Monotone Randomized Apportionment

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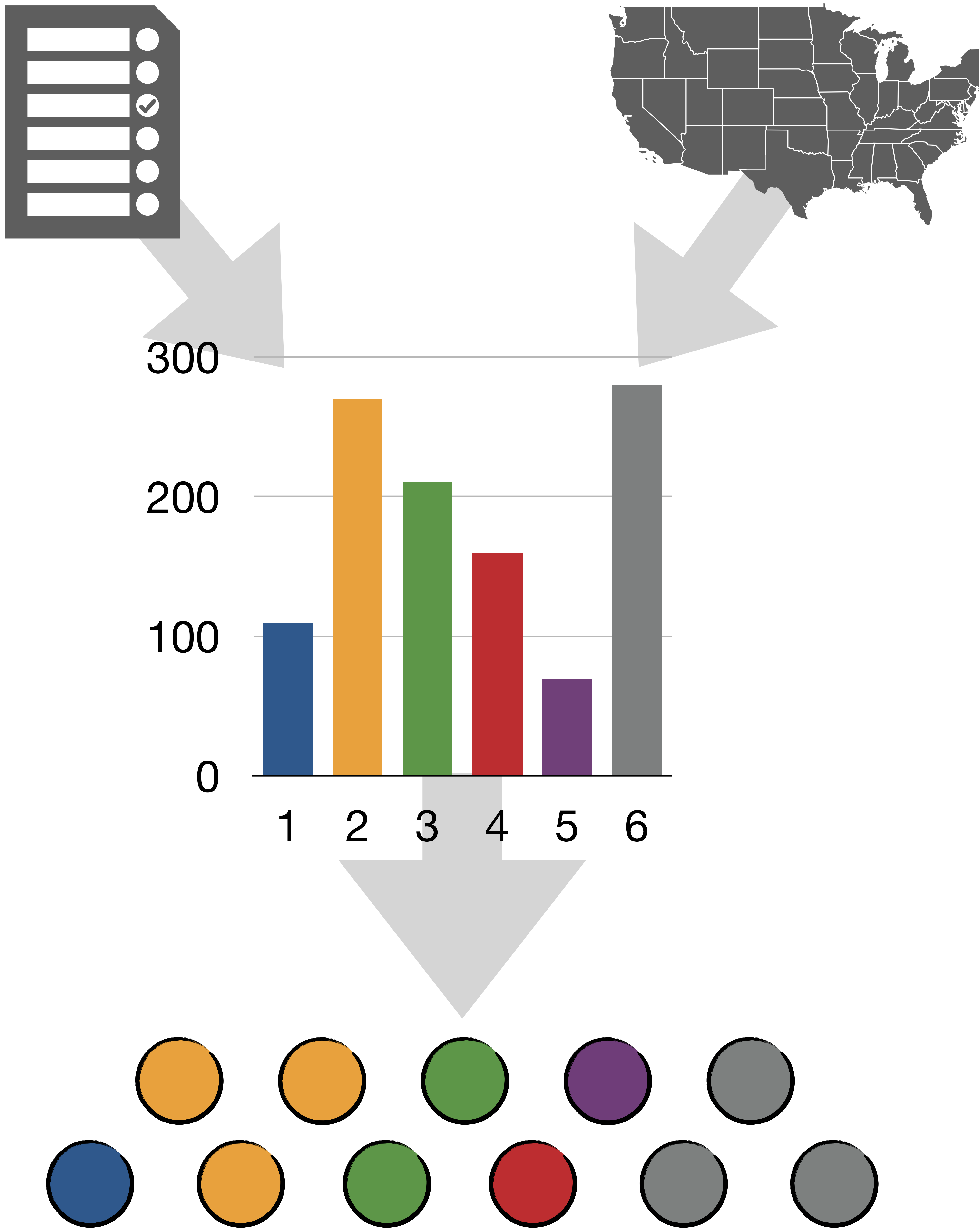
Apportionment

Let n be the number of **parties**.

Input: **vote count** vector $\vec{v} \in \mathbb{R}^n$, **house size** h

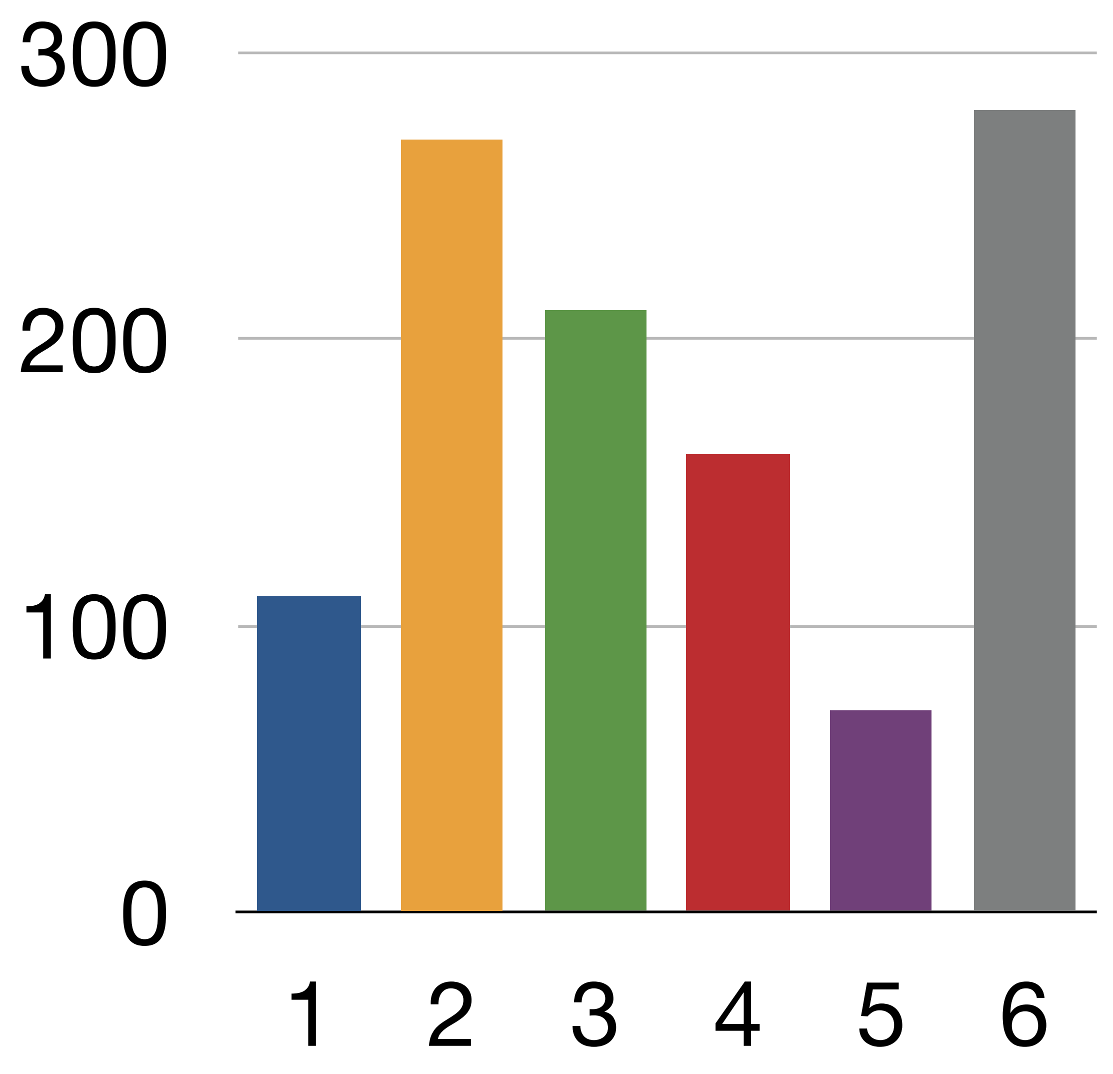
Output: **allocation** vector $\vec{a} \in \mathbb{N}^n$ summing to h



Quota

The **quota** of party $i \in [n]$ is $q_i = \frac{v_i}{\sum_{j \in [n]} v_j} h$.

