5. Lax extensions and predicate liftings

In the previous section we saw that the neighborhood functor does not have a lax extension that preserves diagonals. If we add the requirement that the neighborhoods are monotone, that is we look at the monotone neighborhood functor $\mathcal{M}$, then we have the lax extension $\widetilde{\mathcal{M}}$ that preserves diagonals. In this section we show that some sense of monotonicity is exactly what is needed from a functor in order to have a lax extension that preserves diagonals. Our goal is to prove the following theorem:

Theorem 14. *A finitary functor T_ω has a lax extension that preserves diagonals iff T_ω has a separating set of monotone predicate liftings.*

Proof. This is the overview of the proof which uses results that we establish in the remainder of this section.

For the direction from left to right assume that T_ω has a lax extension L that preserves diagonals. We use the canonical presentation of T_ω from Example 21 together with the natural transformation $\lambda^L : T_\omega\breve{\mathcal{P}} \Rightarrow \breve{\mathcal{P}}T$ from Definition 18 to construct the Moss liftings for T_ω defined as in Definition 23. In Proposition 24 we prove that the Moss liftings are monotone and in Proposition 25 that the set of all Moss liftings is finitely separating. Since T_ω is finitary we can use Proposition 2 to obtain that the Moss liftings are in fact separating for T_ω.

For the direction from right to left assume we have a separating set Λ of monotone predicate liftings for T_ω. By Proposition 1 the monotonicity of each $\lambda \in \Lambda$ entails that we can take $\lambda^\flat : T_\omega \to {}^n\mathcal{N}$ to have codomain ${}^n\mathcal{M}$. We can then apply the initial lift construction from Definition 15 to the set of natural transformations $\Gamma = \{e \circ \lambda^\flat : T_\omega \Rightarrow {}^\omega\mathcal{M}\}_{\lambda\in\Lambda}$, where $e : {}^n\mathcal{M} \Rightarrow {}^\omega\mathcal{M}$ is the embedding as defined in Section 2.4, and obtain a relation lifting $(\widetilde{{}^\omega\mathcal{M}})^\Gamma$ for the functor T_ω. We show in Proposition 16 that the relation lifting $(\widetilde{{}^\omega\mathcal{M}})^\Gamma$ is a lax extension for T that preserves diagonals, since $\widetilde{{}^\omega\mathcal{M}}$ is a lax extension for ${}^\omega\mathcal{M}$ that preserves diagonals and the set of functions $\{e_X \circ \lambda^\flat_X : T_\omega X \Rightarrow {}^\omega\mathcal{M}X\}_{\lambda\in\Lambda}$ is jointly injective at every set X because Λ is assumed to be separating. □

The only part where the assumption that T_ω is finitary is actually used is the application of Proposition 2 in the left to right direction. The construction of the Moss liftings in the left to right direction could be generalized to arbitrary accessible functors if we had allowed for predicate liftings of infinite arity. Thus, one would obtain a version of Theorem 14 for accessible functors.

We now describe the two constructions, initial lift and Moss liftings, that are used in the proof of Theorem 14. The initial lift of a lax extension along a set of natural transformations is taken from [17]. In the proof of Theorem 14 we use it to build a lax extension for T from the lax extension $\widetilde{{}^\omega\mathcal{M}}$ and a separating set of predicate liftings.

Definition 15. Let L be a relation lifting for T, and $\Lambda = \{\lambda : T' \Rightarrow T\}_{\lambda\in\Lambda}$ a set of natural transformations from another functor T' to T. Then we can define a relation lifting L^Λ for T' called the *initial lift of L along Λ* as

$$L^\Lambda R = \bigcap_{\lambda\in\Lambda} \big(\lambda_X \,;\, LR \,;\, \lambda_Y^\circ\big), \quad \text{for all sets } X, Y \text{ and } R : X \nrightarrow Y.$$

Equivalently to the above Definition, one can define $L^\Lambda R : T'X \nrightarrow T'Y$ for an $R : X \nrightarrow Y$ such that

$$(\xi, \upsilon) \in L^\Lambda R \quad \text{iff} \quad \big(\lambda_X(\xi), \lambda_Y(\upsilon)\big) \in LR, \quad \text{for all } \lambda \in \Lambda.$$

Next we show that the initial lift construction preserves laxness and, which is essential for Theorem 14, it also preserves condition (L4), if the set of natural transformations is jointly injective for every set.

Proposition 16. *Let $\Lambda = \{\lambda : T' \Rightarrow T\}_{\lambda\in\Lambda}$ be a set of natural transformations from a functor T' to a functor T and let L be a relation lifting for T. Then L^Λ is a lax extension for T' if L is a lax extension of T. Moreover, L^Λ preserves diagonals, if L preserves diagonals and $\{\lambda_X : T'X \to TX\}_{\lambda\in\Lambda}$ is jointly injective at every set X.*

Proof. It is routine to verify that all the conditions (L1), (L2) and (L3) are preserved by the initial lift construction. That the elements of Λ are natural transformations is only used for the preservation of (L3).

Here we give the proof for the claim that L^Λ preserves diagonals, if L does, and $\{\lambda_X : T'X \to TX\}_{\lambda\in\Lambda}$ is jointly injective at every set X. We first show that if $\{\lambda_X : T'X \to TX\}_{\lambda\in\Lambda}$ is jointly injective at every set X then

$$\bigcap_{\lambda\in\Lambda}\left(\lambda_X \,;\, \lambda_X^\circ\right) = \Delta_{T'X}. \tag{1}$$

For the $\subseteq$-inclusion take $\xi,\xi' \in T'X$ with $(\xi,\xi') \in \bigcap_{\lambda\in\Lambda}(\lambda_X \,;\, \lambda_X^\circ)$. This means that $\lambda_X(\xi) = \lambda_X(\xi')$ for every $\lambda \in \Lambda$. Because the λ_X for $\lambda \in \Lambda$ are jointly injective this implies that $\xi = \xi'$ and hence $(\xi,\xi') \in \Delta_{T'X}$. The $\supseteq$-inclusion follows from the fact that $f \,;\, f^\circ \supseteq \Delta_Y$ for any function $f : X \to Y$.

Now assume that L satisfies (L4) that is $L\Delta_X \subseteq \Delta_{TX}$ for every set X. It follows that $L^\Lambda\Delta_X \subseteq \Delta_{T'X}$ because

$$\begin{aligned}
L^\Lambda\Delta_X &= \bigcap_{\lambda\in\Lambda}\left(\lambda_X \,;\, L\Delta_X \,;\, \lambda_X^\circ\right) && \text{definition}\\
&\subseteq \bigcap_{\lambda\in\Lambda}\left(\lambda_X \,;\, \Delta_{TX} \,;\, \lambda_X^\circ\right) && \text{assumption}\\
&= \bigcap_{\lambda\in\Lambda}\left(\lambda_X \,;\, \lambda_X^\circ\right) && \Delta_{TX}\ \text{neutral element}\\
&= \Delta_{T'X}. && (1)
\end{aligned}$$

This shows that L^Λ satisfies (L4). □

Proposition 16 can be applied with a natural isomorphism between two set functors to show that naturally isomorphic functors possess corresponding lax extensions. We also obtain a lax extension $L_\omega = L^{\{\tau\}}$ of the finitary version T_ω of a functor T with lax extension L. Because the inclusion $\tau : T_\omega \Rightarrow T$ is injective L_ω preserves diagonals whenever L does.

Example 17. Consider the natural transformations $\Diamond, \Box : \mathcal{P} \Rightarrow \mathcal{M}$ with

$$\Diamond_X(U) = \{V \subseteq X \mid U \cap V \neq \emptyset\}, \qquad \Box_X(U) = \{V \subseteq X \mid U \subseteq V\}.$$

These natural transformation are clearly injective at every set X and hence it follows with Proposition 16 that $\widetilde{\mathcal{M}}^{\{\Diamond\}}$ and $\widetilde{\mathcal{M}}^{\{\Box\}}$ are lax extensions of the powerset functor $\mathcal{P}$ that preserve diagonals. Indeed, one can easily verify that they are both equal to the Barr extension $\overline{\mathcal{P}}$ of $\mathcal{P}$.

