# Proof systems for Moss' coalgebraic logic

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**Abstract**

We study Gentzen-style proof theory of the finitary version of the coalgebraic logic introduced by L. Moss. The logic captures the behaviour of coalgebras for a large class of set functors. The syntax of the logic, defined uniformly with respect to a finitary coalgebraic type functor $T$, uses a single modal operator $\nabla_T$ of arity given by the functor $T$ itself, and its semantics is defined in terms of a relation lifting functor $\overline{T}$. An axiomatization of the logic, consisting of modal distributive laws, has been given together with an algebraic completeness proof in work of C. Kupke, A. Kurz and Y. Venema.

In this paper, following our previous work on structural proof theory of the logic in the special case of the finitary powerset functor, we present cut-free, one- and two-sided sequent calculi for the finitary version of Moss' coalgebraic logic for a general finitary functor $T$ in a uniform way. For the two-sided calculi to be cut-free we use a language extended with the boolean dual of the nabla modality.

**Keywords:** modal logic, coalgebraic logic, sequent system, coalgebra, cover modality, Gentzen calculus, completeness.

## 1 Introduction

The theory of coalgebras, introduced by Aczel in the late 1980s [1, 2], is a fast-growing research area in theoretical computer science which provides a unifying framework for state-based evolving systems. The unifying power of the coalgebraic theory of systems lies in that various classes of systems can be formalized as coalgebras of appropriate set (endo)functors $T$ (representing the *type* of the given class); hence their theory can be developed parametrically in $T$. As a key example, any set functor $T$ canonically induces a notion of observational or *behavioural equivalence* between $T$-coalgebras; this notion generalizes the natural notions of bisimilarity which were independently developed for each specific type of system.

In order to describe and reason about the behaviour of systems modelled by coalgebras, specification languages and derivation systems have been introduced, which gave rise to a research programme in its own right, namely *coalgebraic logic*. Coherently with the spirit

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of the coalgebraic theory of systems, coalgebraic logic aims at developing logical formalisms which can be defined and studied uniformly in the functor $T$. Since Kripke models and frames are exactly the $P$-coalgebras, i.e. the coalgebras of the powerset functor, and since *modal logic* is the archetypal bisimulation-invariant logic, the several current proposals for suitable coalgebraic logics, such as based on the so-called predicate lifting modalities [19, 25], or equivalently modalities arising from Stone type dualities [7, 14], or a framework making use of co-equations [3], are set out as generalizations of modal logic in some respects.

In the above mentioned framework of coalgebraic logic based on predicate liftings an extensive work has been done in the direction of proof theory and its applications by L. Schröder and D. Pattinson. Namely, in [20] sequent systems have been defined systematically from a given axiomatization of a logic, and cut admissibility has been proven for strictly one-step complete logics; in [21, 22] one-step completeness has been related to eliminability of cuts and a purely syntactic condition on the modal part of a calculus has been given for the calculus to admit cut elimination.

The research line of the present paper was initiated by Moss [17]; he defined an elegant generalization of modal logic, using the functor $T$ to define a modal connective $\nabla_T$ which takes in input elements $\alpha \in T_\omega \mathcal{L}$ (where $T_\omega$ is the finitary version of $T$, and $\mathcal{L}$ denotes the collection of formulas), and returns formulas $\nabla_T \alpha$.

If $T$ is the powerset functor $P$, its associated connective $\nabla_P$ takes a finite *set* $\alpha$ of formulas and returns a single formula $\nabla_P \alpha$. The semantics of $\nabla_P$ is defined as follows, for an arbitrary Kripke structure $\mathbb{S}$ with accessibility relation $R$:

$$\mathbb{S}, s \Vdash \nabla\alpha \quad \text{if} \quad \begin{array}{l} \text{for all } a \in \alpha \text{ there is a } t \in R[s] \text{ with } \mathbb{S}, t \Vdash a, \text{ and} \\ \text{for all } t \in R[s] \text{ there is an } a \in \alpha \text{ with } \mathbb{S}, t \Vdash a. \end{array} \tag{1}$$

The modal connective $\nabla$ can be seen as a defined connective in the standard modal language:

$$\nabla\alpha = \Box(\bigvee \alpha) \wedge \bigwedge \Diamond\alpha, \tag{2}$$

where $\Diamond\alpha$ denotes the set $\{\Diamond a \mid a \in \alpha\}$. But in fact the following semantic equivalences hold for $\nabla$ interpreted as in (1):

$$\Diamond\alpha \quad \equiv \quad \nabla\{\alpha, \top\} \qquad\qquad \Box\alpha \quad \equiv \quad \nabla\varnothing \vee \nabla\{\alpha\}, \tag{3}$$

which show that the language based on $\nabla$ is an alternative formulation of standard modal logic.

Analogously to the duality between $\Box$ and $\Diamond$, $\nabla$ admits a dual operator $\Delta$, which, interestingly, can be defined in the following, negation-free way:

$$\Delta\alpha = \begin{cases} \nabla\varnothing \vee \bigvee\{\nabla\{a\} \mid a \in \alpha\} \vee \nabla\{\bigwedge \alpha, \top\} & \text{if } \alpha \neq \varnothing \\ \nabla\{\top\} & \text{if } \alpha = \varnothing \end{cases} \tag{4}$$

The $\nabla$-based reformulation of modal logic lends itself naturally to a coalgebraic generalization, thanks to the following fundamental observation, due to Moss [17]: the satisfaction clause (1) is equivalent to

$$\mathbb{S}, s \Vdash \nabla_P \alpha \quad \text{iff} \quad (R[s], \alpha) \in \overline{P}(\Vdash), \tag{5}$$

where $\overline{P}(\Vdash)$ denotes the Egli-Milner (i.e. the $P$-) *relation lifting* of the satisfaction relation $\Vdash$ between states and formulas. Hence, for every set-endofunctor $T$, every $T$-coalgebra $\sigma : X \to TX \times \mathcal{P}(AtProp)$ and every state $s \in X$,

$$\sigma, s \Vdash \nabla_T \alpha \quad \text{iff} \quad (\sigma(s), \alpha) \in \overline{T}(\Vdash), \tag{6}$$

where $\overline{T}(\Vdash) \subseteq TX \times T\mathcal{L}$ denotes the $T$-*relation lifting* of $\Vdash \subseteq X \times \mathcal{L}$ (see also Definition 2.7 for more details and discussion). A well-behaved functorial relation lifting is not available in general: it is available for a large class of set functors, namely for those preserving *weak pullbacks*. (For the same class of functors bisimilarity captures behavioural equivalence.) It leaves out certain interesting functors, for example the double contravariant powerset functor $\breve{P}\breve{P}$, coalgebraically capturing the neighbourhood frames. However, in many cases, such as the related *monotone* neighborhood frames, there is an alternative way of defining relation lifting and subsequently use similar methods to develop a Moss-style logic, as shown in [24, 16].

Moss' finitary coalgebraic logic, despite its nonstandard syntax and limitation to functors preserving weak pullbacks, proved to be an interesting field of study and found applications in logic and automata theory. It is finitely expressive in the sense that it can distinguish any two non-bisimilar states of two finitely branching coalgebras. Its language allows for a disjunctive normal form, which in the case of the powerset functor can be recognized already in the work of K. Fine [8], or in the work of D. Janin and I. Walukiewicz on the automata theory approach to the completeness of modal $\mu$-calculus [9]. The work of D. Janin and I. Walukiewicz inspired Y. Venema [27] to introduce a finitary version of Moss' logic extended with fixpoint operators, and thus to generalize the link between automata theory and fixpoint logics to the coalgebraic level of generality. C. Kupke and Y. Venema also showed that many results of automata theory can be seen as theorems of universal coalgebra [12, 13]. A modular axiomatization of Moss' logics can be given in a uniform way, parametric in the functor $T$, using modal distributive laws. Using this axiomatization an algebraic completeness proof has been given in [11].

In earlier work [6], we set out to develop the Gentzen-style proof theory of $\nabla$-style coalgebraic logic, and we introduced two *sequent calculi*: a one-sided sequent calculus for an expansion of the Boolean propositional language with the modal connective $\nabla_P$; this calculus was shown to be sound, complete w.r.t. the class of all Kripke models, and cut-free; a sound and complete two-sided sequent calculus for the negation-free fragment of the same language, the cut rule of which had been shown to be *not eliminable* (an example of a sequent with no cut-free proof has been given, cf. Example 4.12). Both calculi are generalizable to $\nabla$-style coalgebraic languages for arbitrary weak pullback-preserving $\mathsf{Set}$-endofunctors $T$.

**Contributions of the present paper.** Our main goal is to present a uniform Gentzen-style proof theory for Moss' logic. We introduce one- and two-sided sequent calculi for Moss' logic in a uniform way, parametric in the functor $T$. The calculi share the following structural properties: most importantly, they are cut-free complete. They consist of a suitable fixed propositional part extended with modal rules parametric in the functor $T$. All the rules (except weakening) are invertible (the modal rules are invertible in a weaker sense than the usual invertibility of sequent rules, cf. Lemma 4.5 and 5.2) which allows for a completeness proof. The two-sided calculi are defined for the language using both $\nabla$ and its boolean dual $\Delta$ as primitive modalities: this seems to be necessary to obtain a cut-free presentation. The one-sided calculi are defined in a similar way. For the $\Delta$-free fragment of the language, i.e. with $\nabla$ as the only modal operator, one-sided calculi are still available — they moreover behave better since they satisfy the genuine subformula property.

The present paper extends the earlier work [6] in two directions.
*First direction.* Focusing on the powerset functor $P$, we consider the corresponding coalgebraic language in which both $\nabla_P$ and $\Delta_P$ are taken as primitive, and introduce the *cut-free*, two-sided sequent calculus $S2_P$ (cf. Definition 4.4). The system $S2_P$ is sound and complete w.r.t. the class of Kripke frames (cf. Theorem 4.8). We also introduce the cut-free, one-sided sequent calculus $S1_P$ for the restricted language $\mathcal{L}^*$ of $\mathcal{L}$ in which negation can only be applied to proposition letters, and prove that $S1_P$ is sound and complete w.r.t. Kripke frames.

As was the case in [6], the notion of *slim redistribution* (cf. Definition 3.20) is the main technical ingredient in the formulation of the proof rules in $S1_P$, which guarantees its being generalizable to wide classes of functors. In its specific formulation for the power set functor, a set $\Phi \in PPX$ is a slim redistribution of a set $A \in PPX$ (notation: $\Phi \in SRD(A)$) iff $\bigcup A = \bigcup \Phi$ and $\varphi \cap \alpha \neq \varnothing$ for all $\varphi \in \Phi$ and $\alpha \in A$. However, the technical improvement over [6] is that we refined the notion of slim redistribution to that of *separated slim redistribution* (cf. Definition 3.23), which is the key to our formulation of the cut-free, two-sided system $S2_P$. Here *separated* refers to the fact that in the definition, we separate the formulas stemming from the left- and the right-hand side of the sequent, respectively.
*Second direction.* Focusing on an arbitrary functor $T$ which preserves inclusions and weak pullbacks, and again considering the corresponding coalgebraic language in which both $\nabla_T$ and $\Delta_T$ are taken as primitive, we introduce the *cut-free*, two-sided sequent calculus $\mathrm{G}2_T$ (cf. Definition 5.1) and the *cut-free*, one-sided sequent calculus $\mathrm{G}1_T$ (cf. Definition 5.8) for the corresponding restriction $\mathcal{L}^*$ defined as above; again, both calculi are shown to be sound and complete w.r.t. $T$-coalgebras (cf. Theorem 5.4 and 5.11). As to the relationship between the calculi for $P$ and for $T$, we remark that $S2_P$ and $S1_P$ are *not* mere instantiations of the more general $G2_T$ and $G1_T$ respectively, but are actual simplifications; we refer to Remark 4.2 for more details on this point. We also define the one-sided calculus $G1_{T\nabla}$ (cf. Definition 5.12) for the $\Delta$-free fragment of the language, which is merely the $\Delta$-free fragment of the calculus $G1_T$.

**Structure of the present paper.** In Section 2, we collect some preliminaries on coalgebras (Subsection 2.2), relation lifting (Subsection 2.3), and sequent calculi for propositional logics (Subsection 2.3). Section 3 reviews the finitary version of Moss' coalgebraic logic in its syntax (Subsection 3.1) and semantics (Subsection 3.2), and introduces the main technical ingredients of the paper, namely (separated) slim redistributions (Subsection 3.3). In Section 4, the sequent calculi $S2_P$ and $S1_P$ are introduced and proven to be sound and complete (Subsections 4.1 and 4.2 respectively). In Section 5, the sequent calculi $G2_T$ and $G1_T$ are introduced and proven to be sound and complete (Subsections 5.1 and 5.2 respectively). For the $\Delta$-free fragment of the Moss' language, one-sided calculi are introduced (Subsections 4.3. and 5.3.). In section 6 we briefly discuss finitarity (Subsection 6.1) of the calculi introduced in this paper, and the subformula property (subsection 6.2).

## 2 Preliminaries

In this section we list some notions that we consider to be background knowledge in the remainder of the paper, and we fix notation and terminology.

### 2.1 Categories and Coalgebras

We assume familiarity with basic notions from category theory (such as categories, functors, and natural transformations) [15], and from universal coalgebra [23]. We restrict attention to $\mathsf{Set}$-based coalgebras, where $\mathsf{Set}$ denotes the category with sets as objects and functions as arrows. We let $P$ and $\breve{P}$ denote, respectively, the co- and contravariant power set functor.

**Convention 2.1** Throughout the paper we fix a functor $T : \mathsf{Set} \to \mathsf{Set}$, which we assume to preserve inclusions and weak pullbacks.

**Remark 2.2** Functors preserving inclusions were called *standard* in [17, 11] (definition of standardness is not uniform in the literature, in [4] one more condition is required). However, the restriction that $T$ preserves inclusions is for reasons of presentation only; given an arbitrary set functor $T$, we may find a standard set functor $T'$ such that the restriction of $T$ and $T'$ to all non-empty sets and non-empty functions are naturally isomorphic, as has been shown in [4].

The *finitary version* $T_\omega : \mathsf{Set} \to \mathsf{Set}$ of $T$ is given, on objects, by $T_\omega X := \bigcup\{TY \mid Y \in P_\omega X\}$, where $P_\omega$ denotes the finitary power set functor, and on arrows by $T_\omega f := Tf$. It can be proved that $T_\omega$ also preserves inclusions and weak pullbacks. Given an object $\xi \in T_\omega X$, we let $Base_X(\xi)$ denote the smallest finite subset of $A$ such that $\xi \in TBase_X(\xi)$; in fact, the family of operations $Base_X : T_\omega X \to P_\omega X$ constitutes a natural transformation $Base : T_\omega \overset{\cdot}{\to} P_\omega$.

**Definition 2.3** A *$T$-coalgebra* is a pair $(S, \sigma)$ where $S$ is a set and $\sigma : S \to TS$ is a function; the functor $T$ is called the *type* of the coalgebra. A *morphism of $T$-coalgebras* from $(S, \sigma)$ to $(S', \sigma')$, written $f : (S, \sigma) \to (S', \sigma')$, is a function $f : S \to S'$ such that $Tf \circ \sigma = \sigma \circ f$, that is, the following diagram commutes:

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, "\sigma"'] & S' \arrow[d, "\sigma'"] \\
TS \arrow[r, "Tf"] & TS'
\end{tikzcd}$$

◁

**Example 2.4** Throughout this paper we mostly refer to the class of *extended Kripke polynomial functors* (or to their finitary versions) obtained by the following grammar:

$$T \ := \ Id \mid C \mid P \mid B \mid D \mid T_0 \circ T_1 \mid T_0 \times T_1 \mid T_0 + T_1 \mid T^C,$$

where $C$ is a constant functor (a set), $\circ$, $+$ and $\times$ denote the functor composition, coproduct and product respectively, $T^C$ denotes exponentiation with respect to a set $C$.

- $P$ is the covariant powerset functor, it acts on morphisms as the direct image. *Kripke frames* can be seen as coalgebras for $P$, image finite Kripke frames are coalgebras for its finitary version $P_\omega$. *Kripke models* over a set $\mathsf{Prop}$ can be seen as coalgebras for the functor $P(\mathsf{Prop}) \times P(-)$. *Labelled transition systems* with a set of labels $A$ are coalgebras for the functor $P(-)^A$. Various types of automata can be modeled as coalgebras using this class of functors as well.