An apportionment rule satisfies the **quota** axiom, if $\lfloor q_i \rfloor \leq a_i \leq \lceil q_i \rceil$ holds for all parties.



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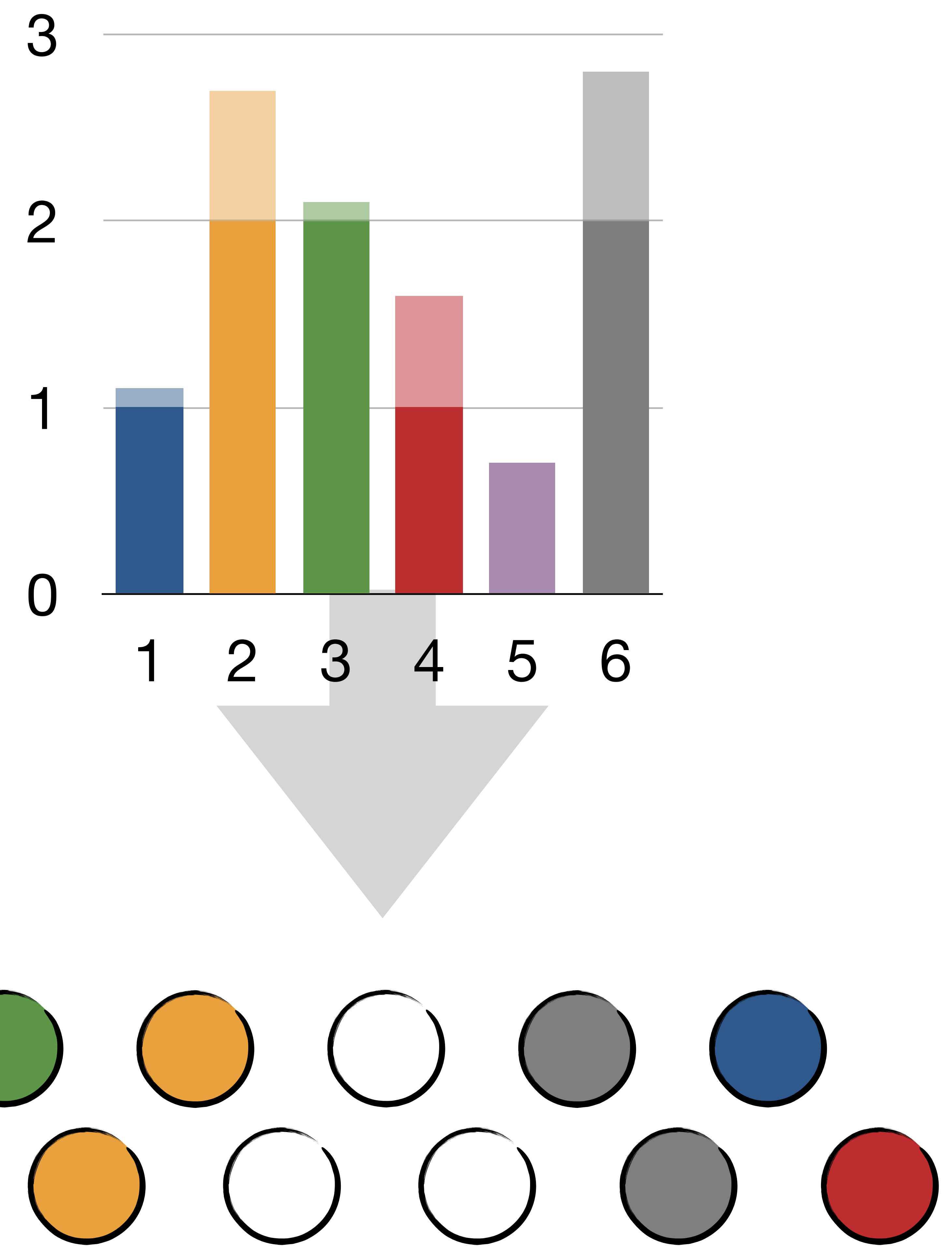
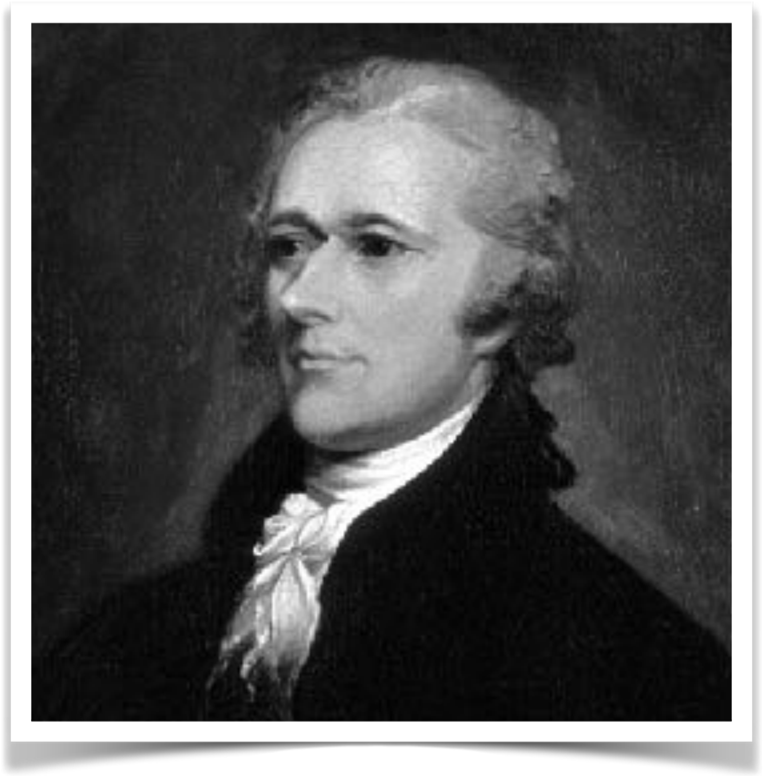
Hamilton's method: First allocate $\lfloor q_i \rfloor$ to every party. Then, allocate remaining seats by largest residues, i.e., $q_i - \lfloor q_i \rfloor$.





