The structure of strategy-proof rules

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Motivation

- Social choice theory studies the aggregation of preferences.

  The Gibbard-Satterthwaite (GS) impossibility puts limits to this endeavor:

$$\left.\begin{array}{r}\text{unrestricted preference domain}\\ \text{strategy-proofness}\\ \text{size of range} \neq 2\\ \text{non-dictatorial}\end{array}\right\} \Rightarrow \emptyset$$

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- We generalize this impossibility to the broader set of non-conditional domains.

- We develop a two-step procedure that serves as a guide for determining the strategy-proof rules on any strict preference domain.

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  Let P_i be the strict preference relation associated with R_i.
- The universal preference domain $\mathcal{R}$ contains all preferences R_i over X.

  $\mathcal{R}_i \subseteq \mathcal{R}$ is a preference domain for agent i.

  $\mathcal{D} \equiv \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_n$ is a domain.

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  A preference profile $R = (R_1, \ldots, R_n) \in \mathcal{D}$ is a list of individual preferences.

- A social choice rule $f : \mathcal{D} \to X$ selects for each R an alternative $f(R) \in X$.

  The range of f is denoted by $r(f)$.

GS Impossibility

1. Strategy-proofness (SP)

   The rule f is manipulable by agent $i \in N$ if there is a preference profile $R \in \mathcal{D}$ and a preference $R'_i \in \mathcal{R}_i$ such that $f(R'_i, R_{-i}) \, P_i \, f(R)$.

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2. Non-dictatorial (ND)

   The rule f is dictatorial if there is an agent $i \in N$ such that for all $R \in \mathcal{D}$ and all $x \in r(f)$, $f(R) \, R_i \, x$.

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   $|r(f)| \neq 2$.

Theorem (Gibbard-Satterthwaite)

If $\mathcal{R}_i = \mathcal{R}$ for all $i \in N$, there is no social choice rule $f : \mathcal{D} \to X$ with $|r(f)| \neq 2$ that is SP and ND.

Preference domain classification

- Set of all ordered pairs of distinct alternatives: $X^* = \{(x, y) \in X^2 \mid x \neq y\}$.

  Ordered pairs of $\mathcal{R}_i$: $S(\mathcal{R}_i) = \{(x, y) \in X^* \mid x \, P_i \, y \text{ for all } R_i \in \mathcal{R}_i\}$.

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- Conditional preference domain: $\mathcal{R}_i$ is not non-conditional.

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Theorem

If $\mathcal{R}_i$ is non-conditional for all $i \in N$, there is no social choice rule $f : \mathcal{D} \to X$ with $|r(f)| \neq 2$ that is SP and ND.

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- **Induction Step**: Let $(x, y) \in S(\mathcal{R}_i)$ and $S(\mathcal{R}'_i) = S(\mathcal{R}_i) \setminus (x, y)$. We have to establish the GS impossibility on $\mathcal{D}' = \mathcal{D}_{-i} \times \mathcal{R}'_i$.

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  Proof by contradiction: There is f on $\mathcal{D}'$ that is SP, ND, and has $|r(f)| \neq 2$.

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  The restriction of $f : \mathcal{D}' \to X$ to $\mathcal{D}$ inherits SP.
  So, the restriction of $f : \mathcal{D}' \to X$ to $\mathcal{D}$ has $|r(f)| = 2$ or is a dictatorial.

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- Given $\mathcal{R}_i$, a conditional preference domain restriction consists of an antecedent $\{(x_i, y_i)\}_{i=1}^{k}$ and a conclusion $(\bar{x}, \bar{y})$ such that all preferences that satisfy the antecedent but not the conclusion are removed from $\mathcal{R}_i$.

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Apply $x \, P \, y \Rightarrow y \, P \, z$ to $\mathcal{R}$

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Algorithm

Any $\mathcal{R}_i \subseteq \mathcal{R}$ can be from $\mathcal{R}$ by applying first a sequence of non-conditional and then a sequence of conditional restrictions. $\mathcal{R}_i$ is non-conditional if and only if non-conditional restrictions are applied exclusively.

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1. Consider the non-ordered pair $\{x, y\}$. If $\mathcal{R}'$ satisfies either the non-conditional restriction (x, y) or (y, x), apply it to $\mathcal{R}_0$. Since $\mathcal{R}'$ does not satisfy either, set $\mathcal{R}_1 = \mathcal{R}_0$. Since $\mathcal{R}_1 \neq \mathcal{R}'$, go to the next step.