- $B$ is the *bag*, or *multiset*, functor. It takes a set $X$ to the set of multisets on $X$ — maps $\mu : X \to \mathbb{N}$ — and acts on morphisms $f : X \to Y$ as follows:

  $$(Bf)\mu(y) := \sum_{f(x)=y} \mu(x).$$

  The finitary version $B_\omega$ takes a set $X$ to the set of multisets $\mu$ on $X$ with *finite support*: the set $\{x \mid \mu(x) > 0\}$ is finite.

- $D$ is the *probability distribution* functor which maps a set $X$ to $DX := \{\delta : X \to [0,1] \mid \sum_{x\in X} \delta(x) = 1\}$, and acts on morphisms as the bag functor does. Coalgebras for the distribution functor are probabilistic Kripke frames.

All the extended Kripke polynomial functors preserve weak pullbacks, and inclusions (only the bag functor has to be "standardized" by representing every $\mu : X \to \mathbb{N}$ by its positive graph $\{[x, \mu(x)] \mid \mu(x) > 0\}$).

As running examples we will use (finitary versions of) the powerset functor, the bag functor, and the binary tree functor $Id \times Id$.

The key notion of equivalence in coalgebra is of two states in two coalgebras being *behaviorally equivalent.*

**Definition 2.5** Two elements (often called states) $s, s'$ in two coalgebras $(S, \sigma)$ and $(S', \sigma')$, respectively, are *behaviorally equivalent* iff there are coalgebra morphisms $f : S \to \mathbb{X}$ and $f' : \mathbb{S}' \to \mathbb{X}$ with a common codomain $\mathbb{X}$ such that $f(s) = f'(s')$. $\lhd$

## 2.2 Relation lifting

As mentioned, in the theory of Moss' coalgebraic logic a key role is played by the categorical notion of *relation lifting* that we will now briefly discuss.

We consider the categories $\mathsf{Set}$ of sets and functions, and $\mathsf{Rel}$ of sets and relations. We treat a relation $R$ from $X$ to $Y$ as an arrow $R : X \longrightarrow\!\!\!\!\!\!\!\!+\;\; Y$ in $\mathsf{Rel}$, but we also deal with it as with the set $R \subseteq X \times Y$ in $\mathsf{Set}$ whenever convenient.

We introduce some notation for relations and functions. The *graph* of a function $f : X \to Y$ is the relation $Grf : X \longrightarrow\!\!\!\!\!\!\!\!+\;\; Y$ defined $Grf := \{(x, f(x)) \in X \times Y \mid x \in X\}$.

The *diagonal* relation on a set $X$ is denoted as $Id_X : X \longrightarrow\!\!\!\!\!\!\!\!+\;\; X$ and defined $Id_X := \{(x, x) \mid x \in X\}$. The *converse* of a relation $R : X \longrightarrow\!\!\!\!\!\!\!\!+\;\; Y$ is the relation $R^{\smile} : Y \longrightarrow\!\!\!\!\!\!\!\!+\;\; X$, defined $R^{\smile} := \{(y,x) \mid (x,y) \in R\}$. Given subsets $Y \subseteq X$, $Y' \subseteq X'$, the *restriction* of $R$ to $Y$ and $Y'$ is given as $R\restriction_{Y\times Y'} := R \cap (Y \times Y')$. The composition of two relations $R : X \relbar\joinrel\mapstochar\joinrel\rightarrow X'$ and $R : X' \relbar\joinrel\mapstochar\joinrel\rightarrow X''$ is denoted by $R;R'$, whereas the composition of two functions $f : X \to X'$ and $f' : X' \to X''$ is denoted by $f' \circ f$ or $f'f$. Thus, we have e.g. $Gr(f' \circ f) = Grf \,;\, Grf'$.

Sets and relations actually form a 2-category $\mathsf{Rel}$: the two-dimensional structure (which is a preorder) on relations is given by *inclusion*: a two-cell

$$\begin{tikzcd} X \arrow[r, bend left, "R"] \arrow[r, bend right, "S"'] \arrow[r, phantom, "\downarrow"] & X' \end{tikzcd}$$

denotes the fact that $R \subseteq S$.

It is easy to see that $Gr(-) : \mathsf{Set} \to \mathsf{Rel}$ is a functor (as it clearly preserves identities and composition).

It has been proved independently by Trnková in [26] and Barr in [5] that a set functor $T$ preserves weak pullbacks if and only if it admits a lifting to a functor $\overline{T}$ on the category $\mathsf{Rel}$:

**Theorem 2.6** *For a functor $T : \mathsf{Set} \to \mathsf{Set}$ the following are equivalent:*

1. *There is a 2-functor $\overline{T} : \mathsf{Rel} \to \mathsf{Rel}$ such that the square*

$$\begin{tikzcd} \mathsf{Rel} \arrow[r, "\overline{T}"] & \mathsf{Rel} \\ \mathsf{Set} \arrow[u, "Gr(-)"] \arrow[r, "T"'] & \mathsf{Set} \arrow[u, "Gr(-)"'] \end{tikzcd} \tag{7}$$

   *commutes.*

2. *The functor $T$ preserves weak pullbacks.*

3. *there is a distributive law $\lambdabar^T : TP \to PT$ of $T$ over, respectively, the monad $P$, and the contravariant power set functor $\breve{P}$. (In particular, $\lambdabar^T$ is a natural transformation $\lambdabar^T : TP \dot{\to} PT$ and $\lambdabar^T : T\breve{P} \dot{\to} \breve{P}T$.)*

The relation lifting arising from the previous theorem is defined as follows:

**Definition 2.7** Given a binary relation $R : X_1 \relbar\joinrel\mapstochar\joinrel\rightarrow X_2$, we define its $T$*-lifting* $\overline{T}R : TX_1 \relbar\joinrel\mapstochar\joinrel\rightarrow TX_2$ as follows:

$$\overline{T}R := \{((T\pi_1^R)\rho, (T\pi_2^R)\rho) \mid \rho \in TR\},$$

where $\pi_i^R : R \to X_i$ denotes the projection functions from $R$ to $X_i$. $\lhd$

**Example 2.8** Fix $R : X \relbar\joinrel\mapstochar\joinrel\rightarrow X'$.

$$\begin{array}{lcl} \overline{Id}R & = & R \\ \overline{C}R & = & Id_C \\ \overline{T_0 \circ T_1}R & = & \overline{T_0}(\overline{T_1}R) \\ \overline{T_0 + T_1}R & = & \overline{T_0}R \uplus \overline{T_1}R \\ \overline{T_0 \times T_1}R & = & \{((\xi_0,\xi_1),(\xi_0',\xi_1')) \mid (\xi_i,\xi_i') \in \overline{T_i}R\} \\ \overline{T^C}R & = & \{(\varphi,\varphi') \mid (\varphi(d),\varphi'(d)) \in \overline{T}R \text{ for each } d \in D\} \end{array} \tag{8}$$

Applying relation lifting to the membership relation $\in$, we obtain an interesting operation: Given a set $X$, we let $\in_X \subseteq X \times PX$ denote the membership relation, restricted to $X$. We define the map $\lambdabar^T_X : TPX \to PTX$ by

$$\lambdabar^T_X(\Phi) := \{\alpha \in TX \mid \alpha \ \overline{T}{\in_X} \ \Phi\},$$

and call elements of $\lambdabar^T_X(\Phi)$ *lifted members* of $\Phi$. The family of maps $\lambdabar^T_X : TPX \to PTX$, natural in $X$, form the distributive law mentioned in the Theorem 2.6 above.

As its role in the distributive law and in what follows in the next section is important, we illustrate the definition of relation lifting by spelling out the definition of the membership relation $\in_A$: $A \longrightarrow\!\!\!\!\!\!\!+\!\!\!\to PA$ lifted by some of the functors introduced in Example 2.4.

**Example 2.9** For the binary tree functor the definition of the lifted membership relation $\overline{Id \times Id} \in : A \times A \longrightarrow\!\!\!\!\!\!\!+\!\!\!\to PA \times PA$ simply says:

$$(a, a')\overline{Id \times Id} \in (A, A') \text{ iff } a \in A \text{ and } a' \in A'.$$

**Example 2.10** For the powerset functor $P$ the definition of the lifted membership relation $\overline{P}(\in) : PA \longrightarrow\!\!\!\!\!\!\!+\!\!\!\to PPA$ boils down to the Egli-Milner lifting of the membership relation:

$$\begin{array}{rll} \alpha\overline{P}(\in)\Phi & \text{iff} & (\forall a \in \alpha)(\exists A \in \Phi)\ a \in A \\ & \text{and} & (\forall A \in \Phi)(\exists a \in \alpha)\ a \in A. \end{array} \tag{9}$$

**Example 2.11** For the bag functor $B$ the lifted membership relation $\overline{B} \in : BA \longrightarrow\!\!\!\!\!\!\!+\!\!\!\to BPA$ looks as follows:

$$\begin{array}{lll} \alpha\overline{B}(\in)\Phi & \text{iff } \exists x \in B(\in) \text{ such that} & \forall a :\ \alpha(a) = \sum\limits_{\{z \mid a = \pi_1^{\in}(z)\}} x(z) \\ & \text{and} & \forall A :\ \Phi(A) = \sum\limits_{\{u \mid A = \pi_2^{\in}(u)\}} x(u). \end{array} \tag{10}$$

which is

$$\begin{array}{lll} \alpha\overline{B}(\in)\Phi & \text{iff } \exists x \in B(\in) \text{ such that} & \forall a :\ \alpha(a) = \sum\limits_{\{A \mid a \in A\}} x(a, A) \\ & \text{and} & \forall A :\ \Phi(A) = \sum\limits_{\{a \mid a \in A\}} x(a, A). \end{array} \tag{11}$$

It is instructive to imagine a witness $x$ as filling a "witness square" of the relation $\in$: columns are labelled by elements of $Base(\Phi)$ and rows by elements of $Base(\alpha)$. We put $x(a, A) = 0$ whenever $a \notin A$ and try to fill the rest of the tab so that the sum of the values in the column of $A \in Base(\Phi)$ is $\Phi(A)$, and the sum of the values in the row of $a \in Base(\alpha)$ is $\alpha(a)$.

Throughout the paper, we will use properties of the relation lifting $\overline{T}R$; unless explicitly stated otherwise, these can always be derived by elementary means from the following fact, gathering the consequences of Theorem 2.6 above (the first four are immediate consequences of the theorem, the rest is not hard to prove.)

**Fact 2.12 (Properties of Relation Lifting)** *The relation lifting* $\overline{T}$ *satisfies the following properties, for all functions* $f : X \to X'$*, all relations* $R, S \subseteq X \times X'$*,* $R' \subseteq X' \times X''$*, and all subsets* $Y \subseteq X$*,* $Y' \subseteq X'$*:*

1. $\overline{T}$ *extends* $T$*:* $\overline{T}(Grf) = Gr(Tf)$*;*
2. $\overline{T}$ *preserves the diagonal (identity):* $\overline{T}(Id_X) = Id_{TX}$*;*
3. $\overline{T}$ *distributes over composition (thus is a functor):* $\overline{T}(R \mathbin{;} S) = \overline{T}(R) \mathbin{;} \overline{T}(S)$*;*
4. $\overline{T}$ *is monotone (preserves inclusions, thus is a 2-functor): if* $R \subseteq S$ *then* $\overline{T}(R) \subseteq \overline{T}(S)$*;*
5. $\overline{T}$ *commutes with relation converse:* $\overline{T}(R^{\smile}) = (\overline{T}R)^{\smile}$*;*
6. $\overline{T}$ *commutes with restriction:* $\overline{T}(R \restriction_{Y \times Y'}) = \overline{T}R \restriction_{TY \times TY'}$ *.*
7. $\overline{T}_\omega$ *coincides with* $\overline{T}$*:* $\overline{T}_\omega R = (\overline{T}R) \restriction_{T_\omega X \times T_\omega X'}$ *.*

**Remark 2.13** The main reason why we restrict our attention to coalgebra types $T$ that preserve weak pullbacks is that for these functors, $\overline{T}$ is a functor, i.e. distributes over relation composition (Theorem 2.6 and Fact 2.12.3). As a consequence of this fact, behavioral equivalence can be captured by the notion of a *bisimulation*.

Given two $T$-coalgebras $(S, \sigma)$ and $(S', \sigma')$, we call a relation $Z \subseteq S \times S'$ a *bisimulation* if $(s, s') \in Z$ implies $(\sigma(s), \sigma'(s')) \in \overline{T}Z$, for all pairs $(s, s') \in S \times S'$. If two states $s$ and $s'$ are linked by some bisimulation, we call them *bisimilar*, notation: $\mathbb{S}, s \leftrightarroweq \mathbb{S}', s'$. Given that the functor $T$ preserves weak pullbacks, one may show that the notions of bisimilarity and behavioral equivalence coincide.

## 2.3 (Propositional) Logic

**Sequent systems** We assume the reader to be familiar with sequent calculi. A *sequent* is pair $(A, B)$ of finite sets of formulas, usually denoted as $A \Longrightarrow B$, and intuitively corresponding to the formula $\bigwedge A \to \bigvee B$. We use standard conventions such as writing $A, B$ instead of $A \cup B$, and $a$ instead of $\{a\}$.

A sequent calculus consists of a collection of *derivation rules*, and in our case these will take the form of pairs consisting of a set of sequents called the *premises* of the rule, and a single sequent called the *conclusion* of the rule. Such a conclusion will be called an *axiom* if the corresponding set of premises is empty. Given such a sequent calculus $G$, a $G$*-derivation* is a well-founded tree, such that each node is labelled by a sequent. Leafs are labelled by axioms, and with each parent node we may associate a rule of which the conclusion labels the parent, and the premises one by one label the children. If the root of such a derivation $\mathcal{D}$ is labelled with a sequent $A \Longrightarrow B$ we say that $\mathcal{D}$ is a $G$*-derivation of/for* $A \Longrightarrow B$. A sequent $A \Longrightarrow B$ is *provable* in $G$, notation: $\vdash_G A \Longrightarrow B$, if there is a $G$-derivation for it. We will write $\vdash$ rather than $\vdash_G$ if this is not likely to cause confusion.

A sequent calculus is *finitary* if all rules have finitely many premises; clearly any derivation in such a system is a finite tree. In a *one-sided* sequent calculus, all sequents are *one-sided*, that is, they have the form $A \Longrightarrow B$ with $B = \varnothing$ (it will be convenient for us to retain the redundancy in this notation).

**Propositional logic** It will be convenient for us to base ourselves on a slightly nonstandard version of propositional logic that is based on taking the *finitary* conjunction ($\bigwedge$) and disjunction symbol ($\bigvee$) of arity $P_\omega$ as primitives, together with the unary negation symbol. That is, given a set $\mathsf{Prop}$ of proposition letters, we define the set $\mathcal{L}_0(\mathsf{Prop})$ of propositional formulas over $\mathsf{Prop}$ by the following grammar:

$$a \ ::= \ p \mid \neg a \mid \bigwedge A \mid \bigvee A,$$

where $p \in \mathsf{Prop}$, and $A \in P_\omega \mathcal{L}_0(\mathsf{Prop})$. We abbreviate $\bot := \bigvee \varnothing$, $\top := \bigwedge \varnothing$ and $a \wedge b := \bigwedge\{a, b\}$.

A propositional sequent $A \Longrightarrow B$ is *valid* if the corresponding formula $\bigwedge A \to \bigvee B$ is a propositional tautology. The sequent systems in this paper will all be based on either the two-sided sequent calculus $G2$ or the one-sided system $G1$.

The sequent calculus $G2$ consists of the (axiom and) rules given in Figure 1.

$$a \Longrightarrow a$$

$$\bigwedge\text{-l}\ \frac{A, B \Longrightarrow C}{A, \bigwedge B \Longrightarrow C} \qquad \bigvee\text{-r}\ \frac{A \Longrightarrow B, C}{A \Longrightarrow \bigvee B, C}$$

$$\bigwedge\text{-r}\ \frac{\{A \Longrightarrow b, C \mid b \in B\}}{A \Longrightarrow \bigwedge B, C} \qquad \bigvee\text{-l}\ \frac{\{A, b \Longrightarrow C \mid b \in B\}}{A, \bigvee B \Longrightarrow C}$$

$$\neg\text{-r}\ \frac{A, a \Longrightarrow C}{A \Longrightarrow C, \neg a} \qquad \neg\text{-l}\ \frac{A \Longrightarrow a, C}{A, \neg a \Longrightarrow C}$$

$$\text{weak-r}\ \frac{A \Longrightarrow C}{A \Longrightarrow C, a} \qquad \text{weak-l}\ \frac{A \Longrightarrow C}{A, a \Longrightarrow C}$$

Figure 1: Sequent system $G2$

It is well known that this calculus is a sound, complete and (obviously) cut-free derivation system for the set of propositionally valid sequents.

