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Constructing crystalline cohomology

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Abstract

Let k be a perfect field of characteristic 0, and let $W = W(k)$ be the ring of Witt vectors of k. Let $S = \mathrm{Spec}(W)$, $S_0 = \mathrm{Spec}(k)$ and let $X_0 \to S_0$ be a smooth and projective scheme. Given a projective space $Y \to S$ and a closed immersion $X_0 \to Y$, we will construct cohomology groups $\mathrm{H}^i_{\mathrm{dR,PD}}(X_0/S; Y)$. We will construct the crystalline cohomology groups $\mathrm{H}^i_{\mathrm{cris}}(X_0/S)$, and show that if $X \to S$ is a smooth lift of $X_0 \to S_0$, that there is a canonical isomorphism $\mathrm{H}^i_{\mathrm{cris}}(X_0/S) \cong \mathrm{H}^i_{\mathrm{dR,PD}}(X_0/S; Y)$. Furthermore, we will show that this isomorphism is *natural* in X_0, which gives us a way of calculating the Frobenius action on crystalline cohomology.

Title: Constructing crystalline cohomology

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Introduction

Let k be a perfect field of characteristic $p > 0$. Suppose we are given a smooth and projective scheme $X_0 \to \operatorname{Spec}(k)$. We will start this thesis by introducing the ring of *Witt vectors* $W(k)$, which is a 'canonical lifting' of k to characteristic 0.

Write $S = \operatorname{Spec}(W)$. One of the original goals of crystalline cohomology is to define a cohomology theory $\mathrm{H}^i_{\mathrm{cris}}(X_0/S)$ with coefficients in W, that agrees with the (algebraic) de Rham cohomology of a lift. More precisely, if there exists a smooth and projective $X \to S$ such that $X_0 = X \otimes_S \operatorname{Spec}(k)$, we want there to be a canonical isomorphism $\mathrm{H}^i_{\mathrm{cris}}(X_0/S) \cong \mathrm{H}^i_{\mathrm{dR}}(X/S)$.

Crystalline cohomology is based on the idea of 'divided powers'. Loosely speaking, if x is an element of a ring, then a divided power of x is a (sometimes formal) element representing the expression $\frac{x^n}{n!}$. The PD is shorthand for 'puissance divisée', which is French for divided powers. We will study rings with divided power structures in chapter 2.

After that, in chapter 3, we will construct a modified version of de Rham cohomology, which we will call *PD-de Rham cohomology* (this name does not seem to appear in the literature). If S has the structure of a PD-scheme, and we are given a closed S-immersion $X \to Y$, we will construct cohomology groups $\mathrm{H}^i_{\mathrm{dR,PD}}(X/S;Y)$. We will see that if X and Y are smooth over S and X is projective, that there is a canonical isomorphism

$$\mathrm{H}^i_{\mathrm{dR,PD}}(X/S;Y) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(X/S) \tag{0.1}$$

The key ingredient here is a 'local Poincaré lemma' that relies on the divided power structures. We can briefly explain the above ideas in the simple case tha $k = \mathbb{F}_p$ (so that $W = \mathbb{Z}_p$), $X_0 = \operatorname{Spec}(\mathbb{F}_p)$ (a closed point) and $Y = \operatorname{Spec}(\mathbb{Z}_p[x])$. Then $X = \mathbb{Z}_p$ is a smooth lift of X_0 to W, and we can consider the classical (algebraic) de Rham chain complex $\Omega^\bullet_{Y/S}$, which is the complex

$$\cdots \to 0 \to \mathbb{Z}_p[x] \xrightarrow{d} \mathbb{Z}_p[x]dx \to 0 \to \ldots .$$

We have a closed immersion $X \to Y$ defined by $\mathbb{Z}_p[x] \to \mathbb{Z}_p$ sending $x \mapsto 0$. This induces a map of chain complexes $\Omega^\bullet_{Y/S} \to \Omega^\bullet_{X/S}$, which is given by the diagram

$$\begin{tikzcd}
\ldots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z}_p[x] \arrow[r, "d"] \arrow[d] & \mathbb{Z}_p[x]dx \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & 0 \arrow[r] & \mathbb{Z}_p \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \ldots
\end{tikzcd}$$

We would like this map to be a quasi-isomorphism, so that it induces the isomorphism (0.1). However, the map d is far from surjective, since the elements x^{pn-1} do not lie in the image for all $n \in \mathbb{N}$. In fact, one easily computes

$$\mathrm{H}^0\left(\Omega^{\bullet}_{Y/S}\right) = \mathbb{Z}_p, \qquad \mathrm{H}^1\left(\Omega^{\bullet}_{Y/S}\right) = \bigoplus_{n\in\mathbb{N}} \left(\left(\mathbb{Z}_p/p^{v_p(n)}\mathbb{Z}_p\right) x^{n-1}dx\right)$$

(here v_p is the valuation on $\mathbb{Z}_p$). Thus the map $\Omega^{\bullet}_{Y/S} \to \Omega^{\bullet}_{X/S}$ fails to be a quasi-isomorphism in degree 1.

We can fix this by "adding all divided powers of the elements defining the ideal of X in Y". In this case, (x) is the ideal defining X, and so we can consider the ring

$$\mathbb{Z}_p\langle x\rangle := \left\{ a_0 + a_1\frac{x}{1!} + \cdots + a_n\frac{x^n}{n!} \;\middle|\; {}_{a_0,\ldots,a_n\in\mathbb{Z}_p}^{n\in\mathbb{N}} \right\} \subseteq \mathbb{Q}_p[x].$$

Then define

$$\Omega^{\mathrm{PD},\bullet}_{Y/S} := \Omega^{\mathrm{PD},\bullet}_{\mathbb{Z}_p\langle x\rangle/\mathbb{Z}_p}$$

Here the PD on the right hand side is notation for a technical definition: We need the differential to be compatible with the PD-structure, in other words we have to enforce $d\left(\frac{x^n}{n!}\right) = \frac{x^{n-1}}{(n-1)!}dx$ as an extra condition on our differential, see Section 2.5.

Then the canonical map $\Omega^{\mathrm{PD},\bullet}_{Y/S} \to \Omega^{\bullet}_{X/S}$ is given by the diagram

$$\begin{tikzcd}
\ldots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z}_p\langle x\rangle \arrow[r, "d"] \arrow[d] & \mathbb{Z}_p\langle x\rangle dx \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & 0 \arrow[r] & \mathbb{Z}_p \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \ldots
\end{tikzcd}$$

In this case, the map d *is* surjective (since $d\left(\frac{x^n}{n!}\right) = \frac{x^{n-1}}{(n-1)!}dx$), and hence the canonical map $\Omega^{\mathrm{PD},\bullet}_{Y/S} \to \Omega^{\bullet}_{X/S}$ is a quasi-isomorphism as desired, and it will induce the quasi-isomorphism (0.1).

We will see that the groups $\mathrm{H}^i_{\mathrm{dR,PD}}(X/S;Y)$ are defined using only the closed fibre $X_0 = X \otimes_S \mathrm{Spec}(k)$, and not X. Thus given an arbitrary projective scheme $X_0 \subseteq \mathbb{P}^n_k$ over $\mathrm{Spec}(k)$, we can pick $Y = \mathbb{P}^n_S$, and using the closed immersion $\mathbb{P}^n_k \to \mathbb{P}^n_S$ we can force a cohomology theory for schemes over $\mathrm{Spec}(k)$ by defining

$$\mathrm{H}^i(X_0/S) := \mathrm{H}^i_{\mathrm{dR,PD}}(X_0/S;\mathbb{P}^n_S)$$

The main ingredient here is that since Y lives over W, we can easily enlarge $\mathcal{I} \subseteq \mathcal{O}_Y$ with divided powers. However, there are several issues. First of all, if X_0 does not lift to a smooth scheme over W it might be the case that the definition is dependent on the choice of projective embedding. But even worse, given a morphism $f\colon X_0 \to X_0'$ and a map $F\colon Y' \to Y$ (compatible with f), it is not at all clear if the induced morphism

$$f^*\colon\, \mathrm{H}^i_{\mathrm{dR,PD}}(X_0'/S, Y') \to \mathrm{H}^i_{\mathrm{dR,PD}}(X_0/S, Y)$$

depends on the choice of the embedding or the map F. Even if X_0 and X_0' lift to some smooth X and X', it need not be the case that f lifts to a map $X \to X'$, and hence even then the map f^* might be dependent on the choice of embedding.

To fix this we will need an intrinsic definition of $\mathrm{H}^i_{\mathrm{dR,PD}}(X_0/S; Y)$ that does not depend on Y. This will be the crystalline cohomology $\mathrm{H}^i_{\mathrm{cris}}(X_0/S)$, and will be defined for all schemes $X_0 \to S$. To do this, we first observe that we should not consider the closed immersion $X \to Y$, but instead the closed immersion to the PD-envelope $X \to \mathcal{D}_{X,\gamma}(Y)$. If Y does not live over $\mathrm{Spec}(W)$, but over $S_n := \mathrm{Spec}(W/p^nW)$ for some $n \in \mathbb{N}$, then this will be thickening of X. Therefore it is natural to study the site $\mathrm{Cris}(X/S)$ of all S-thickenings of X that come with a PD-structure. We will see that it comes with a canonical structure sheaf $\mathcal{O}_{X/S}$, so that we can define the cohomology groups $\mathrm{H}^i_{\mathrm{cris}}(X/S)$ as $\mathrm{H}^i(\mathrm{Cris}(X/S), \mathcal{O}_{X/S})$. In chapter 4 we will study this site and its associated topos $(X/S)_{\mathrm{cris}}$.

Finally in chapter 5 we will prove that if X and Y are smooth over S and X is projective, there exists an isomorphism $\mathrm{H}^i_{\mathrm{cris}}(X/S) \cong \mathrm{H}^i_{\mathrm{dR,PD}}(X/S; Y)$. Furthermore, we will see that this isomorphism is functorial in X. From this it immediately follows that the cohomology groups $\mathrm{H}^i_{\mathrm{dR,PD}}(X/S; Y)$ do not depend on the choice of closed immersion $X \to Y$, and furthermore that the map f^* is also independent on the choice of embedding.

1 Prerequisites

1.1 Witt vectors

Let $p > 0$ be a prime. One of the goals of crystalline cohomology is to give a cohomology theory for schemes of characteristic p with coefficients in a ring of characteristic 0. The Witt vectors will serve as the coefficient ring. They provide a canonical lifting from a perfect field of characteristic p to a ring of characteristic 0.

Let k be a field of characteristic p. Suppose A is a discrete valuation ring of characteristic 0 with residue field k. Denote v for the normalized valuation on A. Note that (the image of) p is contained in the maximal ideal of A, hence $v(p) \geq 1$.

Theorem 1.1. *Let k be a perfect field of characteristic p. Then there exists a complete discrete valuation ring A and a map $\pi\colon A \to k$ such that π induces an isomorphism $A/pA \xrightarrow{\sim} k$. For every ring homomorphism $\phi : k_1 \to k_2$ there exists a unique homomorphism $\Phi : A_1 \to A_2$ making the diagram*

$$\begin{tikzcd} A_1 \arrow[r, "\Phi"] \arrow[d] & A_2 \arrow[d] \\ k_1 \arrow[r, "\phi"] & k_2 \end{tikzcd}$$

commute.

Proof. See [Serre, Chapter II§5]. □

We thus get a functor

$$W\colon \left\{\begin{matrix}\text{perfect fields of}\\ \text{characteristic } p>0\end{matrix}\right\} \to \left\{\begin{matrix}\text{complete DVR's}\\ \text{with } v(p)=1\end{matrix}\right\}.$$

In particular, since any perfect field of characteristic $p > 0$ comes with a Frobenius map sending $x \mapsto x^p$, we get a *lifting of Frobenius*

$$F\colon W(k) \to W(k),$$

satisfying $F(x) = x^p \pmod p$.

Example 1.2. If $k = \mathbb{F}_p$, then $W(k) = \mathbb{Z}_p$.

1.2 Regularity

A useful computational tool for schemes smooth over fields is the notion of a regular sequence. The goal of this section is to generalize some of the theory about regular sequences for schemes over a field to schemes over a discrete valuation ring. We start by introducing the notion of a regular sequence.

Definition 1.3. Let A be a ring. A sequence of elements $f_1, \dots, f_r \in A$ is called a *regular sequence* if

- For each i, the image of f_i is not a zerodivisor in $A/(f_1, \dots, f_{i-1})$,
- $A/(f_1, \dots, f_r)A \neq 0$.

An ideal $I \subseteq A$ is a *regular ideal* if there exists a regular sequence $(f_1, \dots, f_r)$ that generates I.

Definition 1.4. A *regular local ring* is a local ring $(A, \mathfrak{m})$ such that $\mathfrak{m}$ is a regular ideal. A ring A is said to be *regular* if all of its local rings are regular local rings.

Definition 1.5. Let $\iota\colon X \to Y$ be closed immersion with corresponding ideal sheaf $\mathcal{I}$. We say that ι is a *regular immersion* and $\mathcal{I}$ is *regular* if for any $x \in X$ there exists an affine open $x \in U \subseteq Y$ such that $\mathcal{I}(U)$ is a regular ideal in $\mathcal{O}_Y(U)$.

Definition 1.6. A scheme X is said to be regular if for all $x \in X$ the local ring $\mathcal{O}_{X,x}$ is regular.

The notion of a regular scheme is very useful, as it provides us with an easy way of showing a closed immersion is regular.

Lemma 1.7. *Let $\iota\colon X \to Y$ be a closed immersion of regular schemes. Then ι is a regular immersion.*

Proof. See [Stacks, 0E9J] □

It turns out that over a field, any smooth scheme is regular.

Lemma 1.8. *Let k be a field, and $X \to \operatorname{Spec}(k)$ be smooth. Then X is regular.*

Proof. See [Stacks, 056S]. □

We now generalize this to discrete valuation rings, so that we may later apply it to schemes over a ring of Witt vectors.

Proposition 1.9. *Let W be a discrete valuation ring. If $X \to \operatorname{Spec}(W)$ is a smooth morphism, then X is a regular scheme.*

Proof. Since both regularity and smoothness are local conditions, we may assume X is an affine scheme, say $X = \operatorname{Spec}(A)$. Pick a uniformizer π for W. Denote the residue field of W by k and the field of fractions by K. Denote $X_k = X \otimes_{\operatorname{Spec}(W)} \operatorname{Spec}(k)$, similarly for X_K. Note that both $X_k \to \operatorname{Spec}(k)$ and $X_K \to \operatorname{Spec}(K)$ are smooth, so by Lemma 1.8 both X_k and X_K are regular schemes. Therefore the local rings of $A \otimes \operatorname{Spec}(k)$ and $A \otimes \operatorname{Spec}(K)$ are all regular.

Let $\mathfrak{p} \subseteq A$ be a prime ideal. Denote the image of π in A also by π. If $\pi \notin \mathfrak{p}$, then

$$A_{\mathfrak{p}} = \left(A\left[\frac{1}{p}\right]\right)_{\mathfrak{p}} = \left(A \otimes_{\operatorname{Spec}(W)} \operatorname{Spec}(K)\right)_{\mathfrak{p}}$$

is a local ring of $A \otimes \operatorname{Spec}(K)$, hence regular.

Now suppose $\pi \in \mathfrak{p}$. Then $W \to A_{\mathfrak{p}}$ is a local homomorphism of local rings. Furthermore W is Noetherian since it is a discrete valuation ring. Since $W \to A$ is smooth, A is Noetherian, thus $A_{\mathfrak{p}}$ is Noetherian. As W is a discrete valuation ring, it is regular. Furthermore $A/\pi A = A \otimes \operatorname{Spec}(k)$ is regular, hence $A_{\mathfrak{p}}/\pi A_{\mathfrak{p}} = (A/\pi A)_{\mathfrak{p}}$ is regular as well (as it is a local ring in a regular ring). Finally $W \to A_{\mathfrak{p}}$ is flat (as $W \to A$ is smooth). We may thus apply [Stacks, 031E] to conclude that $A_{\mathfrak{p}}$ is regular.

We conclude that all local rings of X are regular, thus X is regular. □

Corollary 1.10. *Let X, Y be smooth schemes over a discrete valuation ring W, and let $\iota\colon X \to Y$ be a closed immersion. Then ι is a regular immersion.*

Proof. This now follows directly from Proposition 1.9 and Lemma 1.7. □

1.3 Completion of schemes

In commutative algebra one has the notion of the completion of a ring in an ideal. We would like to do something similar for schemes, so we introduce a notion of completion for schemes. To do this we need to consider inverse limits of sheaves.

Lemma 1.11. *Let X be a topological space and let $\{\mathcal{F}_i\}_{i \in I}$ be an inverse system of sheaves of abelian groups on X. Then*

$$\mathcal{F}\colon U \mapsto \varprojlim \Gamma(U, \mathcal{F}_i)$$

defines a sheaf on X. Furthermore, $\mathcal{F}$ is the inverse limit of the system $\{\mathcal{F}_i\}$.

Proof. See [Stacks, 009E]. □

Definition 1.12. Let $\iota\colon X \to Y$ be a closed immersion of schemes. Let $\mathcal{I} \subseteq \mathcal{O}_Y$ be the ideal sheaf of X. The *formal completion of Y along X* is the ringed space $Y_{/X}$ with topological space X, and sheaf of topological rings given by

$$\mathcal{O}_{Y_{/X}} := \iota^{-1}\left(\varprojlim \mathcal{O}_Y/\mathcal{I}^n\right).$$

If X is clear from the context, we shall sometimes denote $Y_{/X}$ by $\hat{Y}$.

1.4 Completions and regular sequences

One of the main advantages of using formal completions is that after passing to completions, regular immersions look like inclusions of a point into affine n-space, instead of a complicated map between general rings. The point of this section is to give a precise statement of this phenomenon.

Lemma 1.13. *Let S be an affine scheme, and $\iota\colon X \to Y$ a closed immersion of affine schemes over S, such that $X \to S$ is smooth. Then there exists a section*

$$\mathcal{O}_X \to \mathcal{O}_{Y_{/X}} \tag{1.1}$$

of the projection $\mathcal{O}_{Y_{/X}} \to \mathcal{O}_X$.

Proof. Let $S = \operatorname{Spec}(A)$. Since everything is affine we denote $\mathcal{O}_X, \mathcal{O}_Y$ and $\mathcal{O}_{Y_{/X}}$ for the rings of global sections. Let I be the kernel of $\mathcal{O}_Y \to \mathcal{O}_X$. Since $X \to S$ is smooth, the map $A \to \mathcal{O}_Y/I = \mathcal{O}_X$ is formally smooth. We thus inductively define maps $\mathcal{O}_X = \mathcal{O}_Y/I \to \mathcal{O}_Y/I^n$ as the dotted maps obtained from formal smoothness in the diagram

$$\begin{tikzcd} \mathcal{O}_Y/I \arrow[r] \arrow[dr, dotted] & \mathcal{O}_Y/I^{n-1} \\ A \arrow[u] \arrow[r] & \mathcal{O}_Y/I^n \arrow[u] \end{tikzcd}$$

where we pick the square-zero ideal $I^{n-1}/I^n \subseteq \mathcal{O}_Y/I^n$. These maps are all compatible by construction and we thus obtain a map $\mathcal{O}_X \to \mathcal{O}_{Y_{/X}}$. □

Lemma 1.14. *Let S be an affine scheme, let $X \to Y$ a closed immersion of smooth affine schemes over S. Let $\mathcal{I}$ be the ideal sheaf of X. Let $(f_1, \dots, f_r) \in \mathcal{O}_X(X)$ be a regular sequence generating $\mathcal{I}(Y)$.*

Then the map (1.1) extends to a map

$$\mathcal{O}_X[t_1, \dots, t_r] \to \mathcal{O}_{Y_{/X}}$$

sending $t_i \mapsto f_i$. It induces an isomorphism

$$\mathcal{O}_X[t_1, \dots, t_r]/(t_1, \dots, t_r)^n \xrightarrow{\sim} \mathcal{O}_Y/\mathcal{I}^n$$

for all $n \in \mathbb{N}$.

Proof. Denote $\mathcal{O}_Y = \mathcal{O}_Y(Y)$, denote $\mathcal{O}_X = \mathcal{O}_X(X)$ and $I = \mathcal{I}(Y)$. We only need to show the induced maps

$$\varphi_n\colon \mathcal{O}_X[t_1, \dots, t_r]/(t_1, \dots, t_r)^n \to \mathcal{O}_Y/I^n$$

are isomorphisms, we do so by showing they are both injective and surjective.