For the left-to-right direction of Theorem 14 we use the so called Moss liftings. It is shown in [8] that if we consider the Barr extension of a weak pullback preserving functor then the Moss liftings are monotone predicate liftings. Here we check that the argument also works for arbitrary lax extensions.

The first step in the construction of the Moss liftings is to use the lax extension L of T to define a distributive law between T and the contravariant powerset functor $\breve{\mathcal{P}}$.

Definition 18. Given a lax extension L of a functor T we define for every set X the function

$$\begin{aligned}
&\lambda_X^L : T_\omega\breve{\mathcal{P}}X \to \breve{\mathcal{P}}TX,\\
&\Xi \mapsto \left\{\xi \in TX \,\middle|\, \left(\xi, \tau_{\breve{\mathcal{P}}X}(\Xi)\right) \in L{\in_X}\right\},
\end{aligned}$$

where $\in_X : X \nrightarrow \mathcal{P}X$ denotes the membership relation between elements of X and subsets of X.

One might wonder why we define the type of λ^L with T_ω in its domain but T in its codomain. The reason for this is that we are going to use λ^L in Section 7 to define a semantics for a coalgebraic modal logic. There it connects the finitary syntactic side, hence T_ω, with the unrestricted semantic side. For the purpose of this section and Theorem 14, where we are only concerned with finitary functors, one can disregard any complications arising from the possible difference between T and T_ω.

Proposition 19. *For a lax extension L the mapping $\lambda^L : T_\omega\breve{\mathcal{P}} \Rightarrow \breve{\mathcal{P}}T$ from Definition 18 is a natural transformation.*

Proof. We verify that following diagram commutes for any $f : X \to Y$:

$$\begin{tikzcd} T_\omega \breve{\mathcal{P}} X \arrow[r, "\lambda^L_X"] & \breve{\mathcal{P}} T X \\ T_\omega \breve{\mathcal{P}} Y \arrow[u, "T_\omega \breve{\mathcal{P}} f"] \arrow[r, "\lambda^L_Y"] & \breve{\mathcal{P}} T Y \arrow[u, "\breve{\mathcal{P}} T f"'] \end{tikzcd} \tag{2}$$

First observe that

$$L{\in_X} \mathbin{;} \tau^\circ_{\breve{\mathcal{P}}X} \mathbin{;} (T_\omega \breve{\mathcal{P}} f)^\circ = Tf \mathbin{;} L{\in_Y} \mathbin{;} \tau^\circ_{\breve{\mathcal{P}}Y}. \tag{3}$$

This is shown by the calculation

$$\begin{aligned} L{\in_X} \mathbin{;} \tau^\circ_{\breve{\mathcal{P}}X} \mathbin{;} (T_\omega \breve{\mathcal{P}} f)^\circ &= L{\in_X} \mathbin{;} (T\breve{\mathcal{P}} f)^\circ \mathbin{;} \tau^\circ_{\breve{\mathcal{P}}Y} && \tau \text{ natural} \\ &= L\big({\in_X} \mathbin{;} (\breve{\mathcal{P}} f)^\circ\big) \mathbin{;} \tau^\circ_{\breve{\mathcal{P}}Y} && \text{Proposition 5(ii)} \\ &= L(f \mathbin{;} {\in_Y}) \mathbin{;} \tau^\circ_{\breve{\mathcal{P}}Y} && \text{direct verification} \\ &= Tf \mathbin{;} L{\in_Y} \mathbin{;} \tau^\circ_{\breve{\mathcal{P}}Y}. && \text{Proposition 5(ii)} \end{aligned}$$

To check the commutativity of (2) take an $\Upsilon \in T_\omega \breve{\mathcal{P}} Y$. We need to show that $\breve{\mathcal{P}} T f \circ \lambda^L_Y(\Upsilon) = \lambda^L_X \circ T_\omega \breve{\mathcal{P}} f(\Upsilon)$. This holds because for any $\xi \in TX$ we have that

$$\begin{aligned} \xi \in \lambda^L_X \circ T_\omega \breve{\mathcal{P}} f(\Upsilon) \quad &\text{iff} \quad \big(\xi, \tau_{\breve{\mathcal{P}}X} \circ T_\omega \breve{\mathcal{P}} f(\Upsilon)\big) \in L{\in_X} && \text{definition of } \lambda^L \\ &\text{iff} \quad (\xi, \Upsilon) \in L{\in_X} \mathbin{;} \tau^\circ_{\breve{\mathcal{P}}X} \mathbin{;} (T_\omega \breve{\mathcal{P}} f)^\circ && \text{basic set theory} \\ &\text{iff} \quad (\xi, \Upsilon) \in Tf \mathbin{;} L{\in_Y} \mathbin{;} \tau^\circ_{\breve{\mathcal{P}}Y} && (3) \\ &\text{iff} \quad \big(Tf(\xi), \tau_{\breve{\mathcal{P}}Y}(\Upsilon)\big) \in L{\in_Y} && \text{basic set theory} \\ &\text{iff} \quad Tf(\xi) \in \lambda^L_Y(\Upsilon) && \text{definition of } \lambda^L \\ &\text{iff} \quad \xi \in \breve{\mathcal{P}} T f \circ \lambda^L_Y(\Upsilon). && \text{definition of } \breve{\mathcal{P}} \qquad \square \end{aligned}$$

To define the Moss liftings we need, apart from the natural transformation $\lambda^L : T_\omega \breve{\mathcal{P}} \Rightarrow \breve{\mathcal{P}} T$, a finitary presentation of the functor T_ω. For more about presentations of set functors consult [2]. For more recent work on presentations in a general coalgebraic setting see [1,19].

Definition 20. A *finitary presentation* (Σ, E) of a functor T is a functor Σ of the form

$$\Sigma X = \coprod_{n \in \omega} \Sigma_n \times X^n,$$

where the Σ_n for any $n \in \omega$ are sets, together with a surjective natural transformation $E : \Sigma \Rightarrow T_\omega$.

One can show, as we do in Example 21, that every finitary functor has a finitary presentation. A finitary presentation of T_ω allows us to capture all the information in the sets $T_\omega X$ for a possibly very complex functor T_ω by means of a relatively simple polynomial functor Σ. This is, because for every $\xi \in T_\omega X$ there is some $(r, u) \in \Sigma_n \times X^n$ for an $n \in \omega$ for which $\xi = E_X(r, u)$ and that behaves in a similar way as ξ, since E is a natural transformation. In order to define predicate liftings for an arbitrary functor T it is necessary that we can somehow decompose it into pieces of the form X^n. This is exactly what the polynomial functor of a finitary presentation does.

Example 21. The next example shows that every finitary functor has a finitary presentation. The *canonical presentation* of a finitary functor T_ω is defined such that $\Sigma_n = T_\omega n$ for every cardinal $n \in \omega$ and E is defined at a set X as

$$\begin{aligned} E_X : \coprod_{n \in \omega} T_\omega n \times X^n &\to T_\omega X, \\ (\nu, u) &\mapsto T_\omega u(\nu), \quad \text{where } \nu \in T_\omega n \text{ and } u \in X^n \text{ for an } n \in \omega. \end{aligned}$$

In this definition we take $u \in X^n$ to be a function $u : n \to X$. It is routine to check that this definition indeed provides a finitary presentation of T_ω, meaning that E is a natural transformation and surjective at every set X.

For the next lemma recall from the preliminaries that we use the notation $\tau : T_\omega \Rightarrow T$ for the inclusion of the finitary version of a functor. The lemma shows how a lax extension of T interacts with a finitary presentation of T. This lemma is similar to the forth direction of [8, Lemma 6.3] where this result is proved for the Barr extension. One can use the lax extension L^3_2 of F^3_2 to construct an example which shows that the back direction of [8, Lemma 6.3] does not hold for lax extensions in general.