Alexander Hamilton

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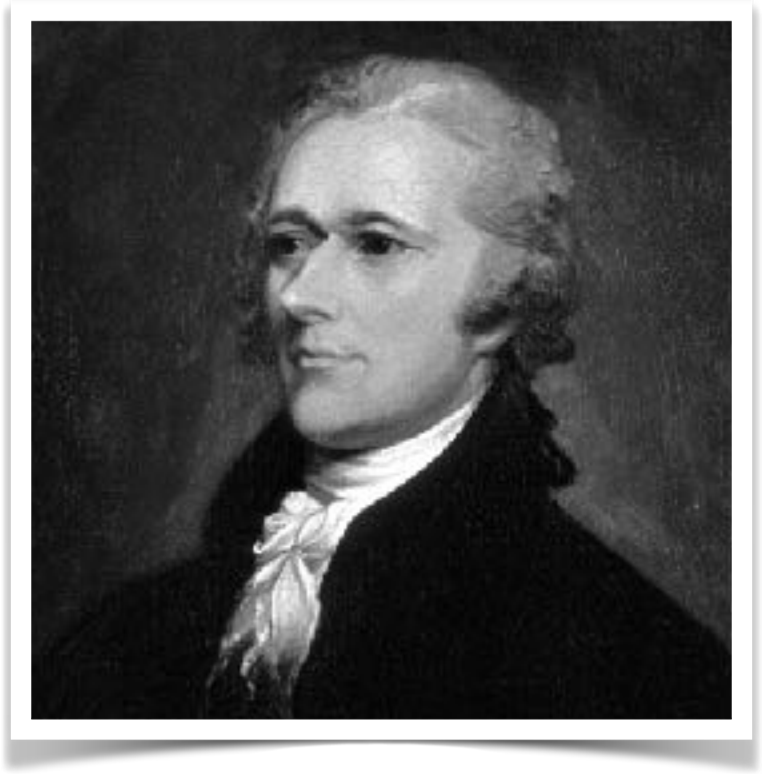

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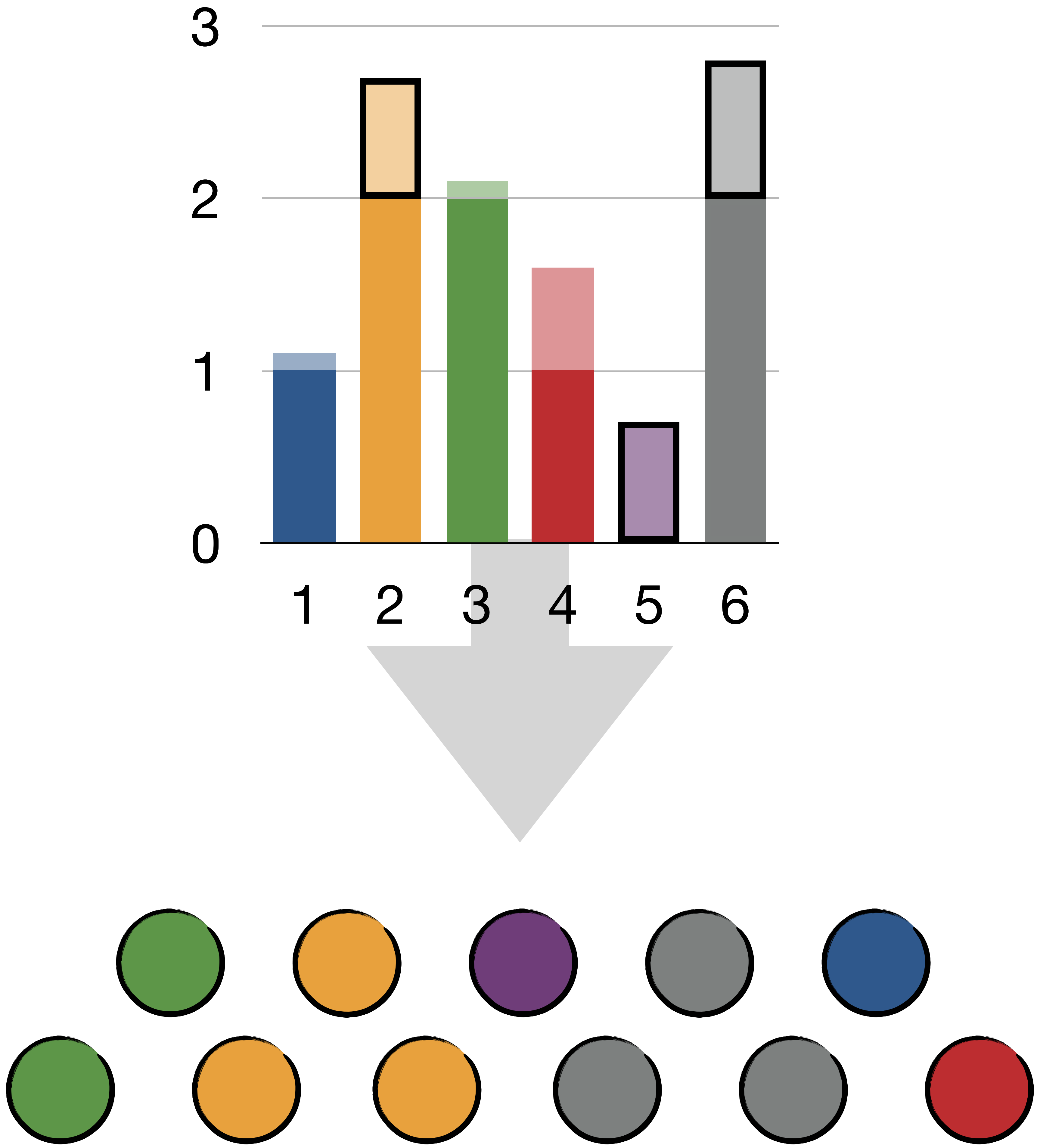
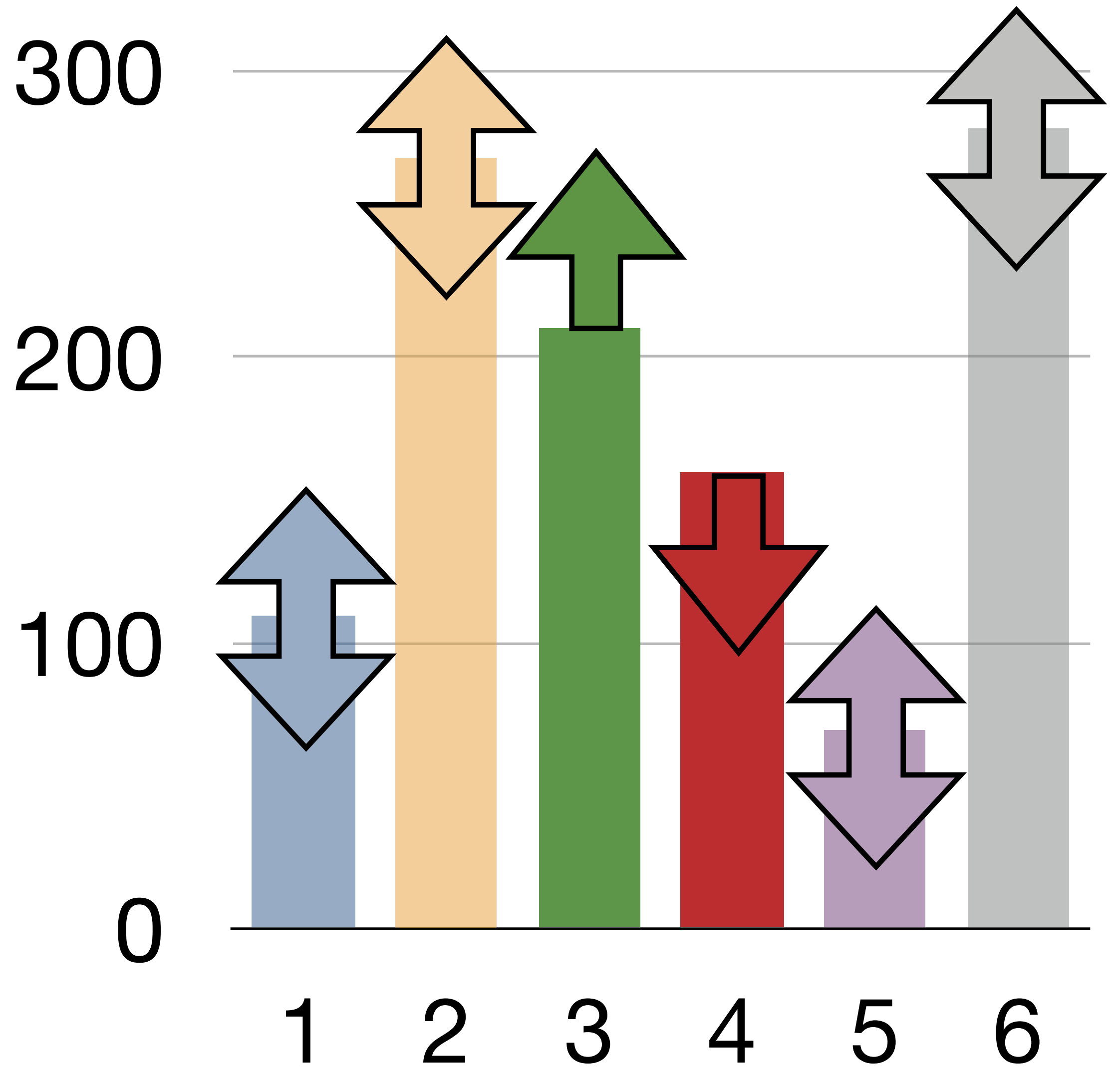