We shall show that they are surjective by induction on n. If $n = 1$ the map is the identity and we are done. Suppose $n > 1$ and $x \in \mathcal{O}_Y/I^n$. By induction there exists $x' \in \operatorname{im}(\varphi_{n-1})$ such that $x' \equiv x \pmod{I^{n-1}}$. Lift x' to $\mathcal{O}_Y/I^n$, we then have that $y = x - x' \in I^{n-1}\mathcal{O}_Y/I^n$. As the f_j generate I there exist $a_\alpha \in \mathcal{O}_Y/I^n$ such that

$$x - x' = \sum_{|\alpha| \geq n-1} a_\alpha f^\alpha,$$

where the sum is over all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $f^\alpha = f_1^{\alpha_1} \cdots f_r^{\alpha_r}$. As the map $\mathcal{O}_X \to \mathcal{O}_Y/I$ is also known to be an isomorphism we can find $a'_\alpha \in \mathcal{O}_X$ such that one has $\varphi_n(a'_\alpha) - a_\alpha \in I$. But then

$$x - x' - \sum_{|\alpha| \geq n-1} \varphi_n(a'_\alpha) f^\alpha = \sum_{|\alpha| \geq n-1} (a_\alpha - \varphi_n(a'_\alpha)) f^\alpha \qquad \in I^n\mathcal{O}_Y/I^n = \{0\},$$

which shows $x \in \operatorname{im}(\varphi_n)$.

To see that it is injective, note that in the proof of [Stacks, 00LN] it is shown that if an expression of the form

$$\sum_{|\alpha| = n} a_\alpha f^\alpha$$

(for some $a_\alpha \in \mathcal{O}_Y$) is in I^{n+1}, then in fact all a_α lie in I. Thus if n is the smallest n such that φ_n is not injective, we would find $a_\alpha \in \mathcal{O}_X$ such that

$$\sum_{|\alpha| = n-1} a_\alpha t^\alpha$$

maps to 0. Therefore all a_α map to I, so since the map (1.1) is a section we may conclude the a_α are all 0. □

Corollary 1.15. *Let k be a perfect field of characteristic $p > 0$ with ring of Witt vectors W. Let $X \to Y$ a closed immersion of smooth schemes over* $\operatorname{Spec}(W)$*. Then for any $x \in X$ there exists an open $U \subseteq X$ and an isomorphism*

$$\mathcal{O}_X(U)[[t_1, \ldots, t_r]] \xrightarrow{\sim} \mathcal{O}_{Y_{/X}}(U).$$

Proof. Pick $x \in X$. Denote $\mathcal{I}$ for the ideal sheaf of X. Since X and Y are smooth over the ring of Witt vectors, by Corollary 1.10, the map $\iota \colon X \to Y$ is a regular immersion. Hence there exists an affine open $V \subseteq Y$ containing x such that $\mathcal{I}(V)$ is generated by a regular sequence $(f_1, \ldots, f_r)$ in $\mathcal{O}_Y(V)$. Set $U = \iota^{-1}V$, then by Proposition 1.14 we get a map

$$\mathcal{O}_X(U)[t_1, \ldots, t_r] \to \mathcal{O}_{Y_{/X}}(U). \tag{1.2}$$

Since induced the maps

$$\mathcal{O}_X(U)[t_1, \ldots, t_r]/(t_1, \ldots, t_r)^n \xrightarrow{\sim} \mathcal{O}_{Y_{/X}}(U)/\mathcal{I}(U)^n$$

are all isomorphisms, by taking the completion on both sides of (1.2) we get the desired isomorphism. $\square$

Thus locally on X the map $X \to Y_{/X}$ is indistinguishable from the map

$$\begin{aligned} X &\to (\mathbb{A}^r \times X)_{/(\{0\}\times X)} \\ x &\mapsto (0, x) \end{aligned}$$

as we claimed earlier.

2 Divided powers

In any $\mathbb{Q}$-algebra A, one can take the (n-th) divided powers of any element $a \in A$, which are defined to be the elements $\gamma_n(a) := \frac{a^n}{n!}$. These elements are very useful as they can often be used to 'integrate' a function. In this chapter we discuss ways of generalizing such a notion to a rings that are not $\mathbb{Q}$-algebras (or even worse: to rings that are of finite characteristic).

2.1 Divided power structures on rings

In some rings, one may sometimes still define divided powers for some elements in A, even though we cannot divide by $n!$ anymore. This is formalised by the following definition.

Definition 2.1. Let A be a ring and $I \subseteq A$ be an ideal. A *divided power structure* on I is a sequence of maps $\gamma_n \colon I \to I$ for all $n \in \mathbb{Z}_{\geq 0}$, such that for all $m > 0, n \geq 0, x, y \in I$ and $a \in A$ one has

$$\gamma_0(x) = 1, \tag{2.1}$$
$$\gamma_1(x) = x, \tag{2.2}$$
$$\gamma_n(x)\gamma_m(x) = \frac{(n+m)!}{n!m!}\gamma_{n+m}(x), \tag{2.3}$$
$$\gamma_n(ax) = a^n\gamma_n(x), \tag{2.4}$$
$$\gamma_n(x+y) = \sum_{i=0}^{n} \gamma_i(x)\gamma_{n-i}(y), \tag{2.5}$$
$$\gamma_n(\gamma_m(x)) = \frac{(nm)!}{n!(m!)^n}\gamma_{nm}(x). \tag{2.6}$$

The triple (A, I, γ) is a *divided power ring* or *PD-ring* (the PD is shorthand for the French term *puissance divisée*). If γ and I are clear from the context, we will sometimes refer to the divided power ring as just A.

Example 2.2. Let A be a $\mathbb{Q}$-algebra. Then

$$\gamma_n(x) = \frac{x^n}{n!},$$

is a divided power structure on A, and (A, A, γ) is a divided power ring.

Remark 2.3. If A a domain of characteristic zero, then the axiom $x\gamma_{n-1}(x) = n\gamma_n(x)$ implies that the divided power structure on any ideal $I \subseteq A$ is unique (if it exists).

Example 2.4. A very important example of a ring with a divided power structure is the *divided power polynomial ring.* Consider for $r \geq 1$ the A-module

$$A\langle x_1, \ldots, x_r\rangle = \bigoplus_{n_1,\ldots,n_r \in \mathbb{Z}^r_{\geq 0}} Ax_1^{[n_1]} \cdots x_t^{[n_t]}.$$

generated by the formal symbols $x_1^{[n_1]} \cdots x_t^{[n_t]}$. We give it the structure of a commutative ring by setting

$$x_i^{[n]} x_i^{[m]} = \frac{(m+n)!}{n!m!} x_i^{[n+m]}.$$

We will often denote $x_i = x_i^{[1]}$. Consider the ideal

$$I = \bigoplus_{n_1+\cdots+n_r \geq 1} Ax_1^{[n_1]} \cdots x_t^{[n_t]}.$$

One easily verifies that the rule

$$\gamma_n(x_i) = x_i^{[n]}$$

extends to a unique PD-structure on I (by simply enforcing the axioms of a divided power structure).

Remark 2.5. Similarly to the case of polynomial rings there is a canonical isomorphism of PD-rings

$$A\langle x_1, \ldots x_{r-1}\rangle\langle x_r\rangle \xrightarrow{\sim} A\langle x_1, \ldots x_r\rangle.$$

Example 2.6. Let k be a perfect field of characteristic p. Let W be the ring of Witt vectors over k. Then W is a discrete valuation ring with maximal ideal $pW \subseteq W$. Since $v_p(n!) \leq n$ for all $n \in \mathbb{N}$, the rule $\gamma_n(x) = \frac{x^n}{n!}$ defines a divided power structure on pW.

We will often refer to the triple (W, pW, γ) by W.

Definition 2.7. Let (A, I, γ) and (B, J, δ) be divided power rings. A ring morphism $\varphi\colon A \to B$ is said to be a *homomorphism of divided powers rings* if $\varphi(I) \subseteq J$ and $\delta_n(\varphi(x)) = \varphi(\gamma_n(x))$.

The *category of divided power rings over* (A, I, γ) is the category whose objects are morphisms $(A, I, \gamma) \to (B, J, \delta)$ and whose morphisms are commutative triangles.

We end this section by quickly discussing the notion of compatibility of PD-structures. This will become important later to define the crystalline site. Let (A, I, γ) be a PD-ring, and let B be an A-algebra.

Definition 2.8. We say that γ *extends to* IB if there exists a PD-structure $\overline{\gamma}$ on IB such that $(A, I, \gamma) \to (B, IB, \overline{\gamma})$ is a morphism of PD-rings.

Definition 2.9. Now let (B, J, δ) be a PD-ring. We say that γ and δ are *compatible* if γ extends to B and $\overline{\gamma} = \delta$ on $IB \cap J$.

2.2 The divided power envelope

In this section we construct the divided power envelope of an ideal in a ring. Loosely speaking, this is the ring obtained by adjoining formal divided powers of all elements in the ideal.

Definition 2.10. Let A, B be rings and $I \subseteq A, J \subseteq B$ be ideals. We define a *morphism of rings with ideals* as a morphism $A \to B$ such that I maps into J. Similar to the above construction we can consider the category of rings with ideals over a base ring (A, I).

Theorem 2.11. *Let (A, I, γ) be a divided power ring, $\varphi\colon A \to B$ a morphism of rings, and $J \subseteq B$ an ideal containing $\varphi(I)$. Then there exists a divided power ring $(D, \bar{J}, \bar{\gamma})$ and a morphism*

$$(A, I, \gamma) \to (D, \bar{J}, \bar{\gamma})$$

satisfying the following universal property:

For any divided power ring (C, K, δ) over (A, I, γ) and any morphism of rings with ideals $(B, J) \to (C, K)$ there is a unique morphism $(D, \bar{J}, \bar{\gamma}) \to (C, K, \delta)$ of divided power rings over (A, I, γ) such that the diagram of rings with ideals

$$\begin{tikzcd} (B, J) \arrow[r] \arrow[d] & (C, K) \\ (D, \bar{J}) \arrow[ur, dashed] & \end{tikzcd}$$

commutes.

Proof. See [Stacks, 07H8]. □

Example 2.12. If $(A, I, \gamma) = (A, (0), 0)$ and $B = A[x_1, \ldots, x_n]$, $J = (x_1, \ldots, x_n)$, then $(D, \bar{J}, \bar{\gamma})$ is the ring from Example 2.4 (for the proof, see [Stacks, 07H5]).

Definition 2.13. The triple $(D, \bar{J}, \bar{\delta})$ is called the *divided power envelope* of J in B relative to (A, I, γ). It is often denoted $D_B(J)$ or $D_{B,\gamma}(J)$, if A and I are clear from the context. It is generated as a B-algebra by the set

$$\{\gamma_n(x) \mid n \in \mathbb{N}, x \in J\}.$$

We will sometimes also denote the ring D with $D_{B,\gamma}(J)$.

We will now summarize some properties of the divided power envelope that we will need in the future.

Remark 2.14. Note that if we apply the universal property to $(D_B(J), \bar{J}, \bar{\gamma})$ itself, we obtain a bijection

$$\operatorname{Hom}\left((B, J), (D_B(J), \bar{J})\right) = \operatorname{Hom}\left((D_B(J), \bar{J}, \bar{\gamma}), (D_B(J), \bar{J}, \bar{\gamma})\right).$$

The identity on the right hand side thus induces a canonical map $B \to D_B(J)$ mapping $J \to \bar{J}$. Using the functoriality of the construction we see that for any morphism $(D_B(J), \bar{J}, \bar{\gamma}) \to (C, K, \delta)$ the corresponding map $(B, J) \to (C, K)$ can be obtained by precomposing with the natural map $(B, J) \to (D_B(J), \bar{J})$.

Lemma 2.15. *Let (A, I, γ) be a PD-ring. Suppose $A \to B$ is a morphism of rings and $J \subseteq B$ such that γ extends to B/J. Then one has an isomorphism*

$$D_{B,\gamma}(J)/\overline{J} \cong B/J.$$

Proof. The universal property applied to (B, J) gives us a morphism $D_{B,\gamma}(J) \to B/J$ sending $\overline{J}$ to 0. This is an inverse to the canoncical map $B/J \to D_{B,\gamma}(J)/\overline{J}$ $\square$

Proposition 2.16. *Let M be an ideal of B such that $M \cdot D_{B,\gamma}(J) = 0$. Then*

$$D_{B,\gamma}(J) \cong D_{B/M}(J/MJ).$$

Proof. Let (C, K, δ) be an arbitrary divided power ring, and $(B, J) \to (C, K)$ a morphism of rings with ideals. By Remark 2.14 we see that the map $(B, J) \to (C, K)$ factors through $(D_B(J), \bar{J})$. As M maps to 0 under $B \to D_B(J)$, it also maps to 0 under $B \to C$, hence the map $(B, J) \to (C, K)$ factors through $(B/M, J/MJ)$. Thus

$$\operatorname{Hom}((B, J), (C, K)) = \operatorname{Hom}((B/M, J/MJ), (C, K)),$$

and we are done by the universal property of $D_{B,\gamma}$. $\square$

Corollary 2.17. *Suppose $m \in \mathbb{N}$ is such that $mB = 0$ and J is an ideal of B generated by q elements. Then*

$$D_{B,\gamma}(J) \cong D_{B/J^{(m-1)q+1}}(\bar{J}).$$

Proof. Let $x_1, \dots, x_q$ be a set of generators for J. Note that $x_1^m = m!\gamma_m(x_1) = 0$ in $D_{B,\gamma}(J)$. Since any element of $J^{(m-1)q+1}$ is generated by elements of the form $x_1^{a_1} \cdots x_q^{a_q}$ for which $a_1 + \cdots + a_q \geq (m-1)q + 1$, by the pigeonhole principle there exists some i such that $a_i \geq m$. Hence all these elements map to 0 in $D_{B,\gamma}(J)$ and $J^{(m-1)q+1} D_{B,\gamma}(J) = 0$. The result then follows directly from Proposition 2.16. $\square$

If $IB \not\subseteq J$, we can still define the divided power envelope as the same ring but with a smaller ideal.

Definition 2.18. In the case that $IB \nsubseteq J$, denote $D_{B,\gamma}(J+IB) = (D, \overline{J+IB}, \overline{\delta})$. We define the *divided power envelope of* J *in* B as the triple $(D, \overline{J}, \overline{\delta}|_{\overline{J}})$, where $\overline{J} \subsetneq \overline{J+IB}$ is the ideal generated by the set

$$\{\overline{\delta}_n(x) \mid n \in \mathbb{N}, x \in J\}.$$

We will again denote $D_{B,\gamma}(J)$ for both the triple $(D, \overline{J}, \overline{\delta}|_{\overline{J}})$ and the ring D.

Note by combining (2.4) and (2.5) one may show that the ring $D_{B,\gamma}(J)$ is still generated as a B-algebra by the elements

$$\{\overline{\delta}_n(x) \mid n \in \mathbb{Z}_{\geq 0}, x \in J\}.$$

2.3 Geometry with divided powers

In this section we extend the definitions of the previous two sections to geometric objects. We mostly follow [Berthelot].

Definition 2.19. A PD-ringed space is a topological space endowed with a sheaf of PD-rings. A PD-scheme is a triple $(X, \mathcal{I}, \gamma)$ such that $\mathcal{I}$ is a quasi-coherent sheaf of ideals and $(\mathcal{O}_X, \mathcal{I}, \gamma)$ is a sheaf of PD-rings on X.

If (A, I, γ) is a PD-ring and $a \in A$, then the localization (A_a, I_a) has a canonical PD structure by setting $\gamma_n(x/a^i) = \gamma_n(x)/a^{in}$. We thus get a sheaf of PD-rings on the spectrum of A, so we may define the (affine) PD-ringed space $\mathrm{Spec}(A, I, \gamma)$.

It turns out that the construction of the divided power envelope also behaves well under localization.

Proposition 2.20. *Let (A, I, γ) be a PD-ring. Let B be an A-algebra and $J \subseteq B$ an ideal containing IB. Let $b \in B$. Then there exists a canonical isomorphism*

$$D_{B,\gamma}(J) \otimes_B B_b \xrightarrow{\sim} D_{B_b,\gamma}(J_b).$$

Proof. This follows directly from [Stacks, 07HD]. □

Let $(S, \mathcal{I}, \gamma)$ be a PD-ringed space. We say that X is a scheme over S if X is a scheme and we have a morphism of ringed spaced $X \to S$.

Corollary 2.21. *Let X be a scheme over S. Then for any quasi-coherent $\mathcal{O}_X$-algebra B and quasi-coherent ideal $J \subseteq B$, the presheaf $D_{B,\gamma}(J)$*

$$\left(\mathcal{D}_{B,\gamma}(J)\right)(U) := D_{B(U),\gamma_U}(J(U))$$

is a sheaf, and a quasi-coherent $\mathcal{O}_X$-algebra.

Suppose we have a closed immersion $X \to Y$ of S-schemes, and $J \subseteq \mathcal{O}_Y$ is the ideal defining X.

Definition 2.22. Define the quasi-coherent $\mathcal{O}_X$-algebra

$$\mathcal{D}_{X,\gamma}(Y) := D_{\mathcal{O}_Y,\gamma}(J).$$

It is called the *divided power envelope of X in Y*. We also define the scheme

$$D_{X,\gamma}(Y) := \mathrm{Spec}_Y(\mathcal{D}_{X,\gamma}(Y))$$

using the notion of relative spectrum, see [Stacks, 01LL].

Remark 2.23. Note that if γ extends to $\mathcal{O}_X$, then by Lemma 2.15 one has a canonical isomorphism $\mathcal{D}_{X,\gamma}(Y)/\overline{J} \cong \mathcal{O}_X$. We thus see that $X \to Y$ factors through a closed immersion $X \to \mathcal{D}_{X,\gamma}(Y)$.

We now introduce the notion of a PD-thickening, this will be important to define the crystalline site later on.

Definition 2.24. We say that U is a thickening of T if $U \to T$ is a closed immersion and the underlying topological spaces of U and T are equal. A PD-thickening is a thickening $U \to T$ and a divided power structure γ on the ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_T$ corresponding to the closed immersion $U \to T$.

Note that given a PD-thickening (U, T, γ) the triple $(T, \mathcal{I}, \gamma)$ is a PD-scheme.

2.4 Completion and divided powers

Let k be a perfect field of characteristic $p > 0$. Set $W = W(k)$ and $S = \mathrm{Spec}(W)$. Denote γ for the canonical PD-structure on W, see Example 2.6. Consider a closed immersion of S-schemes $X \to Y$. Then $D_{X,\gamma}(Y)$ has a natural structure of a W-module, hence we can consider the p-adic completion of the divided power envelope

$$\hat{D}_{X,\gamma}(Y) := \widehat{D_{X,\gamma}(Y)}.$$

On the other hand, we can also complete X along Y and consider the divided power envelope of the completion $D_{X,\gamma}(X_{/Y})$. This will in general not be p-adically complete. We will see its p-adic completion

$$\hat{\mathcal{D}}_{X,\gamma}(Y_{/X}) := \widehat{\mathcal{D}_{X,\gamma}(Y_{/X})}$$

is isomorphic to $\hat{\mathcal{D}}_{X,\gamma}(Y)$.

Set $X_n = X \times_S \mathrm{Spec}(W_n)$ and similarly for Y_n. Let $\mathcal{J} \subseteq \mathcal{O}_Y$ be the ideal sheaf defining X.