Lemma 22. *Let (Σ, E) be a finitary presentation of a functor T with lax extension L, and let $R : X \nrightarrow Y$ be any relation. Then it holds for all $n \in \omega$, $r \in \Sigma_n$, $u \in X^n$ and $v \in Y^n$ that $(\tau_X \circ E_X(r,u), \tau_Y \circ E_Y(r,v)) \in LR$ if $u_i R v_i$ for all $i \in n$.*

Proof. Let $\pi_Y : R \to X$ and $\pi_Y : R \to Y$ be the projections of R. For these it holds that $R = \pi_X^\circ ; \pi_Y$. Because $(u_i, v_i) \in R$ for all $i \in n$ we have that $\rho = (r, ((u_0, v_0), (u_1, v_1), \dots, (u_{n-1}, v_{n-1}))) \in \Sigma R^{gr}$. With the definition of Σ on morphisms it holds that $\Sigma\pi_X(\rho) = (r,u)$ and $\Sigma\pi_Y(\rho) = (r,v)$. Since $\tau \circ E$ is a natural transformation from Σ to T we also get that $\tau_X \circ E_X(r,u) = \tau_X \circ E_X(\Sigma\pi_X(\rho)) = T\pi_X(\tau_R \circ E_R(\rho))$ and $\tau_Y \circ E_Y(r,v) = \tau_Y \circ E_Y(\Sigma\pi_Y(\rho)) = T\pi_Y(\tau_R \circ E_R(\rho))$. It is entailed by these identities that $(\tau_X \circ E_X(r,u), \tau_R \circ E_R(\rho)) \in (T\pi_X)^\circ$ and that $(\tau_R \circ E_R(\rho), \tau_Y \circ E_Y(r,v)) \in T_\omega\pi_Y$. So we obtain

$$\begin{aligned}
\big(\tau_X \circ E_X(r,u), \tau_Y \circ E_Y(r,v)\big) \in (T\pi_X)^\circ ; (T\pi_Y) &\subseteq L\pi_X^\circ ; L\pi_Y && \text{(L3)}\\
&\subseteq L\big(\pi_X^\circ ; \pi_Y\big) = LR. && \text{(L2)}
\end{aligned}$$

Which is what we had to show. □

We can now define the Moss lifting for a functor T by composing the finitary presentation of T_ω with the natural transformation λ^L.

Definition 23. Given a functor T and a lax extension L for T take any finitary presentation (Σ, E) of T_ω according to Definition 20 and let $\lambda^L : T_\omega\breve{\mathcal{P}} \Rightarrow \breve{\mathcal{P}}T$ be the natural transformation of Definition 18. For every $r \in \Sigma_n$ of any $n \in \omega$ the *Moss lifting* of r is an n-ary predicate lifting for T that is defined as

$$\begin{aligned}
&\mu^r : \breve{\mathcal{P}}^n \Rightarrow \breve{\mathcal{P}}T,\\
&\mu^r = \lambda^L \circ E_{\breve{\mathcal{P}}}(r, \cdot).
\end{aligned}$$

This definition yields the following diagram for every set X:

$$\begin{tikzcd}
(\breve{\mathcal{P}}X)^n \arrow[r, "E_{\breve{\mathcal{P}}X}(r,\cdot)"] \arrow[dr, "\mu^r_X"'] & T_\omega\breve{\mathcal{P}}X \arrow[d, "\lambda^L_X"] \\
& \breve{\mathcal{P}}TX
\end{tikzcd}$$

We use Lemma 22 to show that the Moss liftings are monotone.

Proposition 24. *The Moss liftings of a functor T with finitary presentation (Σ, E) and lax extension L are monotone.*

Proof. Take any Moss lifting $\mu^r = \lambda^L \circ E_{\breve{\mathcal{P}}}(r,\cdot) : \breve{\mathcal{P}}^n \Rightarrow \breve{\mathcal{P}}T$ of an $r \in \Sigma_n$ for an $n \in \omega$. Now assume we have $U, U' \in (\breve{\mathcal{P}}X)^n$ for any set X such that $U_i \subseteq U'_i$ for all $i < n$. To prove that μ^r is monotone we need to show that $\mu^r_X(U) \subseteq \mu^r_X(U')$.

So pick any $\xi \in \mu^r_X(U) = \lambda^L_X \circ E_{\breve{\mathcal{P}}X}(r,U)$. By the definition of λ^L this means that $(\xi, \tau_{\breve{\mathcal{P}}X} \circ E_{\breve{\mathcal{P}}X}(r,U)) \in L{\in_X}$. Moreover, we get from the assumption that $U_i \subseteq U'_i$ for all $i \in n$ and Lemma 22 that $(\tau_{\breve{\mathcal{P}}X} \circ E_{\breve{\mathcal{P}}X}(r,U), \tau_{\breve{\mathcal{P}}X} \circ E_{\breve{\mathcal{P}}X}(r,U')) \in L(\subseteq)$. Putting this together yields

$$\begin{aligned}
\big(\xi, \tau_{\breve{\mathcal{P}}X} \circ E_{\breve{\mathcal{P}}X}\big(r,U'\big)\big) \in L{\in_X} ; L(\subseteq) &\subseteq L(\in_X ; \subseteq) && \text{(L2)}\\
&\subseteq L{\in_X}. && \text{(L1)}
\end{aligned}$$

For the last inequality we need that $(\in_X ; \subseteq) \subseteq \in_X$ which is immediate from the definition of subsets. So we have that $(\xi, \tau_{\breve{\mathcal{P}}X} \circ E_{\breve{\mathcal{P}}X}(r,U')) \in L{\in_X}$ and hence by the definition of λ^L that $\xi \in \lambda^L_X \circ \tau_{\breve{\mathcal{P}}X} \circ E_{\breve{\mathcal{P}}X}(r,U') = \mu^r_X(U')$. □

The last thing we have to show is that the set of all Moss liftings is separating. This is the only place in the construction of the Moss liftings where we actually need that the lax extension L preserves diagonals.

Proposition 25. *If L is a lax extension of a functor T that preserves diagonals and let (Σ, E) be a finitary presentation of T. Then the set of all Moss liftings $M = \{\mu^r : \breve{\mathcal{P}}^n \Rightarrow \breve{\mathcal{P}}T \mid r \in \Sigma_n, n \in \omega\}$ is finitely separating.*

Proof. To show that M is separating assume for arbitrary $\xi, \xi' \in TX$ of any finite set X that $(\mu^r)^\flat_X(\xi) = (\mu^r)^\flat_X(\xi')$ for all $r \in \Sigma_n$ of all $n \in \omega$. We need to prove that $\xi = \xi'$. First note that the embedding $\tau_X : T_\omega X \to TX$ is surjective because X is finite.

By the definition of the transposite of a natural transformation it follows from the assumption that for all $n \in \omega$ and $r \in \Sigma_n$

$$\big\{U \in (\breve{\mathcal{P}}X)^n \,\big|\, \xi \in \mu^r_X(U)\big\} = \big\{U \in (\breve{\mathcal{P}}X)^n \,\big|\, \xi' \in \mu^r_X(U)\big\}.$$

This is equivalent to

$$\xi \in \mu^r_X(U) \quad \text{iff} \quad \xi' \in \mu^r_X(U), \quad \text{for all } U \in (\breve{\mathcal{P}}X)^n.$$

Unfolding the definitions of $\mu^r = \lambda^L \circ E_{\breve{\mathcal{P}}}(r, \cdot)$ and $\lambda^L(\Xi) = \{\xi \in TX \mid (\xi, \tau_{\breve{\mathcal{P}}X}(\Xi)) \in L{\in_X}\}$ yields that for all $n \in \omega$, $r \in \Sigma_n$ and $U \in (\breve{\mathcal{P}}X)^n$

$$\big(\xi, \tau_{\breve{\mathcal{P}}X} \circ E_{\breve{\mathcal{P}}X}(r, U)\big) \in L{\in_X} \quad \text{iff} \quad \big(\xi', \tau_{\breve{\mathcal{P}}X} \circ E_{\breve{\mathcal{P}}X}(r, U)\big) \in L{\in_X}.$$