Population Monotonicity

An apportionment rule is **population monotone** if for every vote count vectors v and v' with

- $v_i' > v_i$ and $v_j' < v_j$ it **does not** hold that
- $a_i' < a_i$ and $a_j' > a_j$.



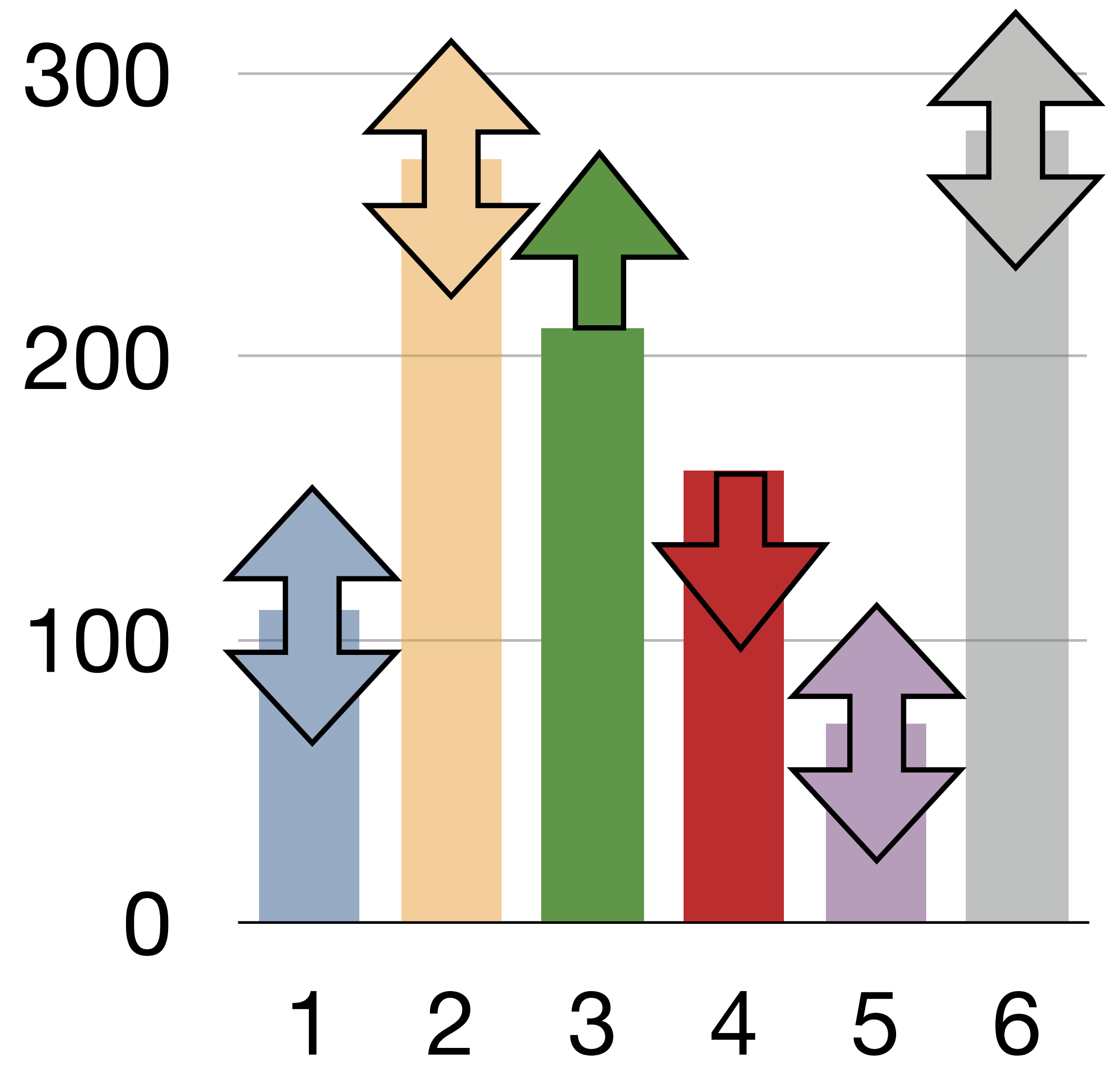
<u>**Impossibility**</u> **(Balinski and Young, 1982):**
There exists no apportionment rule that satisfies **quota** and is **population monotone**.