Lemma 2.25. *Denote with $\mathcal{O}$ either $\mathcal{O}_Y$ or $\mathcal{O}_{Y_{/X}}$. Then the canonical maps*

$$W_n \otimes_W \mathcal{D}_{\mathcal{O},\gamma}(\mathcal{J}) \xrightarrow{\sim} \mathcal{D}_{\mathcal{O}\otimes W_n,\gamma}(\overline{\mathcal{J}})$$

are all isomorphisms, and induce an isomorphism

$$\hat{\mathcal{D}}_{\mathcal{O},\gamma}(\mathcal{J}) \xrightarrow{\sim} \varprojlim_n \mathcal{D}_{\mathcal{O}\otimes W_n,\gamma}(\overline{J}).$$

Proof. Apply [Stacks, 07HE] with $B = \mathcal{O}_Y, I = p\mathcal{O}_Y, I' = p\mathcal{O}_{Y_n}, B' = \mathcal{O}_{Y_n}$ to obtain the first statement. The second statement follows by taking limits on both sides. $\square$

Lemma 2.26. *The canonical map $Y_{/X} \to Y$ induces an isomorphism*

$$\hat{\mathcal{D}}_{X.\gamma}(Y) \xrightarrow{\sim} \hat{\mathcal{D}}_{X,\gamma}(Y_{/X})$$

Proof. The map $Y_{/X} \to Y$ induces isomorphisms $\mathcal{O}_Y/\mathcal{J}^N \xrightarrow{\sim} \mathcal{O}_{Y_{/X}}/\mathcal{J}^N$, hence we get isomorphisms

$$\mathcal{D}_{(\mathcal{O}_Y/\mathcal{J}^N)\otimes W_n}(\mathcal{J}) \xrightarrow{\sim} \mathcal{D}_{\left(\mathcal{O}_{Y_{/X}}/\mathcal{J}^N\right)\otimes W_n}(\mathcal{J}).$$

By Corollary 2.17 we obtain an isomorphism

$$\mathcal{D}_{\mathcal{O}_Y\otimes W_n}(\mathcal{J}) \xrightarrow{\sim} \mathcal{D}_{\mathcal{O}_{Y_{/X}}\otimes W_n}(\mathcal{J}).$$

Taking the limit on both sides gives the required isomorphism after applying the lemma above. $\square$

This motivates the following definition (note that the topological space of $Y_{/X}$ is equal to that of X).

Definition 2.27. Denote f for the map $X \to Y$. We define the ringed space $\hat{D}_{X,\gamma}(Y)$ as the topological space X with sheaf of rings $f^{-1}\hat{\mathcal{D}}_{X,\gamma}(Y)$.

As a corollary we may now compute the divided power envelope locally, in the case that our schemes are smooth.

Theorem 2.28. *Let $X \to Y$ be a closed immersion of smooth schemes over* $\mathrm{Spec}(W)$. *Let γ denote the canonical PD-structure on $pW \subseteq W$. Then locally on X there exists an isomorphism*

$$\hat{D}_{X,\gamma}(Y) \cong \widehat{\mathcal{O}_X\langle t_1,\ldots,t_d\rangle},$$

where the $\hat{}$ means p-adic completion.

Proof. Combine Lemma 2.26 and Corollary 1.15. $\square$

2.5 Divided power differentials

In this sections we construct differentials of PD-rings, which will become very important later.

Definition 2.29. Let (B, J, δ) be a PD-ring and let $A \to B$ be a map of rings. Let M be a B-module. An *A-PD-derivation* is an A-derivation $d\colon B \to M$ satisfying

$$d(\delta_i(x)) = \delta_{i-1}(x)dx$$

for all $x \in J$ and $i \geq 1$.

Example 2.30. Note that in the case that B is a $\mathbb{Q}$-algebra (see Example 2.2) any derivation is a PD-derivation, since by the Leibniz-rule and $\mathbb{Q}$-linearity of d one immediately has

$$d\left(\gamma_i(x)\right) = d\left(\frac{x^i}{i!}\right) = \frac{x^{i-1}}{(i-1)!}dx = \gamma_{i-1}(x)dx.$$

Lemma 2.31. *Let (B, J, δ) be a PD-ring and let $A \to B$ be a map of rings. There exists a B-module $\Omega^{\mathrm{PD}}_{(B,J,\delta)/A}$ equipped with a derivation $d\colon B \to \Omega^{\mathrm{PD}}_{(B,J,\delta)/A}$ such that any other PD-derivation $B \to M$ factors uniquely through d.*

Proof. Define $\Omega^{\mathrm{PD}}_{(B,J,\delta)/A}$ as a quotient of $\Omega_{B/A}$ by all elements that are of the form $d(\delta_i(x)) - \delta_{i-1}(x)dx$ for $x \in J$. □

It turns out that any B-derivation extends uniquely to a derivation on the PD-envelope, as shown by the following proposition.

Proposition 2.32. *Let (A, I, γ) be a PD-ring and let $A \to B$ be a map of rings. Let $J \subseteq B$ be an ideal. Denote with δ the PD-structure on the image of J in $D_{B,\gamma}(J)$. Then there exists an isomorphism*

$$\Omega^{\mathrm{PD}}_{D_{B,\gamma}(J)/A} \xrightarrow{\sim} D_{B,\gamma}(J) \otimes_B \Omega_{B/A}$$

sending $d(\delta_i(x)) \mapsto \delta_{i-1}(x)dx$.

Proof. See [Stacks, 07HW]. □

We will often write $\Omega^{\mathrm{PD}}_{B/A}$ instead of $\Omega^{\mathrm{PD}}_{(B,J,\delta)/A}$, similarly to how we write B for the triple (B, J, δ).

3 PD-de Rham cohomology

In this chapter we will introduce the PD-de Rham cohomology. Let k be a perfect field of characteristic $p > 0$, and let $S = \mathrm{Spec}(W(k))$. Loosely speaking, given a closed immersion of S-schemes $X \to Y$, PD-de Rham cohomology is defined as follows: One starts by taking the divided power envelope $D_{X,\gamma}(Y)$. One then considers its associated de Rham complex $\Omega^{\mathrm{PD}}_{D_{X,\gamma}(Y)/S}$. If we denote with $\hat{X}$ the completion of X along $X \otimes_S \mathrm{Spec}(k)$, we will pull this complex back to $\hat{X}$, p-adically complete the complex, and then take hypercohomology of this complex. For this to work well, our differentials will need to behave well under the various operations, which we will enforce using connections.

3.1 Connections

In this section we briefly introduce the notion of a connection, which is very important for the next two sections.

Definition 3.1. Let S be a scheme and let X be a scheme over S. Let M be a quasicoherent $\mathcal{O}_X$-module. A *connection* is an $\mathcal{O}_S$-linear map

$$\nabla \colon M \to M \otimes_{\mathcal{O}_X} \Omega_{X/S}$$

such that for every local section f of $\mathcal{O}_X$ and m of M one has

$$\nabla(mf) = \nabla(m)f + m \otimes df.$$

Given a scheme X/S, a quasicoherent $\mathcal{O}_X$-module M and a connection ∇, we get induced maps $\nabla \colon M \otimes \Omega^i_{X/S} \to M \otimes \Omega^{i+1}_{X/S}$ defined by

$$\nabla(m df_1 \wedge \dots df_i) = \nabla(m) \wedge df_1 \wedge \cdots \wedge df_i$$

for all local sections m of M and $f_1, \dots, f_i$ of $\mathcal{O}_X$.

Definition 3.2. We say that ∇ is *integrable* if $\nabla^2 = 0$ as a morphism $M \to M \otimes \Omega^2_{X/S}$. If ∇ is integrable then

$$M \otimes_{\mathcal{O}_X} \Omega^\bullet_{X/S} := M \xrightarrow{\nabla} M \otimes_{\mathcal{O}_X} \Omega_{X/S} \xrightarrow{\nabla} M \otimes_{\mathcal{O}_X} \Omega^2_{X/S} \to \dots$$

defines a complex of $\mathcal{O}_S$-modules on X.

3.2 The continuous de Rham complex

We would like to define differentials for the completion of a scheme, however the naive approach yields something 'far too big', as seen by the following example.

Example 3.3. Denote $\widehat{W[x]}$ for completion of $W[x]$ in the ideal (p). Then $\Omega_{W[x]/W}$ is equal to the $W[x]$-module $W[x]dx$. The natural map $W[x] \to \widehat{W[x]}$ induces a map of $\widehat{W[x]}$-modules

$$\widehat{W[x]} \otimes_{W[x]} \Omega_{W[x]/W} \to \Omega_{\widehat{W[x]}/W}.$$

However this is not an isomorphism, as the right hand side is much bigger. For example, the element

$$d\left(\sum_{i=0}^{\infty}(px)^i\right) \in \Omega_{\widehat{W[x]}/W}$$

is not in the image of the left hand side. The reason is that it cannot be identified with the element

$$\left(\sum_{i=0}^{\infty} p^i x^{i-1}\right) \otimes dx$$

as d only commutes with finite sums. Thus $\Omega_{\widehat{W[x]}/W}$ can not be identified with $\hat{\Omega}_{W[x]/W}$, and therefore it is not obvious which of the two definitions is the 'correct' completion of the de Rham complex.

Now suppose X is locally of finite type over S (so that $\Omega_{X/S}$ is quasi-coherent), and let $\mathcal{I} \subseteq \mathcal{O}_X$ be a sheaf of ideals defining a closed subscheme $X_0 \subseteq X$. We give $\mathcal{O}_X$ the structure of a sheaf of topological rings by giving $\mathcal{O}_X(U)$ the $\mathcal{I}(U)$-adic topology. Let $\hat{X}$ be the completion of X along X_0. Then $d\colon \mathcal{O}_X \to \Omega_{X/S}$ is continuous (as by the Leibniz rule $d\left(\mathcal{I}^n\right) \subseteq \mathcal{I}^{n-1}\Omega_{X/S}$), so because $\Omega_{X/S}$ is finitely generated, the map d extends to a unique continuous map

$$\nabla\colon \mathcal{O}_{\hat{X}} \to \mathcal{O}_{\hat{X}} \otimes_{\mathcal{O}_X} \Omega_{X/S}. \tag{3.1}$$

We thus get a natural notion of the completion of the de Rham complex.

Definition 3.4. We define the *continuous de Rham complex (of X along X_0)* as the complex

$$\Omega^{\mathrm{co},\bullet}_{\hat{X}/S}\colon \mathcal{O}_{\hat{X}} \to \mathcal{O}_{\hat{X}} \otimes_{\mathcal{O}_X} \Omega_{X/S} \to \dots$$

of sheaves of $\mathcal{O}_S$-modules on $\hat{X}$ corresponding to the connection (3.1). We will write

$$\Omega^{\mathrm{co},i}_{\hat{X}/S} := \mathcal{O}_{\hat{X}} \otimes_{\mathcal{O}_X} \Omega^i_{X/S}, \qquad \Omega^{\mathrm{co}}_{\hat{X}/S} := \Omega^{\mathrm{co},1}_{\hat{X}/S}.$$

Example 3.5. If we set $S = \mathrm{Spec}(W)$ and $X = \mathrm{Spec}(W[x])$ (in the situation of Example 3.3), and denote $\hat{X}$ for the completion of X along the closed subscheme $X \otimes_S \mathrm{Spec}(k)$

(so that $\mathcal{O}_{\hat{X}} = \widehat{W[x]}$), the complex $\Omega^{\mathrm{co},\bullet}_{\hat{X}/S}$ is given by

$$\cdots \to 0 \to \widehat{W[x]} \to \widehat{W[x]}dx \to 0 \to \dots$$

as desired.

Lemma 3.6. *If X is of finite type over S and X is Noetherian, then $\Omega^{\mathrm{co}}_{\hat{X}/S}$ satisfies the following universal property: For any sheaf $\mathcal{F}$ of p-adically complete $\mathcal{O}_X$-modules and any continuous $\mathcal{O}_S$-derivation $d\colon \mathcal{O}_{\hat{X}} \to \mathcal{F}$ there exists a unique map of $\mathcal{O}_X$-modules $\Omega^{\mathrm{co}}_{\hat{X}/S} \to \mathcal{F}$ making the diagram*

$$\begin{tikzcd} \mathcal{O}_{\hat{X}} \arrow[r, "\nabla"] \arrow[dr, "d"'] & \Omega^{\mathrm{co}}_{\hat{X}/S} \arrow[d, dotted, "\exists!"] \\ & \mathcal{F} \end{tikzcd}$$

commute.

Proof. First note that the conditions imply that $\mathcal{O}_X$ is Noetherian and $\Omega_{X/S}$ is a finitely generated $\mathcal{O}_X$-module, hence by [Stacks, 00MA] we have a canonical isomorphism

$$\Omega^{\mathrm{co}}_{\hat{X}/S} = \mathcal{O}_{\hat{X}} \otimes_{\mathcal{O}_X} \Omega_{X/S} \cong \widehat{\Omega}_{X/S}.$$

The result follows from the universal property of $\Omega_{X/S}$ and the fact that the differential d is continuous. ☐

Corollary 3.7. *Let S be any scheme, let X_0 be any scheme over S and let X, X' be Noetherian schemes of finite type over S containing X_0 as a closed subscheme. Then an isomorphism of ringed spaces $X_{/X_0} \xrightarrow{\sim} X'_{/X_0}$ induces an isomorphism $\Omega^{\mathrm{co}}_{\hat{X}/S} \to \Omega^{\mathrm{co}}_{\hat{X}'/S}$.*

In other words, $\Omega^{\mathrm{co}}_{\hat{X}/S}$ and the complex $\Omega^{\mathrm{co},\bullet}_{\hat{X}/S}$ only depend on ringed space $\hat{X}$ and not on the scheme X.

3.3 The PD-de Rham complex

Suppose $X \to Y$ is a closed immersion of PD-ringed spaces over a PD-ringed space S. In this section we construct a similar de Rham complex for the PD-envelope $D_{X,\gamma}(Y)$. After that, we show that if we complete the complex it only depends on the completion of Y along X.

Let $\mathcal{I} \subseteq \mathcal{O}_Y$ be the ideal sheaf of X. By Proposition 2.32 we get a connection

$$\nabla\colon \mathcal{D}_{X,\gamma}(Y) \to \mathcal{D}_{X,\gamma}(Y) \otimes_{\mathcal{O}_Y} \Omega_{Y/S} \tag{3.2}$$

satifsying

$$\nabla(\gamma_i(x)) := \gamma_{i-1}(x) \otimes dx \tag{3.3}$$

for all local sections x of $\mathcal{I}$.

Definition 3.8. We define the *PD-de Rham complex* as the complex

$$\Omega^{\mathrm{PD},\bullet}_{D_{X,\gamma}(Y)/S} := \mathcal{D}_{X,\gamma}(Y) \to \mathcal{D}_{X,\gamma}(Y) \otimes_{\mathcal{O}_Y} \Omega_{Y/S} \to \dots$$

of sheaves of $\mathcal{O}_S$-modules on Y corresponding to the connection (3.2).

Definition 3.9. Let $\mathcal{I} \subseteq \mathcal{O}_S$ be the ideal on which the PD-structure is defined. We define the *completed PD-de Rham complex*

$$\hat{\Omega}^{\mathrm{co},\mathrm{PD},\bullet}_{D_{X,\gamma}(Y)/S} := \widehat{\Omega^{\mathrm{PD},\bullet}_{D_{X,\gamma}(Y)/S}}$$

of sheaves of $\mathcal{O}_S$-modules on Y where we extend the differentials continuously, and the completions refer to the $\mathcal{I}$-adic completions.

Equivalently, we could have defined the completed PD-de Rham complex as the complex corresponding to the connection

$$\nabla\colon \hat{\mathcal{D}}_{X,\gamma}(Y) \to \hat{\mathcal{D}}_{X,\gamma}(Y) \otimes_{\mathcal{O}_Y} \Omega_{Y/S}$$

obtained by completing (3.2).

Definition 3.10. Let $S = \mathrm{Spec}(\mathcal{O}_S, \mathcal{I}, \gamma)$ be a PD-scheme such that $\mathcal{O}_S$ is $\mathcal{I}$-adically complete. Let $X \to Y$ a closed immersion of S-schemes. Then we define the *PD-de Rham cohomology groups of X relative to Y* as

$$\mathrm{H}^i_{\mathrm{dR},\mathrm{PD}}(X/S;Y) := \mathrm{H}^i(X, \hat{\Omega}^{\mathrm{co},\mathrm{PD},\bullet}_{D_{X,\gamma}(Y)/S})$$

where the completion refers to the $\mathcal{I}$-adic completion.

Remark 3.11. It is good to observe that we can define this for any closed immersion of S-schemes. In particular, X need not be smooth over S. In fact, it follows directly from Definition 2.18 that

$$\mathrm{H}^i_{\mathrm{dR},\mathrm{PD}}(X/S;Y) = \mathrm{H}^i_{\mathrm{dR},\mathrm{PD}}(X_0/S;Y),$$

where $X_0 = X \times_S \mathrm{Spec}(S/\mathcal{I})$. It is also good to observe that if $X_0 \to \mathrm{Spec}(k)$ is smooth and projective, we might not be able to find a smooth lift $X \to S$, but we can always find a smooth Y, as we can set $Y = \mathbb{P}^N_S$ for some N. We thus always get a cohomology theory for X with coefficients in $\mathcal{O}_S$.

Now suppose $S = \mathrm{Spec}(W, pW, \gamma)$ where W is the ring of Witt vectors of a perfect field k of characteristic $p > 0$. Define $X_0 := X \times_S \mathrm{Spec}(k)$, and denote with $\hat{X}$ the completion of X along X_0. From this point on all our sheaves will live on the topological space X_0. If we describe a sheaf that appears to live on $X, Y, \hat{X}, Y_{/X}$ or $Y_{/X_0}$, we are referring to the inverse image living on X_0. Note that $X_0 \to Y$ is a closed immersion, and we may thus consider the completion $Y_{/X_0}$.

Lemma 3.12. *The natural map $Y_{/X} \to Y$ induces an isomorphism*

$$\hat{\Omega}^{\mathrm{co,PD}}_{D_{X,\gamma}(Y)/S} \to \hat{\Omega}^{\mathrm{co,PD}}_{D_{\hat{X},\gamma}(Y_{/X})/S}$$

of sheaves of $\mathcal{O}_S$-modules on X_0.

Proof. Note that since we are extending the PD-structure on $pW \subseteq W$, by Definition 2.18 one has

$$\mathcal{D}_{X,\gamma}(Y) := \mathcal{D}_{X_0,\gamma}(Y),$$
$$\mathcal{D}_{\hat{X},\gamma}(Y_{/X}) := \mathcal{D}_{X_0,\gamma}(Y_{/X}).$$

Hence by Lemma 2.26 we get an isomorphism

$$\mathcal{D}_{X,\gamma}(Y) \xrightarrow{\sim} \mathcal{D}_{\hat{X},\gamma}(Y_{/X}).$$

We thus obtain isomorphisms

$$\mathcal{D}_{X,\gamma}(Y) \otimes_{\mathcal{O}_Y} \Omega^i_{Y/S} \xrightarrow{\sim} \left(\mathcal{D}_{\hat{X},\gamma}(Y_{/X}) \otimes_{\mathcal{O}_{Y_{/X}}} \mathcal{O}_{Y_{/X}}\right) \otimes_{\mathcal{O}_Y} \Omega^i_{Y/S},$$

by associativity of the tensor product on the right hand side we get our desired isomorphism. ☐

Lemma 3.13. *Suppose X and Y are smooth over S. Let $r = \operatorname{codim}(Y, X)$. Then locally on X_0 we have an isomorphism*

$$\hat{\Omega}^{\mathrm{co,PD}}_{D_{\hat{X},\gamma}(Y_{/X_0})/S} \cong \hat{\Omega}^{\mathrm{co,PD}}_{D_{\hat{X},\gamma}((\mathbb{A}^r_X)_{/X_0})/S}.$$

In particular $\hat{\Omega}^{\mathrm{co,PD}}_{D_{\hat{X},\gamma}(Y_{/X})/S}$ locally only depends on X_0 and the completion of Y along X_0.

Proof. Write $\hat{Y}$ for $Y_{/X_0}$. Then note that we can write

$$\hat{\Omega}^{\mathrm{co,PD}}_{D_{\hat{X},\gamma}(Y_{/X})/S} = \hat{\mathcal{D}}_{\hat{X},\gamma}(Y_{/X}) \otimes_{\mathcal{O}_{Y_{/X}}} \Omega^{\mathrm{co}}_{Y_{/X}/S}.$$

The result now follows by combining Theorem 2.28 and Corollary 3.7. ☐

3.4 A local Poincaré lemma over fields of characteristic zero

Let k be a field of characteristic 0. Consider the following algebraic version of the Poincaré lemma.