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2. Consider $\{x, z\}$. If $\mathcal{R}'$ satisfies either the non-conditional restriction (x, z) or (z, x), apply it to $\mathcal{R}_1$. Since $\mathcal{R}'$ satisfies the non-conditional restriction (x, z),

$$\mathcal{R}_2 = \mathcal{R}_1 \setminus \{yzx, zxy, zyx\} = \{xyz, xzy, yxz\}.$$

   Since $\mathcal{R}_2 \neq \mathcal{R}'$, go to the next step.

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2. Consider $\{x, z\}$. If $\mathcal{R}'$ satisfies either the non-conditional restriction (x, z) or (z, x), apply it to $\mathcal{R}_1$. Since $\mathcal{R}'$ satisfies the non-conditional restriction (x, z),

   $$\mathcal{R}_2 = \mathcal{R}_1 \setminus \{yzx, zxy, zyx\} = \{xyz, xzy, yxz\}.$$

   Since $\mathcal{R}_2 \neq \mathcal{R}'$, go to the next step.
3. Consider $\{y, z\}$. Then, $\mathcal{R}_3 = \mathcal{R}_2 \neq \mathcal{R}'$. Thus, go to the next step.

Preference domain classification III

- Check whether the preference domain $\mathcal{R}' = \{xzy, yxz\}$ is non-conditional.

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Since $\mathcal{R}_2 \neq \mathcal{R}'$, go to the next step.
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4. Pick any preference $R_i \in \mathcal{R}_3 \setminus \mathcal{R}'$. Here, $R_i = xyz$. Apply the conditional restriction $x\, P_i\, y \Rightarrow z\, P_i\, y$. Then, $\mathcal{R}_4 = \mathcal{R}_3 \setminus R_i = \mathcal{R}'$. The algorithm stops.

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- Hence, $\mathcal{R}'$ is conditional. It is defined by one non-conditional and one conditional restriction.

SP rules on conditional domains

- Single-peaked preferences on the real line with $x < y < z$:

$$x\,P\,y \Rightarrow y\,P\,z$$

P^1	P^3	P^4	P^6
x	y	y	z
y	x	z	y
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- $S(\{R^1\}) = \{(x, y), (x, z), (y, z)\}$ and $S(\{R^3, R^4, R^6\}) = \{(y, x)\}$.

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Remark

A conditional preference domain becomes non-conditional if the truthfulness of the antecedents of the conditional restrictions that define the preference domain is established.

SP rules on conditional domains

Two-step procedure:

1. For each $i \in N$, assess the truthfulness of the antecedents of the conditional restrictions that define the preference domain.

2. For each combination of non-conditional domains that arises from the first step, apply a subrule that is either dictatorial or strategy-proof of range 2.

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- It is not a complete characterization of all SP rules because the second-step subrules are a function of the information provided in the first step.
- Construct combinations of second-step subrules that are consistent with truthful preference revelation in the first step.

Illustrative example

- Agent 1: $x \, P_1 \, y \Rightarrow y \, P_1 \, z$, that is, $\mathcal{R}_1 = \{xyz, zyx, yzx, yxz\}$.

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xP_1y and xP_2y		xP_1y and yP_2x		yP_1x and xP_2y				yP_1x and yP_2x			
$\mathcal{R}_1'$	$\mathcal{R}_2'$	$\mathcal{R}_1'$	$\mathcal{R}_2'$	$\mathcal{R}_1'$			$\mathcal{R}_2'$	$\mathcal{R}_1'$			$\mathcal{R}_2'$
x	x	x	y	z	y	y	x	z	y	y	y
y	z	y	x	y	z	x	z	y	z	x	x
z	y	z	z	x	x	z	y	x	x	z	z

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xP_1y and xP_2y		xP_1y and yP_2x		yP_1x and xP_2y				yP_1x and yP_2x			
$\mathcal{R}_1'$	$\mathcal{R}_2'$	$\mathcal{R}_1'$	$\mathcal{R}_2'$		$\mathcal{R}_1'$		$\mathcal{R}_2'$		$\mathcal{R}_1'$		$\mathcal{R}_2'$
x	x	x	y	z	y	y	x	z	y	y	y
y	z	y	x	y	z	x	z	y	z	x	x
z	y	z	z	x	x	z	y	x	x	z	z

- Cases 1+2: 3 dictatorial subrules of range 1.