Randomized Apportionment

[Grimmett 04]

Goal: randomized apportionment rule satisfying

- **ex-ante proportionality**, i.e., $\mathbb{E}[a_i] = q_i$
- **ex-post quota**, i.e., $\lfloor q_i \rfloor \leq a_i \leq \lceil q_i \rceil$



<u>Observation:</u> An apportionment rule satisfying **ex-ante proportionality** also satisfies **ex-ante population monotonicity**.

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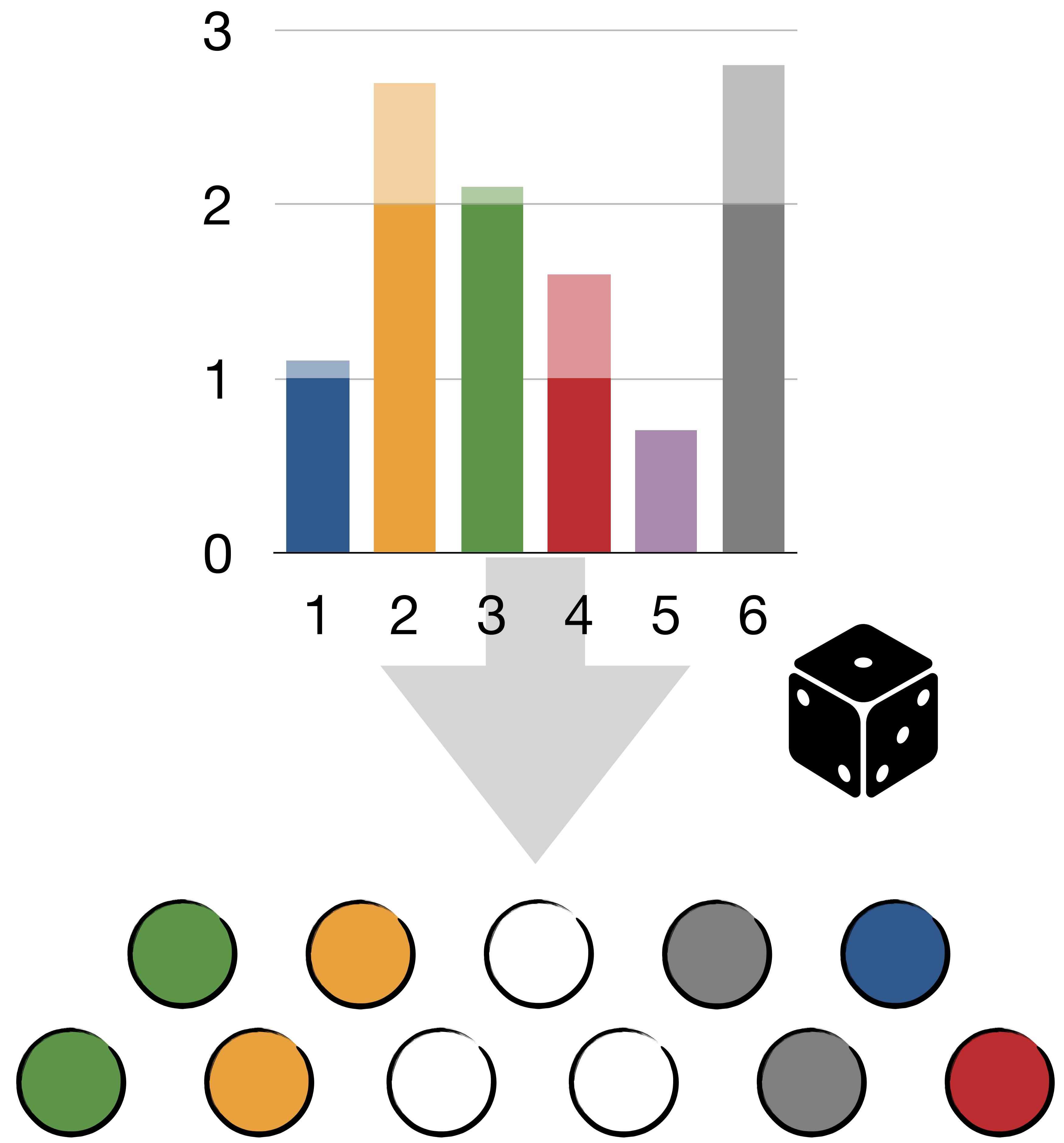
Idea: Give every party $\lfloor q_i \rfloor$ seats and one additional seat with **probability** $p_i = q_i - \lfloor q_i \rfloor$.



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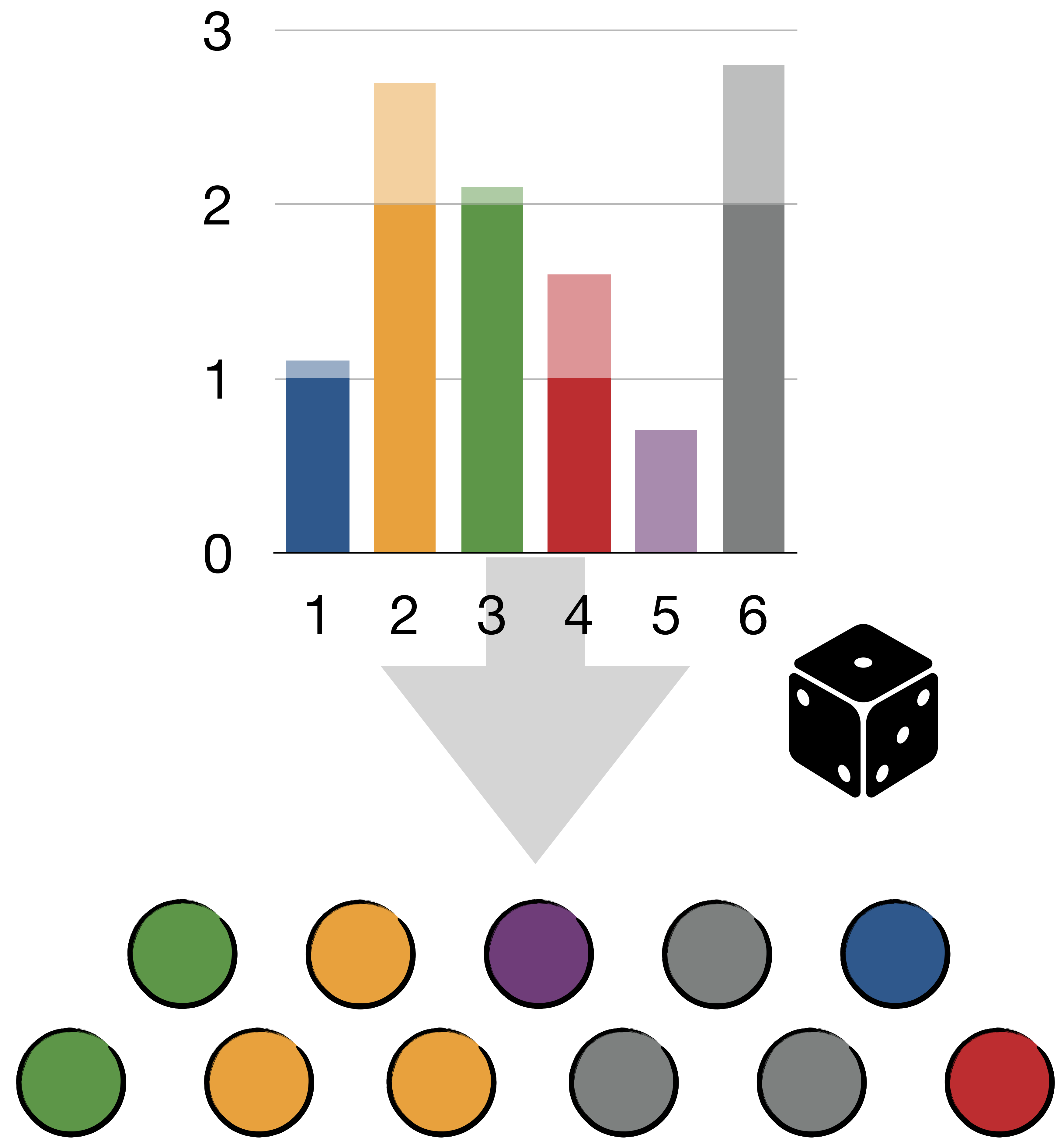