Lemma 3.14 (Poincaré lemma for the affine line over a field)**.** *The natural map of complexes $\iota^*\colon \Omega^\bullet_{k[x]/k} \to \Omega_{k/k}$ induced by $\iota\colon \{0\} \to \mathbb{A}^1$ is a quasi-isomorphism.*

Proof. We explicitly compute the chain complexes:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & k[x] & \xrightarrow{d} & k[x]dx & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & k & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Since the kernel of d is clearly isomorphic to k, it suffices to show d is surjective. The integration map

$$k[x]dx \to k[x]$$
$$\sum_{i=0}^{n} a_i x^i dx \mapsto \sum_{i=0}^{n} \frac{a_i}{i+1} x^{i+1}.$$

is a k-linear right-inverse to d, hence d is surjective. □

We may generalize this as follows.

Lemma 3.15 (Poincaré lemma for the relative affine line over fields)**.** *Let A be a k-algebra. The natural map $\Omega^\bullet_{A[x]/k} \to \Omega_{A/k}$ induced by the map $A[x] \to A$ sending $x \mapsto 0$ is a quasi-isomorphism.*

Proof. Omitted. The reader may want to read ahead and compare with Lemma 3.22, the proof is almost identical. □

We now generalize to arbitrary schemes, we need to complete for the statement to remain true.

Theorem 3.16. *Suppose $X \to Y$ is a closed immersion of smooth schemes over $S = \mathrm{Spec}(k)$. Then the natural map*

$$\Omega^{\mathrm{co},\bullet}_{Y_{/X}/S} \to \Omega^\bullet_{X/S}$$

is a quasi-isomorphism of chain complexes on X.

Proof. The statement is local on X, so we may assume X and Y are affine schemes. Since X and Y are smooth, they are regular over S, hence $X \to Y$ is a regular immersion. Hence we may apply Lemma 1.14 to reduce to the case that $\mathcal{O}_{Y_{/X}} = \mathcal{O}_X[\![t_1, \ldots, t_r]\!]$. By Corollary 3.7 we may assume that $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(A[x_1, \ldots, x_n])$. Consider the sequence

$$\Omega^\bullet_{A[x_1,\ldots,x_n]/k} \to \Omega^\bullet_{A[x_1,\ldots,x_{n-1}]/k} \to \cdots \to \Omega^\bullet_{A[x_1]/k} \to \Omega^\bullet_{A/k}.$$

By Lemma 3.15 each of the individual maps is a quasi-isomorphism, hence so is the composition. Now complete in the ideal $(x_1, \ldots, x_n)$ obtain the required quasi-isomorphism $\Omega^{\mathrm{co},\bullet}_{A[\![x_1,\ldots,x_n]\!]/k} \to \Omega^\bullet_{A/k}$. □

3.5 A local Poincaré lemma over the Witt vectors

Let k be a perfect field of characteristic $p > 0$, let $W = W(k)$. We want to do generalize Theorem 3.16 to schemes over the Witt vectors. Note however, that the natural generalization of Lemma 3.14 is wrong since the complex

$$\dots \to 0 \to W[x] \to W[x]dx \to 0 \to \dots$$

has a nonzero first cohomology group: the element $x^{np-1}dx$ does not lie in the image for all $n \in \mathbb{N}$. We can fix this by replacing $W[x]$ with its PD-envelope $W\langle x\rangle$.

Lemma 3.17 (Poincaré lemma for the affine line over the Witt vectors)**.** *The natural map of chain complexes $\Omega^{\mathrm{PD},\bullet}_{W\langle x\rangle/W} \to \Omega^\bullet_{W/W}$ is a quasi-isomorphism.*

Proof. The easiest proof is to just copy the proof of Lemma 3.14 and use the divided powers $\gamma_i(x)$ to divide by $i+1$. We will give a different proof by constructing an explicit chain homotopy, which will be easier to generalize.

Consider the map $\iota^\bullet\colon \Omega^\bullet_{W/W} \to \Omega^{\mathrm{PD},\bullet}_{W\langle x\rangle/W}$ given the natural section $W \to W\langle x\rangle$ of π in degree 0 and 0 in all other degrees. Note $\pi^\bullet \circ \iota^\bullet = \mathrm{id}_{\Omega_{W/W}}$ so it suffices to construct a chain homotopy S from $\mathrm{id}_{\Omega^{\mathrm{PD}}_{W\langle x\rangle/W}}$ to $\iota^\bullet \circ \pi^\bullet$.

Define $S_1\colon W\langle x\rangle dx \to W\langle x\rangle$ sending $\gamma_i(x)dx \mapsto \gamma_{i+1}(x)$ and extending linearly, and $S_n = 0$ for all $n \neq 1$. Note that if

$$f = \sum_{i=0}^{n} a_i\gamma_i(x)$$

one has

$$\begin{aligned}(S_1 \circ d)(f) &= S\left(\sum_{i=0}^{n} a_{i-1}\gamma_{i-1}(x)dx\right) = \sum_{i=1}^{n} a_i\gamma_i(x) = f - \pi\circ\iota(f), \qquad (3.4)\\ (d\circ S_1)(fdx) &= d\left(\sum_{i=0}^{n} a_i\gamma_{i+1}(x)\right) = \sum_{i=0}^{n} a_i\gamma_i(x)dx = f.\end{aligned}$$

From this it follows easily that S provides a chain homotopy from $\mathrm{id}_{\Omega^{\mathrm{PD}}_{W\langle x\rangle/W}}$ to $\iota^\bullet \circ \pi^\bullet$. $\square$

Following the case over fields we now try to generalize Lemma 3.15. This is just a lot of bookkeeping to keep track of the coefficients of $\Omega_{A/W}$ appearing everywhere, but in essence the proof is identical to the above example. We start by describing $\Omega_{A[x]/W}$ in terms of $\Omega_{A/W}$.

Definition 3.18. Define the cochain complex $(K^\bullet, d_K)$ by

$$\begin{aligned} K^n &:= \left(A[x] \otimes_A \Omega^n_{A/W}\right) \oplus \left(A[x]dx \otimes_A \Omega^{n-1}_{A/W}\right) \\ d^n_K \colon K^n &\to K^{n+1} \\ (f \otimes \omega, 0) &\mapsto (f \otimes d\omega, df \otimes \omega) \\ (0, gdx \otimes \eta) &\mapsto (0, -gdx \otimes d\eta) \end{aligned}$$

Lemma 3.19. *There exists an isomorphism of chain complexes*

$$\varphi^\bullet \colon K^\bullet \xrightarrow{\sim} \Omega^\bullet_{A[x]/W}$$

given by

$$\begin{aligned} \varphi^n \colon K^n &\to \Omega^n_{A[x]/W} \\ (f \otimes \omega, 0) &\mapsto f \cdot \omega \\ (0, gdx \otimes \eta) &\mapsto gdx \wedge \eta \end{aligned}$$

Proof. One verifies easily that the maps φ^n commute with the differentials, and hence we get a morphism of complexes $\varphi^\bullet \colon K^\bullet \to \Omega^\bullet_{A[x]/W}$.

Since the inclusion $A \to A[x]$ is smooth, by [Stacks, 04B2] we have an exact sequence of $A[x]$-modules

$$0 \to A[x] \otimes_A \Omega_{A/W} \to \Omega_{A[x]/W} \to A[x]dx \to 0.$$

Since the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A[x] \otimes_A \Omega_{A/W} & \longrightarrow & K^1 & \longrightarrow & A[x]dx & \longrightarrow & 0 \\ & & \big\downarrow \mathrm{id} & & \big\downarrow \varphi^1 & & \big\downarrow \mathrm{id} & & \\ 0 & \longrightarrow & A[x] \otimes_A \Omega_{A/W} & \longrightarrow & \Omega_{A[x]/W} & \longrightarrow & A[x]dx & \longrightarrow & 0 \end{array}$$

commutes we see that φ^1 is an isomorphism. Now let $n > 1$. Note that

$$\begin{aligned} \wedge^n_{A[x]} K^1 &= \wedge^n_{A[x]} \left(\left(A[x] \otimes_A \Omega_{A/W}\right) \oplus A[x]dx\right) \\ &= \bigoplus_{p+q=n} \left(\wedge^p_{A[x]} \left(A[x] \otimes_A \Omega_{A/W}\right)\right) \otimes \left(\wedge^q_{A[x]} A[x]dx\right) \\ &= \bigoplus_{p+q=n} \left(A[x] \otimes_A \Omega^p_{A/W}\right) \otimes \left(\wedge^q_{A[x]} A[x]dx\right) \\ &= \left(A[x] \otimes_A \Omega^n_{A/W}\right) \oplus \left(A[x]dx \otimes_A \Omega^{n-1}_{A/W}\right) \\ &= K^n, \end{aligned}$$

where in the fourth equality we used that $A[x]dx$ is a free $A[x]$-module of rank 1 hence

$$\wedge^q_{A[x]} A[x]dx = \begin{cases} A[x] & q = 0, \\ A[x]dx & q = 1, \\ 0 & q \geq 2. \end{cases}$$

One easily verifies that $\varphi^n = \wedge^n \varphi^1$. We conclude that φ^n is an isomorphism as well. $\square$

Using this we may describe the PD-de Rham complex $\Omega^{\mathrm{PD}}_{A\langle x\rangle/W}$ in terms of $\Omega_{A/W}$ as well.

Definition 3.20. Define the cochain complex $(P^\bullet, d_P)$ by

$$\begin{aligned} P^n &:= \left(A\langle x\rangle \otimes_A \Omega^n_{A/W}\right) \oplus \left(A\langle x\rangle dx \otimes_A \Omega^{n-1}_{A/W}\right) \\ d^n_P &\colon P^n \to P^{n+1} \\ (f \otimes \omega, 0) &\mapsto (f \otimes d\omega, df \otimes \omega) \\ (0, gdx \otimes \eta) &\mapsto (0, -gdx \otimes d\eta) \end{aligned}$$

Corollary 3.21. *The map $\varphi^\bullet$ extends to an isomorphism $\varphi^\bullet \colon P^\bullet \to \Omega^{\mathrm{PD},\bullet}_{A\langle x\rangle/W}$.*

Proof. Since $P^n = A\langle x\rangle \otimes_{A[x]} K^n$, this follows by Proposition 2.32 (the reader must verify that d^1_P is a PD-differential, this follows since $d \colon A\langle x\rangle \to A\langle x\rangle dx$ and $d \colon A \to \Omega_{A/W}$ are). $\square$

We now generalize Lemma 3.15.

Proposition 3.22 (Poincaré lemma for the relative affine line over the Witt vectors)**.** *Let $W \to A$ be an arbitrary ring map. Then the canonical map*

$$\Omega^{\mathrm{PD},\bullet}_{A\langle x\rangle/W} \to \Omega^\bullet_{A/W}$$

is a quasi-isomorphism of chain complexes.

Proof. Any $f \in A\langle x\rangle$ has a unique presentation

$$f = \sum_{i=0}^{k} a_i \gamma_i(x) \in A\langle x\rangle$$

We will write $f_0 := a_0$ (the coefficient of $\gamma_0(x)$). Consider the map $\pi^\bullet$ given by

$$\begin{aligned} \pi^n &\colon P^n \to \Omega^n_{A/W} \\ (f \otimes \omega, 0) &\mapsto f_0 \cdot \omega \\ (0, gdx \otimes \eta) &\mapsto 0. \end{aligned}$$

Since the diagram

$$\begin{tikzcd}
P^\bullet \arrow[rr, "\varphi^\bullet", "\sim"'] \arrow[dr, "\pi^\bullet"'] & & \Omega^{\mathrm{PD},\bullet}_{A\langle x\rangle/W} \arrow[dl] \\
& \Omega^\bullet_{A/W} &
\end{tikzcd}$$

commutes it suffices to show $\pi^\bullet$ is a quasi-isomorphism. Consider the map $\iota^\bullet$ given by

$$\begin{aligned}
\iota^n \colon \Omega^n_{A/W} &\to P^n \\
\omega &\mapsto (1 \otimes \omega, 0)
\end{aligned}$$

Since $\pi^\bullet \circ \iota^\bullet = \mathrm{id}^\bullet_{\Omega_{A/W}}$ it suffices to show that $\iota^\bullet \circ \pi^\bullet$ induces the identity after taking cohomology. We will do so by constructing an explicit chain homotopy $\mathrm{id} \sim \iota^\bullet \circ \pi^\bullet$.

We start by computing for all $n \in \mathbb{Z}_{\geq 0}$ the image

$$\iota^n \circ \pi^n \left(f \otimes \omega, g(x)dx \otimes \eta\right) = \left(f_0 \otimes \omega, 0\right).$$

Let $S\colon A\langle x\rangle dx \to A\langle x\rangle$ be the unique A-linear map satisfying $S(\gamma_i(x)dx) = \gamma_{i+1}(x)$. Observe

$$\begin{aligned}
(d \circ S)(fdx) &= fdx \\
(S \circ d)(f) &= f - f_0
\end{aligned}$$

for all $f, g \in A\langle x\rangle$ (compare with (3.4) in Lemma 3.17).

Define the maps

$$\begin{aligned}
h_n \colon P^n &\to P^{n-1} \\
(f \otimes \omega, gdx \otimes \eta) &\mapsto (S(g) \otimes \eta, 0).
\end{aligned}$$

We claim they provide a chain homotopy from id to $\iota^\bullet \circ \pi^\bullet$. Indeed, one computes

$$\begin{aligned}
dh_n(f \otimes \omega, gdx \otimes \eta) &= d\left(S(gdx) \otimes \eta, 0\right) \\
&= (S(gdx) \otimes d\eta, gdx \otimes \eta) \\
h_{n+1}d(f \otimes \omega, gdx \otimes \eta) &= h_{n+1}(f \otimes d\omega, df \otimes \omega - gdx \otimes d\eta) \\
&= ((f - f_0) \otimes \omega - S(gdx) \otimes d\eta, 0),
\end{aligned}$$

so that

$$(dh_n + h_{n+1}d)(f \otimes \omega, \eta \otimes gdx) = ((f - f_0) \otimes \omega, gdx \otimes \eta)$$

which is precisely equal to

$$\left(\mathrm{id} - \iota^\bullet \circ \pi^\bullet\right)\left(f \otimes \omega, gdx \otimes \eta\right).$$

We have thus constructed a chain homotopy from $\iota^\bullet \circ \pi^\bullet$ to the identity. Therefore $\pi^\bullet$ is a quasi-isomorphism which is what we needed to show. $\square$

Theorem 3.23. *Let $S = \operatorname{Spec}(W, pW, \gamma)$. Let $X \to Y$ be a closed immersion of S-schemes, let $X_0 = X \times_S \operatorname{Spec}(k)$ and denote with $\hat{X}$ the completion of X along X_0. If X and Y are smooth over S, the map*

$$\hat{\Omega}^{\mathrm{co,PD},\bullet}_{D_{\hat{X},\gamma}(Y)/S} \to \Omega^{\mathrm{co},\bullet}_{\hat{X}/S} \tag{3.5}$$

induced by the morphism of ringed spaces $\hat{X} \to Y$ is a quasi-isomorphism of chain complexes of $\mathcal{O}_S$-modules on X_0.

Example 3.24. Denote with K the fraction field of W. Suppose $X = \operatorname{Spec}(W)$ and $Y = \operatorname{Spec}(W[x])$, where $X \to Y$ corresponds to the map $W[x] \to W$ sending $x \mapsto 0$. Note that $\mathcal{O}_{\hat{X}} = \mathcal{O}_X$, and $\mathcal{D}_{X,\gamma}(Y) = W\langle x\rangle$. Furthermore clearly $\mathcal{O}_{Y_{/X}} = W[\![x]\!]$, and $\hat{\mathcal{D}}_{X,\gamma}(Y_{/X}) = \hat{\mathcal{D}}_{X,\gamma}(Y) = \widehat{W\langle x\rangle}$. The latter can be described as the ring of all formal power series

$$\sum_{i=0}^{\infty} a_i \frac{x^i}{i!} \in K[\![x]\!], \qquad a_i \in W,$$

with the property that $\lim_{i\to\infty} a_i = 0$ in W.

Proof of Theorem 3.23. The proof is in essence the same as the proof of Theorem 3.16. Note that since we have a map, the statement can be checked locally on X_0. Just as in the case over fields, by applying Theorem 1.15 we may reduce to the case that $X = \operatorname{Spec}(A)$ and that $Y_{/X_0} = \operatorname{Spec} \hat{A}[\![x_1, \dots, x_r]\!]$, with the natural map given by the projection to A.

By Lemma 3.12 and Lemma 3.13 we may thus assume that $Y = \operatorname{Spec} A[x_1, \dots, x_r]$. Observing that $\Omega^{\mathrm{co}}_{\hat{X}/S} = \hat{\Omega}_{X/S}$ we may reduce to showing that

$$\Omega^{\mathrm{PD},\bullet}_{D_{X,\gamma}(Y)/S} \to \Omega^{\bullet}_{X/S}$$

already induces a quasi-isomorphism.

Consider the sequence

$$\Omega^{\mathrm{PD},\bullet}_{A\langle x_1,\dots,x_r\rangle/W} \to \Omega^{\mathrm{PD},\bullet}_{A\langle x_1,\dots,x_{r-1}\rangle/W} \to \cdots \to \Omega^{\mathrm{PD},\bullet}_{A\langle x_1\rangle/W} \to \Omega^{\bullet}_{A/W}$$

constructed using Remark 2.5. Each individual map is a quasi-isomorphism by the above proposition (replacing A with $A\langle x_1, \dots, x_i\rangle$), and hence the composition of all the maps is. But that is precisely the map we needed to be a quasi-isomorphism. □

We are now almost ready to relate $\mathrm{H}^i_{\mathrm{dR,PD}}(X/S; Y)$ and $\mathrm{H}^i_{\mathrm{dR}}(X/S)$. To do this we need a technical lemma about de Rham cohomology.

Lemma 3.25. *Let $S = \operatorname{Spec}(W, pW, \gamma)$ and $S_0 = \operatorname{Spec}(k)$. Suppose X is a projective S-scheme. Let $\hat{X}$ be the completion of $\hat{X}$ along $X \times_S S_0$. Then there is a canonical isomorphism*

$$\mathrm{H}^i_{\mathrm{dR}}(X/S) \cong \mathrm{H}^i(\hat{X}, \Omega^{\mathrm{co},\bullet}_{\hat{X}/S}).$$

Proof. Let $X_0 = X \times_S S_0$. Since X is projective we have $\Omega^{\mathrm{co}}_{\hat{X}/S} = \hat{\Omega}_{X/S}$. By considering the long exact sequence of cohomology associated to the short exact sequence of chain complexes

$$0 \to \hat{\Omega}^{\geq n}_{X/S} \to \hat{\Omega}^{\bullet}_{X/S} \to \hat{\Omega}^{\leq n-1}_{X/S} \to 0$$

we see that it thus suffices to show that $\mathrm{H}^i(X, \mathcal{F}) \cong \mathrm{H}^i(\hat{X}, \hat{\mathcal{F}})$ for any coherent $\mathcal{O}_X$-module $\mathcal{F}$. By the theorem of formal functions we have $\mathrm{H}^i(X, \mathcal{F}) \cong \varprojlim \mathrm{H}^i(X, \mathcal{F}/p^n\mathcal{F})$. Since $\mathrm{Supp}\,(\mathcal{F}/p^n\mathcal{F}) \subseteq X_0$, we have $\mathrm{H}^i(X, \mathcal{F}/p^n\mathcal{F}) \cong \mathrm{H}^i(\hat{X}, \hat{\mathcal{F}}/p^n\hat{\mathcal{F}})$. The result follows from showing that $\mathrm{H}^i(\hat{X}, \hat{\mathcal{F}}) \cong \varprojlim \mathrm{H}^i(\hat{X}, \hat{\mathcal{F}}/p^n\hat{\mathcal{F}})$. This follows from the fact that $\mathrm{R}^1 \varprojlim \mathrm{H}^i(\hat{X}, \hat{\mathcal{F}}/p^n\hat{\mathcal{F}}) = 0$, which is a consequence of the fact that the system is Mittag-Leffler, as $\mathrm{H}^i(\hat{X}, \hat{\mathcal{F}}/p^n\hat{\mathcal{F}})$ is a finitely generated W-module for all n since $\mathcal{F}$ is coherent and X is projective. □

Corollary 3.26. *Let $S = \mathrm{Spec}(W, pW, \gamma)$. Let $X \to Y$ be a closed immersion of S-schemes, let $X_0 = X \times_S \mathrm{Spec}(k)$. If $X \to S$ is projective, and X and Y are smooth over S, there exists a canonical isomorphism*

$$\mathrm{H}^i_{\mathrm{dR,PD}}(X/S; Y) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(X/S).$$

Proof. Denote with $\hat{X}$ the completion of X along X_0. Since $X \to S$ is projective

$$\mathrm{H}^i(X, \Omega^{\bullet}_{X/S}) \xrightarrow{\sim} \mathrm{H}^i(\hat{X}, \Omega^{\mathrm{co},\bullet}_{\hat{X}/S})$$

by the above lemma. By Theorem 3.23, we have a canonical isomorphism

$$\mathrm{H}^i_{\mathrm{dR,PD}}(X/S; Y) \xrightarrow{\sim} \mathrm{H}^i(\hat{X}, \Omega^{\mathrm{co},\bullet}_{\hat{X}/S}).$$

Combining the two isomorphisms, the result follows. □

We have thus reached our goal: If $X_0 \to \mathrm{Spec}(k)$ lifts to some smooth $X \to S$ we have a canonical isomorphism

$$\mathrm{H}^i_{\mathrm{dR,PD}}(X_0/S; Y) \cong \mathrm{H}^i_{\mathrm{dR}}(X/S)$$

(recall Remark 3.11). In particular, if X_0 lifts then the PD-de Rham cohomology groups of X are independent of Y.