Turning to one-sided sequent calculi for propositional logic, we first need to redesign the language by restricting the use of the negation symbol to proposition letters. That is, given a set $\mathsf{Prop}$ of proposition letters, we define the set $\mathcal{L}_0^*(\mathsf{Prop})$ of propositional formulas *in negation normal form*, briefly: nnf-formulas, as follows:

$$a \ ::= \ p \mid \neg p \mid \bigwedge A \mid \bigvee A,$$

where $p \in \mathsf{Prop}$, and $A \in P_\omega \mathcal{L}_0(\mathsf{Prop})$. A *literal* is a formula of the form $p$ or $\neg p$, with $p \in \mathsf{Prop}$. Clearly every formula in $\mathcal{L}_0(\mathsf{Prop})$ is equivalent to a formula in negation normal form. The one-sided sequent calculus $G1$ consists of the axiom scheme rules given in Figure 2. This calculus is a sound, complete and (obviously) cut-free derivation system for the set of propositionally valid one-sided $\mathcal{L}_0^*$-sequents.

$$p, \neg p \Longrightarrow \varnothing$$

$$\bigwedge\text{-l}\ \frac{A, B \Longrightarrow \varnothing}{A, \bigwedge B \Longrightarrow \varnothing} \qquad \bigvee\text{-l}\ \frac{\{A, b \Longrightarrow \varnothing \mid b \in B\}}{A, \bigvee B \Longrightarrow \varnothing} \qquad \text{weak-l}\ \frac{A \Longrightarrow \varnothing}{A, a \Longrightarrow \varnothing}$$

Figure 2: Sequent system $G1$

## 3 Moss' coalgebraic logic

In this section we introduce the finitary version of Moss' coalgebraic language. In the sequel, we deal with syntax mainly, and hence we are mostly concerned with the *finitary* versions of the coalgebra functors. In order not to clutter up our notation too much, we often write $T$ instead of $T_\omega$ for brevity.

### 3.1 Syntax

The formulas of our language are inductively given as follows.

**Definition 3.1** Given a set $\mathsf{Prop}$ of variables, the set $\mathcal{L}(\mathsf{Prop})$ of Moss formulas in $\mathsf{Prop}$ is given by the following grammar:

$$a \ ::= \ p \mid \neg a \mid \bigwedge A \mid \bigvee A \mid \nabla\alpha \mid \Delta\alpha,$$

where $p \in \mathsf{Prop}$, $A \in P_\omega\mathcal{L}(\mathsf{Prop})$ and $\alpha \in T_\omega\mathcal{L}(\mathsf{Prop})$. The fragment $\mathcal{L}^*(\mathsf{Prop})$ of $\mathcal{L}$-formulas in *negation normal form* is defined as follows:

$$a \ ::= \ p \mid \neg p \mid \bigwedge A \mid \bigvee A \mid \nabla\alpha \mid \Delta\alpha,$$

that is, we only allow negations in front of proposition letters. $\lhd$

We will omit explicit reference to the set of proposition letters, for instance writing $\mathcal{L}$ rather than $\mathcal{L}(\mathsf{Prop})$, if $\mathsf{Prop}$ is either well known or not important.

**Remark 3.2** In fact there are quite substantial differences between the language we just defined and Moss' original language. First of all, Moss' language does not have explicit proposition letters. Second, it is infinitary in nature, not only allowing infinite disjunctions but also expressions of the form $\nabla\alpha$ with $\alpha$ an element of $T\mathcal{L}$ rather than of $T_\omega\mathcal{L}$. And finally, in his original language the modality $\Delta$ does not occur. We feel justified to still refer to our syntax as (a variant of) Moss' language because of the characteristic role of the $\nabla$ modality.

The connective $\Delta$ should be understood as the *Boolean dual* of $\nabla$, in the same sense that $\bigwedge$ and $\bigvee$ are Boolean duals. For this purpose, consider the negation connective as a map $\neg : \mathcal{L} \to \mathcal{L}$ mapping formulas to formulas. Applying the functor to this map we obtain a function $T\neg : T\mathcal{L} \to T\mathcal{L}$, so that for any $\alpha \in \mathcal{L}$, the expressions $\nabla(T\neg)\alpha$ and $\Delta(T\neg)\alpha$ are well-formed formulas. The point is now that the formula $\Delta\alpha$ will be equivalent to the formula

$\neg\nabla(T\neg)\alpha$ — following the analogy, observe that $\bigwedge A \equiv \neg\bigvee(P\neg)A$. The $\Delta$ is discussed in detail in [10].

Despite its unconventional appearance, the language $\mathcal{L}$ admits fairly standard definitions of most syntactic notions. The point of restricting Moss' modality to the set $T_\omega\mathcal{L}$ is that the formulas $\nabla\alpha$ and $\Delta\alpha$ have a finite, clearly defined set of *immediate* subformulas. To see this, recall from the preliminaries, that if $\alpha$ belongs to the set $T_\omega\mathcal{L}$, then the set $\mathit{Base}(\alpha) \in P_\omega\mathcal{L}$ is the *smallest* (finite) subset $X \subseteq \mathcal{L}$ such that $\alpha \in T_\omega X$. This observation underlies the following syntactic definitions.

**Definition 3.3** The set $\mathit{Sfor}(a)$ of *subformulas* of a Moss formula $a$ is inductively defined as follows:

$$\begin{array}{rcll} \mathit{Sfor}(p) & := & \{p\} & \\ \mathit{Sfor}(\neg a) & := & \{\neg a\} \cup \mathit{Sfor}(a) & \\ \mathit{Sfor}(\odot A) & := & \{\odot A\} \cup \bigcup_{a\in A} \mathit{Sfor}(a) & (\odot \in \{\bigwedge, \bigvee\}) \\ \mathit{Sfor}(\heartsuit\alpha) & := & \{\heartsuit\alpha\} \cup \bigcup_{a\in \mathit{Base}(\alpha)} \mathit{Sfor}(a). & (\heartsuit \in \{\nabla, \Delta\}) \end{array}$$

The elements of $\mathit{Base}(\alpha) \subseteq \mathit{Sfor}(\nabla\alpha)$ will be called the *immediate* subformulas of $\nabla\alpha$.

The *(modal) depth* $d(a)$ of a Moss formula $a$ is inductively defined as follows:

$$\begin{array}{rcll} d(p) & := & 0 & \\ d(\neg a) & := & d(a) & \\ d(\odot A) & := & \max(d[A]) & (\odot \in \{\bigwedge, \bigvee\}) \\ d(\heartsuit\alpha) & := & 1 + \max\big(d[\mathit{Base}(\alpha)]\big) & (\heartsuit \in \{\nabla, \Delta\}) \end{array}$$

$\lhd$

It is not hard to see that $\mathit{Sfor}(a)$ is a finite set and that $d : \mathcal{L} \to \omega$ is well-defined map assigning to each formula $a$ a natural number $d(a)$.

**Convention 3.4** Since in this paper we will not only be dealing with formulas and sets of formulas, but also with elements of the sets $T_\omega\mathcal{L}$, $P_\omega T_\omega\mathcal{L}$ and $T_\omega P_\omega\mathcal{L}$, it will be convenient to use the *naming convention* of Figure 3. Observe that, similar to taking negation as a map $\neg : \mathcal{L} \to \mathcal{L}$, we may see the boolean connectives $\bigvee$ and $\bigwedge$ as maps from finite sets of formulas

| Set | Elements |
|---|---|
| Prop | $p, q, \ldots$ |
| $\mathcal{L}$ | $a, b, \ldots$ |
| $T_\omega\mathcal{L}$ | $\alpha, \beta, \ldots$ |
| $P_\omega\mathcal{L}$ | $A, B, \ldots$ |
| $T_\omega P_\omega\mathcal{L}$ | $\Phi, \Psi, \ldots$ |
| $P_\omega T_\omega\mathcal{L}$ | $\Gamma, \Theta, \ldots$ |

Figure 3: Naming convention

to formulas, $\bigvee, \bigwedge : P_\omega \mathcal{L} \to \mathcal{L}$. Applying the functor to these maps, we obtain functions $T\bigvee, T\bigwedge : T_\omega P_\omega \mathcal{L} \to T\mathcal{L}$. In particular, for any object $\Phi \in T_\omega P_\omega \mathcal{L}$, we obtain well-formed formulas of the form $\nabla(T\bigvee)\Phi$ and $\nabla(T\bigwedge)\Phi$.

### 3.2 Semantics

Since we included explicit proposition letters in our language, we have to interpret our formulas in $T$-*models*, that is, $T$-coalgebras that are endowed with a valuation function interpreting the proposition letters.

**Definition 3.5** A *valuation* on a $T$-coalgebra $(S, \sigma)$ is a valuation $V : \mathsf{Prop} \to PS$; the induced structure $(S, \sigma, V)$ will be called a $T$-*model* or *coalgebraic model.* For such a model, the satisfaction relation $\Vdash_{\sigma,V} \subseteq S \times \mathcal{L}$ is defined by the following induction on the complexity of formulas:

$$\begin{array}{lll} s \Vdash_{\sigma,V} p & \text{if} & s \in V(p), \\ s \Vdash_{\sigma,V} \neg a & \text{if} & s \nVdash_{\sigma,V} a, \\ s \Vdash_{\sigma,V} \bigwedge A & \text{if} & s \Vdash_{\sigma,V} a \text{ for all } a \in A, \\ s \Vdash_{\sigma,V} \bigvee A & \text{if} & s \Vdash_{\sigma,V} a \text{ for some } a \in A, \\ s \Vdash_{\sigma,V} \nabla\alpha & \text{if} & (\sigma(s), \alpha) \in \overline{T}(\Vdash_{\sigma,V}), \\ s \Vdash_{\sigma,V} \Delta\alpha & \text{if} & (\sigma(s), \alpha) \notin \overline{T}(\nVdash_{\sigma,V}), \end{array}$$

If $s \Vdash_{\sigma,V} a$ we say that $a$ is *true*, or *holds* at $s$ in $\mathbb{S}$, and we usually write $\mathbb{S}, s \Vdash a$, where $\mathbb{S}$ denotes the $T$-model $(S, \sigma, V)$. When no confusion is likely we may write $s \Vdash a$ instead of $\mathbb{S}, s \Vdash a$. ◁

**Remark 3.6** For those readers that are worried about the correctness of this definition we note that given the properties of relation lifting, the clause for the $\nabla$ modality may be replaced with the following:

$$s \Vdash_{\sigma,V} \nabla\alpha \text{ if } (\sigma(s), \alpha) \in \overline{T}(\Vdash_{\sigma,V} \restriction_{S \times Base(\alpha)}),$$

which reveals the inductive nature of the definition: in order to know whether $\nabla\alpha$ is true at a point $s$, we only need to know the meaning of the immediate subformulas of $\nabla\alpha$. A similar observation can be made about $\Delta$.

The semantics of the $\Delta$-operator is perhaps easier to understand by observing that the formula $\Delta\alpha$ is *false* at a state $s$ iff the pair $(\sigma(s), \alpha)$ belongs to the lifted version $\overline{T}\nVdash$ of the complement $\nVdash$ of the satisfaction relation.

**Example 3.7** For the binary tree functor $Id \times Id$, the semantics of nabla is as follows: given $\alpha = (a_0, a_1)$, a coalgebra $\sigma : X \to X \times X$ and a state $s$ with $\sigma(s) = (t_0, t_1)$

$$s \Vdash \nabla(a_0, a_1) \text{ iff } t_0 \Vdash a_0 \text{ and } t_1 \Vdash a_1.$$

**Example 3.8** For the powerset functor $P$, the semantics of nabla in a coalgebra $\sigma : X \to PX$ and a state $s$ is

$$\begin{array}{lll} s \Vdash \nabla\alpha & \text{iff} & (\forall a \in \alpha)(\exists t \in \sigma(s))\ s \Vdash a \\ & \text{and} & (\forall t \in \sigma(s))(\exists a \in \alpha)\ s \Vdash a. \end{array} \tag{12}$$

**Example 3.9** For the bag (multiset) functor $B$ nabla works as follows. Recall from the previous section that we may represent a multiset $\alpha : \mathcal{L} \to \mathbb{N}$ by its positive graph, it is a usefull notation to write it as follows: $\{a^{\alpha(a)} \mid a \in \mathcal{L}\}$. Given a coalgebra $\sigma : X \to BX$ and a state $s$, the condition for $s \Vdash_\sigma \nabla\alpha$ unravels as follows (for the definition of lifting in this case see Example 2.11):

$$\begin{array}{lll} s \Vdash_\sigma \nabla\alpha & \text{iff } \exists x \in B(\Vdash) \text{ such that} & \forall t : \ \sigma(s)(t) = \sum\limits_{\{a \ \mid \ t \Vdash a\}} x(t,a) \\ & \text{and} & \forall a : \ \alpha(a) = \sum\limits_{\{t \ \mid \ t \Vdash a\}} x(t,a). \end{array} \tag{13}$$

For example, if $\sigma(s)(t_1) = 5,\ \sigma(s)(t_2) = 3$ and $\sigma(s)(t_3) = 2$ and $t_1 \Vdash a$ and $t_2, t_3 \Vdash b$, it is easy to see that

$$s \Vdash \nabla\{a^5, b^5\}.$$

Two important observations about finitary Moss' logic are that it is *adequate* with respect to behavioral equivalence (or, equivalently, bisimilarity), and *expressive* when we confine attention to finitely branching coalgebras. For this purpose, given two $T$-models $\mathbb{S}$ and $\mathbb{S}'$, with states $s$ and $s'$, respectively, we write $\mathbb{S}, s \equiv_{\mathcal{L}} \mathbb{S}', s'$ to indicate that for all $\mathcal{L}$-formulas $a$ we have $\mathbb{S}, s \Vdash a$ iff $\mathbb{S}', s' \Vdash a$. We call a coalgebra $(S, \sigma)$ *finitely branching* if $\sigma : S \to T_\omega S$, that is, the range of $\sigma$ is included in the set $T_\omega S$.

**Fact 3.10** *Let $\mathbb{S} = (S, \sigma, V)$ and $\mathbb{S}' = (S', \sigma', V')$ be two $T$-models with states $s$ and $s'$, respectively.*

1. *$\mathcal{L}$ is adequate: if $\mathbb{S}, s \leftrightarrow \mathbb{S}', s'$ then $\mathbb{S}, s \equiv_{\mathcal{L}} \mathbb{S}', s'$*
2. *$\mathcal{L}$ is finitely expressive: if $\mathbb{S}, s \equiv_{\mathcal{L}} \mathbb{S}', s'$ then $\mathbb{S}, s \leftrightarrow \mathbb{S}', s'$, provided that $\sigma$ and $\sigma'$ are finitely branching.*

*The same holds for the $\Delta$-free fragment of the language.*

**Definition 3.11** We say that a formula $a$ entails a formula $b$, notation: $a \leq b$, if for every coalgebraic model $\mathbb{S} = (S, \sigma, V)$, and any $s \in S$, we have that whenever $\mathbb{S}, s \Vdash a$ then $\mathbb{S}, s \Vdash b$. We say that two formulas $a$ and $b$ are *equivalent*, notation: $a \equiv b$, if $a \leq b$ and $b \leq a$. A formula $a$ is *valid*, notation: $\models a$, if it holds at every state of every coalgebraic model. $\lhd$

**Example 3.12** It is easy to prove that nabla is a monotone modality in the following sense:

$$\alpha \overline{T}(\leq)\beta \text{ entails } \nabla\alpha \leq \nabla\beta.$$

This is the rule $\nabla 1$ of the the derivation system **M** [11].

**Example 3.13** Recall from the discussion in Convention 3.4 that for every $\Phi \in T_\omega P_\omega \mathcal{L}$, we may consider the (correctly defined) formula $\nabla(T\bigvee)\Phi$. For such a $\Phi$ it is straightforward to verify that the set $\lambda^T(\Phi) \in PT\mathcal{L}$ actually belongs to the set $P_\omega T_\omega \mathcal{L}$, so that $\Phi$ has finitely many lifted members, all belonging to the set $T_\omega\mathcal{L}$. This means that the expression $\bigvee\{\nabla\alpha \mid \alpha \in \lambda^T(\Phi)\}$ is actually a well-formed formula. It happens to be the case that

$$\nabla(T\bigvee)\Phi \equiv \bigvee\{\nabla\alpha \mid \alpha \in \lambda^T(\Phi)\}. \tag{14}$$

The corresponding equation is, under the name ($\nabla 3$), a key axiom of the derivation system **M** [11].