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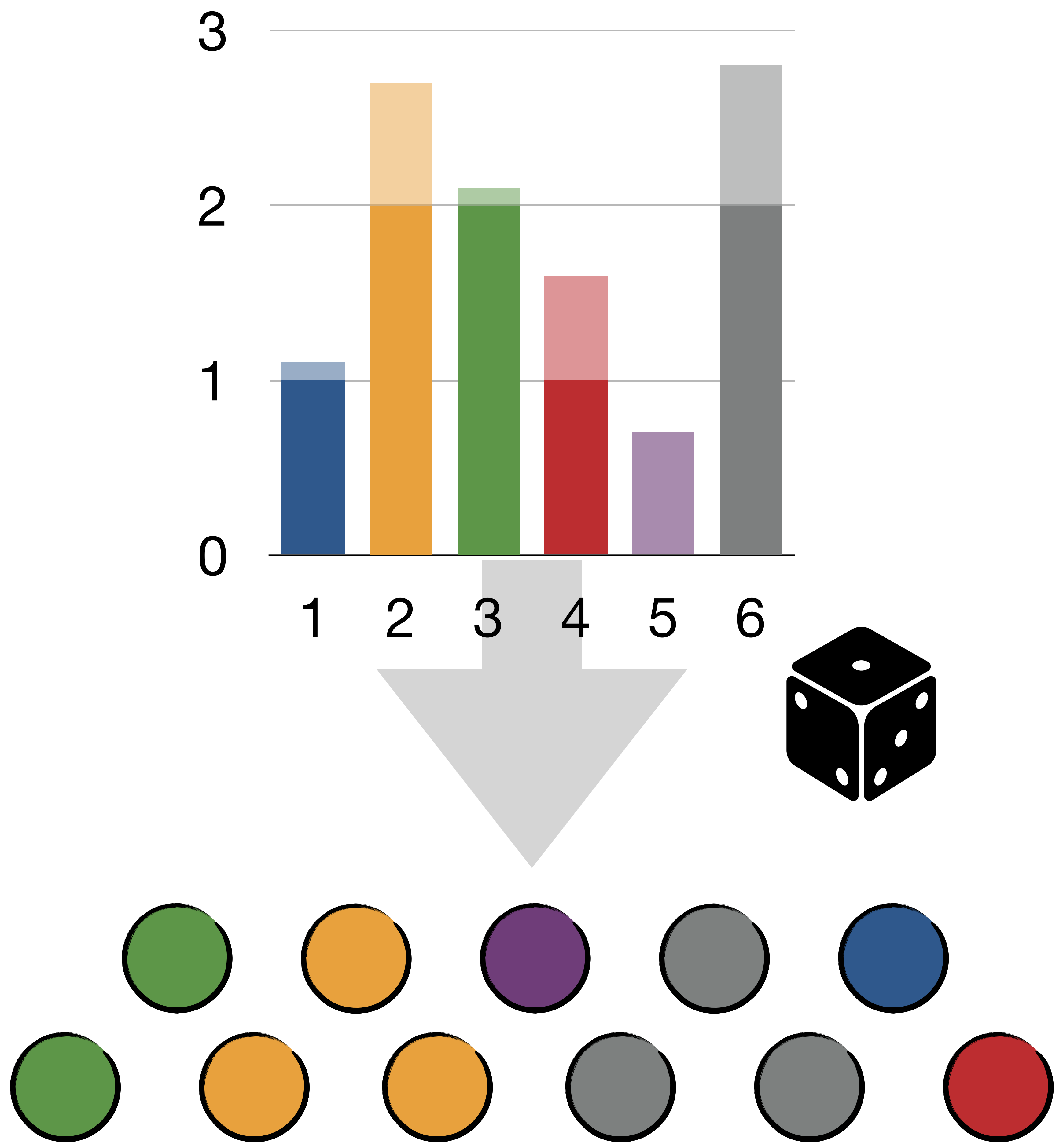
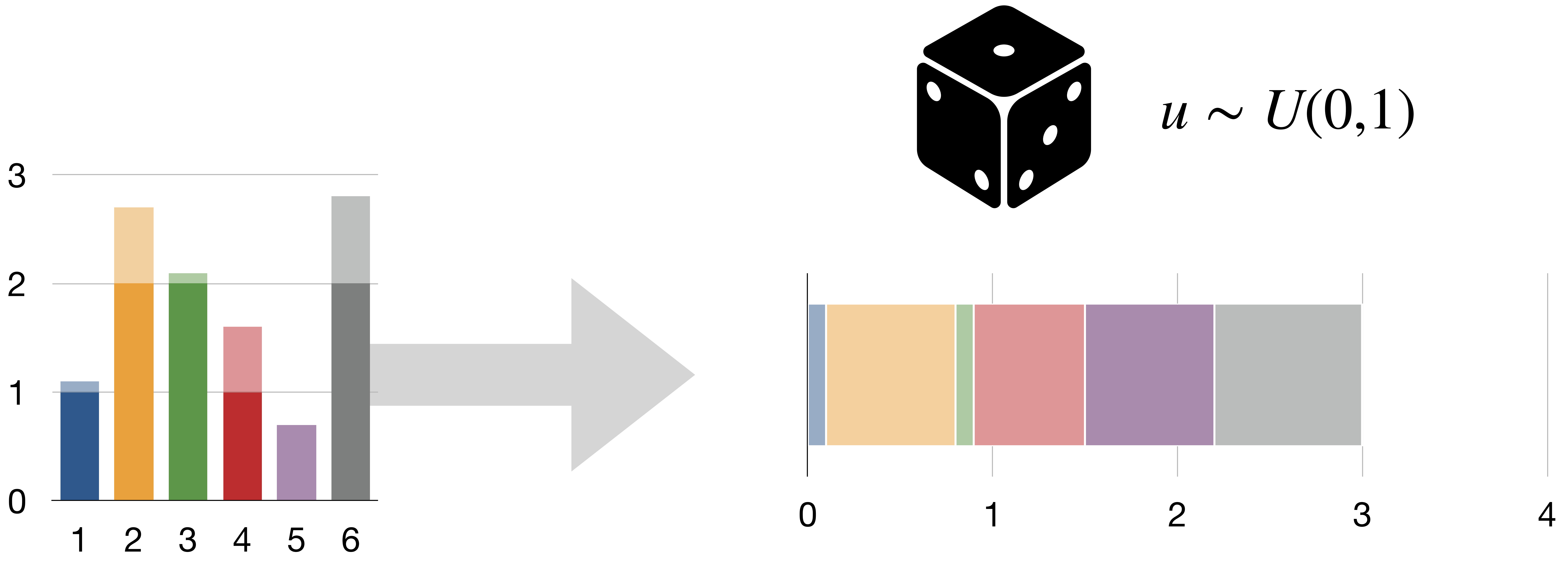
A **rounding rule** maps residues $\overrightarrow{p} \in [0,1)^n$ to a **random set** $S \subset [n]$ of size $k := \sum_{i \in [n]} p_i$ such that:

$$\mathbb{P}[i \in S] = p_i$$



Grimmett's Rounding Rule

[Grimmett 04]