4 The crystalline topos

Fix a PD-scheme $S = (S, \mathcal{I}, \gamma)$ with $p\mathcal{O}_S \subseteq \mathcal{I}$ which will serve as our base scheme. Let X be an S-scheme to which γ extends. The two most important examples to keep in mind are $S = \mathrm{Spec}(W)$ or $S = \mathrm{Spec}(W/p^nW)$ and X a smooth scheme over $\mathrm{Spec}(W/pW)$, where W is the ring of Witt vectors of some finite field k of characteristic $p > 0$.

In this chapter we start by introducing the crystalline site $\mathrm{Cris}(X/S)$. Then we will study its associated topos $(X/S)_{\mathrm{cris}}$, which we will use to define the cohomology groups $\mathrm{H}^i_{\mathrm{cris}}(X/S)$.

After that, we will see that the crystalline topos is functorial in X. Finally we will study the relation between the functoriality of the Zariski and the crystalline topos.

4.1 The crystalline site

We start by defining the crystalline site. We will define both a small and a big crystalline site. The big site is a technical notion that will help us make the associated topos of sheaves functorial in both X and S.

Definition 4.1. The *big crystalline site of X relative to S*, denoted $\mathrm{CRIS}(X/S)$, is the site defined as follows.

- The *objects* of $\mathrm{CRIS}(X/S)$ are PD-thickenings (U, T, δ) where $U \to X$ is a morphism of schemes and $U \to T$ is a closed S-immersion.
- The *morphisms* $(U, T, \delta) \to (U', T', \delta')$ in $\mathrm{CRIS}(X/S)$ are commutative squares

$$\begin{tikzcd} X & U \arrow[l] \arrow[r] \arrow[d, hook] & T \arrow[d] \\ & U' \arrow[ul] \arrow[r] & T' \end{tikzcd}$$

  such that the map $T \to T'$ is an S-PD-morphism $(T, \mathcal{I}, \delta) \to (T', \mathcal{I}', \delta')$.
- The *coverings* of $(U, T, \delta) \in \mathrm{CRIS}(X/S)$ are collections of morphisms

$$\{(U_i, T_i, \delta_i) \to (U, T, \delta)\}_{i \in I}$$

  such that U_i is the scheme-theoretic inverse image of U under the map $T_i \to T$ for all i, and $\{T_i \to T\}_{i \in I}$ is a Zariski-covering of T.

We will often refer to an object (U, T, δ) as just T.

Definition 4.2. The *small crystalline site of X relative to S*, denoted $\mathrm{Cris}(X/S)$, is the full subcategory of $\mathrm{CRIS}(X/S)$ whose objects are the (U, T, δ) such that $U \to X$ is an open immersion, and whose coverings are the families of morphisms that are coverings in $\mathrm{CRIS}(X/S)$.

Definition 4.3. Given a commutative square

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array} \tag{4.1}$$

where S and S' are both PD-schemes and $S' \to S$ is a morphism of PD-schemes, we define a morphism

$$\mathrm{CRIS}(X'/S') \to \mathrm{CRIS}(X/S)$$

by sending $(U \to X', T, \delta) \mapsto (U \to X' \to X, T, \delta)$.

Note that we cannot define a morphism between the associated small crystalline sites. We will see later that it will be possible to define a morphism between the associated small topoi, fortunately.

4.2 The crystalline topos

In this section we discuss some properties of the crystalline topos needed to define crystalline cohomology. We start by explicitly describing sheaves on $\mathrm{Cris}(X/S)$.

Proposition 4.4. *A sheaf $\mathcal{F}$ on $\mathrm{Cris}(X/S)$ is uniquely determined by*

- *for every $(U, T, \delta) \in \mathrm{Cris}(X/S)$ a Zariski sheaf $\mathcal{F}_T$ on T and*
- *for every $f\colon (U', T', \delta') \to (U, T, \delta)$ in $\mathrm{Cris}(X/S)$ a map $c_f\colon f^{-1}\mathcal{F}_T \to \mathcal{F}_{T'}$*

satisfying the following conditions

- *for every other map $g\colon (U'', T'', \delta'') \to (U', T', \delta')$ one has $c_g \circ g^{-1}c_f = c_{g\circ f}$ and*
- *if $T' \to T$ is an open immersion, then c_f is an isomorphism.*

Proof. The constructions are described in [Stacks, 07IN], the proof for the equivalence is identical to that in [Stacks, 0213]. □

Example 4.5. If one chooses $\mathcal{F}_T = \mathcal{O}_T$ in the above proposition we obtain a sheaf (of rings) on $\mathrm{Cris}(X/S)$. This is called the *structure sheaf* and denoted by $\mathcal{O}_{X/S}$. Similarly for $\mathrm{CRIS}(X/S)$.

We write $(X/S)_{\mathrm{cris}}$ for the topos of sheaves on $\mathrm{Cris}(X/S)$, and $(X/S)_{\mathrm{CRIS}}$ for the topos of sheaves on $\mathrm{CRIS}(X/S)$.

Lemma 4.6. *Given a commutative square as in (4.1) the associated morphism of sites* $\mathrm{CRIS}(X'/S') \to \mathrm{CRIS}(X/S)$ *is cocontinuous, and therefore defines a morphism of topoi* $f_{\mathrm{CRIS}}\colon (X'/S')_{\mathrm{CRIS}} \to (X/S)_{\mathrm{CRIS}}$.

Proof. We leave cocontinuity for the reader to verify. The fact that it is a morphism of topoi then follows from [Stacks, 00XO]. □

Lemma 4.7. *The inclusion* $\mathrm{Cris}(X/S) \to \mathrm{CRIS}(X/S)$ *induces a morphism of topoi*

$$i\colon (X/S)_{\mathrm{cris}} \to (X/S)_{\mathrm{CRIS}}.$$

Furthermore, there exists a morphism of topoi

$$\pi\colon (X/S)_{\mathrm{CRIS}} \to (X/S)_{\mathrm{cris}}$$

such that $\pi \circ i$ *is the identity on* $(X/S)_{\mathrm{cris}}$*, and* $\pi_* = \iota^{-1}$.

Proof. This follows since the inclusion $\mathrm{Cris}(X/S) \to \mathrm{CRIS}(X/S)$ preserves nonempty limits, is fully faithful, continuous, and cocontinuous. For more details, see [Stacks, 07IJ]. □

Definition 4.8. Given a commutative square (4.1) we define a morphism of topoi

$$f_{\mathrm{cris}}\colon (X'/S')_{\mathrm{cris}} \to (X/S)_{\mathrm{cris}}$$

by $f_{\mathrm{cris}} = \pi \circ f_{\mathrm{CRIS}} \circ i$.

Using that $\pi \circ i = \mathrm{id}$ it follows that the diagram

$$\begin{tikzcd}
(X'/S')_{\mathrm{cris}} \arrow[r, "f_{\mathrm{cris}}"] \arrow[d, "i"] & (X/S)_{\mathrm{cris}} \arrow[d, "i"] \\
(X'/S')_{\mathrm{CRIS}} \arrow[r, "f_{\mathrm{CRIS}}"] & (X/S)_{\mathrm{CRIS}}
\end{tikzcd}$$

commutes.

Warning. In general, it seems unlikely that the diagram

$$\begin{tikzcd}
(X'/S')_{\mathrm{CRIS}} \arrow[r, "f_{\mathrm{CRIS}}"] \arrow[d, "\pi"] & (X/S)_{\mathrm{CRIS}} \arrow[d, "\pi"] \\
(X'/S')_{\mathrm{cris}} \arrow[r, "f_{\mathrm{cris}}"] & (X/S)_{\mathrm{cris}}
\end{tikzcd}$$

commutes.

4.3 Crystalline cohomology

We would like to define the cohomology of $(X/S)_{\mathrm{cris}}$ using abstract nonsense about cohomology on topoi. To do this we need a notion of global sections on $(X/S)_{\mathrm{cris}}$. On the Zariski (or étale) topos one does this by simply evaluating a sheaf in the final object of the site. However, $\mathrm{Cris}(X/S)$ does in general not have a final object.

To work around this will need to take global sections over other sheaves. This is some abstract nonsense, and is actually easier to treat in generality, so fix a site $\mathcal{C}$ with topos $\mathcal{T}$. Recall the definition of a representable sheaf.

Definition 4.9. A sheaf $\mathcal{F} \in \mathcal{T}$ is said to be representable if there exists $V \in \mathcal{C}$ such that $\mathcal{F} \cong \mathrm{Hom}(-, V)$.

Definition 4.10. Fix an object $T \in \mathcal{T}$. Define the functor

$$\begin{aligned} \Gamma(T, -)\colon \mathcal{T} &\to \mathrm{Set} \\ \mathcal{F} &\mapsto \mathrm{Hom}_{\mathcal{T}}(T, \mathcal{F}). \end{aligned}$$

Note that in the case that T is represented by U by the Yoneda lemma one has

$$\Gamma(\mathrm{Hom}(-, U), \mathcal{F}) \cong \mathcal{F}(U) = \Gamma(U, \mathcal{F}).$$

Thus our definition makes sense: We simply extended $\Gamma(-,-)$ from the representable sheaves to all sheaves.

Lemma 4.11. *The sheaf $U \mapsto \{0\}$ is a final object in $\mathcal{T}$.*

Note that in the Zariski site $(X/S)_{\mathrm{zar}}$, it is represented by the open set X. This motivates the following definition.

Definition 4.12. Let E be the final object of $\mathcal{T}$. Then we define $\Gamma(\mathcal{T}, -) := \Gamma(E, -)$. Similarly we write $\mathrm{H}^i(\mathcal{T}, -) := \mathrm{R}^i\Gamma(\mathcal{T}, -) = \mathrm{R}^i\Gamma(E, -)$.

Using this we can define the crystalline cohomology of X over S.

Definition 4.13. The *crystalline cohomology of X over S* is defined as

$$\mathrm{H}^i_{\mathrm{cris}}(X/S) := \mathrm{H}^i((X/S)_{\mathrm{cris}}, \mathcal{O}_{X/S}).$$

Let us now give a different definition using the language of derived functors, this will be useful for computing it later on.

Proposition 4.14. *The functor*

$$\begin{aligned} \mathrm{Cris}(X/S) &\to X_{\mathrm{zar}} \\ (U, T, \delta) &\mapsto U \end{aligned}$$

defines a morphism of topoi

$$u_{X/S}\colon (X/S)_{\mathrm{cris}} \to \mathrm{Sh}\,(X_{\mathrm{zar}})\,.$$

It is explicitly given as follows.

- *For* $\mathcal{F} \in (X/S)_{\mathrm{cris}}$ *and* $j\colon U \to X$ *an open immersion*

$$(u_{X/S*}(\mathcal{F}))(U) = \Gamma((U/S)_{\mathrm{cris}}, j_{\mathrm{cris}}^{-1}\mathcal{F}).$$

- *For* $F \in \mathrm{Sh}(X_{\mathrm{zar}})$ *and* $(U, T, \delta) \in \mathrm{Cris}(X/S)$

$$(u_{X/S}^{-1}(F))(U, T, \delta) = F(U).$$

Proof. See [Berthelot, Proposition 5.18]. □

Warning. Given a diagram (4.1), the diagram

$$\begin{tikzcd} (X'/S')_{\mathrm{cris}} \arrow[r, "f_{\mathrm{cris}}"] \arrow[d, "u_{X'/S'}"] & (X/S)_{\mathrm{cris}} \arrow[d, "u_{X/S}"] \\ \mathrm{Sh}\,(X'_{\mathrm{zar}}) \arrow[r, "f_{\mathrm{zar}}"] & \mathrm{Sh}\,(X_{\mathrm{zar}}) \end{tikzcd} \tag{4.2}$$

generally does not commute.

Note that $\Gamma((X/S)_{\mathrm{cris}}, -) = \Gamma(X_{\mathrm{zar}}, -) \circ u_{X/S}$. Thus one can compute crystalline cohomology in two steps:

$$\mathrm{R}\Gamma((X/S)_{\mathrm{cris}}, \mathcal{F}) = \mathrm{R}\Gamma(X_{\mathrm{zar}}, \mathrm{R}u_{X/S*}\mathcal{F}).$$

Finally we have a version of $u_{X/S}$ for the big crystalline site.

Proposition 4.15. *The functor*

$$\begin{aligned} \mathrm{CRIS}(X/S) &\to \mathrm{Sh}\,(\mathrm{Sch}/X) \\ (U, T, \delta) &\mapsto U \end{aligned}$$

defines a morphism of topoi

$$U_{X/S}\colon (X/S)_{\mathrm{cris}} \to \mathrm{Sh}\,(\mathrm{Sch}/X)\,.$$

Proof. This is the case as the functor is cocontinuous. □

4.4 Crystals and connections

In this section we introduce two important concepts, the notion of a *crystal* and of a *connection* on the crystalline site. Later on, in chapter 5.3, we will see that it are precisely the crystals (with a connection) for which the crystalline cohomology agrees with the Zariski cohomology of a suitable complex.

Definition 4.16. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{X/S}$-modules on $\mathrm{Cris}(X/S)$.

- We say that $\mathcal{F}$ is *locally quasi-coherent* if $\mathcal{F}_T$ is a quasi-coherent sheaf on T for every $(U, T, \delta) \in \mathrm{Cris}(X/S)$.
- We say that $\mathcal{F}$ is a *crystal* if $\mathcal{F}$ is locally quasi-coherent and all of the comparison maps c_f in Proposition 4.4 are isomorphisms.
- If $\mathcal{F}$ is a crystal, we say that $\mathcal{F}$ is *finite* if $\mathcal{F}_T$ is a coherent sheaf for each $(U, T, \delta) \in \mathrm{Cris}(X/S)$.

Equivalently $\mathcal{F}$ is a crystal if it is a quasi-coherent $\mathcal{O}_{X/S}$-module on the ringed site $\mathrm{Cris}(X/S)$ (to be defined later).

Example 4.17. The rule $\mathcal{F}(U, T, \delta) := \mathcal{O}_U(U)$ defines a sheaf on $\mathrm{Cris}(X/S)$, however it is not locally quasi-coherent and therefore not a crystal.

We now very quickly introduce differentials on the crystalline site. For more details, see [Stacks, 07IW].

Definition 4.18. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{X/S}$-modules on $\mathrm{Cris}(X/S)$. A PD-S-derivation is a map of sheaves $D\colon \mathcal{O}_{X/S} \to \mathcal{F}$ such that for any $(U, T, \delta) \in \mathrm{Cris}(X/S)$ and $V \subseteq S$ open containing the image of T, the map

$$D\colon \mathcal{O}_T(T) \to \mathcal{F}(T)$$

is a PD-$\mathcal{O}_V(V)$-derivation.

Lemma 4.19. *There exists a locally quasi-coherent $\mathcal{O}_{X/S}$-module $\Omega^{\mathrm{PD}}_{X/S}$ and a universal PD-S-derivation $d_{X/S}\colon \mathcal{O}_{X/S} \to \Omega^{\mathrm{PD}}_{X/S}$.*

Proof. It is given by

$$\left(\Omega^{\mathrm{PD}}_{X/S}\right)_T = \Omega^{\mathrm{PD}}_T$$

for all $T \in \mathrm{Cris}(X/S)$. $\square$

It is good to note that $\Omega^{\mathrm{PD}}_{X/S}$ is in general not a crystal (but it is locally quasi-coherent). We denote $\Omega^{\mathrm{PD},i}_{X/S} = \bigwedge^i \Omega^{\mathrm{PD}}_{X/S}$.

Definition 4.20. Let $\mathcal{F}$ be an $\mathcal{O}_{X/S}$-module on $\mathrm{Cris}(X/S)$. A *connection* is a map of abelian sheaves

$$\nabla\colon \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^{\mathrm{PD}}_{X/S}$$

such that

$$\nabla(fs) = f\nabla(s) + s \otimes df$$

for all local sections $f \in \mathcal{F}$ and $s \in \mathcal{O}_{X/S}$. Given a connection we obtain maps

$$\nabla\colon \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^{\mathrm{PD},i}_{X/S} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^{\mathrm{PD},i+1}_{X/S}.$$

A connection is *integrable* if $\nabla^2 = 0$. If ∇ is integrable we obtain the *de Rham complex*

$$\mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^{\mathrm{PD},1}_{X/S} \to \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^{\mathrm{PD},2}_{X/S} \to \dots$$

on $\mathrm{Cris}(X/S)$.

Lemma 4.21. *Let $\mathcal{F}$ be a crystal on* $\mathrm{Cris}(X/S)$*. Then $\mathcal{F}$ comes with a canonical integrable connection.*

Proof. See [Stacks, 07J6]. □

4.5 Localization on the crystalline topos

When computing sheaf cohomology on a scheme, one can often use information about $\mathcal{F}|_U$ in $\mathrm{Sh}(U_{\mathrm{zar}})$ for an open $U \subseteq X$. On the crystalline topos we can similarly consider the localized topos $(X/S)_{\mathrm{cris}}|_{\tilde{D}}$ for $D \in \mathrm{Cris}(X/S)$.

We will later use the localization for two purposes: It will allow us to compare crystalline cohomology with the Zariski cohomology of a well-chosen complex depending on D. Furthermore, it turns out that for a well-chosen localization we can eliminate the functoriality problem observed in (4.2).

To do al this we need to understand localizations of the crystalline topos, which we will do in this section. We start by generalizing Proposition 4.4 to localizations.

Lemma 4.22. *Let $T = (U, T, \delta)$ be an object of* $\mathrm{Cris}(X/S)$*. Let $\tilde{T} = \mathrm{Hom}_{\mathrm{Cris}(X/S)}(-, T)$ be the image in $(X/S)_{\mathrm{cris}}$ of T under the Yoneda embedding. Then an element of the topos $(X/S)_{\mathrm{cris}}|_{\tilde{T}}$ is uniqely determined by*

- *for every morphism $u\colon T' \to T$ in* $\mathrm{Cris}(X/S)$ *a sheaf $\mathcal{F}_u$ on T and*
- *for every morphism $u \to v$ (commutative triangle over T) a map $\mathcal{F}_v \to \mathcal{F}_u$*

that are compatible in the sense of Proposition 4.4.

Proof. By [Stacks, 00Y1] we may identify $(X/S)_{\mathrm{cris}}|_{\tilde{T}}$ with $\mathrm{Sh}(\mathrm{Cris}(X/S)/T)$. Hence this follows directly from Proposition 4.4. □

From this it is easy to see that the pullback corresponding to the morphism of topoi

$$j_T\colon (X/S)_{\mathrm{cris}}|_{\tilde{T}} \to (X/S)_{\mathrm{cris}}$$

can be computed as follows: For any $\mathcal{F} \in (X/S)_{\mathrm{cris}}$ and $u\colon T' \to T$ in $\mathrm{Cris}(X/S)$ the pullback can be computed as $j_T^{-1}(\mathcal{F})_u = \mathcal{F}_{T'}$.

Recall we had a morphism of topoi $u_{X/S}\colon (X/S)_{\mathrm{cris}} \to \mathrm{Sh}(X_{\mathrm{zar}})$, we now give a local version of it.