Because $\tau_{\breve{\mathcal{P}}X} \circ E_{\breve{\mathcal{P}}X}$ is surjective, and the variables n, r and U quantify over the full domain of $E_{\breve{\mathcal{P}}X} : \coprod_{n\in\omega}(\Sigma_n \times (\breve{\mathcal{P}}X)^n) \to T\breve{\mathcal{P}}X$, it follows that

$$(\xi, \Xi) \in L{\in_X} \quad \text{iff} \quad \big(\xi', \Xi\big) \in L{\in_X}, \quad \text{for all } \Xi \in T\mathcal{P}X. \tag{4}$$

To get $\xi = \xi'$ from (4) consider the map

$$s_X : X \to \mathcal{P}X,$$
$$x \mapsto \{x\}.$$

Because of (L3) we have that $(\xi, Ts_X(\xi)) \in Ts_X \subseteq Ls_X$. Moreover we clearly have that $s_X \subseteq {\in_X}$ and because of (L1) it follows that $(\xi, Ts_X(\xi)) \in L{\in_X}$. With (4) we get that $(\xi', Ts_X(\xi)) \in L{\in_X}$. Then we compute

$$\begin{aligned}
\big(\xi, \xi'\big) \in Ls_X \,;\, L{\ni_X} &\subseteq L(s_X \,;\, \ni_X) && \text{(L2)}\\
&= L\Delta_X && s_X \,;\, \ni_X = \Delta_X\\
&\subseteq \Delta_{TX}. && \text{(L4)}
\end{aligned}$$

From this it follows that $\xi = \xi'$, which finishes the proof. □

6. The logic of a lax extension

In this section we show how to define a semantics for the cover modality on T-coalgebras, using a lax extension L of T. For this purpose we fix a lax extension L of a functor T that from now on is assumed to preserve inclusions. This assumption guarantees that the usual definitions and proofs by induction on the complexity of formulas work well for a syntax that is based on the functor T. We do not lose any generality by making this assumption because as we discussed in the preliminaries every set functor is isomorphic to one that preserves inclusions. We also fix a countable set P of propositional variables.

The language of the cover modality is defined as follows:

Definition 26. The language $\mathcal{L}_L$ is defined as the set of all formulas that are generated by the grammar:

$$a ::= p \mid \neg a \mid \bigvee A \mid \nabla\alpha$$

where $p \in \mathsf{P}$, $A \in \mathcal{P}_\omega\mathcal{L}_L$ and $\alpha \in T_\omega\mathcal{L}_L$.

We use $\bigwedge A$, for an $A \in \mathcal{P}_\omega\mathcal{L}_L$, as an abbreviation for $\neg\bigvee\{\neg a \mid a \in A\}$.

The set $\mathcal{L}_L$ only depends on the functor T and not on the lax extensions L of T. We keep the subscript L however because we think of the $\mathcal{L}_L$ as a formal language together with its semantics on T-coalgebras which depends on the lax extension L. To define a semantics for the language $\mathcal{L}_L$ on T-coalgebras we first have to give an interpretation for the propositional variables. This is done by adding a valuation to T-coalgebras yielding T-models.

Definition 27. A *T-model* $\mathbb{X} = (\xi, V_{\mathbb{X}})$ is a T-coalgebra $\xi : X \to TX$ together with a *valuation* $V_{\mathbb{X}}$, that is a function $V_{\mathbb{X}} : \mathsf{P} \to \mathcal{P}X$. Morphisms and bisimulations between T models have to preserve the values of propositional variables. A morphism f from a T-model $\mathbb{X} = (\xi, V_{\mathbb{X}})$ to a T-model $\mathbb{Y} = (\upsilon, V_{\mathbb{Y}})$ is a T-coalgebra morphism f from ξ to υ such that $f^{-1}[V_{\mathbb{Y}}(p)] = V_{\mathbb{X}}(p)$ for all $p \in \mathsf{P}$. An L-bisimulation between $\mathbb{X}$ and $\mathbb{Y}$ is an L-bisimulation $R : X \nrightarrow Y$ between ξ and υ such that whenever $(x, y) \in R$ we have that

$$x \in V_{\mathbb{X}}(p) \quad \text{iff} \quad y \in V_{\mathbb{Y}}(p), \quad \text{for all } p \in \mathsf{P}.$$

We use the lax extension L to define a semantics of $\mathcal{L}_L$ on T-models.

Definition 28. Using the fixed lax extension L for the functor T we can define the *semantics* for the language $\mathcal{L}_L$ on T-models. For a T-model $\mathbb{X} = (\xi, V_{\mathbb{X}})$ define the satisfaction relation $\Vdash_{\mathbb{X}} : X \nrightarrow \mathcal{L}_L$ by recursion as

$$\begin{array}{lll} x \Vdash_{\mathbb{X}} p & \text{iff} \quad x \in V_{\mathbb{X}}(p), & p \in \mathsf{P} \\ x \Vdash_{\mathbb{X}} \neg a & \text{iff} \quad \text{not } x \Vdash_{\mathbb{X}} a, & a \in \mathcal{L}_L \\ x \Vdash_{\mathbb{X}} \bigvee A & \text{iff} \quad x \Vdash_{\mathbb{X}} a \text{ for some } a \in A, & A \in \mathcal{P}_\omega \mathcal{L}_L \\ x \Vdash_{\mathbb{X}} \nabla\alpha & \text{iff} \quad \big(\xi(x), \alpha\big) \in L\Vdash_{\mathbb{X}}. & \alpha \in T_\omega \mathcal{L}_L \end{array}$$

If $x \Vdash_{\mathbb{X}} a$ holds then we also say that the formula a is satisfied or true at the state x of the model $\mathbb{X}$. Since the model $\mathbb{X}$ is usually clear from the context we can omit the subscript $\mathbb{X}$ and just write $x \Vdash a$ if a is true at x.

We also define the extension $[\![a]\!]$ of a formula as the set of states, in some given model $\mathbb{X}$, where a is true. More precisely:

$$\begin{aligned} &[\![\cdot]\!] : \mathcal{L}_L \to \mathcal{P}X, \\ &a \mapsto \{x \in X \mid x \Vdash_{\mathbb{X}} a\}. \end{aligned}$$

Remark 29. The clauses in Definition 28 are not stated in a correct recursive way. In the recursive clause for the cover modality we make use of the unrestricted satisfaction relation $\Vdash_{\mathbb{X}}$ that has yet to be defined. We can only suppose that $\Vdash_{\mathbb{X}} \restriction_{X \times S}$ is already defined, where $S \subseteq \mathcal{L}_L$ is a set of formulas of less complexity than $\nabla\alpha$ with $\nabla\alpha \in TS \subseteq T\mathcal{L}_L$. The actual recursive definition is that $x \Vdash_{\mathbb{X}} \nabla\alpha$ iff $(\xi(x), \alpha) \in L(\Vdash_{\mathbb{X}} \restriction_{X \times S})$. We need a little argument why this is equal to the clause given above. Because T preserves inclusions we can use Proposition 6 to get that $(\xi(x), \alpha) \in L(\Vdash_{\mathbb{X}} \restriction_{X \times S}) = (L\Vdash_{\mathbb{X}}) \restriction_{TX \times TS}$ which is equivalent to $(\xi(x), \alpha) \in L\Vdash_{\mathbb{X}}$.