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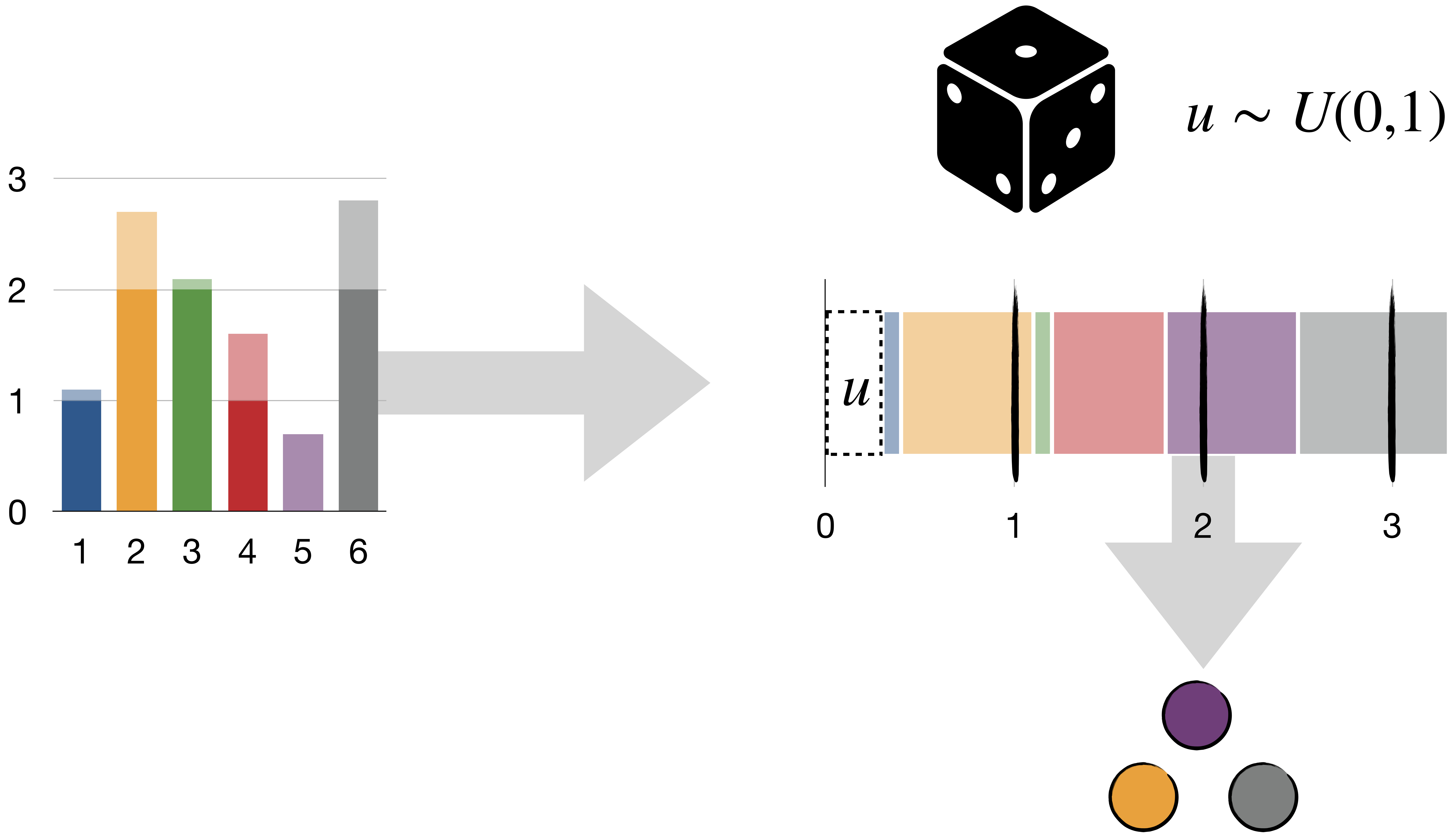
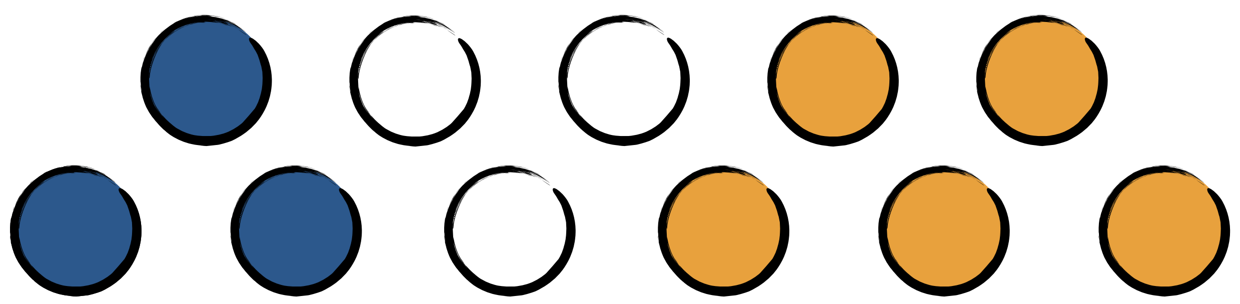
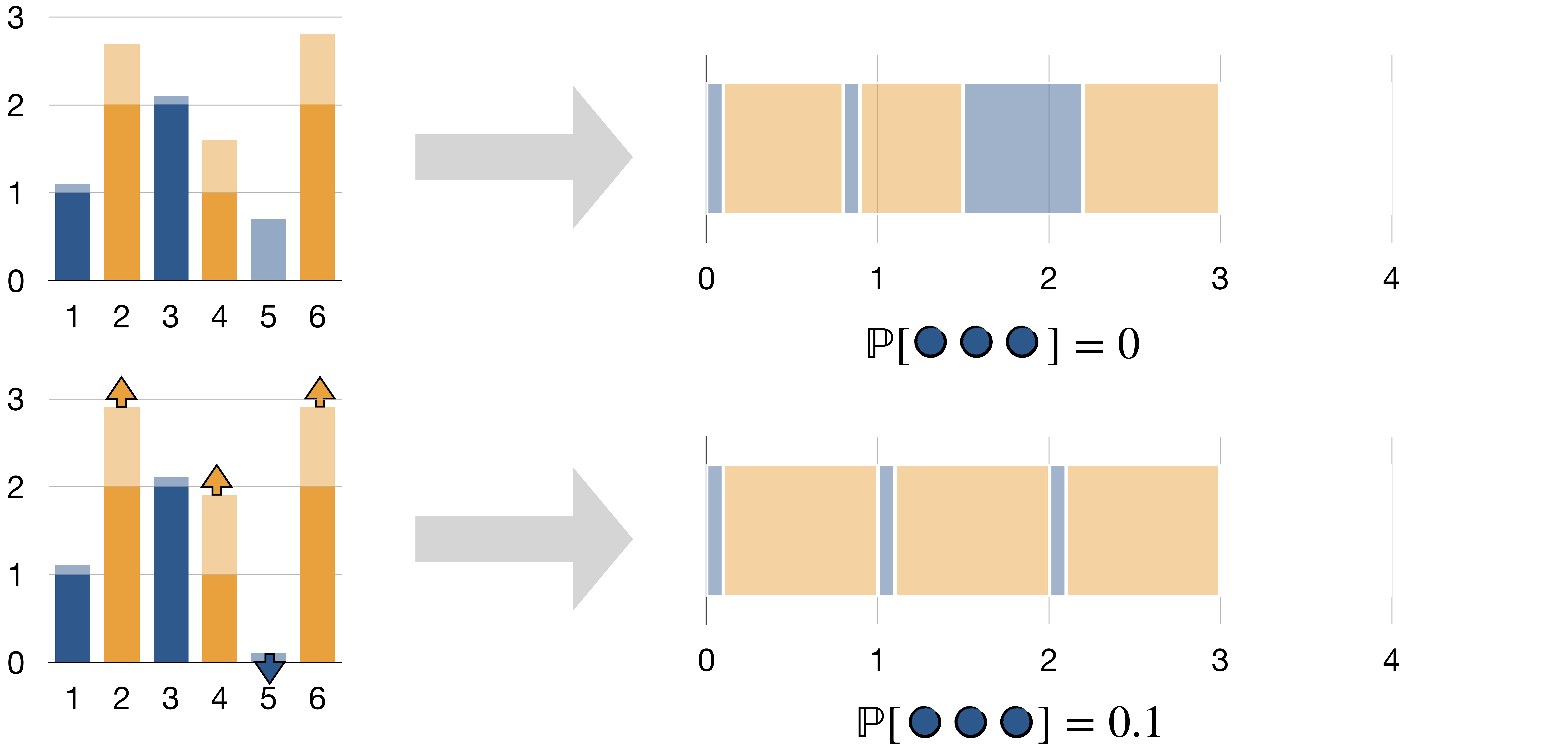