Proposition 4.23. *Let $T = (U, T, \delta) \in \mathrm{Cris}(X/S)$. Then there is a commutative diagram of topoi*

$$\begin{tikzcd}
(X/S)_{\mathrm{cris}}|_{\tilde{T}} \arrow[r, "j_T"] \arrow[d, "\varphi_T"] & (X/S)_{\mathrm{cris}} \arrow[d, "u_{X/S}"] \\
\mathrm{Sh}\,(U_{\mathrm{zar}}) \arrow[r, "\iota_U"] & \mathrm{Sh}\,(X_{\mathrm{zar}})
\end{tikzcd}$$

The map φ_T is defined as follows:

- *Given a sheaf $\mathcal{F} \in \mathrm{Cris}(X/S)|_{\tilde{T}}$*

$$\varphi_{T,*}(\mathcal{F}) = \mathcal{F}_{\mathrm{id}_T}.$$

- *Given a sheaf F on $U_{\mathrm{zar}} = T_{\mathrm{zar}}$ and $u\colon T' \to T$*

$$(\varphi_T^{-1}(F))_u = u^*(F).$$

Proof. The diagram is obtained from the diagram of sites

$$\begin{tikzcd}
\mathrm{Cris}(X/S)/T \arrow[r, "j_T"] \arrow[d] & \mathrm{Cris}(X/S) \arrow[d] \\
U_{\mathrm{zar}} \arrow[r] & X_{\mathrm{zar}}
\end{tikzcd}$$

For details, see [Berthelot, Proposition 5.26]. $\square$

It is also shown in [Berthelot, Proposition 5.26] that $\varphi_{T,*}$ is exact and takes injectives to injectives.

4.6 Restriction of cohomology

The point of this section is to introduce a map that allows us to compare crystalline cohomology with Zariski cohomology.

Let $\mathcal{F}$ be a crystal in $(X/S)_{\text{cris}}$, and $T = (U, T, \delta) \in \text{Cris}(X/S)$. Then the general base change map [Stacks, 07A7] associated to the commutative diagram of topoi

$$\begin{tikzcd} (X/S)_{\text{cris}}|_{\tilde{T}} \arrow[r, "j_T"] \arrow[d, "\varphi_T"] & (X/S)_{\text{cris}} \arrow[d, "u_{X/S}"] \\ \text{Sh}\,(U_{\text{zar}}) \arrow[r, "\iota_U"] & \text{Sh}\,(X_{\text{zar}}) \end{tikzcd}$$

(see Proposition 4.23) is a map

$$\mathrm{R}u_{X/S,*}\mathcal{F}\big|_U \to \varphi_{T,*}j_T^{-1}\mathcal{F} = \mathcal{F}_T. \tag{4.3}$$

(note $\varphi_{T,*}$ and j_T^* are exact). We will refer to it as the *local restriction of cohomology* map. We may compose with the base change map associated to the diagram

$$\begin{tikzcd} \text{Sh}\,(U_{\text{zar}}) \arrow[r, "\iota_U"] \arrow[d] & \text{Sh}\,(X_{\text{zar}}) \arrow[d] \\ \text{Sh}\,(S_{\text{zar}}) \arrow[r] & \text{Sh}\,(S_{\text{zar}}) \end{tikzcd}$$

to obtain a map

$$\mathrm{R}\Gamma((X/S)_{\text{cris}}, \mathcal{F}) \to \mathrm{R}\Gamma(U_{\text{zar}}, \mathcal{F}_T). \tag{4.4}$$

This last map we call the *restriction of cohomology* map.

The reason for the name is that if one identifies $\Gamma(T, \mathcal{F}) = \Gamma(T, \mathcal{F}_T)$ (note φ_T is exact), it agrees with the map

$$\mathrm{R}\Gamma((X/S)_{\text{cris}}, \mathcal{F}) \to \mathrm{R}\Gamma(T, \mathcal{F})$$

induced by the morphism of sheaves $\tilde{T} \to E$ (here E is the final object of $(X/S)_{\text{cris}}$). This interpretation could be considered restriction of cohomology from the entire site $\text{Cris}(X/S)$ to the site $\text{Cris}(X/S)/T$.

Since base change maps behave nicely under composition [Stacks, 0E46], the restriction of cohomology map agrees with the base change map associated to the diagram

$$\begin{tikzcd} (X/S)_{\text{cris}}|_{\tilde{T}} \arrow[r, "j_T"] \arrow[d, "(U\to S)\circ\varphi_T"] & (X/S)_{\text{cris}} \arrow[d, "\Gamma((X/S),-)"] \\ \text{Sh}\,(S_{\text{zar}}) \arrow[r] & \text{Sh}\,(S_{\text{zar}}) \end{tikzcd}$$

4.7 Localization and functoriality

In this section we show that by localizing we can construct a version of the diagram (4.2) that does commute.

Definition 4.24. Given a commutative square (4.1), elements $D \in \mathrm{Cris}(X/S)$ and $D' \in \mathrm{Cris}(X'/S')$, and a map $D' \to D$ making the diagram

$$\begin{tikzcd} D' \arrow[r, "F"] \arrow[d] & D \arrow[d] \\ X' \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd} \tag{4.5}$$

commute, the cocontinuous functor

$$\begin{aligned} \mathrm{Cris}(X/S)/D &\to \mathrm{Cris}(X'/S')/D' \\ D &\mapsto D \times_T T' \end{aligned}$$

(the fibred product is in the category of PD-schemes, see [Stacks, 07ME]) defines a morphism of topoi

$$f_{\mathrm{local}} \colon (X'/S')_{\mathrm{cris}}|_{\tilde{D}'} \to (X/S)_{\mathrm{cris}}|_{\tilde{D}}$$

The remainder of this section is devoted to showing that this morphism f_{local} agrees with both $f_{\mathrm{cris}} \colon \mathrm{Cris}(X'/S') \to \mathrm{Cris}(X/S)$ and $F_{\mathrm{zar}} \colon \mathrm{Sh}(D'_{\mathrm{zar}}) \to \mathrm{Sh}(D_{\mathrm{zar}})$, thus providing a way to compare these last two morphisms. Showing that it agrees with F_{zar} is straightforward.

Proposition 4.25. *Given a diagram (4.5), the diagram*

$$\begin{tikzcd} (X'/S')_{\mathrm{cris}}|_{\tilde{D}'} \arrow[r, "f_{\mathrm{local}}"] \arrow[d, "\varphi_{D'}"] & (X/S)_{\mathrm{cris}}|_{\tilde{D}} \arrow[d, "\varphi_D"] \\ \mathrm{Sh}\left(D'_{\mathrm{zar}}\right) \arrow[r, "F_{\mathrm{zar}}"] & \mathrm{Sh}\left(D_{\mathrm{zar}}\right) \end{tikzcd}$$

commutes.

Proof. We can define F_{zar} using the morphism of sites

$$\begin{aligned} D_{\mathrm{zar}} &\to D'_{\mathrm{zar}} \\ U &\mapsto U \times_X D'. \end{aligned}$$

The proposition follows as the diagram on the underlying sites commutes. □

To show it agrees with f_{cris}, we need to compare with the big crystalline site.

Definition 4.26. Given a diagram (4.5), the cocontinuous functor

$$\begin{aligned} \mathrm{CRIS}(X/S)/D &\to \mathrm{CRIS}(X'/S')/D' \\ D &\mapsto D \times_T T' \end{aligned} \tag{4.6}$$

defines a morphism of topoi

$$f_{\mathrm{LOCAL}} \colon (X'/S')_{\mathrm{CRIS}}|_{\tilde{D}'} \to (X/S)_{\mathrm{CRIS}}|_{\tilde{D}}$$

As the functor is also continuous, it follows that given a sheaf $\mathcal{F}' \in (X'/S')_{\mathrm{CRIS}}$ and $T \in \mathrm{CRIS}(X/S)/D$, one has

$$\left(f_{\mathrm{LOCAL},*}\mathcal{F}'\right)(T) = \mathcal{F}'(T \times_D D'). \tag{4.7}$$

Lemma 4.27. *Given a diagram (4.5), the diagram*

$$\begin{tikzcd} (X'/S')_{\mathrm{CRIS}}|_{\tilde{D}'} \arrow[r, "f_{\mathrm{LOCAL}}"] \arrow[d, "j_{D'}"] & (X/S)_{\mathrm{CRIS}}|_{\tilde{D}} \arrow[d, "j_D"] \\ (X'/S')_{\mathrm{CRIS}} \arrow[r, "f_{\mathrm{CRIS}}"] & (X/S)_{\mathrm{CRIS}} \end{tikzcd}$$

commutes.

Proof. Consider the functor

$$\begin{aligned} \mathrm{CRIS}(X'/S')/D' &\to \mathrm{CRIS}(X/S)/D \\ (T' \to D') &\mapsto (T' \to D' \to D) \end{aligned} \tag{4.8}$$

It is clearly cocontinuous, and a left adjoint to (4.6). Hence both functors define the same morphism of topoi by [Stacks, 00XY]. But it is easy to see that (4.8) makes the diagram of the underlying sites commute. $\square$

Corollary 4.28. *Given $\mathcal{F} \in (X/S)_{\mathrm{cris}}$ and $T' \in \mathrm{Cris}(X'/S')/D'$, we have*

$$f_{\mathrm{LOCAL}}^{-1}(\mathcal{F})(T' \to D') = \mathcal{F}(T' \to D' \to D) \tag{4.9}$$

Proposition 4.29. *Given a diagram (4.5), the diagram*

$$\begin{tikzcd} (X'/S')_{\mathrm{cris}}|_{\tilde{D}'} \arrow[r, "f_{\mathrm{local}}"] \arrow[d, "j_{D'}"] & (X/S)_{\mathrm{cris}}|_{\tilde{D}} \arrow[d, "j_D"] \\ (X'/S')_{\mathrm{cris}} \arrow[r, "f_{\mathrm{cris}}"] & (X/S)_{\mathrm{cris}} \end{tikzcd}$$

commutes.

Proof. Consider the diagram

$$\begin{tikzcd}
(X'/S')_{\mathrm{cris}}|_{\tilde{D}'} \arrow[r, "f_{\mathrm{local}}"] \arrow[d, "i"] \arrow[ddd, bend right, "j'_D"'] & (X/S)_{\mathrm{cris}}|_{\tilde{D}} \arrow[d, "i"] \arrow[ddd, bend left, "j_D"] \\
(X'/S')_{\mathrm{CRIS}}|_{\tilde{D}'} \arrow[r, "f_{\mathrm{LOCAL}}"] \arrow[d, "j_{D'}"] & (X/S)_{\mathrm{CRIS}}|_{\tilde{D}} \arrow[d, "j_D"] \\
(X'/S')_{\mathrm{CRIS}} \arrow[r, "f_{\mathrm{CRIS}}"] & (X/S)_{\mathrm{CRIS}} \arrow[d, "\pi"] \\
(X'/S')_{\mathrm{cris}} \arrow[u, "i"] \arrow[r, "f_{\mathrm{cris}}"] & (X/S)_{\mathrm{cris}}
\end{tikzcd}$$

one must be careful as we do not know if everything commutes. By definition of f_{cris} we have $f_{\mathrm{cris}} \circ j'_D = \pi \circ f_{\mathrm{CRIS}} \circ i \circ j'_D$. Since the morphisms between the underlying sites agree, $i \circ j'_D = j'_D \circ i$. Applying Lemma 4.27 we obtain $f_{\mathrm{cris}} \circ j'_D = \pi \circ j_D \circ f_{\mathrm{LOCAL}} \circ i$. Note $f_{\mathrm{LOCAL}} \circ i = i \circ f_{\mathrm{local}}$ as the morphisms between the sites agree, hence $f_{\mathrm{cris}} \circ j'_D = \pi \circ j_D \circ i \circ f_{\mathrm{local}}$. The result follows from the fact that $j_D = \pi \circ j_D \circ i$, which is a consequence of the fact that $j_D \circ i = i \circ j_D$ and $\pi \circ i = \mathrm{id}$. ☐

4.8 The ringed topoi structure

We end this chapter by upgrading our previous constructions to ringed topoi.

Definition 4.30. We define the ringed topos $((X/S)_{\mathrm{cris}}, \mathcal{O}_{X/S})$ where $\mathcal{O}_{X/S}$ is the structure sheaf from Example 4.5. Similarly we define the ringed topos $((X/S)_{\mathrm{CRIS}}, \mathcal{O}_{X/S}^{\mathrm{CRIS}})$. For $T \in \mathrm{Cris}(X/S)$, the identity on $\mathcal{O}_{X/S}(T)$ provides us with a map

$$i^\sharp \colon \left(i^{-1}\mathcal{O}_{X/S}^{\mathrm{CRIS}}\right)(T) \to \mathcal{O}_{X/S}$$

Using the fact that $\pi_* = i^{-1}$, we see that i and π get the structure of morphisms of ringed topoi satisfying $\pi \circ i = \mathrm{id}$.

Definition 4.31. Given a diagram (4.1) and $(U' \to X', T', \delta') \in \mathrm{CRIS}(X'/S')$, the identity $\mathcal{O}_{T'}(T') \to \mathcal{O}_{T'}(T')$ gives a map

$$f_{\mathrm{CRIS}}^\sharp \colon \left(f_{\mathrm{CRIS}}^{-1}\mathcal{O}_{X/S}\right)(T') \to \mathcal{O}_{X'/S'}(T').$$

This gives f_{CRIS} the structure of a morphism of ringed topoi. We give f_{cris} the structure of a morphism of ringed topoi by declaring $f_{\mathrm{cris}} = \pi \circ f_{\mathrm{CRIS}} \circ i$.

Definition 4.32. Given a commutative diagram (4.5) and $T \in \mathrm{Cris}(X/S)/D$, the projection $T \times_D D' \to T$ provides us with a canonical map

$$f_{\mathrm{local}}^\sharp \colon \mathcal{O}_{X/S}(T) \to \mathcal{O}_{X'/S'}(T \times_D D') = \left(f_{\mathrm{local},*}\mathcal{O}_{X'/S'}\right)(T)$$

This gives $f_{\mathrm{local},*}$ the structure of a morphism of ringed topoi.

Definition 4.33. Since $\varphi_{D,*}\mathcal{O}_{X/S} = \mathcal{O}_D$, the identity gives φ_D the structure of a morphism of ringed topoi.

Remark 4.34. The maps

$$\begin{aligned} \varphi_{D,*}f_{\mathrm{local}}^{\sharp}\colon \varphi_{D,*}\mathcal{O}_{X/S} &\to \varphi_{D,*}f_{\mathrm{local},*}\mathcal{O}_{X'/S'} \\ F_{\mathrm{zar}}^{\sharp}\colon \mathcal{O}_D &\to F_{\mathrm{zar},*}\mathcal{O}_{D'} \end{aligned}$$

agree, as the projection $D \times_D D' \to D$ agrees with $F\colon D' \to D$. Thus the diagram of ringed topoi

$$\begin{tikzcd} (X'/S')_{\mathrm{cris}}|_{\tilde{D}'} \arrow[r, "f_{\mathrm{local}}"] \arrow[d, "\varphi_{D'}"] & (X/S)_{\mathrm{cris}}|_{\tilde{D}} \arrow[d, "\varphi_D"] \\ \mathrm{Sh}\,(X'_{\mathrm{zar}}) \arrow[r, "F_{\mathrm{zar}}"] & \mathrm{Sh}\,(X_{\mathrm{zar}}) \end{tikzcd}$$

commutes.

Lemma 4.35. *Suppose we are given a commutative diagram (4.5). Consider the map*

$$\varphi\colon f_{\mathrm{LOCAL}}^{-1}\mathcal{O}_{X'/S'}^{\mathrm{CRIS}} \to \mathcal{O}_{X/S}^{\mathrm{CRIS}}$$

of sheaves on $\mathrm{CRIS}(X'/S')/D'$ *induced by the identity map* $\mathcal{O}_{T'} \to \mathcal{O}_{T'}$ *for all objects* $T' \in \mathrm{Cris}(X'/S')/D'$ *(see (4.9)). Consider also the map*

$$\psi\colon \mathcal{O}_{X/S} \to f_{\mathrm{LOCAL},*}\mathcal{O}_{X'/S'}$$

of sheaves on $\mathrm{CRIS}(X/S)/D$ *induced by the projections* $T \times_D D' \to T$ *for all objects* $T \in \mathrm{Cris}(X/S)/D$ *(see (4.7). Then* φ *and* ψ *agree under the adjunction between* f_{LOCAL}^{-1} *and* $f_{\mathrm{LOCAL},*}$.

Proof. It suffices to show that the composition

$$f_{\mathrm{LOCAL}}^{-1}\mathcal{O}_{X/S} \xrightarrow{f_{\mathrm{LOCAL}}^{-1}\psi} f_{\mathrm{LOCAL}}^{-1}f_{\mathrm{LOCAL},*}\mathcal{O}_{X'/S'} \xrightarrow{\alpha} \mathcal{O}_{X'/S'}$$

is equal to φ. Given an object $T' \in \mathrm{CRIS}(X'/S')/D'$, the map α is defined by

$$\alpha_{T'\to D'}\colon \mathcal{O}_{X'/S'}(T' \times_D D' \to D') \to \mathcal{O}_{X'/S'}(T' \to D')$$

induced by the map $h\colon T' \to T' \times_D D'$ coming from the diagram

$$\begin{tikzcd} T' \arrow[r, dotted, "h"] \arrow[rr, bend left, "\mathrm{id}_{T'}"] \arrow[dr] & T' \times_D D' \arrow[r] \arrow[d] & T' \arrow[d] \\ & D' \arrow[r] & D \end{tikzcd}$$

But then this is obvious, as $f_{\mathrm{LOCAL}}^{-1}\psi$ is induced by the projection $T' \times_D D' \to T'$. $\square$

This motivates the following definition.

Definition 4.36. Given a diagram (4.5) we give f_{LOCAL} the structure of a morphism of ringed topoi by setting $f_{\mathrm{LOCAL}}^{\sharp} = \psi$ (the map from the previous lemma).

Corollary 4.37. *Given a diagram (4.5), the diagram of ringed topoi*

$$\begin{array}{ccc}
(X'/S')_{\mathrm{cris}}|_{\tilde{D}'} & \xrightarrow{f_{\mathrm{local}}} & (X/S)_{\mathrm{cris}}|_{\tilde{D}} \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\
(X'/S')_{\mathrm{CRIS}}|_{\tilde{D}'} & \xrightarrow{f_{\mathrm{LOCAL}}} & (X/S)_{\mathrm{CRIS}}|_{\tilde{D}} \\
\downarrow{\scriptstyle j_{D'}} & & \downarrow{\scriptstyle j_D} \\
(X'/S')_{\mathrm{CRIS}} & \xrightarrow{f_{\mathrm{CRIS}}} & (X/S)_{\mathrm{CRIS}}
\end{array}$$

commutes.

Proof. The definition by ψ ensures us that the top square commutes, while the definition by φ ensures us that the lower square commutes. □

Corollary 4.38. *Given a diagram (4.5), the diagram of ringed topoi*

$$\begin{array}{ccc}
(X'/S')_{\mathrm{cris}}|_{\tilde{D}'} & \xrightarrow{f_{\mathrm{local}}} & (X/S)_{\mathrm{cris}}|_{\tilde{D}} \\
\downarrow{\scriptstyle j_{D'}} & & \downarrow{\scriptstyle j_D} \\
(X'/S')_{\mathrm{cris}} & \xrightarrow{f_{\mathrm{cris}}} & (X/S)_{\mathrm{cris}}
\end{array}$$

commutes.

Proof. Argue as in Proposition 4.29, using the previous corollary. □

5 Comparing crystalline and PD-de Rham cohomology

Let p be a prime number. In this chapter we will compare crystalline cohomology with the PD-de Rham cohomology. We will first do this with coefficients in a ring on which p is nilpotent. Then we will then obtain a statement about coefficients in a p-adically complete ring by completing and using a form of the theorem of formal functions.

5.1 Comparison in finite characteristic

We generalize [Stacks, 07MW]. Let (A, I, γ) be a PD-ring such that p is nilpotent in A. Write $S = \operatorname{Spec}(A)$ and $S_0 = A/I$. Let Y be a smooth scheme over S and let X be a scheme over S_0. Let $\iota\colon X \to Y$ be a closed immersion over S. Consider the divided power envelope $D_{X,\gamma}(Y) = (D, \mathcal{J}, \delta)$. The goal of this section is to prove the following theorem.