Remark 30. We give the semantics for $\mathcal{L}_L$ in a direct way with a recursive definition of the satisfaction relation. Alternatively one can also use an initial algebra approach where the natural transformation λ^L from Definition 18 plays a crucial role. See for instance [7] for this approach. The crucial observation there is that the semantics of the cover modality can be given by

$$[\![\nabla\alpha]\!] = \breve{\mathcal{P}}\xi \circ \lambda^L_X \circ \big(T_\omega [\![\cdot]\!]\big)(\alpha). \tag{5}$$

We will later make use of this fact later so let us see why it holds:

$$\begin{array}{llr} x \in [\![\nabla\alpha]\!] & \text{iff} \quad x \Vdash_{\mathbb{X}} \nabla\alpha & \text{Definition of } [\![\cdot]\!] \\ & \text{iff} \quad \big(\xi(x), \alpha\big) \in L\Vdash_{\mathbb{X}} & \text{semantics of } \nabla \\ & \text{iff} \quad \big(\xi(x), \alpha\big) \in L\big(\in_X ; [\![\cdot]\!]^\circ\big) & \in_X ; [\![\cdot]\!]^\circ = \Vdash_{\mathbb{X}} \\ & \text{iff} \quad \big(\xi(x), \alpha\big) \in L(\in_X) ; \big(T[\![\cdot]\!]\big)^\circ & \text{Proposition 5 (ii)} \\ & \text{iff} \quad \big(\xi(x), \tau_{\mathcal{L}_L}(\alpha)\big) \in L(\in_X) ; \big(T[\![\cdot]\!]\big)^\circ & \alpha \in T_\omega \mathcal{L}_L \subseteq T\mathcal{L}_L \\ & \text{iff} \quad \big(\xi(x), T[\![\cdot]\!] \circ \tau_{\mathcal{L}_L}(\alpha)\big) \in L(\in_X) & \text{set theory} \\ & \text{iff} \quad \big(\xi(x), \tau_{\breve{\mathcal{P}}X} \circ T_\omega [\![\cdot]\!](\alpha)\big) \in L(\in_X) & \tau \text{ natural trans.} \\ & \text{iff} \quad \xi(x) \in \lambda^L_X\big(\big(T[\![\cdot]\!]\big)(\alpha)\big) & \text{Definition 18} \\ & \text{iff} \quad x \in \breve{\mathcal{P}}\xi \circ \lambda^L_X \circ \big(T[\![\cdot]\!]\big)(\alpha). & \text{Definition of } \breve{\mathcal{P}} \end{array}$$

Now we are going to look at the expressive power of $\mathcal{L}_L$ with respect to states in T-models. For this we start with a definition.

Definition 31. Two states x in a T-model $\mathbb{X} = (\xi, V_{\mathbb{X}})$ and y in a T-model $\mathbb{Y} = (\upsilon, V_{\mathbb{Y}})$ are *modally equivalent* iff they satisfy the same formulas, that is:

$$x \Vdash_{\mathbb{X}} a \quad \text{iff} \quad y \Vdash_{\mathbb{Y}} a, \quad \text{for all } a \in \mathcal{L}_L.$$

The next proposition shows that whenever two states are L-bisimilar then they are modally equivalent. One can also prove a partial converse to this: Whenever two states in T_ω-coalgebras are modally equivalent then they are bisimilar. Because the proof of this result is technically a bit tedious we omit it here and refer the interested reader to [11].

Proposition 32 *(Adequacy). Given a state x_0 in a T-model $\mathbb{X} = (\xi, V_{\mathbb{X}})$ and a state y_0 in a T-model $\mathbb{Y} = (\upsilon, V_{\mathbb{Y}})$, if x_0 and y_0 are L-bisimilar then x_0 and y_0 are modally equivalent.*

Proof. Let R be an L-bisimulation between $\mathbb{X}$ and $\mathbb{Y}$ with $(x_0, y_0) \in R$ and let $\Phi \subseteq \mathcal{L}_L$ be the set of formulas on which bisimilar points agree, that is

$$\Phi := \left\{ a \in \mathcal{L}_L \;\middle|\; x \Vdash_{\mathbb{X}} a \text{ iff } y \Vdash_{\mathbb{Y}} a, \text{ for all } (x, y) \in R \right\}.$$

With this definition of Φ it is obvious that

$$R \,;\, \Vdash_{\mathbb{Y}} \restriction_{Y \times \Phi} \subseteq \Vdash_{\mathbb{X}} \restriction_{X \times \Phi},$$

and in the other direction that

$$R^{\circ} \,;\, \Vdash_{\mathbb{X}} \restriction_{X \times \Phi} \subseteq \Vdash_{\mathbb{Y}} \restriction_{Y \times \Phi}. \tag{6}$$

We are now going to prove that $\Phi = \mathcal{L}_L$. This entails that for every pair $(x, y) \in R$ the state x satisfies the same formulas as the state y. So in particular x_0 and y_0 satisfy the same formulas because $(x_0, y_0) \in R$.

We show with induction on the complexity of a formula $a \in \mathcal{L}_L$ that $a \in \Phi$. The base case $a = p \in \mathsf{P}$ follows directly from the condition that R is an L-bisimulation between the T-models $\mathbb{X}$ and $\mathbb{Y}$ and hence preserves the values of propositional variables. The Boolean cases are standard so we focus on the case where $a = \nabla\alpha$ for some $\alpha \in T_\omega \mathcal{L}_L$. The induction hypothesis is that $\alpha \in T_\omega \Phi$. We have to show that $x \Vdash_{\mathbb{X}} \nabla\alpha$ iff $y \Vdash_{\mathbb{Y}} \nabla\alpha$ for all $(x, y) \in R$.

So assume that $x \Vdash_{\mathbb{X}} \nabla\alpha$. By the definition of the satisfaction relation that means $(\xi(x), \alpha) \in L{\Vdash_{\mathbb{X}}}$ and because $\alpha \in T_\omega \Phi \subseteq T\Phi$ in particular that $(\xi(x), \alpha) \in (L{\Vdash_{\mathbb{X}}}) \restriction_{TX \times T\Phi} = L({\Vdash_{\mathbb{X}}} \restriction_{X \times \Phi})$ where the last equality holds by Proposition 6. Because R is an L-bisimulation we have that $(\upsilon(y), \xi(x)) \in LR^{\circ}$, and so we get

$$\begin{aligned}
\big(\upsilon(y), \alpha\big) \in LR^{\circ} \,;\, L({\Vdash_{\mathbb{X}}} \restriction_{X \times \Phi}) &\subseteq L\big(R^{\circ} \,;\, {\Vdash_{\mathbb{X}}} \restriction_{X \times \Phi}\big) && \text{(L2)} \\
&\subseteq L({\Vdash_{\mathbb{Y}}} \restriction_{Y \times \Phi}) && \text{(6) and (L1)} \\
&= (L{\Vdash_{\mathbb{Y}}}) \restriction_{TY \times T\Phi} && \text{Proposition 6} \\
&\subseteq L{\Vdash_{\mathbb{Y}}}.
\end{aligned}$$

This shows that $y \Vdash_{\mathbb{Y}} \nabla\alpha$. The other direction from $y \Vdash_{\mathbb{Y}} \nabla\alpha$ to $x \Vdash_{\mathbb{X}} \nabla\alpha$ is proved analogously. □

7. Translating the logics of lax extensions and predicate liftings

In this section we investigate how $\mathcal{L}_L$ relates to coalgebraic languages given by a set of predicate liftings. We show that $\mathcal{L}_L$ is intertranslatable with the logical language associated to its Moss liftings and that if L preserves diagonals and T preserves finite sets then every language of any set of predicate liftings can be translated into $\mathcal{L}_L$. We first define the syntax and semantics of the logic $\mathcal{L}_\Lambda$ of a set of predicate liftings.

Definition 33. Given a set of predicate liftings Λ for the functor T we define the language $\mathcal{L}_\Lambda$ by the grammar:

$$a ::= p \mid \neg a \mid \bigvee A \mid [\lambda]\bar{a}$$

where $p \in \mathsf{P}$, $A \in \mathcal{P}_\omega \mathcal{L}_L$, $\lambda \in \Lambda$, and $\bar{a} = (a_0, \ldots, a_{\mathsf{ar}(\lambda)-1}) \in (\mathcal{L}_\Lambda)^{\mathsf{ar}(\lambda)}$.