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- Agent 1: $x \, P_1 \, y \Rightarrow y \, P_1 \, z$, that is, $\mathcal{R}_1 = \{xyz, zyx, yzx, yxz\}$.
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xP_1y and xP_2y		xP_1y and yP_2x		yP_1x and xP_2y				yP_1x and yP_2x			
$\mathcal{R}_1'$	$\mathcal{R}_2'$	$\mathcal{R}_1'$	$\mathcal{R}_2'$	$\mathcal{R}_1'$			$\mathcal{R}_2'$	$\mathcal{R}_1'$			$\mathcal{R}_2'$
x	x	x	y	z	y	y	x	z	y	y	y
y	z	y	x	y	z	x	z	y	z	x	x
z	y	z	z	x	x	z	y	x	x	z	z

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- Cases 3+4: 3 dictatorial subrules of range 1 and 2 SP rules of range 2, the range being $\{x, z\}$ or $\{y, z\}$.

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xP_1y and xP_2y		xP_1y and yP_2x		yP_1x and xP_2y				yP_1x and yP_2x			
$\mathcal{R}_1'$	$\mathcal{R}_2'$	$\mathcal{R}_1'$	$\mathcal{R}_2'$	$\mathcal{R}_1'$			$\mathcal{R}_2'$	$\mathcal{R}_1'$			$\mathcal{R}_2'$
x	x	x	y	z	y	y	x	z	y	y	y
y	z	y	x	y	z	x	z	y	z	x	x
z	y	z	z	x	x	z	y	x	x	z	z

- Cases 1+2: 3 dictatorial subrules of range 1.
- Cases 3+4: 3 dictatorial subrules of range 1 and 2 SP rules of range 2, the range being $\{x, z\}$ or $\{y, z\}$.
- In total, there $3 \times 3 \times 5 \times 5 = 225$ possible combinations of SP subrules.

  We have excluded 287 rules or 56% of all rules.

Conclusion

Our paper contributes to the existing literature on SP in three ways.

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- Most importantly, the two-step procedure serves as a guide for determining the structure of SP rules in conditional domains.

Applications:

- Alcalde-Unzu and Vorsatz (2018)
- Alcalde-Unzu, Gallo, and Vorsatz (2023).

Example: Alcalde-Unzu and Vorsatz (2018)

- Alternatives are real numbers with $x < y < z$.

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- Agent 1 is located at x, agent 2 at y, and agent 3 at z (publicly known).

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- Agent 1 is located at x, agent 2 at y, and agent 3 at z (publicly known).
- Each agent has single-peaked or single-dipped preferences (private info).

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- Preference domains are conditional.

Agents 1 and 3

P^1	P^6
x	z
y	y
z	x

Agent 2

P^3	P^4	P^2	P^5
y	y	x	z
x	z	z	x
z	x	y	y

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1. Ask each agent whether she has single-peaked or single-dipped preferences.

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Agents 1 and 3			Agent 2			
P^1	P^6		P^3	P^4	P^2	P^5
x	z		y	y	x	z
y	y		x	z	z	x
z	x		z	x	y	y

1. Ask each agent whether she has single-peaked or single-dipped preferences.
2. Apply our impossibility result. For example,
   a. If all agents have single-dipped preferences: majority voting between x and z.
   b. If only agent 2 has single-peaked preferences $\Rightarrow y$.
   c. If agents 1 and 3 have single-peaked preferences $\Rightarrow y$.
   d. If agent 1 (but not agent 3) has single-peaked preferences $\Rightarrow x$.
   e. If agent 3 (but not agent 1) has single-peaked preferences $\Rightarrow z$.

Example: Alcalde-Unzu, Gallo, and Vorsatz (2023)

- Alternatives are numbers on the real line.
- Each agent has single-peaked or single-dipped preferences (public info).
- The location of an agent's peak/dip is private info.
- Preference domains are conditional.
- Two-step procedure.
  1. Ask the single-peaked agents about the location of their peak. For each profile of reported peaks at most two alternatives are preselected.
  2. All agents vote on the two preselected alternatives.

We obtain a closed-form solution that generalizes the median voter schemes (all agents have single-peaked preferences) and voting by collections of left-decisive sets (all agents have single-dipped preferences)

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