For a good understanding of the semantics of our language, and of the sequent calculi to be defined later on, we need to discuss the relation between $\nabla$ and $\Delta$ in somewhat more detail. We already mentioned that $\Delta$ can be seen as the *Boolean dual* of $\nabla$ — this can now be made precise, see Fact 3.15. In addition however, perhaps surprisingly, the semantics of $\Delta$ can also be expressed in terms of $\nabla$ *without* the use of the negation. For this purpose, we need the following definition.

**Definition 3.14** For $\alpha$ in $T_\omega\mathcal{L}$, we put

$$\begin{array}{rcl} \mathcal{D}(\alpha) & := & \big\{\Phi \in TPBase(\alpha) \mid (\alpha, \Phi) \notin \overline{T}{\not\in}\big\}, \\ L_T(\alpha) & := & \{(T\bigwedge)\Phi \mid \Phi \in \mathcal{D}(\alpha)\}, \\ R_T(\alpha) & := & \{(T\bigvee)\Phi \mid \Phi \in \mathcal{D}(\alpha)\}. \end{array}$$

$\lhd$

**Fact 3.15** *[10] Let $\mathbb{S} = (S, \sigma, V)$ be a coalgebraic model, and $s$ a state in $\mathbb{S}$. For any $\alpha \in T_\omega\mathcal{L}$ the following equivalences hold:*

$$\mathbb{S}, s \Vdash \Delta\alpha \quad \textit{iff} \quad \mathbb{S}, s \Vdash \neg\nabla(T\neg)\alpha \quad \textit{iff} \quad \mathbb{S}, s \Vdash \nabla\beta \textit{ for some } \beta \in L_T(\alpha) \tag{15}$$

*and*

$$\mathbb{S}, s \Vdash \nabla\alpha \quad \textit{iff} \quad \mathbb{S}, s \Vdash \neg\Delta(T\neg)\alpha \quad \textit{iff} \quad \mathbb{S}, s \Vdash \Delta\beta \textit{ for all } \beta \in R_T(\alpha). \tag{16}$$

While for the powerset functor we consequently simplify the definition of $L_T$ and $R_T$, for the other functors the "double negation" definition of the set $\mathcal{D}(\alpha)$ might seem cryptic. Therefore, recalling the definition of the relation lifting, we illustrate the above definition with the bag functor $B$:

**Example 3.16** Consider a multiset $\{a^n\}$. Then

$$R_B(\{a^n\}) = \{(B\bigvee)\Phi \mid \Phi : P\{a \vee b\} \to \mathbb{N},\ \neg(\{a^n\}\overline{B}(\notin)\Phi)\},$$

where the condition for $\Phi$ actually means that either $\Phi(\varnothing) \neq n$ or $\Phi(\{a\}) > 0$. Thus $R_B(\{a^n\}) = \{\{\perp^k\}|k \neq n\} \cup \{\{a^m\}|m > 0\}$.

Observe that the set $R_B((a)^n)$ in this example is *infinite*.

**Remark 3.17** Clearly, on the basis of the previous fact, we may see $\Delta$ as a *definable* connective in the language with $\nabla$ as the only modality, by putting $\Delta\alpha := \neg\nabla(T\neg)\alpha$.

The reader may also be tempted to conclude, that we can define $\Delta$ in terms of the connectives $\nabla, \bigwedge$ and $\bigvee$ alone (that is, without using negation), via $\Delta\alpha := \bigvee\{\nabla\beta \mid \beta \in L_T(\alpha)\}$. The problem is, however, that unless the functor $T$ restricts to finite sets, the sets $\mathcal{D}(\alpha)$ and $L_T(\alpha)$ may be *infinite* (as $R_B(\{a^n\})$ in the example above); in that case, the proposed defining expression $\bigvee\{\nabla\beta \mid \beta \in L_T(\alpha)\}$ is not a well-formed formula of our language. Nevertheless, in case $T$ does restrict to finite sets, the formulas $\Delta\alpha$ and $\bigvee\{\nabla\beta \mid \beta \in L_T(\alpha)\}$ are equivalent, and so are $\nabla\alpha$ and $\bigwedge\{\Delta\beta \mid \beta \in R_T(\alpha)\}$.

We now turn to the notion of a valid sequent, which we define in a completely standard way. Intuitively, a sequent $A \Longrightarrow B$ is valid iff the formula $\bigwedge A \to \bigvee B$ holds in every state of every coalgebraic model.

**Definition 3.18** A sequent $A \Longrightarrow B$ is *valid*, notation: $\models A \Longrightarrow B$, if for every coalgebraic model $\mathbb{S} = (S, \sigma, V)$, and any $s \in S$, the following holds: whenever $\mathbb{S}, s \Vdash a$ for all formulas $a \in A$, then there is at least one formula $b \in B$ such that $\mathbb{S}, s \Vdash b$. If the above condition fails we say that the sequent $A \Longrightarrow B$ is *refuted* at $s$ in $\mathbb{S}$, and we say that $A \Longrightarrow B$ is *refutable* if it can be refuted somewhere. $\lhd$

Clearly then a sequent is not valid iff it is refutable.

### 3.3 Slim redistributions

An important role in this paper is played by the notion of a *slim redistribution*, and its variant of a *separated slim redistribution.* Slim redistributions are the key to understand how $\nabla$ interacts with $\bigwedge$; and as we will see, separated slim redistributions enable us to use the same idea in a setting of two-sided sequent systems.

**Remark 3.19** Formulated specifically for the power set functor, that is, in the case that $T = P$, a set $\Phi \in TPX$ is a slim redistribution of a set $\Gamma \in PTX$ iff $\bigcup\Gamma = \bigcup\Phi$ and $F \cap \alpha \neq \varnothing$ for all $F \in \Phi$ and $\gamma \in \Gamma$. (We keep $T$ and $P$ separated in the notation in order to facilitate the abstraction to the general case.) Borrowing some intuition from topology, these two conditions tell us that on the one hand every given $\gamma \in \Gamma$ is 'covered' by $\Phi$ (in the sense that $\gamma \subseteq \bigcup\Phi$) in such a way that every $F \in \Phi$ has nonempty intersection with $\gamma$. On the other hand, the requirement that $\bigcup\Phi \subseteq \bigcup\Gamma$ is a minimality condition on $\Phi$, taking care that every such $\Phi$ can be effectively constructed from $\Gamma$ by scrambling and suitably reorganizing its 'ingredients'.

The above relation between $\Phi$ and $\Gamma$ can be reformulated in terms of the lifted membership relation $\overline{T}\!\in$. Recall that $\gamma \in TX$ is a lifted member of $\Phi \in TPX$ if $\gamma \subseteq \bigcup\Phi$ and $\gamma \cap F \neq \varnothing$ for all $F \in \Phi$. Hence we will say that $\Phi$ is a redistribution of $\Gamma$ if $\Gamma \subseteq \lambda^P(\Phi)$ (every element of $\Gamma$ is a lifted member of $\Phi$), and that $\Phi$ is slim if $\Phi \subseteq TP(\bigcup\Gamma)$ ($\Phi$ is built from the ingredients of $\Gamma$). It is this formulation that can be generalized to the case of an arbitrary set functor.

**Definition 3.20** A set $\Phi \in TPX$ is a *redistribution* of $\Gamma \in PTX$ if $\Gamma \subseteq \lambda^T_X(\Phi)$. In case $\Gamma \in P_\omega T_\omega X$, we call a redistribution $\Phi$ *slim* if $\Phi \in T_\omega P_\omega(\bigcup_{\alpha \in \Gamma} \mathit{Base}(\alpha))$. The set of slim redistributions of $\Gamma$ is denoted as $\mathit{SRD}(\Gamma)$.

For brevity, in the sequel we will often write $\mathcal{B}(\Gamma)$ instead of $\bigcup_{\gamma \in \Gamma} \mathit{Base}(\gamma)$. $\lhd$

**Example 3.21** Consider the powerset functor: the following are simple examples of the set of slim redistributions

- $\mathit{SRD}(\{\{a_0\} \dots \{a_n\}\}) = \{\{a_0, \dots, a_n\}\}$
- $\mathit{SRD}(\{\{a\}\}) = \{\{a\}\}$
- $\mathit{SRD}(\{\varnothing\}) = \{\varnothing\}$
- $\mathit{SRD}(\{\alpha, \varnothing\}) = \varnothing$ if $\alpha \neq \varnothing$
- $\mathit{SRD}(\varnothing) = P\varnothing$

It holds in general that

$$\mathit{SRD}(\varnothing) = T\varnothing.$$

**Example 3.22** To see how slim redistributions are involved in the interaction between $\nabla$ and $\bigwedge$, consider an arbitrary collection of formulas $\bigwedge\{\nabla\gamma \mid \gamma \in \Gamma\}$ for some finite set $\Gamma \subseteq T_\omega\mathcal{L}$. A straightforward argument suffices to show that the following holds, for any $T$-model $\mathbb{S}$ and any state in $\mathbb{S}$:

$$\mathbb{S}, s \Vdash \nabla\gamma \text{ for all } \gamma \in \Gamma \text{ iff } \mathbb{S}, s \Vdash \nabla(T\textstyle\bigwedge)\Phi \text{ for some } \Phi \in \mathit{SRD}(\Gamma). \tag{17}$$

In the case that $T$ restricts to finite sets, the set $\mathit{SRD}(\Gamma)$ is finite, and we may formulate (17) as an equivalence of formulas:

$$\bigwedge\{\nabla\gamma \mid \gamma \in \Gamma\} \equiv \bigvee\{\nabla(T\textstyle\bigwedge)\Phi \mid \Phi \in \mathit{SRD}(\Gamma)\}.$$

Formulated as the derivation rule ($\nabla$2), the equivalence (17) provides one of the key principles underlying the derivation system **M** [11].

In the setting of two-sided sequent calculi, we need to slightly modify the notion of a slim redistribution. For an intuitive explanation of the required modification, suppose that we are dealing with a sequent of the form $\{\nabla\gamma \mid \gamma \in \Gamma\} \implies \{\Delta\theta \mid \theta \in \Theta\}$, and that we want to introduce a proof rule to reduce this sequent. Using the fact that $\Delta$ is the Boolean dual of $\nabla$, we may think of this sequent as the one-sided $\{\nabla\gamma \mid \gamma \in \Gamma\}, \{\nabla(T\neg)\theta \mid \theta \in \Theta\} \implies \varnothing$, corresponding to the conjunction $\bigwedge\Big(\{\nabla\gamma \mid \gamma \in \Gamma\} \cup \{\nabla(T\neg)\theta \mid \theta \in \Theta\}\Big)$. A natural way to reduce this sequent would be to use the equivalence (17), working with the set $\mathit{SRD}\big(\Gamma \cup (T\neg)[\Theta]\big)$. However, this leads to problems in case the sets $\Gamma$ and $(T\neg)[\Theta]$ overlap: we need to remember which side of the sequent the objects $\gamma$ (left) or $(T\neg)\theta$ (right) originally came from. Formally, our way of dealing with this kind of sequent is the following.

**Definition 3.23** Given $\Gamma, \Theta \in P_\omega T_\omega \mathcal{L}$, let

$$\mathcal{B}^0(\Gamma) := \{(a, 0) \mid a \in \mathcal{B}(\Gamma)\}, \quad \mathcal{B}^1(\Theta) := \{(b, 1) \mid b \in \mathcal{B}(\Theta)\},$$

and let the assignments $a \mapsto (a, 0)$ and $b \mapsto (b, 1)$ define the maps $f_0 : \mathcal{B}(\Gamma) \to \mathcal{B}^0(\Gamma)$ and $f_1 : \mathcal{B}(\Theta) \to \mathcal{B}^1(\Theta)$ respectively. We identify the disjoint union $\mathcal{B}(\Gamma) \uplus \mathcal{B}(\Theta)$ with $\mathcal{B}^0(\Gamma) \cup \mathcal{B}^1(\Theta)$ with $f_0, f_1$ as the injection maps. Then define the set of *separated slim redistributions of* $\Gamma$ *and* $\Theta$ as follows:

$$SSRD(\Gamma, \Theta) := \Big\{ \Phi \in TP(\mathcal{B}(\Gamma) \uplus \mathcal{B}(\Theta)) \mid \ \forall \alpha \in \Gamma \, (Tf_0)\alpha \, \overline{T}{\in} \, \Phi) \\ \& \ \forall \beta \in \Theta \, (Tf_1)\beta \, \overline{T}{\in} \, \Phi) \Big\}.$$

Given a $\Phi \in SSRD(\Gamma, \Theta)$, any $A \in Base(\Phi)$ is a subset of $\mathcal{B}(\Gamma) \uplus \mathcal{B}(\Theta)$, and so we may define the sets $A_L, A_R \in P_\omega \mathcal{L}$ by putting

$$A_L := (\breve{P} f_0)A \text{ and } A_R := (\breve{P} f_1)A, \tag{18}$$

where $\breve{P}$ denotes the contravariant power set functor. $\lhd$

In order to provide some intuition concerning the above definition of the sets $A_L$ and $A_R$, we mention (without proof) that for any $\Phi \in SSRD(\Gamma, \Theta)$, the sets $(T\breve{P}f_0)\Phi$ and $(T\breve{P}f_1)\Phi$ are slim redistributions of $\Gamma$ and $\Theta$, respectively, and for any $A \in Base(\Phi)$, the sets $A_L$ and $A_R$ are elements of $Base((T\breve{P}f_0)\Phi)$ and $Base((T\breve{P}f_1)\Phi)$, respectively. As a consequence, we find that $A_L \subseteq \mathcal{B}(\Gamma)$ and $A_R \subseteq \mathcal{B}(\Theta)$; in particular, if we think of the *sequent* $A_L \Longrightarrow A_R$ as being formed by $\{\nabla\alpha \mid \alpha \in \Gamma\} \Longrightarrow \{\Delta\beta \mid \beta \in \Theta\}$, we may observe that material originating from the given side (left or right) of the latter sequent will end again on that same side.

The definition of *SSRD* and its intuition coming from dealing with sequents which have two sides is further motivated by the following key example:

**Example 3.24** A key example of a separated slim redistribution of sets $\Gamma, \Theta \in T_\omega P_\omega \mathcal{L}$ arises semantically. Fix a model $\mathbb{S}$ and a state $s$ in $\mathbb{S}$. Consider, for any state $t$ of $\mathbb{S}$, the set

$$Q_t := \big\{ f_0 a \mid a \in \mathcal{B}(\Gamma) \text{ and } \mathbb{S}, t \Vdash a \big\} \cup \big\{ f_1 b \mid b \in \mathcal{B}(\Theta) \text{ and } \mathbb{S}, t \not\Vdash b \big\}.$$

This defines a map $Q : S \to P(\mathcal{B}(\Gamma) \uplus \mathcal{B}(\Theta))$, and hence applying the functor we obtain a map $TQ : TS \to TP(\mathcal{B}(\Gamma) \uplus \mathcal{B}(\Theta))$ Then for the object $\Phi_s := (TQ)(\sigma(s)) \in TP(\mathcal{B}(\Gamma) \uplus \mathcal{B}(\Theta))$ one may prove

$$\Phi_s \in SSRD(\Gamma, \Theta) \text{ iff } \mathbb{S}, s \Vdash \nabla\alpha \text{ for all } \alpha \in \Gamma \text{ and } \mathbb{S}, s \not\Vdash \Delta\beta \text{ for all } \beta \in \Theta. \tag{19}$$

We do not prove this fact here since it resembles the proof of Lemma 5.2.

**Example 3.25** For the binary tree functor $Id \times Id$, consider a nonempty set of pairs, e.g. $\Gamma = \{(a_0, a_1), (b_0, b_1)\}$, then $SRD(\Gamma) = \{(A, B) \mid \{a_0, b_0\} \subseteq A, \{a_1, b_1\} \subseteq B\}$ where $A, B \in P_\omega(Base(\alpha) \cup Base(\beta))$.

Similarly for $SSRD(\Gamma, \Sigma)$ in case one of the sets is nonempty. $SRD(\varnothing) = SSRD(\varnothing, \varnothing) = \varnothing \times \varnothing = \varnothing$.