The semantics of $\mathcal{L}_\Lambda$ is given by the predicate liftings in Λ:

Definition 34. The *semantics* for the language $\mathcal{L}_\Lambda$ on a T-model $\mathbb{X} = (\xi, V_{\mathbb{X}})$ is given by the satisfaction relation $\Vdash_{\mathbb{X}} : X \nrightarrow \mathcal{L}_\Lambda$ that is defined recursively with the same clauses as in Definition 28 for propositional variables and Boolean connectives and the following clause for the modality of $\lambda \in \Lambda$:

$$x \Vdash_{\mathbb{X}} [\lambda]\bar{a} \quad \text{iff} \quad \xi(x) \in \lambda_X\big([\![a_0]\!], \ldots, [\![a_{\mathsf{ar}(\lambda)-1}]\!]\big) \quad \bar{a} \in (\mathcal{L}_\Lambda)^{\mathsf{ar}(\lambda)}$$

where as above $[\![\cdot]\!]$ is the extension of a formula:

$$\begin{aligned}
&[\![\cdot]\!] : \mathcal{L}_\Lambda \to \mathcal{P}X, \\
&a \mapsto \{x \in X \mid x \Vdash_{\mathbb{X}} a\}.
\end{aligned}$$

The semantics of $[\lambda]$ could be defined directly over the map $[\![\cdot]\!]$ as

$$[\![[\lambda]\bar{a}]\!] = \breve{\mathcal{P}}\xi \circ \lambda_X\big([\![a_0]\!], \ldots, [\![a_{\mathsf{ar}(\lambda)-1}]\!]\big).$$

We are interested in translations between the different languages $\mathcal{L}_L$, for a lax extension L of a functor T, and $\mathcal{L}_\Lambda$ for a set of predicate liftings Λ for T.

Definition 35. Let T be a fixed functor and let $\mathcal{L}_1$ and $\mathcal{L}_2$ be any of the languages $\mathcal{L}_L$, for a lax extension L of T, or $\mathcal{L}_\Lambda$, for a set Λ of predicate liftings of T. A *translation* from $\mathcal{L}_1$ to $\mathcal{L}_2$ is a function $\tau : \mathcal{L}_1 \to \mathcal{L}_2$ such that for all $a \in \mathcal{L}_1$ the formulas a and $\tau(a)$ are equivalent, which means that for all states x of all T-models $\mathbb{X}$ we have that

$$x \Vdash_{\mathbb{X}} a \quad \text{iff} \quad x \Vdash_{\mathbb{X}} \tau(a).$$

For a translation $\tau : \mathcal{L}_1 \to \mathcal{L}_2$ we also call $\mathcal{L}_1$ the *source language* and $\mathcal{L}_2$ the *target language*.

Since the Boolean part is the same in all the coalgebraic logics that we discuss the only difficulty for constructing translations is to translate the modalities of source language into a formula of the target language. But this can already be done on a one-step level where only one layer of modalities is considered.

Definition 36. The set of Boolean formulas $\mathsf{Bool}(C)$ over a set of formulas C is a build with the grammar:

$$a ::= c \mid \neg a \mid \bigvee A$$

where $c \in C$ and $A \in \mathcal{P}_\omega \mathsf{Bool}(C)$.

A *one-step formula* of the language $\mathcal{L}_L$ is an element of

$$\mathcal{L}_L^1 = \mathsf{Bool}\big(\big\{\nabla\alpha \bigm| \alpha \in T_\omega \mathsf{Bool}(\mathsf{P})\big\}\big).$$

These formulas are modal formulas that contain only one layer of modalities and where every propositional variable is in the scope of one modality.

Similarly an *one-step formula* of the language $\mathcal{L}_\Lambda$ is an element of

$$\mathcal{L}_\Lambda^1 = \mathsf{Bool}\big(\big\{[\lambda]\bar{a} \bigm| \bar{a} \in \mathsf{Bool}(\mathsf{P})^{\mathsf{ar}(\lambda)}, \lambda \in \Lambda\big\}\big).$$

Definition 37. The *Boolean semantics* of formulas in $\mathsf{Bool}(C)$ in a set X for a valuation $V : \mathsf{P} \to \mathcal{P}X$ is given by the function:

$$\begin{aligned}
&\mathsf{Bool}(V) : \mathsf{Bool}(C) \to \mathcal{P}X,\\
&\mathsf{Bool}(V)(p) = V(p),\\
&\mathsf{Bool}(V)(\neg a) = X \setminus \mathsf{Bool}(V)(a),\\
&\mathsf{Bool}(V)\Big(\bigvee A\Big) = \bigcup\big\{\mathsf{Bool}(V)(a) \bigm| a \in A\big\}.
\end{aligned}$$

The *one-step semantics* of a one-step formula $a \in \mathcal{L}_L^1$ in a set X for a valuation $V : \mathsf{P} \to \mathcal{P}TX$ is defined as $[\![a]\!]_V^1 = \mathsf{Bool}(h)(a) \in \mathcal{P}TX$, where

$$\begin{aligned}
&h : \big\{\nabla\alpha \bigm| \alpha \in T_\omega \mathsf{Bool}(\mathsf{P})\big\} \to \mathcal{P}TX,\\
&h(\nabla\alpha) = \lambda_X^L \circ \big(T_\omega \mathsf{Bool}(V)\big)(\alpha).
\end{aligned}$$

Similarly, one defines the *one-step semantics* for a one-step formula $a \in \mathcal{L}_\Lambda^1$ in a set X for a valuation V is defined as $[\![a]\!]_V^1 = \mathsf{Bool}(h)(a) \in \mathcal{P}TX$, where

$$\begin{aligned}
&h : \big\{[\lambda]\bar{a} \bigm| \bar{a} \in \mathsf{Bool}(\mathsf{P})^{\mathsf{ar}(\lambda)}, \lambda \in \Lambda\big\} \to \mathcal{P}TX,\\
&h\big([\lambda]\bar{a}\big) = \lambda_X\big(\mathsf{Bool}(V)(a_0), \ldots, \mathsf{Bool}(V)(a_{\mathsf{ar}(\lambda)-1})\big).
\end{aligned}$$

Two one-step formulas a and b, from either $\mathcal{L}_L^1$ or $\mathcal{L}_\Lambda^1$, are *one-step equivalent* if $[\![a]\!]_V^1 = [\![b]\!]_V^1$ for all sets X and valuations $V : \mathsf{P} \to \mathcal{P}X$.

A *one-step translation* from $\mathcal{L}_L$ to a language $\mathcal{L}$ assigns a one-step formula $b \in \mathcal{L}$ to every $\nabla\alpha$ with $\alpha \in T_\omega \mathsf{P}$ such that b is one-step equivalent to $\nabla\alpha$.

A *one-step translation* from $\mathcal{L}_\Lambda$ to a language $\mathcal{L}$ assigns a one-step equivalent one-step formula $b \in \mathcal{L}$ to every formula of the form $[\lambda](p_0, \ldots, p_{\mathsf{ar}(\lambda)-1})$, where $p_0, \ldots, p_{\mathsf{ar}(\lambda)-1}$ are propositional variables.

One can easily check that the one-step semantics corresponds with the actual semantics of $\mathcal{L}_L$ and $\mathcal{L}_\Lambda$ respectively in the sense that for every one-step formula a from either $\mathcal{L}_L$ or $\mathcal{L}_\Lambda$ and every T-model (ξ, V)

$$[\![a]\!] = \breve{\mathcal{P}}\xi\big([\![a]\!]_V^1\big).$$

Let us see a way how we can construct a translation from the full source language from a one-step translation of its modalities. For this we first put the formulas of the source language in bijective correspondence $\sigma : \mathsf{P} \cong \mathcal{L}$ with the set of propositional letters. Now, given a one-step translation from $\mathcal{L}_L$ to another language $\mathcal{L}$ one can define a translation $\tau : \mathcal{L}_L \to \mathcal{L}$ recursively by:

$$\tau(p) = p, \qquad \tau(\neg a) = \neg\tau(a),$$
$$\tau\Big(\bigvee A\Big) = \bigvee \mathcal{P}_\omega \tau(A), \qquad \tau(\nabla\alpha) = b[\tau \circ \sigma]$$

where b is the one-step translation of $\nabla T_\omega \sigma^{-1}(\alpha)$ and $b[\tau\circ\sigma]$ the formula b where all propositional letters p have been simultaneously substituted with $\tau\circ\sigma(p)$. It is clear that one can check that this translation preserves truth.

Similarly, we can construct a translation from $\mathcal{L}_\Lambda$ to $\mathcal{L}$ given a one-step translation for formulas of the form $[\lambda]\bar{a}$ for $\lambda \in \Lambda$ and $\bar{a} \in \mathsf{P}^{\mathsf{ar}(\lambda)}$. The crucial modal step becomes

$$\tau\big([\lambda]\bar{a}\big) = b[\tau\circ\sigma],$$

where b is the one-step translation of the formula $[\lambda](\sigma^{-1}(a_0), \ldots, \sigma^{-1}(a_{\mathsf{ar}(\lambda)-1}))$.