Theorem 5.1. *Under the above assumptions, there exists an isomorphism*

$$\mathrm{H}^i_{\mathrm{cris}}(X/S) \xrightarrow{\sim} \mathrm{H}^i\left(X_{\mathrm{zar}}, \Omega^{\mathrm{PD},\bullet}_{D_{X,\gamma}(Y)/S}\right).$$

We start by showing that there exists an object $(X, D_{X,\gamma}(Y), \delta)$ in $\mathrm{Cris}(X/S)$.

Lemma 5.2. *The scheme $D_{X,\gamma}(Y)$ is a PD-thickening of X.*

Proof. Since X lives over S_0 one has $I\mathcal{O}_X = 0$, hence by [Stacks, 07H1] we know that γ extends to $\mathcal{O}_X$. It suffices to show the natural map $j\colon X \to D_{X,\gamma}(Y)$ obtained from Remark 2.23 is a thickening. Denote $\iota\colon X \to Y$ for the closed immersion, and $\pi\colon D_{X,\gamma}(Y) \to Y$ for the map obtained from [Stacks, 01LP], so that we have a diagram

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "j"'] & Y \\ & D_{X,\gamma}(Y) \arrow[u, "\pi"'] \end{tikzcd}$$

Let $\{U_i \to Y\}$ be a covering of open affines. Denote $V_i = \iota^{-1}(U_i)$ and $W_i = \pi^{-1}(U_i)$. Clearly the V_i are affine as ι is a closed immersion. By [Stacks, 01LP] one has that $W_i = \operatorname{Spec} \mathcal{D}_{X,\gamma}(Y)(U_i)$, which is also affine. By commutativity of the diagram one has

$j^{-1}(W_i) = V_i$. Hence the covering $\{W_i \to D_{X,\gamma}(Y)\}$ satisfies the property of the covering in [Stacks, 0BPF], so we may reduce to the affine case.

Thus consider a ring map $A \to B$ and ideal $J \subseteq B$. By Corollary 2.17 the ideal $\overline{J}$ is nilpotent in $D_{B,\gamma}(J)$, so it suffices to show that $B/J \to D_{B,\gamma}(J)/J$ is an isomorphism, which is precisely the statement of Lemma 2.15. $\square$

Lemma 5.3. *Suppose X and Y are affine. Let $\mathcal{F}$ be a crystal in $(X/S)_{\mathrm{cris}}$. Then the restriction of cohomology map (4.4) induces an isomorphism*

$$\mathrm{R}\Gamma\left((X/S)_{\mathrm{cris}}, \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^{\mathrm{PD},\bullet}_{X/S}\right) \xrightarrow{\sim} \mathrm{R}\Gamma\left(X_{\mathrm{zar}}, \iota^{-1}\mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega^{\mathrm{PD},\bullet}_{D_{X,\gamma}(Y)/S}\right).$$

Proof. Write $\mathcal{D} = \mathcal{D}_{X,\gamma}(Y)$ and $D = D_{X,\gamma}(Y)$. Let $T = (X, D, \delta) \in \mathrm{Cris}(X/S)$ be the PD-thickening from Lemma 5.2. By Lemma 4.21 the crystal $\mathcal{F}$ comes with a connection ∇. This restricts to give a connection

$$\nabla_D \colon \mathcal{F}_D \to \mathcal{F}_D \otimes_{\mathcal{D}} \Omega^{\mathrm{PD},\bullet}_{D/S}$$

(details omitted).

The restriction of cohomology map gives a map

$$\mathrm{R}\Gamma\left((X/S)_{\mathrm{cris}}, \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^{\mathrm{PD},\bullet}_{X/S}\right) \xrightarrow{\sim} \mathrm{R}\Gamma\left(D_{\mathrm{zar}}, \mathcal{F}_D \otimes_{\mathcal{D}} \Omega^{\mathrm{PD},\bullet}_{D/S}\right)$$

(the complexes are constructed using the connections ∇ and ∇_D respectively). Now identify $D_{\mathrm{zar}} = X_{\mathrm{zar}}$, since $\mathcal{F}$ is a crystal one has $\mathcal{F}_D = \iota^{-1}\mathcal{F}_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_{X,\gamma}(Y)$, hence we get the required map. Remains to show it is an isomorphism.

Note that since p is nilpotent in A, it is nilpotent in $\mathcal{O}_Y$ and thus in $\mathcal{D}_{X,\gamma}(Y)$. Therefore $\mathcal{D}_{X,\gamma}(Y)/p^e\mathcal{D}_{X,\gamma}(Y) = \mathcal{D}_{X,\gamma}(Y)$ for large e. Hence under the equivalence of categories in [Stacks, 07L5] (see [Stacks, 07JG]) for the construction) $\mathcal{F}$ corresponds to the pair $(M, \nabla) = (\mathcal{F}_D, \nabla_D)$. Hence the map is an isomorphism by [Stacks, 07LH]. $\square$

Lemma 5.4. *Let $\mathcal{F}$ be a crystal in $(X/S)_{\mathrm{cris}}$. Then the restriction of cohomology map (4.4) induces an isomorphism*

$$\mathrm{R}\Gamma\left((X/S)_{\mathrm{cris}}, \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^{\mathrm{PD},\bullet}_{X/S}\right) \xrightarrow{\sim} \mathrm{R}\Gamma\left(X_{\mathrm{zar}}, \iota^{-1}\mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega^{\mathrm{PD},\bullet}_{D_{X,\gamma}(Y)/S}\right).$$

Proof. Write $\mathcal{D} = \mathcal{D}_{X,\gamma}(Y)$ and $D = D_{X,\gamma}(Y)$. Repeating the argument in Lemma 5.3, but now applying the local restriction of cohomology map (4.3) we obtain a map

$$\mathrm{R}u_{X/S,*}\left(\mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^{\mathrm{PD},\bullet}_{X/S}\right) \to \iota^{-1}\mathcal{F}_D \otimes_{\mathcal{D}} \Omega^{\mathrm{PD},\bullet}_{D/S}$$

in $D(\mathrm{Ab}(X_{\mathrm{zar}}))$. It suffices to show this map is an isomorphism, the isomorphism of the statement can then be obtained by composing with $\mathrm{R}\Gamma(X_{\mathrm{zar}}, -)$.

But we may check this map is an isomorphism locally. Let $U \subseteq X$ be an affine open such that $\iota(U)$ is contained in an affine open of Y. By [Stacks, 07KL] one has

$$\mathrm{R}u_{X/S,*}\left(\mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^{\mathrm{PD},\bullet}_{X/S}\right)\Big|_U = \mathrm{R}u_{U/S,*}\left(\mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^{\mathrm{PD},\bullet}_{X/S}|_{(U/S)_{\mathrm{cris}}}\right).$$

It thus suffices to show the map

$$\mathrm{R}u_{U/S,*}\left(\mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^{\mathrm{PD},\bullet}_{X/S}|_{(U/S)_{\mathrm{cris}}}\right) \to \mathcal{F}_D \otimes_{\mathcal{D}} \Omega^{\mathrm{PD},\bullet}_{D/S}|_U$$

is an isomorphism. Since U is affine this can be verified on global sections, in which case it reduces to the statement of Lemma 5.3. □

Theorem 5.5 (Comparison in finite characteristic)**.** *Let $\mathcal{F}$ be a crystal in $(X/S)_{\mathrm{cris}}$. Then there exists an isomorphism*

$$\mathrm{R}\Gamma\left((X/S)_{\mathrm{cris}}, \mathcal{F}\right) \xrightarrow{\sim} \mathrm{R}\Gamma\left(X_{\mathrm{zar}}, \iota^{-1}\mathcal{F}_Y \otimes_{\mathcal{O}_Y} \Omega^{\mathrm{PD},\bullet}_{D_{X,\gamma}(Y)/S}\right) \tag{5.1}$$

Proof. By [Stacks, 07LM] we may construct an isomorphism

$$\mathrm{R}\Gamma\left((X/S)_{\mathrm{cris}}, \mathcal{F}\right) \xrightarrow{\sim} \mathrm{R}\Gamma\left((X/S)_{\mathrm{cris}}, \mathcal{F} \otimes_{\mathcal{O}_{X/S}} \Omega^{\mathrm{PD},\bullet}_{X/S}\right) \tag{5.2}$$

Composing with the map from Lemma 5.4 we obtain the required isomorphism. □

Setting $\mathcal{F} = \mathcal{O}_{X/S}$ we obtain Theorem 5.1.

5.2 Functoriality

The goal of this chapter is to formulate and proof a statement saying the isomorphism from Theorem 5.5 is 'functorial' in X and Y. I was unable to find the proofs for these reults written out explicitly anywhere. It was inspired by [CoRe, Theorem 3.3].

Suppose X, Y, S and X', Y', S' satisfy the conditions from Theorem 5.1, and we are given a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{G} & Y \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S \end{array} \tag{5.3}$$

Then we have a commutative diagram of (ringed) topoi

$$\begin{array}{ccc} (X'/S')_{\mathrm{cris}} & \xrightarrow{g_{\mathrm{cris}}} & (X/S)_{\mathrm{cris}} \\ \big\downarrow{\scriptstyle \Gamma((X'/S')_{\mathrm{cris}},-)} & & \big\downarrow{\scriptstyle \Gamma((X/S)_{\mathrm{cris}},-)} \\ \mathrm{Sh}\left(S'_{\mathrm{zar}}\right) & \xrightarrow{f_{\mathrm{zar}}} & \mathrm{Sh}\left(S_{\mathrm{zar}}\right) \end{array} \tag{5.4}$$

see [Stacks, 07MH]. The very general base change map [Stacks, 0735], composed with the map

$$\mathrm{L}g_{\mathrm{cris}}^{*}\mathcal{O}_{X/S} \to g_{\mathrm{cris}}^{*}\mathcal{O}_{X/S} \xrightarrow{g_{\mathrm{cris}}^{\sharp}} \mathcal{O}_{X'/S'} \tag{5.5}$$

provides us with a canonical map

$$\mathrm{L}f_{\mathrm{zar}}^{*}\mathrm{R}\Gamma((X/S)_{\mathrm{cris}}, \mathcal{O}_{X/S}) \to \mathrm{R}\Gamma\left((X'/S')_{\mathrm{cris}}, \mathcal{O}_{X'/S'}\right) \tag{5.6}$$

On the other hand, the map $G\colon Y' \to Y$ induces a map $G\colon \mathcal{D}_{X',\gamma}(Y') \to \mathcal{D}_{X,\gamma}(Y)$ by the universal property of the divided power envelope. This gives us a map

$$g_{\mathrm{zar}}^{*}D_{X,\gamma}(Y) \to D_{X',\gamma}(Y')$$

of sheaves on $X_{\mathrm{zar}} = Y_{\mathrm{zar}}$. Together with the canonical map $g_{\mathrm{zar}}^{*}\Omega_{Y/S} \to \Omega_{Y'/S'}$ we obtain a map of de Rham complexes

$$g_{\mathrm{zar}}^{*}\Omega_{D_{X,\gamma}(Y)/S}^{\mathrm{PD},\bullet} \to \Omega_{D_{X',\gamma}(Y')/S'}^{\mathrm{PD},\bullet} \tag{5.7}$$

The very general base change map [Stacks, 0735] corresponding to the commutative diagram of topoi

$$\begin{tikzcd} \mathrm{Sh}\left(X'_{\mathrm{zar}}\right) \arrow[r, "g_{\mathrm{zar}}"] \arrow[d] & \mathrm{Sh}\left(X_{\mathrm{zar}}\right) \arrow[d] \\ \mathrm{Sh}\left(S'_{\mathrm{zar}}\right) \arrow[r, "f_{\mathrm{zar}}"] & \mathrm{Sh}\left(S_{\mathrm{zar}}\right) \end{tikzcd} \tag{5.8}$$

composed with the map (5.7) gives a map

$$\mathrm{L}f_{\mathrm{zar}}^{*}\mathrm{R}\Gamma\left(X_{\mathrm{zar}}, \Omega_{D_{X,\gamma}(Y)/S}^{\mathrm{PD},\bullet}\right) \to \mathrm{R}\Gamma\left(X'_{\mathrm{zar}}, \Omega_{D_{X',\gamma}(Y')/S'}^{\mathrm{PD},\bullet}\right) \tag{5.9}$$

The goal of this section is to proof the following theorem.

Theorem 5.6. *Suppose X, Y, S and X', Y', S' satisfy the conditions from Theorem 5.1, and we are given a commutative diagram (5.3). Then the diagram*

$$\begin{tikzcd} \mathrm{L}f_{\mathrm{zar}}^{*}\mathrm{R}\Gamma((X/S)_{\mathrm{cris}}, \mathcal{O}_{X/S}) \arrow[r, "(5.6)"] \arrow[d, "(5.1)", "\wr"'] & \mathrm{R}\Gamma\left((X'/S')_{\mathrm{cris}}, \mathcal{O}_{X'/S'}\right) \arrow[d, "(5.1)", "\wr"'] \\ \mathrm{L}f_{\mathrm{zar}}^{*}\mathrm{R}\Gamma\left(X_{\mathrm{zar}}, \Omega_{D_{X,\gamma}(Y)/S}^{\mathrm{PD},\bullet}\right) \arrow[r, "(5.9)"] & \mathrm{R}\Gamma\left(X'_{\mathrm{zar}}, \Omega_{D_{X',\gamma}(Y')/S'}^{\mathrm{PD},\bullet}\right) \end{tikzcd}$$

commutes.

Proof. We use the following convention: If a morphism is given by a commutative diagram, we mean that it is induced by the base change map of the corresponding diagram. Write

$D = D_{X,\gamma}(Y), D' = D_{X',\gamma}(Y')$. Note that we have a commutative diagram (4.5). The diagram

$$\begin{tikzcd}
\mathrm{L}f_{\mathrm{zar}}^{*}\mathrm{R}\Gamma((X/S)_{\mathrm{cris}}, \Omega_{X/S}^{\mathrm{PD},\bullet}) \arrow[r, "(5.6)"] \arrow[d, "(5.2)", "\wr"'] & \mathrm{R}\Gamma\left((X'/S')_{\mathrm{cris}}, \Omega_{X'/S'}^{\mathrm{PD},\bullet}\right) \arrow[d, "(5.2)", "\wr"'] \\
\mathrm{L}f_{\mathrm{zar}}^{*}\mathrm{R}\Gamma((X/S)_{\mathrm{cris}}, \mathcal{O}_{X/S}) \arrow[r, "(5.6)"] & \mathrm{R}\Gamma\left((X'/S')_{\mathrm{cris}}, \mathcal{O}_{X'/S'}\right)
\end{tikzcd}$$

where the top map is made by extending (5.5) to the complexes, commutes, so we only need to show the diagram

$$\begin{tikzcd}
\mathrm{L}f_{\mathrm{zar}}^{*}\mathrm{R}\Gamma((X/S)_{\mathrm{cris}}, \Omega_{X/S}^{\mathrm{PD},\bullet}) \arrow[r, "(5.6)"] \arrow[d, "(4.4)", "\wr"'] & \mathrm{R}\Gamma\left((X'/S')_{\mathrm{cris}}, \Omega_{X'/S'}^{\mathrm{PD},\bullet}\right) \arrow[d, "(4.4)", "\wr"'] \\
\mathrm{L}f_{\mathrm{zar}}^{*}\mathrm{R}\Gamma\left(X_{\mathrm{zar}}, \Omega_{D/S}^{\mathrm{PD},\bullet}\right) \arrow[r, "(5.9)"] & \mathrm{R}\Gamma\left(X'_{\mathrm{zar}}, \Omega_{D'/S'}^{\mathrm{PD},\bullet}\right)
\end{tikzcd}$$

commutes (recall (4.4) is the restriction of cohomology map), where the vertical maps are induced by the restriction on cohomology maps. Consider the commutative diagrams of ringed topoi

$$\begin{tikzcd}
(X'/S')_{\mathrm{cris}}|_{\tilde{D}'} \arrow[r, "g_{\mathrm{local}}"] \arrow[d, "\varphi_{D'}"] \arrow[dr, phantom, "(5.10)"] & (X/S)_{\mathrm{cris}}|_{\tilde{D}} \arrow[r, "j_D"] \arrow[d, "\varphi_D"] & (X/S)_{\mathrm{cris}} \arrow[dd, "\Gamma"] \\
\mathrm{Sh}\,(X'_{\mathrm{zar}}) \arrow[r, "g_{\mathrm{zar}}"] \arrow[d] \arrow[dr, phantom, "(5.8)"] & \mathrm{Sh}\,(X_{\mathrm{zar}}) \arrow[d] \arrow[r, phantom, "(4.4)"] & \\
\mathrm{Sh}\,(S'_{\mathrm{zar}}) \arrow[r, "f_{\mathrm{zar}}"] & \mathrm{Sh}\,(S_{\mathrm{zar}}) \arrow[r, "\mathrm{id}"] & \mathrm{Sh}\,(S_{\mathrm{zar}})
\end{tikzcd}$$

((4.4) is the restriction of cohomology map), and

$$\begin{tikzcd}
(X'/S')_{\mathrm{cris}}|_{\tilde{D}'} \arrow[r, "j_{D'}"] \arrow[d, "(X'_{\mathrm{zar}} \to S'_{\mathrm{zar}}) \circ \varphi_{D'}"'] \arrow[dr, phantom, "(4.4)"] & (X'/S')_{\mathrm{cris}} \arrow[r, "g_{\mathrm{cris}}"] \arrow[d, "\Gamma"] \arrow[dr, phantom, "(5.4)"] & (X/S)_{\mathrm{cris}} \arrow[d, "\Gamma"] \\
\mathrm{Sh}\,(S'_{\mathrm{zar}}) \arrow[r, "\mathrm{id}"] & \mathrm{Sh}\,(S'_{\mathrm{zar}}) \arrow[r, "F_{\mathrm{zar}}"] & \mathrm{Sh}\,(S_{\mathrm{zar}})
\end{tikzcd}$$

By Corollary 4.38 the outer squares of the two diagrams agree. Hence the (very general) base change maps associated to the two sqares are the same. By [Stacks, 0E46] and [Stacks, 0E47] we can compute the base change maps as the composition of the base change maps of the individual squares. We thus get a commutative diagram

$$
\begin{tikzcd}
\mathrm{L}f_{\mathrm{zar}}^{*}\mathrm{R}\Gamma\left((X/S)_{\mathrm{cris}}, \Omega_{X/S}^{\mathrm{PD},\bullet}\right) \arrow[r, "\sim", "(4.4)"'] \arrow[dd, "(5.4)"] \arrow[ddr, "\text{base change map}" sloped] & \mathrm{L}f_{\mathrm{zar}}^{*}\mathrm{R}\Gamma\left(X_{\mathrm{zar}}, \Omega_{D/S}^{\mathrm{PD},\bullet}\right) \arrow[d, "(5.8)"] \\
 & \mathrm{R}\Gamma\left(X'_{\mathrm{zar}}, \mathrm{L}g_{\mathrm{zar}}^{*}\Omega_{D/S}^{\mathrm{PD},\bullet}\right) \arrow[d, "(5.10)"] \\
\mathrm{R}\Gamma\left((X'/S')_{\mathrm{cris}}, \mathrm{L}g_{\mathrm{cris}}^{*}\Omega_{X/S}^{\mathrm{PD},\bullet}\right) \arrow[r, "\sim", "(4.4)"'] \arrow[d, "g_{\mathrm{cris}}^{\sharp}"] & \mathrm{R}\Gamma\left(X'_{\mathrm{zar}}, \left(\mathrm{L}g_{\mathrm{cris}}^{*}\Omega_{X/S}^{\mathrm{PD},\bullet}\right)_{D'}\right) \arrow[d, "g_{\mathrm{cris}}^{\sharp}"] \\
\mathrm{R}\Gamma\left((X'/S')_{\mathrm{cris}}, \Omega_{X'/S'}^{\mathrm{PD},\bullet}\right) \arrow[r, "\sim", "(4.4)"'] & \mathrm{R}\Gamma\left(X'_{\mathrm{zar}}, \Omega_{D'/S'}^{\mathrm{PD},\bullet}\right)
\end{tikzcd}
$$

where the top square is obtained by comparing the base change maps and the bottom square is induced by the map

$$g_{\mathrm{cris}}^{*}\Omega_{X/S}^{\mathrm{PD},\bullet} \to \Omega_{X'/S'}$$

induced by $g_{\mathrm{cris}}^{\sharp}$. It thus suffices to show that the map

$$\Gamma\left(X'_{\mathrm{zar}}, \mathrm{L}g_{\mathrm{zar}}^{*}\Omega_{D/S}^{\mathrm{PD},\bullet}\right) \xrightarrow{(5.10)} \mathrm{R}\Gamma\left(X'_{\mathrm{zar}}, \left(\mathrm{L}g_{\mathrm{cris}}^{*}\Omega_{X/S}^{\mathrm{PD},\bullet}\right)_{D'}\right) \xrightarrow{g_{\mathrm{cris}}^{\sharp}} \mathrm{R}\Gamma\left(X'_{\mathrm{zar}}, \Omega_{D'/S'}^{\mathrm{PD},\bullet}\right)$$

agrees with the map induced by (5.7). This will follow from the fact that the base change map

$$g_{\mathrm{zar}}^{-1}\varphi_{D*}\mathcal{O}_{X/S} \to \varphi_{D'*}g_{\mathrm{local}}^{-1}\mathcal{O}_{X/S}$$

coming from the diagram of topoi

$$
\begin{tikzcd}
(X'/S')_{\mathrm{cris}}|_{\tilde{D}'} \arrow[r, "g_{\mathrm{local}}"] \arrow[d, "\varphi_{D'}"] & (X/S)_{\mathrm{cris}}|_{\tilde{D}} \arrow[d, "\varphi_{D}"] \\
\mathrm{Sh}\left(X'_{\mathrm{zar}}\right) \arrow[r, "g_{\mathrm{zar}}"] & \mathrm{Sh}\left(X_{\mathrm{zar}}\right)
\end{tikzcd}
\tag{5.10}
$$

agrees with the map $g_{\mathrm{zar}}^{\sharp}\colon g_{\mathrm{zar}}^{-1}\mathcal{O}_{D} \to \mathcal{O}_{D'}$. We leave this for the reader to verify. □

5.3 Comparison in characteristic zero

In this section we upgrade our results to characteristic 0, to obtain the comparison theorem we want. Let (A, I, γ) be a PD-ring such that $p \in I$. We also assume A is Noetherian and I-adically complete. Let $S = \mathrm{Spec}(A)$ and $S_0 = A/I$. Let Y be a smooth and projective scheme over S and let X be a scheme over S_0. Let $\iota\colon X \to Y$ be a closed immersion over S.