**Example 3.26** We list some simple examples of *SSRD* for the powerset functor, some of them appear later in the examples of proofs:

- $\mathit{SSRD}(\varnothing, \varnothing) = P\varnothing$
- $\mathit{SSRD}(\{\varnothing\}, \{\varnothing\}) = \{\varnothing\}$
- $\mathit{SSRD}(\{\alpha, \varnothing\}, \Sigma) = \varnothing$ whenever $\alpha \neq \varnothing$, or $\Sigma \neq \varnothing$ and $\Sigma \neq \{\varnothing\}$
- $\mathit{SSRD}(\varnothing, \{\{a\}\}) = \{\{(a, 1)\}\}$

**Example 3.27** For the bag functor $B$ we compute $\mathit{SSRD}(\{a^5, b^5\}, \{\bot^n\})$ where $n \neq 10$: it is empty, there is no such *SSRD*. For suppose there is some $\Phi : P_\omega\{(a, 0), (b, 0)(\bot, 1)\} \to \mathbb{N}$ with $\{(a, 0)^5, (b, 0)^5\} \overline{B}{\in} \Phi$ and at the same time $\{(\bot, 1)^n\} \overline{B}{\in} \Phi$. This would mean we have at the same time witnesses $x_0$ and $x_1$ in $B(\in)$ for these two facts. Imagine we are filling the "witness squares" simultaneously: the bases of the two multisets are disjoint, thus for each $A \in P_\omega\{(a, 0), (b, 0)(\bot, 1)\}$ not containing $(\bot, 1)$ we have to fill $x_1(a, A) = 0$ and $x_1(b, A) = 0$, and for each $A$ not containing $(a, 0)$ nor $(b, 0)$ we similarly have to fill $x_0(\bot, A) = 0$, and consequently for any of these sets we must have $\Phi(A) = 0$. Now for the rest of sets $A \in P_\omega\{(a, 0), (b, 0)(\bot, 1)\}$ — those containing $(\bot, 1)$ and at the same time containing $(a, 0)$ or $(b, 0)$ — it holds that

$$\sum x_0(\bot, A) = \sum x_1(a, A) + \sum x_1(b, A) = 10 = \sum \Phi(A)$$

which gives $\sum \Phi(A) = 10$ contradicting the fact that $\{(\bot, 1)^n\} \overline{B}{\in} \Phi$ and $n \neq 10$.

## 4 The case of the power set functor

Throughout this section we assume that $T = P$, that is we are dealing with the power set functor. As a consequence, $T_\omega$ is the finitary power set functor $P_\omega$.

We will first introduce a two-sided sequent system $S2_P$ for the full language $\mathcal{L}_P$, and then a one-sided system $S1_P$ for the variant $\mathcal{L}_P^*$ of the language where the use of negation is restricted to atomic formulas. These two systems are *simplified versions* of the corresponding instances of the proof systems $G2_T$, $G1_T$, which we will define in the next section.

### 4.1 The sequent calculus $S2_P$

In order to define the two-sided calculus $S2_P$, we introduce the following variants of the sets $L_T(\alpha)$ and $R_T(\alpha)$ of Definition 3.14.

**Definition 4.1** For every $\alpha \in T_\omega\mathcal{L}$ let us define

$$L'_P\alpha = \begin{cases} \{\{\varnothing\} \cup \{\{a\} \mid a \in \alpha\} \cup \{\{\bigwedge \alpha, \top\}\}\} & \text{if } \alpha \neq \varnothing \\ \{\{\top\}\} & \text{if } \alpha = \varnothing \end{cases} \tag{20}$$

$$R'_P\alpha = \begin{cases} \{\{\varnothing\} \cup \{\{a\} \mid a \in \alpha\} \cup \{\{\bigvee \alpha, \bot\}\}\} & \text{if } \alpha \neq \varnothing \\ \{\{\bot\}\} & \text{if } \alpha = \varnothing \end{cases} \tag{21}$$

◁

**Remark 4.2** Note that $L'_P(\alpha)$ and $R'_P(\alpha)$ are *not* the sets we obtain by instantiating $T = P$ in the definition of $L_T(\alpha)$ and $R_P(\alpha)$ of Definition 3.14. We leave it for the reader to verify that $L'_P(\alpha) \subseteq L_P(\alpha)$, and that in general it is a proper subset. It is in this sense that the system $S2_P$ simplifies the calculus $G2_T$ defined in the next section.

The importance of these notions is that they allow us to interdefine $\nabla$ and $\Delta$, without the use of the negation operator.

**Lemma 4.3** *For every $\alpha, \beta \in T_\omega\mathcal{L}$,*

$$\Delta\alpha \equiv \bigvee\{\nabla\beta \mid \beta \in L'_P\alpha\} \quad \nabla\alpha \equiv \bigwedge\{\Delta\beta \mid \beta \in R'_P\alpha\}.$$

**Proof.** We omit the proof of this Lemma, which is a routine verification. QED

We are now ready to introduce the calculus $S2_P$. It extends the propositional calculus $G2$ with two kinds of modal proof rules. The rules $\Delta$-l and $\nabla$-r can be seen as the incarnation of the previous lemma into proof rules. For a motivation of the last rule of the system, $P(\nabla\Delta)$, which reduces the modal depth of sequents, we refer to the subsection on separated slim redistributions.

**Definition 4.4** The sequent calculus $S2_P$ for the language $\mathcal{L}_P$ is defined as the extension of the propositional calculus $G2$ with the following modal rules:

$$\Delta\text{-l}\ \frac{\{A, \nabla\beta \Longrightarrow B \mid \beta \in L'_P\alpha\}}{A, \Delta\alpha \Longrightarrow B} \qquad \nabla\text{-r}\ \frac{\{A \Longrightarrow \Delta\beta, B \mid \beta \in R'_P\alpha\}}{A \Longrightarrow \nabla\alpha, B}$$

$$P(\nabla\Delta)\ \frac{\{A^\Phi_L \Longrightarrow A^\Phi_R \mid \Phi \in \mathit{SSRD}(\Gamma,\Sigma)\}}{\{\nabla\alpha \mid \alpha \in \Gamma\} \Longrightarrow \{\Delta\beta \mid \beta \in \Sigma\}}\ \forall\Phi.\ A^\Phi \in \Phi$$

The rule $P(\nabla\Delta)$ is to be read as follows: given $\Gamma, \Sigma \in P_\omega T_\omega\mathcal{L}$, if for every $\Phi \in \mathit{SSRD}(\Gamma, \Sigma)$ there exists some $A^\Phi \in \Phi$ such that $A^\Phi_L \Longrightarrow A^\Phi_R$, then $\{\nabla\alpha \mid \alpha \in \Gamma\} \Longrightarrow \{\Delta\beta \mid \beta \in \Gamma\}$. $\lhd$

The following lemma states soundness of the $P(\nabla\Delta)$ rule: whenever the side condition is satisfied and the resulting assumptions are valid sequents, the conclusion is a valid sequent as well. But it also states a form of invertibility of the rule: whenever the conclusion is valid we can find suitable valid assumption sequents fulfilling the side condition. The lemma is for convenience stated as a contraposition:

**Lemma 4.5** *The following are equivalent for all $\Gamma, \Sigma \in P_\omega T_\omega\mathcal{L}$ and all finite sets $C, D$ of proposition letters:*

1. *the sequent $\{\nabla\alpha \mid \alpha \in \Gamma\}, C \Longrightarrow D, \{\Delta\beta \mid \beta \in \Sigma\}$ is not valid.*

2. *the sequent $C \Longrightarrow D$ is not valid, and for some $\Phi \in \mathit{SSRD}(\Gamma, \Sigma)$, $A_L \Longrightarrow A_R$ is not valid for every $A \in \Phi$.*

**Proof.** Let us show that (1) implies (2): By assumption, $C \Longrightarrow D$ is not valid and there exists a model $\mathbb{S}$ and a state $s$ in $\mathbb{S}$ such that $\mathbb{S}, s \Vdash \nabla\alpha$ for every $\alpha \in \Gamma$ and $\mathbb{S}, s \nVdash \Delta\beta$ for every $\beta \in \Sigma$. For every $t \in R[s]$, let $A_t := \{f_0 a \mid a \in \bigcup\Gamma \text{ and } \mathbb{S}, t \Vdash a\} \cup \{f_1 b \mid b \in \bigcup\Sigma \text{ and } \mathbb{S}, t \nVdash b\}$, and let $\Phi_s = \{A_t \mid t \in R[s]\}$. By definition, $A_L \Longrightarrow A_R$ is not valid for all $A \in \Phi_s$, and we have already checked that $\Phi_s \in SSRD(\Gamma, \Sigma)$ (see Example 3.24).

Conversely, assume that $C \Longrightarrow D$ is not valid (hence, $C \cap D = \varnothing$) and that there exists some $\Phi \in SSRD(\Gamma, \Sigma)$ such that for every $A \in \Phi$, $A_L \Longrightarrow A_R$ is not satisfied at some state $s_A$ in some model $\mathbb{S}_A$. Then consider the model $\mathbb{S}$ which consists of the disjoint union of the models $\mathbb{S}_A$ plus one extra point $s$ such that $R[s] = \{s_A \mid A \in \Phi\}$ and $p \in V(s)$ iff $p \in C$. It is routine to verify that $\mathbb{S}, s$ satisfies $\{\nabla\alpha \mid \alpha \in \Gamma\} \cup C \cup \{\neg\Delta\beta \mid \beta \in \Sigma\} \cup \{\neg p \mid p \in D\}$. QED

Theorem 4.8 below states the soundness and completeness of $S2_P$ (with respect to the standard semantics). Since $S2_P$ is formulated without the cut rule, once completeness has been established, it immediately follows that the cut rule is redundant.

To be able to perform an inductive argument we introduce a measure on sequents. The measure is induced by the modal depth of formulas as given in Definition 3.3, and by a measure of left and right formula occurrences in the sequent.

**Definition 4.6** By induction on the complexity of a formula $a$ in $\mathcal{L}_P$ we define the following measures $k_l(a), k_r(a)$:

$$\begin{array}{lcl@{\qquad\qquad}lcl@{\qquad\qquad}l}
k_l(p) & := & 1 & k_r(p) & := & 1 & \\
k_l(\neg a) & := & 1 + k_r(a) & k_r(\neg a) & := & 1 + k_l(a) & \\
k_l(\odot A) & := & 1 + \sum\limits_{a \in A} k_l(a) & k_l(\odot A) & := & 1 + \sum\limits_{a \in A} k_l(a) & (\odot \in \{\bigwedge, \bigvee\}) \\
k_l(\nabla\alpha) & := & 2 & k_r(\nabla\alpha) & := & 3 & \\
k_l(\Delta\alpha) & := & 3 & k_r(\Delta\alpha) & := & 2 &
\end{array}$$

Finally, given a sequent $A \Longrightarrow B$, we let

$$m(A \Longrightarrow B) := \Big(\max(d[A \cup B]), \sum_{a \in A} k_l(a) + \sum_{b \in B} k_r(b)\Big)$$

denote its measure. ◁

Intuitively, $k_l(a)$ and $k_r(a)$ measure the boolean complexity of $a$ occurring on the left side (right side) of a sequent, where formulas of the form $\nabla\alpha$ and $\Delta\alpha$ count as proposition letters with a slightly enlarged weight. The full measure of a sequent being a pair of natural numbers, we will order the set $\omega \times \omega$ lexicographically, that is:

$$(m, n) < (m, n') \text{ if } m < m' \text{ or } m = n \text{ and } n < n'.$$

As we will see in the proof below, with this definition we ensure that for each rule the complexity of any of its assumptions is strictly less than that of its conclusion.

**Lemma 4.7** *For each rule of the calculus $S2_P$, the measure of any of its assumptions is strictly less than that of its conclusion.*

**Proof.** Let us denote $m(A \Longrightarrow B)$ by $(k, l)$. We show that in a backward application of the $P(\nabla, \Delta)$ rule, $k$ strictly decreases, and that for all the other rules $k$ does not increase, while $l$ strictly decreases.

Leaving the propositional rules as an exercise for the reader, we consider the modal rules explicitly. For the $\Delta$-l rule, denote $\sum_{a \in A} k_l(a) + \sum_{b \in B} k_r(b) = n$. Now $m(\Delta\alpha, A \Longrightarrow B) = 3 + n$ while $m(\nabla\beta, A \Longrightarrow B) = 2 + n$. Since the modal depth clearly remains unchanged, this suffices for the $\Delta$-l rule. Clearly, the case of the $\nabla$-r rule is similar.

Consider a backward application of the $P(\nabla, \Delta)$ rule. From the definition of $\Phi \in SSRD(\Gamma, \Sigma)$ and of $A^\Phi_L$ and $A^\Phi_R$ it is clear that the modal depth of each assumption (that is, $\max(d[A^\Phi_L \cup A^\Phi_R]))$ is strictly less then that of the conclusion. QED

**Theorem 4.8 (Soundness and Completeness of $S2_P$)** *For every $\mathcal{L}_P$-sequent $A \Longrightarrow B$,*

$$\vdash_{S2_P} A \Longrightarrow B \textit{ iff } \models A \Longrightarrow B.$$

**Proof.** The proof of soundness is standard, by induction on the depth of the derivation of $A \Longrightarrow B$. The only cases of interest are when the last rule applied is (a) $\Delta$-l, (b) $\nabla$-r or (c) $P(\nabla\Delta)$. The cases (a) and (b) readily follow from Lemma 4.3. Case (c) follows from the implication $(1 \Longrightarrow 2)$ of Lemma 4.5.

As for completeness, it is shown by induction on the $m(A \Longrightarrow B)$. If $m(A \Longrightarrow B) = (0, 0)$ then $A = B = \varnothing$ and the sequent is not valid (and by soundness it is also not provable.)

The induction step for a sequent $A \Longrightarrow B$ with $m(A \Longrightarrow B) \neq (0, 0)$ distinguishes two cases: either the sequent is of the form $\{\nabla\alpha \mid \alpha \in \Gamma\}, C \Longrightarrow D, \{\Delta\beta \mid \beta \in \Sigma\}$ where $C, D$ are sets of proposition letters (which we call a reduced form), or it is not.

**Case 1.** The sequent $A \Longrightarrow B$ is not of the reduced form. Observe that the propositional rules and the $\nabla$-r and $\Delta$-l rules are invertible, that is, they preserve validity backwards (the latter two by Lemma 4.3). Moreover, with each backwards application of any of the mentioned rules, the value of $m(A \Longrightarrow B)$ strictly decreases by Lemma 4.7. It is then clear then we can apply the mentioned rules backwards until no such rule is applicable, and this procedure terminates in a set of sequents $\{A_i \Longrightarrow B_i \mid i \in I\}$, each of them (a) being of the reduced form, (b) having a strictly smaller measure than the sequent we started with, and (c) being valid. To show that the sequent $A \Longrightarrow B$ is provable, we apply the induction hypothesis to the sequents $A_i \Longrightarrow B_i$, and finish the proof with the applications of the appropriate rules forward.

**Case 2.** The sequent $A \Longrightarrow B$ is of the reduced form, say $\{\nabla\alpha \mid \alpha \in \Gamma\}, C \Longrightarrow D, \{\Delta\beta \mid \beta \in \Sigma\}$ where $C, D$ are sets of atoms. From the assumptions and the implication $(2 \Rightarrow 1)$ of Lemma 4.5, we obtain one of the following two subcases.

*Subcase 2a.* The propositional sequent $C \Longrightarrow D$ is valid, in which case $p \in C \cap D$ for some proposition letter $p$. Then we can start a derivation with the axiom $p \Longrightarrow p$ as its only leaf followed by applications of the weakening rules to derive the full sequent $A \Longrightarrow B$.