We now show that the logic $\mathcal{L}_L$ is intertranslatable with the logic of the Moss lifting of Λ and any finitary presentation of T as defined in Definition 23 in the previous section. This translation is completely straightforward and does not involve any Boolean connectives. So one can see the logic of the Moss liftings $\mathcal{L}_M$ to be just a different, more standard, syntactic presentation of $\mathcal{L}_L$.

Proposition 38. *The language $\mathcal{L}_L$ is intertranslatable, on a one-step level, with the language $\mathcal{L}_M$ where M is the set of Moss liftings for L and any finitary presentation (Σ, E) of T_ω.*

Proof. Recall from Definition 23 that the Moss lifting μ^r for an $r \in \Sigma_n$ of a finitary presentation (Σ, E) was defined as $\mu^r = \lambda^L \circ E_{\breve{\mathcal{P}}}(r, \cdot)$.

If we are now given a modality $[\mu^r](p_0, \ldots, p_{n-1})$ in $\mathcal{L}_M$ of an $r \in \Sigma_n$ we translate it to the formula $\nabla E_{\mathsf{P}}(r, (p_0, \ldots, p_{n-1}))$. Conversely if we are given a $\nabla\alpha$ in $\mathcal{L}L$, for some $\alpha \in T_\omega\mathsf{P}$, we translate it to $[\mu^r](p_0, \ldots, p_{n-1})$ for some $(r, (p_0, \ldots, p_{n-1})) \in \Sigma\mathsf{P}$ with $E_{\mathsf{P}}(r, (p_0, \ldots, p_{n-1})) = \alpha$. Such an object always exists because E is surjective. To show that this defines a one-step translation in either direction we need to show that

$$\big[\!\big[[\mu^r](p_0, \ldots, p_{n-1})\big]\!\big]^1_V = \big[\!\big[\nabla E_{\mathsf{P}}\big(r, (p_0, \ldots, p_{n-1})\big)\big]\!\big]^1_V$$

for all valuations $V : \mathsf{P} \to \mathcal{P}X$. This is the case because of

$$\begin{aligned}
\big[\!\big[[\mu^r](p_0, \ldots, p_{n-1})\big]\!\big]^1_V &= \mu^r_X\big(V(p_0), \ldots, V(p_{n-1})\big) && \text{Definition 37}\\
&= \lambda^L_X \circ E_{\breve{\mathcal{P}}X}\Big(r, \big(V(p_0), \ldots, V(p_{n-1})\big)\Big) && \text{Definition 23}\\
&= \lambda^L_X \circ (T_\omega V) \circ E_{\mathsf{P}}\big(r, (p_0, \ldots, p_{n-1})\big) && E \text{ natural}\\
&= \big[\!\big[\nabla E_{\mathsf{P}}\big(r, (p_0, \ldots, p_{n-1})\big)\big]\!\big]^1_V. && \text{Definition 37} \qquad \Box
\end{aligned}$$

The next proposition states that the one step-semantics of a one-step formula is essentially just a predicate lifting.

Proposition 39. *Every Boolean formula $a(p_0, \ldots, p_{n-1}) \in \mathsf{Bool}(\mathsf{P})$ over n propositional variables induces a natural transformation $\varphi^a : \breve{\mathcal{P}}^n \Rightarrow \breve{\mathcal{P}}$ such that for all valuations $V : \mathsf{P} \to \mathcal{P}X$ with $V(p_i) = U_i$ for $i < n$*

$$\varphi^a_X(U_0, \ldots, U_{n-1}) = \mathsf{Bool}(V)(a).$$

Every one-step formula $a(p_0, \ldots, p_{n-1}) \in \mathcal{L}^1_\Lambda$ with n propositional variables induces a natural transformation $\rho^a : \breve{\mathcal{P}}^n \Rightarrow \breve{\mathcal{P}}T$ such that for all valuations $V : \mathsf{P} \to \mathcal{P}X$ with $V(p_i) = U_i$ for $i < n$

$$\rho^a_X(U_0, \ldots, U_{n-1}) = [\![a]\!]^1_V.$$

Proof. One needs to check that the assignments for φ^a_X and ρ^a_X are well-defined, which is obvious, and that they are natural. Naturality for φ^a is the fact that the Boolean operations on set algebras are preserved under inverse images of functions. For the naturality of ρ^a we additionally need that the modalities of $\mathcal{L}_\Lambda$ are preserved under inverse images, which is the case because their semantics is given by predicate liftings. $\Box$

By Yoneda's Lemma predicate liftings can be shown to correspond to subsets of $T2^n$, where $2 = \{\top, \bot\}$ is the two element set. This crucial observation was first published in [16].

Proposition 40. *The n-ary predicate liftings for a functor T are in bijective correspondence to the elements of $\breve{\mathcal{P}}T2^n$. A predicate lifting $\lambda : \breve{\mathcal{P}}^n \Rightarrow \breve{\mathcal{P}}T$ corresponds to the set $\varphi_{2^n}(\pi_i^{-1}[\{\top\}])_{i<n} = \varphi_{\breve{\mathcal{P}}n}(\{S \subseteq n \mid i \in S\})_{i<n}$.*

Proof. First observe that there is a natural isomorphism:

$$i_X : (\breve{\mathcal{P}}X)^n \cong \left(2^X\right)^n \cong 2^{X\times n} \cong \left(2^n\right)^X = \mathsf{Hom}\left(X, 2^n\right).$$

Precomposing a predicate lifting with the inverse of i shows that predicate liftings can be seen as natural transformations from the contravariant hom-functor $\mathsf{Hom}(-, 2^n)$ to the functor $\breve{\mathcal{P}}T$. Now the statement follows from an application of Yoneda's Lemma for contravariant functors [9, Section III.2] and the observation that $(\pi_i^{-1}[\{\top\}])_{i<n} \in (\breve{\mathcal{P}}2^n)^n$ is the image of the identity id_{2^n} under the inverse of i_{2^n}. $\square$

Our goal is to prove Theorem 43 which states that every predicate lifting can be translated into a one-step formula in the language $\mathcal{L}_\Lambda$ of a separating set of predicate liftings for a functor that preserves finite sets. We first prove two lemmas.

Lemma 41. *Let $S \in (\mathcal{P}X)^n$ be separating. Then for any $U \subseteq X$ there is a Boolean formula $a(p_0, \dots, p_{n-1}) \in \mathcal{L}^0$ such that for the induced $\varphi^a : \breve{\mathcal{P}}^n \Rightarrow \breve{\mathcal{P}}$ it holds that $\varphi^a_X(S) = U$.*

Proof. Because the tuple S is separating we have for any $x \in U$ and $x' \in X \setminus U$ an $i_x^{x'} < n$ such that either $x \in S(i_x^{x'})$ and $x' \notin S(i_x^{x'})$ or that $x \notin S(i_x^{x'})$ and $x' \in S(i_x^{x'})$. Let $\pm S(i_x^{x'})$ be $S(i_x^{x'})$ in the former case and $X \setminus S(i_x^{x'})$ in the latter. Hence we have that $x \in \pm S(i_x^{x'})$ and $x' \notin \pm S(i_x^{x'})$.