Set $S_n = S/I^{n+1}$ and $Y_n = Y \otimes_S S_n$. Sometimes we will consider $\mathcal{O}_S$ or $\mathcal{O}_{S_n}$ as a sheaf on X, but we will drop the pushforward notation. Finally consider the object $D_n = (X, D_{X,\gamma}(Y_n), \delta)$ in $\mathrm{Cris}(X/S)$, note that by (a more general) Lemma 2.25 we have $D_n = \iota^{-1} D_{X,\gamma}(Y) \otimes \mathcal{O}_{S_n}$. Denote with $\hat{Y}$ the completion of Y along Y_0.

Consider the inverse system of maps

$$\nabla_n \colon \mathcal{F}_{Y_n} \to \mathcal{F}_{Y_n} \otimes_{\mathcal{O}_{Y_n}} \Omega^{\mathrm{PD}}_{D_n/S}$$

obtained from Lemma 4.21. Since $Y \to \mathrm{Spec}(A)$ is projective, by Grothendieck's existence theorem [Stacks, 0885] we obtain a quasi-coherent $\mathcal{O}_{\hat{Y}}$-module $\mathcal{F}_{\hat{Y}} := \varprojlim \mathcal{F}_{Y_n}$ and a map

$$\nabla \colon \mathcal{F}_{\hat{Y}} \to \varprojlim \mathcal{F}_{Y_n} \otimes_{\mathcal{O}_{Y_n}} \Omega^{\mathrm{PD}}_{D_n/S}.$$

To compute the right hand side, we need a technical lemma.

Lemma 5.7. *Let A be a Noetherian ring. Let $I \subseteq A$ be an ideal. Suppose A is I-adically complete. Let M be a finite A-module and let N be any A-module. Then there exists a canonical isomorphism*

$$M \otimes_A \hat{N} \xrightarrow{\sim} \widehat{M \otimes_A N}.$$

Proof. Thanks to [Brandenburg]. Consider the natural transformation

$$(-) \otimes_A \hat{N} \to \varprojlim((-) \otimes_A N/I^n N)$$

Note that both functors are right-exact (the inverse system on the right hand side is always surjective). Thus picking a free presentation $A^s \to A^r \to M$ (note A is Noetherian so the kernel of $A^r \to M$ is also finitely generated) we obtain a diagram

$$\begin{tikzcd} \hat{N}^s \arrow[r] \arrow[d] & \hat{N}^r \arrow[r] \arrow[d] & M \otimes_A \hat{N} \arrow[r] \arrow[d] & 0 \arrow[d] \\ \hat{N}^s \arrow[r] & \hat{N}^r \arrow[r] & \widehat{M \otimes_A N} \arrow[r] & 0 \end{tikzcd}$$

By the five lemma the map $M \otimes_A \hat{N} \to \widehat{M \otimes_A N}$. is an isomorphism. □

Lemma 5.8. *There exists a canonical isomorphism*

$$\varprojlim \mathcal{F}_{Y_n} \otimes_{\mathcal{O}_{Y_n}} \Omega^{\mathrm{PD}}_{D_n/S} \xrightarrow{\sim} \mathcal{F}_{\hat{Y}} \otimes_{\mathcal{O}_{\hat{Y}}} \hat{\Omega}^{\mathrm{PD,co}}_{D_{X,\gamma}(Y)/S}.$$

of sheaves on $\hat{Y}$.

Proof. The question is local on Y, so assume Y is affine. First note that by Lemma 5.7 (with $A = \mathcal{O}_{\hat{Y}}$, note $\mathcal{F}_{\hat{Y}}$ is finite)

$$\varprojlim \mathcal{F}_{Y_n} \otimes_{\mathcal{O}_{Y_n}} \Omega^{\mathrm{PD}}_{D_n/S} = \mathcal{F}_{\hat{Y}} \otimes_{\mathcal{O}_{\hat{Y}}} \left(\varprojlim \Omega^{\mathrm{PD}}_{D_n/S} \right).$$

Let $\mathcal{D}_n = \mathcal{D}_{X,\gamma}(Y_n)$. By Proposition 2.32 we have $\Omega^{\mathrm{PD}}_{D_n/S} = \mathcal{D}_n \otimes_{Y_n} \Omega_{Y_n/S}$. Since the map $Y \to \mathrm{Spec}(A)$ is projective, $\Omega_{Y/S}$ is a finite $\mathcal{O}_Y$-module. Furthermore, since $\Omega_{Y_n/S} \otimes S_{n-1} = \Omega_{Y/S} \otimes S_{n-1}$, we clearly have $\varprojlim \Omega_{Y_n/S} = \Omega_{\hat{Y}/S}$. Since $\Omega_{\hat{Y}/S}$ is finite (as $Y \to S$ is projective) we have

$$\begin{aligned}
\varprojlim \Omega^{\mathrm{PD}}_{D_n/S} &= \varprojlim \left(\mathcal{D}_n \otimes_{\mathcal{O}_{Y_n}} \Omega_{Y_n/S}\right) \\
&= \varprojlim \mathcal{D}_n \otimes_{\mathcal{O}_{\hat{Y}}} \Omega_{\hat{Y}/S} && \text{(Lemma 5.7)} \\
&= \hat{\mathcal{D}}_{X,\gamma}(Y) \otimes_{\mathcal{O}_{\hat{Y}}} \Omega_{\hat{Y}/S} && \text{(Lemma 2.25)} \\
&= \hat{\Omega}^{\mathrm{co,PD}}_{D_{X,\gamma}(Y)/S}, && \text{(Lemma 5.7)}
\end{aligned}$$

which suffices. $\square$

We thus obtain a connection

$$\nabla \colon \mathcal{F}_{\hat{Y}} \to \mathcal{F}_{\hat{Y}} \otimes_{\mathcal{O}_{\hat{Y}}} \hat{\Omega}^{\mathrm{PD,co}}_{D_{X,\gamma}(Y)/S},$$

and we can form the associated complex

$$\mathcal{F}_{\hat{Y}} \otimes_{\mathcal{O}_{\hat{Y}}} \hat{\Omega}^{\mathrm{PD,co},\bullet}_{D_{X,\gamma}(Y)/S}.$$

As a formal consequence of the previous lemma we have a canonical isomorphism

$$\varprojlim \mathcal{F}_{Y_n} \otimes_{\mathcal{O}_{Y_n}} \Omega^{\mathrm{PD},\bullet}_{D_n/S} \xrightarrow{\sim} \mathcal{F}_{\hat{Y}} \otimes_{\mathcal{O}_{\hat{Y}}} \hat{\Omega}^{\mathrm{PD,co},\bullet}_{D_{X,\gamma}(Y)/S}. \tag{5.11}$$

This allows us to give a completed version of Theorem 5.5.

Theorem 5.9. *Let $\mathcal{F}$ be a finite crystal in $(X/S)_{\mathrm{cris}}$. Denote j for the canonical map of topological spaces $X \to \hat{Y}$. Then there exists an isomorphism*

$$\mathrm{R}\Gamma\left((X/S)_{\mathrm{cris}}, \mathcal{F}\right) \xrightarrow{\sim} \mathrm{R}\Gamma\left(X_{\mathrm{zar}}, j^{-1}\mathcal{F}_{\hat{Y}} \otimes_{\mathcal{O}_{\hat{Y}}} \hat{\Omega}^{\mathrm{PD,co},\bullet}_{D_{X,\gamma}(Y)/S}\right).$$

Proof. By [Stacks, 07MV] we have

$$\mathrm{R}\Gamma\left((X/S)_{\mathrm{cris}}, \mathcal{F}\right) = \mathrm{R}\varprojlim \mathrm{R}\Gamma(\mathrm{Cris}(X/S_n), \mathcal{F}_n).$$

Using Theorem 5.5 we have a canonical isomorphism

$$\mathrm{R}\Gamma(\mathrm{Cris}(X/S_n), \mathcal{F}_n) \xrightarrow{\sim} \mathrm{R}\Gamma\left(X_{\mathrm{zar}}, \iota_{Y_n}^{-1}\mathcal{F}_{Y_n} \otimes_{\mathcal{O}_{Y_n}} \Omega^{\mathrm{PD},\bullet}_{D_n/S_n}\right)$$

where ι_{Y_n} denotes the closed immersion $X \to Y_n$. By Theorem 5.6 the maps defining the inverse limit agree, hence we have a canonical isomorphism.

$$\varprojlim \mathrm{R}\Gamma(\mathrm{Cris}(X/S_n), \mathcal{F}_n) \xrightarrow{\sim} \varprojlim \mathrm{R}\Gamma\left(X_{\mathrm{zar}}, \iota_{Y_n}^{-1}\mathcal{F}_{Y_n} \otimes_{\mathcal{O}_{Y_n}} \Omega^{\mathrm{PD},\bullet}_{D_n/S_n}\right)$$

By [Stacks, 0D6K] we may interchange $\mathrm{R}\Gamma(X_{\mathrm{zar}}, -)$ and $\mathrm{R}\varprojlim$ to obtain an isomorphism

$$\mathrm{R}\Gamma\left((X/S)_{\mathrm{cris}}, \mathcal{F}\right) \xrightarrow{\sim} \mathrm{R}\Gamma\left(X_{\mathrm{zar}}, \mathrm{R}\varprojlim \iota_{Y_n}^{-1}\mathcal{F}_{Y_n} \otimes_{\mathcal{O}_{Y_n}} \Omega^{\mathrm{PD},\bullet}_{D_n/S_n}\right).$$

Since $\iota_{Y_n}^{-1}$ is exact, by (5.11) it suffices to show

$$\mathrm{R}\varprojlim \mathcal{F}_{Y_n} \otimes_{\mathcal{O}_{Y_n}} \Omega^{\mathrm{PD},\bullet}_{D_n/S_n} = \varprojlim \mathcal{F}_{Y_n} \otimes_{\mathcal{O}_{Y_n}} \Omega^{\mathrm{PD},\bullet}_{D_n/S_n}.$$

By Leray's acyclicity lemma [Stacks, 015E], it suffices to show

$$\mathrm{R}\varprojlim \mathcal{F}_{Y_n} \otimes_{\mathcal{O}_{Y_n}} \Omega^{\mathrm{PD},i}_{D_n/S_n} = \varprojlim \mathcal{F}_{Y_n} \otimes_{\mathcal{O}_{Y_n}} \Omega^{\mathrm{PD},i}_{D_n/S_n}$$

for all $i \in \mathbb{Z}_{\geq 0}$. Applying [Stacks, 0BKY] we may reduce to the case that Y is affine. This will follow immediately if we can show that all transition maps in the inverse system are surjective (then the system is Mittag-Leffler by [Stacks, 0596], now use [Stacks, 07KW]). Since $\mathcal{F}$ is a crystal, we have $\mathcal{F}_{Y_n} = \mathcal{F}_{Y_{n+1}} \otimes \mathcal{O}_{S_n}$, hence it suffices to show

$$\Omega^{\mathrm{PD},i}_{D_{n+1}/S_{n+1}} \to \Omega^{\mathrm{PD},i}_{D_n/S_n}$$

is surjective. We will show this in the case $i = 1$, the other cases follow directly. Let $\mathcal{D}_n = \mathcal{D}_{X,\gamma}(Y_n)$ and write

$$\Omega^{\mathrm{PD}}_{D_n/S_n} = \mathcal{D}_n \otimes_{\mathcal{O}_{Y_n}} \Omega_{Y_n/S_n}.$$

The map $\mathcal{D}_{n+1} \to \mathcal{D}_n$ is surjective by [Stacks, 07HE], and $\Omega_{Y_{n+1}/S_{n+1}} \to \Omega_{Y_n/S_n}$ is surjective by [Stacks, 00RR]. The result follows. $\square$

Corollary 5.10. *Let k be a perfect field of characteristic $p > 0$, and let $W = W(k)$ be the ring of Witt vectors of k. Let $S = \mathrm{Spec}(W)$, $S_0 = \mathrm{Spec}(k)$. Let X_0 be a scheme over S_0, and let Y be a smooth and projective scheme over S. Given a closed immersion $X_0 \to Y$, there exists an isomorphism*

$$\mathrm{H}^i_{\mathrm{cris}}(X_0/S) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR,PD}}(X_0/S; Y).$$

Furthermore, the isomorphism is functorial, that is given a diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "F"] & S \arrow[d] \\ X_0' \arrow[r] & Y' \arrow[r] & S' \end{tikzcd}$$

the diagram

$$\begin{tikzcd} \mathrm{H}^i_{\mathrm{cris}}(X_0/S) \arrow[r, "\sim"] & \mathrm{H}^i_{\mathrm{dR,PD}}(X_0/S; Y) \\ \mathrm{H}^i_{\mathrm{cris}}(X_0'/S') \arrow[r, "\sim"] \arrow[u, "f^*"] & \mathrm{H}^i_{\mathrm{dR,PD}}(X_0'/S'; Y') \arrow[u, "F^*"] \end{tikzcd}$$

commutes.

Proof. The construction of the isomorphism is obtained by setting $\mathcal{F} = \mathcal{O}_{X/S}$ in the above theorem. Functoriality follows directly from Theorem 5.6. $\square$

Corollary 5.11. *Let k be a perfect field of characteristic $p > 0$, and let $W = W(k)$ be the ring of Witt vectors of k. Let $S = \mathrm{Spec}(W)$, $S_0 = \mathrm{Spec}(k)$. Let X be a smooth and projective scheme over S and let $X_0 = X \times_S S_0$. Then there exists an isomorphism*

$$\mathrm{H}^i_{\mathrm{cris}}(X_0/S) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(X/S).$$

Proof. This follows from Corollary 3.26. $\square$

Populaire samenvatting

In de algebraïsche meetkunde is men op zoek naar het antwoord op de volgende vraag: Zij $f_1, \ldots, f_r$ polynomen in n variabelen $x_1, \ldots, x_n$. Zij k een lichaam (bijvoorbeeld $k = \mathbb{R}$ of $k = \mathbb{C}$), en zij

$$Z_k(f_1, \ldots, f_r) = \{(a_1, \ldots, a_n) \in k^n \mid f_j(a_1, \ldots, a_n) = 0, \ \forall j \in \{1, \ldots, r\}\}$$

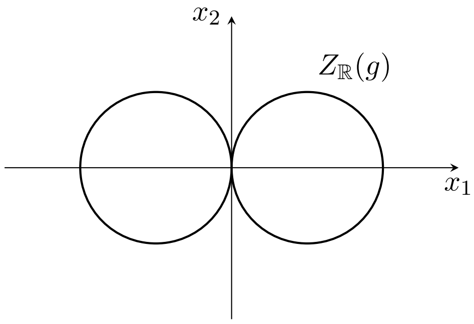
Wat kunnen we zeggen over $Z_k(f_1, \ldots, f_r)$?

Als bijvoorbeeld $r = 1$ en $f_1(x_1, x_2) = x_1^2 + x_2^2 - 1$, dan is $Z_{\mathbb{R}}(f)$ de eenheidscirkel in de $\mathbb{R}^2$. Als $k = \mathbb{R}$, dan heeft deze verzameling veel meetkundige eigenschappen. Een vraag die je bijvoorbeeld kunt stellen is "hoeveel gaten zitten er in deze variëteit". Dit is natuurlijk niet een heel precieze vraag, maar in veel gevallen is er wel een goed antwoord op te geven. Voor de cirkel is het evident dat er alleen één 1-dimensionaal gat in zit.

Voor het polynoom

$$g(x_1, x_2) = ((x_1 - 1)^2 + x_2^2 - 1)((x_1 + 1)^2 + x_2^2 - 1)$$

is de verzameling $Z_{\mathbb{R}}(g)$ een figuur 8:



en intuïtief zou je kunnen zeggen dat er nu twee ééndimensionale gaten zijn. Om deze vraag precies te stellen en te kunnen beantwoorden geef je $Z_{\mathbb{R}}(g)$ de structuur van een *topologische ruimte*. Het 'gaten tellen' kan je dan vervangen door het uitrekenen van de *cohomologiegroepen* van $Z_{\mathbb{R}}(g)$. Dit wordt in meer algemeenheid bestudeert in de algebraïsche topologie.

Maar er is ook een heel ander aspect aan deze theorie. Als k een eindig lichaam is (bijvoorbeeld $k = \mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$, met p een priemgetal), dan is $\mathbb{Z}_k(f_1, \ldots, f_r)$ een eindige verzameling. Het is immers een deelverzameling van de eindige verzameling

k^n. Een natuurlijke vraag om stellen is dan: Wat kunnen we zeggen over de grootte $\#Z_k(f_1, \ldots, f_r)$?

Stel bijvoorbeeld $p = 7$ en $k = \mathbb{Z}/7\mathbb{Z}$. Beschouw weer de verzameling $Z_k(f_1)$ met $f_1(x_1, x_2) = x_1^2 + x_2^2 - 1$. Dan is

$$Z_k(f_1) = \{(0,1), (1,0), (0,6), (6,0), (2,5), (5,2), (5,5), (2,2)\}$$

en $\#Z_k(f_1) = 8$.

Een diep resultaat in de algebraïsche meetkunde is dat er een verband bestaat tussen de grootte van $\mathbb{Z}_{\mathbb{F}_p}(f_1, \ldots, f_r)$ (de algebraïsche informatie), en de cohomologiegroepen van $\mathbb{Z}_{\mathbb{C}}(f_1, \ldots, f_r)$ (de meetkundige informatie). De precieze formulering van dit verband is beschreven in de *Weil-vermoedens.*

De Weil-vermoedens zijn inmiddels bewezen. Een belangrijk ingrediënt in het bewijs is het bestaan van een *Weilcohomologie-theorie.* Dit is een cohomologie-theorie die werkt over de eindige lichamen $\mathbb{F}_p$ in plaats van over $\mathbb{C}$. In andere woorden, dit is een manier om 'gaten te tellen' in ruimtes met eindig veel punten, waar je geen zinnig plaatje meer kunt tekenen.

Er bestaan twee belangrijke Weilcohomologie-theorieën: Étale cohomolgie en kristallijne cohomologie. In deze scriptie worden de beginselen van de theorie van kristalllijne cohomologie behandeld.

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