*Subcase 2b.* For each $\Phi \in SSRD(\Gamma, \Sigma)$ there exists some set $A^\Phi \in \Phi$ such that the sequent $A^\Phi_L \Longrightarrow A^\Phi_R$ is valid. Since by Lemma 4.7 we have $m(A^\Phi_L \Longrightarrow A^\Phi_R) < m(A \Longrightarrow B)$, we can apply the induction hypothesis and conclude that for each $\Phi \in \mathit{SSRD}(\Gamma, \Sigma)$ the sequent $A_L^\Phi \Longrightarrow A_R^\Phi$ is provable. Then we may obtain a derivation for our sequent by prolonging all these derivations with an application of $P(\nabla\Delta)$, so as to obtain a proof of $\{\nabla\alpha \mid \alpha \in \Gamma\} \Longrightarrow \{\Delta\beta \mid \beta \in \Sigma\}$, followed by applications of left and right weakening rules to add the elements in $C$ and $D$ respectively. QED

**Example 4.9** We illustrate the modal rules of the calculus with the following simple proof of one of the original nabla axioms:

$$\nabla\text{-r}\,\dfrac{\nabla\text{-r}\,\dfrac{P(\nabla,\Delta)\,\dfrac{\varnothing}{\varnothing \Longrightarrow \Delta\{\bot\}, \Delta\varnothing} \qquad P(\nabla,\Delta)\,\dfrac{\varnothing \Longrightarrow \top}{\varnothing \Longrightarrow \Delta\{\top\}} \qquad P(\nabla,\Delta)\,\dfrac{\varnothing \Longrightarrow \top \qquad \varnothing \Longrightarrow \top, \bot}{\varnothing \Longrightarrow \Delta\{\top, \bot\}}}{\varnothing \Longrightarrow \Delta\{\bot\}, \nabla\{\top\}}}{\varnothing \Longrightarrow \nabla\varnothing, \nabla\{\top\}}$$

Here the following is needed to see that the above is a correct proof:

- $R'_P(\varnothing) = \{\{\bot\}\}$ for the last inference
- $R'_P(\{\top\}) = \{\varnothing, \{\top\}, \{\top, \bot\}\}$ for the one but last inference
- $\mathit{SSRD}(\varnothing, \{\{\bot\}, \varnothing\}) = \varnothing$
- $\mathit{SSRD}(\varnothing, \{\{\top\}\}) = \{\{(\top, 1)\}\}$
- If $\Phi \in \mathit{SSRD}(\varnothing, \{\{\top, \bot\}\})$ then either $\{(\top, 1)\} \in \Phi$ or $\{(\top, 1), (\bot, 1)\} \in \Phi$

**Example 4.10** As another example we prove the sequent $\nabla\{a\}, \nabla\{b\} \Longrightarrow \nabla\{a, b\}$. In this case we use the fact that $R'_P(\{a, b\}) = \{\varnothing, \{a\}, \{b\}, \{a \vee b, \bot\}\}$. To derive this sequent we have to prove the following four sequents:
First,

$$P(\nabla, \Delta)\,\dfrac{\varnothing}{\nabla\{a\}, \nabla\{b\} \Longrightarrow \Delta\varnothing}$$

is provable because $\mathit{SSRD}(\{\{a\}, \{b\}\}, \{\varnothing\}) = \varnothing$.
Second,

$$P(\nabla, \Delta)\,\dfrac{a, b \Longrightarrow a}{\nabla\{a\}, \nabla\{b\} \Longrightarrow \Delta\{a\}}$$

is provable because the only member of $\mathit{SSRD}(\{\{a\}, \{b\}\}, \{\{a\}\})$ is $\{\{(a, 0), (b, 0), (a, 1)\}\}$.
Third,

$$P(\nabla, \Delta)\,\dfrac{a, b \Longrightarrow b}{\nabla\{a\}, \nabla\{b\} \Longrightarrow \Delta\{b\}}$$

similarly as above.
And fourth,

$$P(\nabla,\Delta)\,\frac{a,b\Longrightarrow a\vee b\qquad a,b\Longrightarrow a\vee b,\bot}{\nabla\{a\},\nabla\{b\}\Longrightarrow\Delta\{a\vee b,\bot\}}$$

is correct because for each $\Phi\in \mathit{SSRD}(\{\{a\},\{b\}\},\{\{a\vee b,\bot\}\})$ we have either $\{(a,0),(b,0),(a\vee b,1)\}\in \mathit{Base}(\Phi)$ or $\{(a,0),(b,0),(a\vee b,1),(\bot,1)\}\in \mathit{Base}(\Phi)$, thus fulfilling the side condition of the rule.

**Example 4.11**

$$\dfrac{P(\nabla,\Delta)\,\dfrac{\varnothing}{\nabla\varnothing\Longrightarrow\Delta\{a,\bot\}}\qquad P(\nabla,\Delta)\,\dfrac{a\Longrightarrow a\qquad a\Longrightarrow a,\bot}{\nabla\{a\}\Longrightarrow\Delta\{a,\bot\}}}{\nabla\varnothing\vee\nabla\{a\}\Longrightarrow\Delta\{a,\bot\}}$$

where in the left branch the set $\mathit{SSRD}(\{\varnothing\},\{\{a,\bot\}\})=\varnothing$. In the other branch observe that for each $\Phi\in \mathit{SSRD}(\{\{a\}\},\{\{a,\bot\}\})$ we have either $\{(a,0),(a,1)\}\in \mathit{Base}(\Phi)$ or $\{(a,0),(a,1),(\bot,1)\}\in \mathit{Base}(\Phi)$.

**Example 4.12** As the last example we give a cut free proof of the sequent, which has been shown in Remark 27. of [6] to have no cut free proof in the calculus introduced there. The following four proof trees prove four assumptions of the last $\nabla$-r inference:

$$\nabla\text{-r}\ \dfrac{\dfrac{\varnothing}{\nabla\{p\vee q\}\Longrightarrow\Delta\varnothing}\qquad\dfrac{\varnothing}{\nabla\{p\vee q\}\Longrightarrow\Delta\varnothing,\Delta\{q,\top\}}\qquad\dfrac{\varnothing}{\nabla\{p\vee q\}\Longrightarrow\Delta\varnothing,\Delta\{q\}}}{\nabla\{p\vee q\}\Longrightarrow\Delta\varnothing,\nabla\{q\}}$$

$$\nabla\text{-r}\ \dfrac{\dfrac{\dfrac{\varnothing}{\nabla\{p\vee q\}\Longrightarrow\Delta\varnothing}}{\nabla\{p\vee q\}\Longrightarrow\Delta\varnothing,\Delta\{p\}}\qquad\dfrac{p\vee q\Longrightarrow p,q}{\nabla\{p\vee q\}\Longrightarrow\Delta\{p\},\Delta\{q\}}\qquad\dfrac{p\vee q\Longrightarrow p,q,\top\quad p\vee q\Longrightarrow p,q\quad p\vee q\Longrightarrow p,\top}{\nabla\{p\vee q\}\Longrightarrow\Delta\{p\},\Delta\{q,\top\}}}{\nabla\{p\vee q\}\Longrightarrow\Delta\{p\},\nabla\{q\}}$$

$$\nabla\text{-r}\ \dfrac{\dfrac{\dfrac{\varnothing}{\nabla\{p\vee q\}\Longrightarrow\Delta\varnothing}}{\nabla\{p\vee q\}\Longrightarrow\Delta\varnothing,\Delta\{\top\}}\qquad\dfrac{p\vee q\Longrightarrow\top,q}{\nabla\{p\vee q\}\Longrightarrow\Delta\{\top\},\Delta\{q\}}\qquad\dfrac{p\vee q\Longrightarrow\top,q}{\nabla\{p\vee q\}\Longrightarrow\Delta\{\top\},\Delta\{q,\top\}}}{\nabla\{p\vee q\}\Longrightarrow\Delta\{\top\},\nabla\{q\}}$$

$$\nabla\text{-r}\ \dfrac{\dfrac{\dfrac{\varnothing}{\nabla\{p\vee q\}\Longrightarrow\Delta\varnothing}}{\nabla\{p\vee q\}\Longrightarrow\Delta\varnothing,\Delta\{p\vee\top,\bot\}}\qquad\dfrac{\begin{array}{c}p\vee q\Longrightarrow\top\vee p,q\\ p\vee q\Longrightarrow\top\vee p,q,\bot\end{array}}{\nabla\{p\vee q\}\Longrightarrow\Delta\{p\vee\top,\bot\},\Delta\{q\}}\qquad\dfrac{\begin{array}{c}p\vee q\Longrightarrow\top\vee p,q\\ p\vee q\Longrightarrow\top\vee p,q,\bot\end{array}}{\nabla\{p\vee q\}\Longrightarrow\Delta\{p\vee\top,\bot\},\Delta\{q,\bot\}}}{\nabla\{p\vee q\}\Longrightarrow\Delta\{p\vee\top,\bot\},\nabla\{q\}}$$

where in the last proof tree we use a compact notation for space reasons: the two top sequents in the middle constitute the two assumptions of the inference, the same applies to the two top sequents on the right.

Next, one $\nabla$-r inference applied to the four conclusions of the four proof trees above yields:

$$\nabla\{p\vee q\}\Longrightarrow\nabla\{p,\top\},\nabla\{q\}.$$

The conditions are not hard to check, e.g.

$$R_T(\{p,\top\})=\{\varnothing,\{p\},\{\top\},\{p\vee\top,\bot\}\}$$

and in the last proof tree each $\Phi\in \mathit{SSRD}(\{\{p\vee q\}\},\{\{p\vee\top,\bot\},\{q,\bot\}\})$ must contain either $\{(p\vee q,0),(\top\vee p,1),(q,1)\}$ or $\{(p\vee q,0),(\top\vee p,1),(q,1),(\bot,1)\}$.

## 4.2 The sequent calculus $S1_P$

Let us now introduce the cut free, one-sided sequent proof system $S1_P$, which arises from the classical propositional left-sided calculus $G1$ (introduced in Section 2) by adding modal rules.

The language for this calculus is the restriction $\mathcal{L}^*$ of $\mathcal{L}$ where negations can only be applied to proposition letters. Every formula in $\mathcal{L}$ is semantically equivalent to some formula of $\mathcal{L}^*$. One way to see this is to recall that the nabla operator is interdefinable with the standard modal operators $\Box$ and $\Diamond$, and that every formula of the basic modal language is equivalent to a formula in which negations can only be applied to proposition letters. Sequents for this calculus are of form $A \Longrightarrow \varnothing$, where $A$ is a finite set of formulas in $\mathcal{L}^*$.

**Definition 4.13** The calculus $S1_P$, operating on $\mathcal{L}^*_P$-sequents, is obtained by extending the calculus $G1$ with the following modal rules:

$$\Delta\text{-l}\,\frac{\{A, \nabla\beta \Longrightarrow \varnothing \mid \beta \in L'_P\alpha\}}{\Delta\alpha, A \Longrightarrow \varnothing}$$

$$P(\nabla)\,\frac{\{A^\Phi \Longrightarrow \varnothing \mid \Phi \in \mathit{SRD}(\Gamma)\}}{\{\nabla\alpha \mid \alpha \in \Gamma\} \Longrightarrow \varnothing}\,\forall\Phi.A^\Phi \in \Phi$$

The rule $P(\nabla)$ is to be read as follows: given $\Gamma \in P_\omega T_\omega \mathcal{L}$, if for every $\Phi \in \mathit{SRD}(\Gamma)$ there exists some $A^\Phi \in \Phi$ such that $A^\Phi \Longrightarrow \varnothing$, then $\{\nabla\alpha \mid \alpha \in \Gamma\} \Longrightarrow \varnothing$. $\lhd$

Recall that a sequent $A \Longrightarrow \varnothing$ is *valid* if $A$ is not satisfiable, i.e. for every model $\mathbb{S}$ and every state $s$ in $\mathbb{S}$, there exists some $a \in A$ such that $\mathbb{S}, s \not\Vdash a$. Then the next lemma provides the soundness and a form of semantic invertibility of the rule $P(\nabla)$:

**Lemma 4.14** *The following are equivalent for every $\Gamma \in P_\omega T_\omega \mathcal{L}$ and every collection $C$ of literals:*

1. *$\{\nabla\alpha \mid \alpha \in \Gamma\} \cup C$ is satisfiable;*

2. *$C$ is satisfiable and there is some $\Phi \in \mathit{SRD}(\Gamma)$ such that every $A \in \Phi$ is satisfiable.*

**Proof.** The proof of this lemma can be verified directly, or by unravelling the meaning of equation (17) in Example 3.22 for the case $T = P$. QED

The following theorem states the soundness and completeness of $S1_P$.

**Theorem 4.15 (Soundness and completeness for $S1_P$)** *For each $\mathcal{L}^*$-sequent $A \Longrightarrow \varnothing$,*

$$\vdash_{S1_P} A \Longrightarrow \varnothing \quad \textit{iff} \quad \models A \Longrightarrow \varnothing.$$

**Proof.** The soundness follows from the soundness of the calculus $S2_P$, because the one-sided rules are in fact just instances of the corresponding two-sided rules.

As for completeness, we use the measure $m(A \Longrightarrow \varnothing)$ defined previously, and reason by induction on this measure. It obviously decreases with each backward application of a one-sided rule.

If $m(A \Longrightarrow \varnothing) = (0,0)$, then $A = \varnothing$ and the sequent is not valid (and by soundness also not derivable.)

Suppose that $m(A \Longrightarrow \varnothing) > (0,0)$. As in the two-sided case, the propositional rules and the $\Delta$-l rule are invertible. Thus the only interesting case is to consider a sequent which is of the form $\{\nabla\alpha \mid \alpha \in \Gamma\} \cup C \Longrightarrow \varnothing$ where $C$ is a finite collection of literals. From the assumptions it follows that the set $\{\nabla\alpha \mid \alpha \in \Gamma\} \cup C$ is not satisfiable, and to by the implication ($2 \Longrightarrow 1$) of Lemma 4.14, there are two possibilities: either $C$ is not satisfiable, or for every $\Phi \in \mathit{SRD}(\Gamma)$ there exists some $A^\Phi \in \Phi$ which is not satisfiable.

If $C$ is not satisfiable, then there is some atom $p \in C$ with $\neg p \in C$. We produce a derivation, starting with the axiom $p, \neg p \Longrightarrow \varnothing$, and continuing with applications of the weakening rule to derive $\{\nabla\alpha \mid \alpha \in \Gamma\} \cup C \Longrightarrow \varnothing$.

On the other hand, suppose that $C$ is satisfiable and that for every $\Phi \in \mathit{SRD}(\Gamma)$ there exists some $A^\Phi \in \Phi$ such that $\models A^\Phi \Longrightarrow \varnothing$. Then by induction hypothesis we have $\vdash_{S1_P} A^\Phi \Longrightarrow \varnothing$ for every $\Phi \in \mathit{SRD}(\Gamma)$. Then a derivation for our sequent consists of the prolongation of all these derivations, taken together, with an application of $P(\nabla)$, so as to obtain a proof of $\{\nabla\alpha \mid \alpha \in \Gamma\} \Longrightarrow \varnothing$, followed by applications of weakening to add the elements in $C$. QED

**Example 4.16** We illustrate the modal rules of the one-sided calculus with a simple proof:

$$P(\nabla,\Delta)\,\frac{\bot \Longrightarrow \varnothing}{\nabla\{\bot\} \Longrightarrow \varnothing}$$

where the only member of $\mathit{SRD}(\{\{\bot\}\})$ is $\{\{\bot\}\}$.

## 4.3 The $\Delta$-free fragment

Since the modality $\Delta_P$ is definable in the $\Delta_P$-free fragment of $\mathcal{L}^*$ (recall Lemma 4.3) it makes sense to restrict ourselves to this fragment without loosing expressive power. The setting we have used so far is modular: we can obtain a complete one-sided calculus $S1_{P\nabla}$ for the Delta-free fragment of the language simply omitting the $\Delta$-l rule in the previous calculus $S1_P$:

**Definition 4.17** The calculus $S1_{P\nabla}$, operating on $\Delta_P$-free $\mathcal{L}^*_P$-sequents, is obtained by extending the calculus $G1$ with the following modal rule:

$$P(\nabla)\,\frac{\{A^\Phi \Longrightarrow \varnothing \mid \Phi \in \mathit{SRD}(\Gamma)\}}{\{\nabla\alpha \mid \alpha \in \Gamma\} \Longrightarrow \varnothing}\;\forall\Phi.A^\Phi \in \Phi$$

◁

The proof of the following theorem is then immediately obtained from the earlier proofs:

**Theorem 4.18 (Soundness and completeness for $S1_{P\nabla}$)** *For each $\Delta$-free $\mathcal{L}^*$-sequent $A \Longrightarrow \varnothing$,*

$$\vdash_{S1_P} A \Longrightarrow \varnothing \ \text{ iff } \ \models A \Longrightarrow \varnothing.$$

# 5 The general case

We have now arrived at the main section of our paper, where we will discuss sequent calculi for finitary Moss' coalgebraic language in the case of a general finitary coalgebra functor $T$ (satisfying our default constraints of preserving inclusions and weak pullbacks). More specifically, we will define two-sided sequent system $G2_T$ for the full language, and a one-sided system $G1_T$ for the version of the language where negation may only occur in front of proposition letters.

## 5.1 The two-sided sequent calculus $G2_T$

For the introduction of the two-sided system $G2_T$, the reader should recall that we defined the sets $L_T\alpha$ and $R_T\alpha$ in Definition 3.14 (and explained the intuition behind them in Fact 3.15), and that we discussed the notion of a separated slim redistribution at the end of section 3.