Now define the formulas $l_x^{x'}$ for all $x \in U$ and $x' \in X \setminus U$ by

$$l_x^{x'} = \begin{cases} p_{i_x^{x'}} & \text{if } x \in S(i_x^{x'}) \text{ and } x' \notin S(i_x^{x'}), \\ \neg p_{i_x^{x'}} & \text{if } x \in X \setminus S(i_x^{x'}) \text{ and } x' \notin X \setminus S(i_x^{x'}). \end{cases}$$

Because the $i_x^{x'} < n$ for all $x \in U$ and $x' \in X \setminus U$ there are only finitely many distinct literals $l_x^{x'}$. Hence, the following is a well-defined Boolean formula:

$$a = \bigvee_{x\in U} \bigwedge_{x'\in X\setminus U} l_x^{x'}.$$

This formula induces a natural transformation $\varphi^a : \breve{\mathcal{P}} \Rightarrow \breve{\mathcal{P}}$ for which instantiated at the set X we have that

$$\varphi^a_X(S) = \bigcup_{x\in U} \bigcap_{x'\in X\setminus U} \pm S\left(i_x^{x'}\right) = U.$$

This is what we had to show. $\square$

Lemma 42. *The tuple $(\pi_i^{-1}[\{\top\}])_{i<n} \in (\breve{\mathcal{P}}2^n)^n$ is separating.*

Proof. Pick any two distinct sequences $\sigma, \sigma' \in 2^n$. Because they are distinct we can without loss of generality assume that $\pi_i(\sigma) = \sigma_i = \top$ whereas $\pi_i(\sigma') = \sigma'_i = \bot$ for some $i < n$. Hence $\sigma \in \pi_i^{-1}[\{\top\}]$ and $\sigma' \notin \pi_i^{-1}[\{\top\}]$. This shows that $(\pi_i^{-1}[\{\top\}])_{i<n} \in (\breve{\mathcal{P}}2^n)^n$ is separating. $\square$

Theorem 43. *Let Λ be a finitely separating set of predicate liftings for a finite set preserving functor T. Every n-ary predicate lifting $\delta : \breve{\mathcal{P}}^n \to \breve{\mathcal{P}}T$ is induced by a one-step formula a in the language $\mathcal{L}_\Lambda$.*

Proof. We construct the one-step formula a in the language $\mathcal{L}_\Lambda$ such that $\rho^a_{2^n}(\pi_i^{-1}[\{\top\}])_{i<n} = \delta_{2^n}(\pi_i^{-1}[\{\top\}])_{i<n} \in \breve{\mathcal{P}}T2^n$ which by Proposition 40 ensures that the predicate lifting ρ^a is equal to δ.

That Λ is finitely separating entails at the point 2^n that the set

$$Q = \left\{\lambda_{2^n}(U) \in \breve{\mathcal{P}}T2^n \;\middle|\; \lambda \in \Lambda, U \in \left(\breve{\mathcal{P}}2^n\right)^{\mathsf{ar}(\lambda)}\right\} \subseteq \mathcal{P}T2^n$$

is separating. This set Q is finite because it is a subset of $\mathcal{P}T2^n$ which is finite by the assumption that T preserves finite sets. Hence there is a finite number m and finitely many $\lambda^{(i)} \in \Lambda$ and $U^{(i)} \in (\breve{\mathcal{P}}2^n)^{\mathsf{ar}(\lambda^{(i)})}$ for all $i < m$ such that

$$Q = \left\{\lambda^{(i)}_{2^n}\left(U^{(i)}\right) \in \breve{\mathcal{P}}T2^n \;\middle|\; i < m\right\}.$$

So we can consider the tuple $S = (\lambda^{(0)}_{2^n}(U^{(0)}), \dots, \lambda^{(m-1)}_{2^n}(U^{(m-1)}))$ which is separating because Q is. Thus it follows from Lemma 41 that there is a Boolean formula b such that $\varphi^b_{2^n}(S) = \delta_{2^n}(\pi_i^{-1}[\{\top\}])_{i<n}$.

Next consider the $U^{(i)} = (U_0^{(i)}, \dots, U_{\mathrm{ar}(\lambda^{(i)})-1}^{(i)}) \in (\breve{\mathcal{P}}2^n)^{\mathrm{ar}(\lambda^{(i)})}$ for all $i < m$. For every $j < \mathrm{ar}(\lambda^{(i)})$ we can use Lemma 41 and Lemma 42 to get a Boolean formula $c_j^{(i)}$ with

$$\big(\varphi^{c_j^{(i)}}\big)_{2^n}\big(\pi_i^{-1}\big[\{\top\}\big]\big)_{i<n} = U_j^{(i)}.$$

Now define the one-step formula

$$a = b\big(\big[\lambda^{(0)}\big]\big(c_0^{(0)}, \dots, c_{\mathrm{ar}(\lambda')-1}^{(0)}\big), \dots, \big[\lambda^{(m-1)}\big]\big(c_0^{(m-1)}, \dots, c_{\mathrm{ar}(\lambda^{(m-1)})-1}^{(m-1)}\big)\big).$$

Putting all the equalities from above together we obtain for the natural transformation induced by a that $\rho_{2^n}^a(\pi_i^{-1}[\{\top\}])_{i<n} = \delta_{2^n}(\pi_i^{-1}[\{\top\}])_{i<n}$. This concludes the proof. $\square$

In the notation of the theorem it is clear that the formula a is one-step equivalent to $[\rho](p_0, \dots, p_{n-1})$ because the induced natural transformation φ^a is equal to ρ. Hence $a \in \mathcal{L}_\Lambda$ provides a one-step translation of $[\rho](p_0, \dots, p_{n-1})$ and we obtain the following corollary

Corollary 44. *Let Θ and Λ be sets of predicate liftings for a finite set preserving functor T such that Λ is separating. Then there is a translation from $\mathcal{L}_\Theta$ to $\mathcal{L}_\Lambda$.*

We can combine this result with the observation from Proposition 38 that $\mathcal{L}_L$ for a lax extension L is intertranslatable with $\mathcal{L}_M$ where M is the set of Moss liftings for L. By Proposition 25 these Moss lifting are finitely separating if L preserves diagonals. So we obtain the following two corollaries.

Corollary 45. *If L is a lax extension that preserves diagonals of a functor T that preserves finite sets then $\mathcal{L}_{L'}$, for any other lax extensions L' of T, and $\mathcal{L}_\Lambda$, for any set of predicate liftings Λ for T, can be translated to $\mathcal{L}_L$.*

Corollary 46. *If L and L' are two lax extensions of T that preserve diagonals and T preserves finite sets then $\mathcal{L}_L$ and $\mathcal{L}_{L'}$ are intertranslatable.*

8. Conclusions and open questions

In this paper we showed that lax extensions that preserve diagonals can be used in the theory of coalgebra to give a relational characterization of behavioral equivalence. This together with the fact that lax extensions can be used to define the semantics of an adequate cover modality indicates that lax extensions provide an adequate generalization for the role that the Barr extension of weak pullback preserving functors has played so far in the theory of coalgebras and coalgebraic modal logic. In this way the use of relation liftings in the theory of coalgebras can be extended to set functors that do not preserve weak pullbacks but nevertheless admit a lax extension that preserves diagonal relations.

The importance of lax extensions that preserve diagonals would motivate the study of their properties in their own right. A pressing question, that we were unable to answer, concerns the uniqueness of such lax extensions. We do not know of an example of a functor with two distinct lax extension that preserve diagonals. It would be interesting to find such an example or otherwise prove that any set functor has at most one lax extensions that preserve diagonals.

A negative result of this paper is that the neighborhood functor does not allow for a relation lifting that captures behavioral equivalence. This shows that there are limits to the use of relation liftings in the theory of coalgebras. A goal for further research would be to determine which functors have a relation lifting that captures behavioral equivalence. All the examples of such functors that we know of also have a lax extension that preserves diagonals. So it might turn out that, whenever a functor allows for a relational characterization of behavioral equivalence it has a lax extension that preserves diagonals.

A further, probably easier, problem would be to characterize the functors that have a lax extension that preserves diagonals. Our Theorem 14 is a first step in this direction but it only applies to finitary functors and the condition it gives, namely that the functor has a separating set of monotone predicate liftings, is not more fundamental than what it is supposed to characterize. It might be interesting to look for a more elementary definition for the kind of monotonicity a functor needs to posses in order to allow for a separating set of monotone predicate liftings or, respectively, for a lax extension preserving diagonals. Moreover, it would be nice to have a canonical way for obtaining a lax extension that preserves diagonals for all functors that possess one, similar to the definition of the Barr extension for weak pullback preserving functors.

We also presented the logic of a lax extension and showed that if the lax extension preserves diagonals and if the set functor preserves finite sets then other standard coalgebraic logics can be translated into it. We plan to write a follow-up paper in which we study some of the logical properties of the cover modality of a lax extension and prove a uniform interpolation for its logic. Some of these results can already be found in [11].

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