**Definition 5.1** The sequent calculus $G2_T$ for the language $\mathcal{L}$ is the extension of the propositional calculus $G2$ with the following modal rules:

$$\Delta_T\text{-l}\ \frac{\{A, \nabla\beta \Longrightarrow B \mid \beta \in L_T\alpha\}}{\Delta\alpha, A \Longrightarrow B} \qquad \nabla_T\text{-r}\ \frac{\{A \Longrightarrow \Delta\beta, B \mid \beta \in R_T\alpha\}}{A \Longrightarrow \nabla\alpha, B}$$

$$T(\nabla\Delta)\ \frac{\{A^\Phi_L \Longrightarrow B^\Phi_R \mid \Phi \in SSRD(\Gamma, \Theta)\}}{\{\nabla\alpha \mid \alpha \in \Gamma\} \Longrightarrow \{\Delta\beta \mid \beta \in \Theta\}}\ \forall\Phi.A^\Phi \in Base(\Phi)$$

Here the rule $T(\nabla\Delta)$ is to be read as follows: if, for every $\Phi \in SSRD(\Gamma, \Theta)$, we can find an $A^\Phi \in Base(\Phi)$ such that the sequent $A^\Phi_L \Longrightarrow A^\Phi_R$ is derivable, then we may also derive the sequent $\{\nabla\alpha \mid \alpha \in \Gamma\} \Longrightarrow \{\Delta\beta \mid \beta \in \Theta\}$. ◁

Clearly $G2_T$ is a cut-free system. Our contribution here will be to show that it provides a sound and complete system for the valid $\mathcal{L}$-sequents.

As in the special case of the power set functor, the following lemma embodies the soundness and invertibility of the rules $\nabla_T$-r and $\Delta_T$-l.

**Lemma 5.2** *The following are equivalent, for all $\Gamma, \Theta \in P_\omega T_\omega \mathcal{L}$ and all finite sets $P, Q$ of proposition letters:*

1. *the sequent $\{\nabla\alpha \mid \alpha \in \Gamma\}, P \Longrightarrow Q, \{\Delta\beta \mid \beta \in \Theta\}$ is refutable;*
2. *the sequent $P \Longrightarrow Q$ is refutable, and there is some $\Phi \in SSRD(\Gamma, \Theta)$ such that for every $A \in Base(\Phi)$, the sequent $A_L \Longrightarrow A_R$ is refutable.*

**Proof.** The direction ($1 \Rightarrow 2$) can be proved exactly as the analogous statement of Lemma 4.5.

For the opposite direction, assume that $P \Longrightarrow Q$ is refutable (implying that $P \cap Q = \varnothing$), and that there exists some $\Phi \in \mathit{SSRD}(\Gamma, \Theta)$, such that for every $A \in \mathit{Base}(\Phi)$, the sequent $A_L \Longrightarrow A_R$ is refuted at some state $t_A$ in some model $\mathbb{S}_A = (S_A, \sigma_A, V_A)$. Let $X$ denote the set $\{t_A \mid A \in \mathit{Base}(\Phi)\}$, then we may see $t$ as a map $t : \mathit{Base}(\Phi) \to X$.

Define the coalgebraic model $\mathbb{S} := (S, \sigma, V)$ by taking some fresh root $r$ and putting

$$\begin{array}{rcl}
S & := & \{r\} \uplus \displaystyle\biguplus_{A \in \mathit{Base}(\Phi)} S_A, \\
\sigma(x) & := & \begin{cases} \sigma_A(x) & \text{if } x \in S_A, \\ (Tt)(\Phi) & \text{if } x = r, \end{cases} \\
V(p) & := & \displaystyle\bigcup_{A \in \mathit{Base}(\Phi)} V_A(p) \cup \{r \mid p \in P\}.
\end{array}$$

It easily follows from the theory of universal coalgebra that for all $A \in \mathit{Base}(\Phi)$, the inclusion map $\iota_A : S_A \to S$ is a coalgebra homomorphism (in other words, $(S_A, \sigma_A)$ is a subcoalgebra of $(S, \sigma)$), so that for all formulas $a$ it holds that

$$\text{if } s \in S_A, \text{ then } \mathbb{S}_A, s \Vdash a \text{ iff } \mathbb{S}, s \Vdash a. \tag{22}$$

We claim that the sequent $\{\nabla\alpha \mid \alpha \in \Gamma\}, P \Longrightarrow Q, \{\Delta\beta \mid \beta \in \Theta\}$ is refuted at $r$ in $\mathbb{S}$. Since $P \cap Q = \varnothing$, it is immediate by the definition of $V$ that

$$\mathbb{S}, r \Vdash p \text{ for all } p \in P \text{ and } \mathbb{S}, r \nVdash q \text{ for all } q \in Q. \tag{23}$$

We now verify that

$$\mathbb{S}, r \Vdash \nabla\alpha \text{ for each } \alpha \in \Gamma. \tag{24}$$

For this purpose, observe that for each $a \in \mathcal{B}(\Gamma)$, and each $A \in \mathit{Base}(\Phi)$, it follows from the definition of the map $t : \mathit{Base}(\Phi) \to X$ that $f_0 a \in A$ implies $\mathbb{S}_A, t_A \Vdash a$. Hence by (22) and the definition of $A_L$, for all $a \in \mathcal{B}(\Gamma)$ we obtain that $f_0(a) \in A$ implies $\mathbb{S}, t_A \Vdash a$. This condition can be formulated concisely as follows:

$$Gr(f_0)\,;\in\,;Gr(t) \ \subseteq\ \Vdash^{\smile}. \tag{25}$$

From this it follows by the properties of relation lifting (see Fact 2.12) that

$$Gr(Tf_0)\,;\overline{T}{\in}\,;Gr(Tt) \ \subseteq\ \overline{T}{\Vdash^{\smile}}. \tag{26}$$

In diagrams in $\mathsf{Rel}$ the above reasoning is illustrated as follows: the diagram

$$\begin{tikzcd}
\mathcal{B}(\Gamma) \arrow[r, "Gr(f_0)", "/" marking] \arrow[rrr, bend right, "\Vdash^{\smile}"', "/" marking] & \mathcal{B}(\Gamma) \uplus \mathcal{B}(\Theta) \arrow[r, "\in", "/" marking] \arrow[r, phantom, "\downarrow"'] & P(\mathcal{B}(\Gamma) \uplus \mathcal{B}(\Theta)) \arrow[r, "Gr(t)", "/" marking] & S
\end{tikzcd}$$

lifts, by the properties of the relation lifting, to

$$\begin{tikzcd}
T\mathcal{B}(\Gamma) \arrow[r, "Gr(Tf_0)", "/" marking] \arrow[rrr, bend right, "/" marking, "\overline{T}\Vdash^{\smile}"'] & T(\mathcal{B}(\Gamma) \uplus \mathcal{B}(\Theta)) \arrow[r, "\overline{T}\in", "/" marking, "\downarrow"'] & TP(\mathcal{B}(\Gamma) \uplus \mathcal{B}(\Theta)) \arrow[r, "Gr(Tt)", "/" marking] & S
\end{tikzcd}$$

Now observe that since $\alpha \in \Gamma$ and $\Phi \in \mathit{SSRD}(\Gamma, \Theta)$, we have $(Tf_0)\alpha \ \overline{T}{\in}\ \Phi$, while by definition of $\sigma$ we have $(Tt)\Phi = \sigma(r)$. This means that the pair $(\alpha, \sigma(r))$ belongs to the relation $Gr(Tf_0) \,;\, \overline{T}{\in} \,;\, Gr(Tt)$, and so by (26) we find that $\sigma(r)\ \overline{T}{\Vdash}\ \alpha$. Hence by definition of the satisfaction relation we obtain that $\mathbb{S}, r \Vdash \nabla\alpha$, as required.

It remains to check that

$$\mathbb{S}, r \not\Vdash \Delta\beta \text{ for each } \beta \in \Theta, \tag{27}$$

but we leave this as an exercise for the reader, the proof being completely analogous to that of (24). QED

Moving to our soundness and completeness result, we first note that the definition of a measure of an $\mathcal{L}_T$-sequent can be defined in *exactly* the same way as in the special case where $T = P$, see Definition 4.6. With a little more work than in the special case, we can prove that our proof rules (read backwards) decrease complexity.

**Lemma 5.3** *For each rule of the calculus* $G2_T$*, the measure of any of its assumptions is strictly less than that of its conclusion.*

**Proof.** Consider a backward application of a $G2_T$-rule $R$, with conclusion $A \Longrightarrow B$, and denote $m(A \Longrightarrow B) = (k, l)$.

As in the special case of the power set functor, it is easy to see that if $R$ is a Boolean rule (that is, a rule of the system $G2$), then $k$ will not increase, and $l$ strictly decreases. The propositional rules, and the modal rules $\Delta$-l and $\nabla$-r can be treated exactly as in the proof of Lemma 4.7, so we omit the details.

This leaves the case of the rule $T(\nabla\Delta)$. Assume that the conclusion of this rule is the sequent $\{\nabla\alpha \mid \alpha \in \Gamma\} \Longrightarrow \{\Delta\beta \mid \beta \in \Theta\}$, and take some $\Phi \in \mathit{SSRD}(\Gamma, \Theta)$ and some $C \in \mathit{Base}(\Phi)$. In the paragraph below Definition 3.23 we observed that $C_L \subseteq \mathcal{B}(\Gamma)$ and $C_R \subseteq \mathcal{B}(\Theta)$. From this it is straightforward to derive that for each formula $a$ in $C_L$ we have $d(a) < \max\big(d[\{\nabla\gamma \mid \gamma \in \Gamma\}]\big)$, and similarly for formulas $b \in C_R$. Then it easily follows that for each $C \in \mathit{Base}(\Phi)$ we have $m(C_L \Longrightarrow C_R) < m(\{\nabla\alpha \mid \alpha \in \Gamma\} \Longrightarrow \{\Delta\beta \mid \beta \in \Theta\})$, as required. QED

On the basis of the Lemmas 5.2 and 5.3 above, the proof of our soundness and completeness theorem is now straightforward (in fact, it is completely analogous to that of Theorem 4.8).

**Theorem 5.4 (Soundness and completeness of $G2_T$)** *For every* $\mathcal{L}$*-sequent* $A \Longrightarrow B$*,*

$$\vdash A \Longrightarrow B \ \ \textit{iff} \ \models A \Longrightarrow B.$$

**Proof.** As before, the soundness proof is standard, by induction on the $G2_T$-derivation of $A \Longrightarrow B$. In case the last rule applied was $\Delta_T$-l, the result follows from Fact 3.15; the case of $\nabla_T$-r follows by symmetry. The soundness of the rule $T(\nabla\Delta)$ follows from the implication $1 \Longrightarrow 2$ of Lemma 5.2.

We may prove completeness by induction on the complexity of the sequent $A \Longrightarrow B$, exactly as in the proof of Theorem 4.8. We leave the details to the reader. QED

**Remark 5.5** It is straightforward to check that the above proofs also work for the negation-free variant $G2_T^-$ of $G2_T$. This sequent system, tailored towards the negation-free fragment $\mathcal{L}^-$ of the language, is obtained simply by omitting the proof rules for the negation ($\neg$-l and $\neg$-r) from the system $G2_T$. In other words, we have a soundness and completeness result for $G2_T^-$, stating that every negation-free sequent is derivable in $G2_T^-$ iff it is valid. Note that in the case of the power set functor, the negation-free fragment of the logic corresponds to *positive modal logic*, see [18].

We illustrate the proof system with presenting it for two concrete examples of functors - the binary tree functor $Id \times Id$ and the multiset functor $B$.

**The binary tree functor.** Consider $T = Id \times Id$.

**Example 5.6** The following is an example of a proof of the sequent $\nabla(a, b), \nabla(c, d) \Longrightarrow \nabla(a \wedge c, b \wedge d)$ in the calculus. For each $x, y \in P_\omega\{a \wedge c, b \wedge d\}$ such that $a \wedge c \in x$ consider the following proof

$$T(\nabla, \Delta)\ \cfrac{\cfrac{a, c \Longrightarrow a \wedge c}{a, c \Longrightarrow \bigvee x}}{\nabla(a, b), \nabla(c, d) \Longrightarrow \Delta(\bigvee x, \bigvee y)}$$

For each $x, y \in P_\omega\{a \wedge c, b \wedge d\}$ such that $b \wedge d \in y$ consider the following proof

$$T(\nabla, \Delta)\ \cfrac{\cfrac{b, d \Longrightarrow b \wedge d}{b, d \Longrightarrow \bigvee y}}{\nabla(a, b), \nabla(c, d) \Longrightarrow \Delta(\bigvee x, \bigvee y)}$$

Now one application of the $T(\nabla, \Delta)$ rule yields the sequent:

$$\nabla(a, b), \nabla(c, d) \Longrightarrow \nabla(a \wedge c, b \wedge d).$$

In the above proofs, unraveling the definition, we obtain

$$R_T(a \wedge c, b \wedge d) = \{(\bigvee x, \bigvee y) \mid x, y \in P_\omega\{a \wedge c, b \wedge d\}, \text{ and } a \wedge c \in x \text{ or } b \wedge d \in y\}.$$

Moreover, each $\Phi \in SSRD(\{(a, b), (c, d)\}, \{(\bigvee x, \bigvee y)\}$ is of the form $(A, B)$ where

$$\{(a, 0), (c, 0), (\bigvee x, 1)\} \subseteq A \text{ and } \{(b, 0), (d, 0), (\bigvee y, 1)\} \subseteq B.$$

Thus, modulo weakening inferences if necessary, the above sequents provide the required proof.

**The bag (multiset) functor.** Consider $T$ is $B$.

**Example 5.7** Recall examples of $\mathit{SSRD}$ in Example 3.27 and $R_B$ in Example 3.16. We have

$$\frac{\varnothing}{\nabla\{a^5, b^5\} \Longrightarrow \Delta\{\bot^n\}}\ n \neq 10$$

where $\mathit{SSRD}(\Gamma, \Sigma) = \varnothing$, reasoning is similar to that in Example 3.27.

$$\frac{\varnothing}{\nabla\{a^5, b^5\} \Longrightarrow \Delta\{(a \vee b)^m\}}\ m \neq 10$$

where similar reasoning applies.

$$\frac{a, b \Longrightarrow a \vee b}{\nabla\{a^5, b^5\} \Longrightarrow \Delta\{(a \vee b)^m\}}\ m = 10$$

where any $\Phi \in \mathit{SSRD}(\Gamma, \Sigma)$ is of type $\Phi :\ P(\{a \vee b\} \uplus \{a, b\}) \to \mathbb{N}$, and from it being a separated slim redistribution we obtain that it contains in its base either of the sets $\{(a, 0), (b, 0), (a \vee b, 1)\}, \{(a, 0), (a \vee b, 1)\}, \{(b, 0), (a \vee b, 1)\}$, we have illustrated the proof w.l.o.g. using the set $\{a, b, a \vee b\}$. (If $\Phi$ sends all of the above mentioned sets to zero, it is not possible that $\{a^5, b^5\}$ and $\{(a \vee b)^{10}\}$ are lifted members of $\Phi$ at the same time.).

Now the sequent

$$\nabla\{a^5, b^5\} \Longrightarrow \nabla\{(a \vee b)^{10}\}$$

is provable from the sequents above using a $\nabla$-r inference: observe that

$$R_B((a \vee b)^{10}) = \{(B \bigvee)\Phi \mid \Phi : P\{a \vee b\} \to \mathbb{N},\ \neg((a \vee b)^{10}\overline{B}(\notin)\Phi)\},$$

where the condition for $\Phi$ actually means that either $\Phi(\varnothing) \neq 10$ or $\Phi(\{a \vee b\}) > 0$. Thus $R_B(\{(a \vee b)^{10}\}) = \{\{\bot^n\} | n \neq 10\} \cup \{\{(a \vee b)^m\} | m > 0\}$. Each of the required sequents is proved as in one of the cases above — we have split the case $m > 0$ in two according to whether $m = 10$ or not.

## 5.2 The one-sided sequent calculus $G1_T$

As in the special case of the power set functor, we are also interested in a one-sided version of the sequent calculus $G2_T$. As before we need to restrict the use of negation to atomic formulas, working with the language $\mathcal{L}^*$ instead of with $\mathcal{L}$. Using the fact that $\nabla$ and $\Delta$ are each other's Boolean duals, it is not difficult to see that every formula $a$ in $\mathcal{L}$ can be rewritten into an equivalent formula $a^* \in \mathcal{L}^*$.

The one-sided sequent system for this language is defined as follows.

**Definition 5.8** The sequent calculus $G1_T$ for the language $\mathcal{L}^*$ is obtained by extending the calculus $G1$ with the following modal rules:

$$\Delta\text{-l}\ \frac{\{A, \nabla\beta \Longrightarrow \varnothing \mid \beta \in L_T\alpha\}}{\Delta\alpha, A \Longrightarrow \varnothing}$$

$$T(\nabla)\,\frac{\{A^{\Phi} \Longrightarrow \varnothing \mid \Phi \in \mathit{SRD}(\Gamma)\}}{\{\nabla\alpha \mid \alpha \in \Gamma\} \Longrightarrow \varnothing}\; A^{\Phi} \in \mathit{Base}(\Phi)$$

The rule $T(\nabla)$ is to be read as follows: given $\Gamma \in P_\omega T_\omega \mathcal{L}$, if for every $\Phi \in \mathit{SRD}(\Gamma)$ there exists some $A^{\Phi} \in \Phi$ such that $A^{\Phi} \Longrightarrow \varnothing$, then $\{\nabla\alpha \mid \alpha \in \Gamma\} \Longrightarrow \varnothing$. $\lhd$

As in the special case of the power set functor, Lemma 5.10 below expresses the soundness and a form of semantic invertibility of the rule $T(\nabla)$. We use the following lemma to simplify the task of proving the Lemma 5.10: instead of proving it directly, we link the current case to the previous case of the two-sided calculus:

**Lemma 5.9** *For any $\Gamma \in P_\omega T_\omega \mathcal{L}$, the map $TPf_0$ restricts to a bijection between the sets $\mathit{SRD}(\Gamma)$ and $\mathit{SSRD}(\Gamma, \varnothing)$.*

**Proof.** We first prove that $TPf_0$ maps slim redistributions of $\Gamma$ to separated slim redistributions of $(\Gamma, \varnothing)$. For this purpose, fix a $\Phi \in \mathit{SRD}(\Gamma)$. Then by definition, $\Phi$ is an element of $TP(\mathcal{B}(\Gamma))$, so that we immediately find $\mathit{Base}(\Phi) \subseteq P(\mathcal{B}(\Gamma))$. Since $\mathit{Base} : T_\omega \stackrel{\cdot}{\to} P_\omega$ is a natural transformation, this means that $\mathit{Base}((TPf_0)\Phi) = (PPf_0)(\mathit{Base}(\Phi)) \subseteq (PPf_0)(P\mathcal{B}(\Gamma)) \subseteq P(\mathcal{B}(\Gamma) \uplus \mathcal{B}(\varnothing))$, as a straightforward verification reveals. From this it is immediate that $(TPf_0)\Phi \in TP(\mathcal{B}(\Gamma) \uplus \mathcal{B}(\varnothing))$, as required.

Thus in order to prove that $(TPf_0)\Phi \in \mathit{SSRD}(\Gamma, \varnothing)$, it suffices to check that for every $\gamma \in \Gamma$ we have $(Tf_0)\gamma \,\overline{T}{\in}\, (TPf_0)\Phi$. Fix such a $\gamma$; then by $\Phi \in \mathit{SRD}(\Gamma)$ we obtain that $\gamma \in \lambda^T(\Phi)$, and since $\lambda^T : TP \stackrel{\cdot}{\to} PT$ is a natural transformation, we have $\lambda^T((TPf_0)\Phi) = (PTf_0)(\lambda^T(\Phi))$. From this it is immediate that $(Tf_0)\gamma \in \lambda^T((TPf_0)\Phi))$, which means that $(Tf_0)\gamma \,\overline{T}{\in}\, (TPf_0)\Phi$ indeed. This finishes the proof that $TPf_0$ maps slim redistributions of $\Gamma$ to separated slim redistributions of $(\Gamma, \varnothing)$.

It remains to prove the statement of the Lemma. First note that since $f_0$ restricts to a bijection between the sets $\mathcal{B}(\Gamma)$ and $\mathcal{B}(\Gamma) \uplus \mathcal{B}(\varnothing)$, it has an inverse $g : \mathcal{B}(\Gamma) \uplus \mathcal{B}(\varnothing) \to \mathcal{B}(\Gamma)$. We leave it for the reader to verify that for any $\Xi \in \mathit{SSRD}(\Gamma, \varnothing)$, the object $(TPg)\Xi$ is a slim redistribution of $\Gamma$. Given the fact that $TPf_0$ restricts to a bijection from $T_\omega P_\omega \mathcal{B}(\Gamma)$ and $T_\omega P_\omega \mathcal{B}(\Gamma) \uplus \mathcal{B}(\varnothing)$, the statement of the Lemma follows. QED

**Lemma 5.10** *The following are equivalent for every $\Gamma \in P_\omega T_\omega \mathcal{L}$ and every collection $C$ of literals:*

1. *$\{\nabla\alpha \mid \alpha \in \Gamma\} \cup C$ is satisfiable;*
2. *$C$ is satisfiable and for some $\Phi \in \mathit{SRD}(\Gamma)$, every $B \in \mathit{Base}(\Phi)$ is satisfiable.*

**Proof.** Fix a collection $\Gamma \in P_\omega T_\omega \mathcal{L}$ and a set $C$ of literals. Let $P, Q$ be the unique sets of proposition letters such that $C = P \cup \{\neg q \mid q \in Q\}$. We will prove the Lemma by showing both statements 1 and 2 above to be equivalent to the statements 3 and 4 below:

3. $\{\nabla\gamma \mid \gamma \in \Gamma\}, P \Longrightarrow Q$ is refutable;
4. $P \Longrightarrow Q$ is refutable, and there is some $\Phi \in \mathit{SSRD}(\Gamma, \Theta)$ such that for every $A \in \mathit{Base}(\Phi)$, the sequent $A_L \Longrightarrow A_R$ is refutable.

The equivalence ($1 \Leftrightarrow 3$) is immediate, and the equivalence ($3 \Leftrightarrow 4$) is an instantiation of Lemma 5.2 above. It remains to prove the equivalence of the statements 2 and 4.

($2 \Rightarrow 4$) Suppose that statement 2 holds, then clearly the sequent $P \Longrightarrow Q$ is refutable. Define $\Xi := (TPf_0)\Phi$, then we have $\Xi \in SSRD(\Gamma, \varnothing)$ by Lemma 5.9, and $Base(\Xi) = (PPf_0)Base(\Phi)$ since $Base : T_\omega \stackrel{\cdot}{\to} P_\omega$ is a natural transformation. Hence an arbitrary $A \in Base(\Xi)$ is of the form $(Pf_0)B$ with $B \in Base(\Phi)$. Then it is easy to see that $A_L = B$ and $A_R = \varnothing$, so that the refutability of $A$ is immediate by the satisfiability of $B$ (which holds by assumption).

($4 \Rightarrow 2$) If statement 4 holds, then the set $C$ is obviously satisfiable. By Lemma 5.9 there is a slim redistribution $\Phi$ of $\Gamma$ such that $\Xi = (TPf_0)\Phi$. We claim that every $B \in Base(\Phi)$ is satisfiable. To see this, fix a set $B \in Base(\Phi)$. As before we have $Base(\Xi) = (PPf_0)Base(\Phi)$, so that we find $(Pf_0)B \in Base(\Xi)$. Then by assumption the sequent $((Pf_0)B)_L \Longrightarrow ((Pf_0)B)_R$ is refutable; but it is easy to see that $((Pf_0)B)_L = B$ and $((Pf_0)B)_R = \varnothing$. Now the satisfiability of $B$ is immediate by the refutability of the sequent $B \Longrightarrow \varnothing$. QED

On the basis of the Lemmas above, the proof of the following soundness and completeness result is a straightforward variation of earlier proofs. We omit the details.

**Theorem 5.11 (Soundness and completeness for $G1_T$)** *For each $\mathcal{L}^*$-sequent $A \Longrightarrow \varnothing$,*

$$\vdash_{G1_T} A \Longrightarrow \varnothing \quad \textit{iff} \quad \models A \Longrightarrow \varnothing.$$

## 5.3 The Δ-free fragment

For the functors preserving finite sets, the modality $\Delta_T$ is definable in the $\Delta_T$-free fragment of $\mathcal{L}^*$, while for other functors it is not (recall Fact 3.15 and the discussion below in Remark 3.17). However, since already the $\Delta$-free fragment of Moss' logic is expressive (recall Fact 3.10), it makes sense to restrict ourselves to this fragment of the Moss' language even in the case of a general functor $T$, without loosing expressive power.

Again, by modularity of the calculi defined so far, we can obtain a complete one-sided calculus $G1_{T\nabla}$ for the $\Delta$-free fragment of the language simply omitting the $\Delta$-l rule in the previous calculus $G1_T$:

**Definition 5.12** The sequent calculus $G1_{T\nabla}$ for the $\Delta$-free fragment of the language $\mathcal{L}^*$ is obtained by extending the calculus $G1$ with the following modal rule:

$$T(\nabla)\,\frac{\{A^\Phi \Longrightarrow \varnothing \mid \Phi \in SRD(\Gamma)\}}{\{\nabla\alpha \mid \alpha \in \Gamma\} \Longrightarrow \varnothing}\,A^\Phi \in Base(\Phi)$$

◁

The proof of the following theorem is a variation of the previous proof, we omit the details here as well.

**Theorem 5.13 (Soundness and completeness for $G1_{T\nabla}$)** *For each $\mathcal{L}^*$-sequent $A \Longrightarrow \varnothing$ not containing $\Delta$,*

$$\vdash_{G1_{T\nabla}} A \Longrightarrow \varnothing \quad \textit{iff} \quad \models A \Longrightarrow \varnothing.$$

# 6 Concluding remarks

We have presented structurally well-behaved proof systems for the finitary Moss logics. By structurally well behaved we mainly mean that the calculi are complete without the cut rule. There are two issues we have touched in the paper which are worth mentioning separately: the finitarity vs. infinitarity of the rules in various cases, and the subformula property which often follows merely from the absence of the cut rule.

## 6.1 Finitarity

The sources of possible infinitary behaviour of the calculi are the rules $\nabla_T$-r, $\Delta_T$-l and $T(\nabla\Delta)$ whose arities are bounded by cardinality of the sets $R_T, L_T$ and $SSRD$. The question is, when are the following sets *finite*?:

- $R_T(\alpha)$ for a given $\alpha \in T_\omega\mathcal{L}$
- $L_T(\alpha)$ for a given $\alpha \in T_\omega\mathcal{L}$
- $SSRD(\Gamma, \Sigma)$ for given $\Gamma, \Sigma \in P_\omega T_\omega \mathcal{L}$

One clear answer is that for all functors $T$ preserving finite sets, all the above sets are finite and therefore *both* the two- and one-sided calculi $G1_T$ and $G2_T$ are finitary (the same holds for the axiomatization given in [11].) This includes the calculi $S1_P$ and $S2_P$.

Preservation of finite sets is actually quite rare: from the functors we consider in examples, the finitary powerset functor as well as the binary tree functor preserve finite sets, while the bag functor and the probabilistic distribution functors do not, but not even the polynomial functors do in general (e.g. functors involving an infinite set constant functor or an infinite sum, often included in the definition of polynomial functors, do not preserve finite sets).

**The Bag functor.** Let's have a closer look at the finitary bag functor $B$. Consider a finite multiset $\Phi \in BP_\omega\mathcal{L}$. Then $\bigcup Base(\Phi)$ is finite — those are members of $\mathcal{L}$ occurring in some $A \in P_\omega\mathcal{L}$ with $\Phi(A) > 0$. Imagine you are to fill the "witness square" of the relation $\in \subseteq \bigcup Base(\Phi) \times \Phi$. There are only finitely many ways of filling this square (because there are only finitely many ways how to write a natural number as a finite sum of natural numbers). Therefore there are only finitely many lifted members of $\Phi$.

Fix a finite set $\Gamma \in P_\omega B\mathcal{L}$. We can show, that for each $\alpha \in \Gamma$ there are only finitely many $\Phi$ such that $\alpha \ \overline{B}{\in}\ \Phi$ by a similar argument as given above. This shows that the set $SRD(\Gamma)$ is finite, and consequently the one-sided calculus for the $\Delta$-*free* fragment $G1_{\nabla B}$ is finitary.

Example 3.16 shows that the sets $R_B$ and $L_B$ are often infinite, therefore the calculi involving $\Delta$ are infinitary in general.

**The probabilistic distribution functor.** It is not hard to create an analogous example which shows that the two sets $R_{\mathcal{D}_\omega}$ and $L_{\mathcal{D}_\omega}$ might be infinite in the case of the functor $\mathcal{D}_\omega$. But even the set of slim redistribution might be infinite in this case: consider $SRD(\{a^{0.5}, b^{0.5}\})$ — since there are infinitely many possibilities how to write 0.5 as a sum there are infinitely many possible witnesses $x \in \mathcal{D}(\in)$ for infinitely many $\Phi$ such that $\{a^{0.5}, b^{0.5}\}\overline{\mathcal{D}} \in \Phi$. Therefore all the calculi for $\mathcal{D}_\omega$ are infinitary.

## 6.2 Subformula property

Given our notion of subformulas (recall Definition 3.3), all the propositional rules as well as the $T(\nabla\Delta)$ rules satisfy the subformula property: the assumptions of a rule contain only genuine subformulas of the conclusion. Thus the $\Delta$-free calculi $G1_{T\nabla}$ and $S1_{P\nabla}$ enjoy the subformula property.

It is, however, not the case of the calculi containing rules $\nabla_T$-r, $\Delta_T$-l, since they involve the sets $R_T(\alpha)$ and $L_T(\alpha)$ which use the operations $T\bigwedge$ and $T\bigvee$ on objects $\Phi$ in $TPBase(\alpha)$. This, on one hand, means that the calculi containing those rules do not enjoy the genuine subformula property, but, on the other hand, this means that the material which can occur in a proof of a sequent is not arbitrary — it is at least bounded by the sets $R_T(\alpha)$ and $L_T(\beta)$ for formulas $\nabla\alpha$ and $\Delta\beta$ occurring on the right (resp. left) context of sequents in the proof, modulo negations.

Let us have a closer look: consider e.g. a proof of a sequent $\Gamma \Rightarrow \Sigma, \nabla\alpha$. If $\nabla\alpha$ was not introduced by weakening, it had to be introduced by a $\nabla_T$-r rule, with immediate predecessors $\Delta(T\bigvee)\Phi$ with $\Phi$ in $TPBase(\alpha)$. Those of $\Delta(T\bigvee)\Phi$ not introduced by weakening had to be introduced by a $T(\nabla\Delta)$ rule where the assumption may contain formulas from $Base((T\bigvee)\Phi)$. It is not hard to see that $Base((T\bigvee)\Phi)$ contains only finite disjunctions of formulas from $Base(\alpha)$: denote by $Base(\alpha)^\vee$ the set of finite disjunctions of formulas from $Base(\alpha)$. Now

$$\bigvee : PBase(\alpha) \to Base(\alpha)^\vee,$$

therefore

$$T\bigvee : TPBase(\alpha) \to TBase(\alpha)^\vee,$$

and $(T\bigvee)\Phi$ is in fact in $TBase(\alpha)^\vee$ and since

$$Base : TBase(\alpha)^\vee \to PBase(\alpha)^\vee$$

we conclude that $Base((T\bigvee)\Phi)$ is in $PBase(\alpha)^\vee$, which means it contains only finite disjunctions of subformulas of $\nabla\alpha$. If we pursuit this reasoning further we may conclude that any provable sequent is provable from axioms containing only (atomic) subformulas of the given sequent.

The main goal of this paper has been to introduce a uniform proof theory for the finitary Moss' logic, parametric in the functor $T$. We succeeded in defining cut-free, one- and two-sided calculi for the logics. For the two-sided cacluli, using the full language with both $\nabla$ and $\Delta$ seems necessary to obtain a cut-free presentation. An advantage of the two-sided presentation is, apart from its naturality, the fact that we can use its negation free fragment to capture proof theory of the positive fragment of the Moss' logic. The one-sided calculi for the $\Delta$-free fragment are in general better behaved — they enjoy the genuine subformula property and are finitary for some functors for which the general calculi are infinitary (e.g. the bag functor.)

Since, in this paper, we mainly concentrated on a uniform approach parametric in the functor, we have not touched decidability and complexity issues. We do not expect results concerning decidability and complexity can be obtained in a uniform way for all the functors we consider, we think that a finer case-analysis will be necessary. We leave those issues for further research.

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