A New Apportionment Paradox





Monotone
Rounding Rules

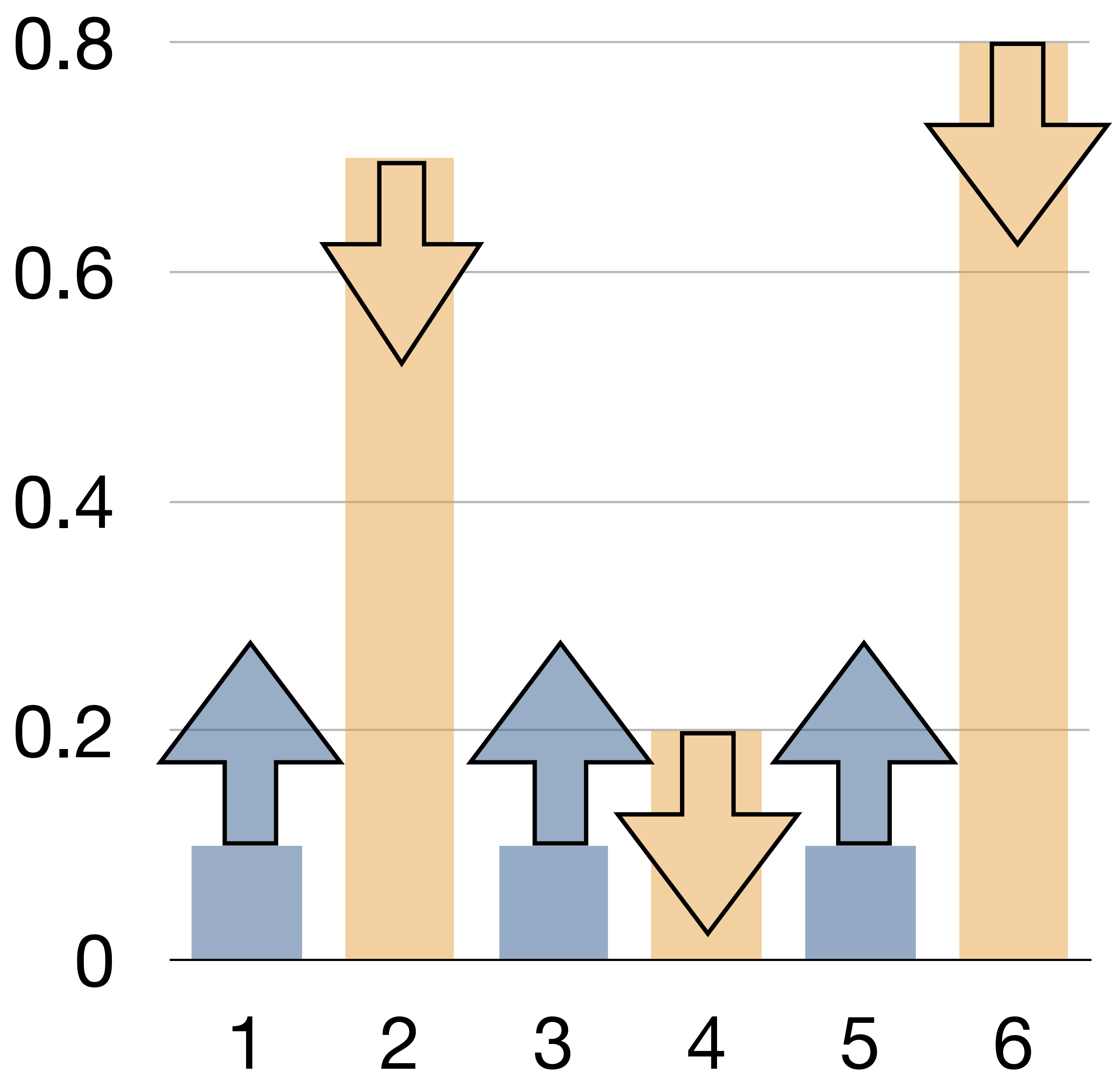
Selection Monotonicity

Let $\overrightarrow{p}$ and $\overrightarrow{p}'$ be two residue vectors summing to k and T be a coalition of k parties such that

- $p_i' \geq p_i$ for $i \in T$, and
- $p_i' \leq p_i$ for $i \notin T$.

A rounding rule satisfies **selection monotonicity** if

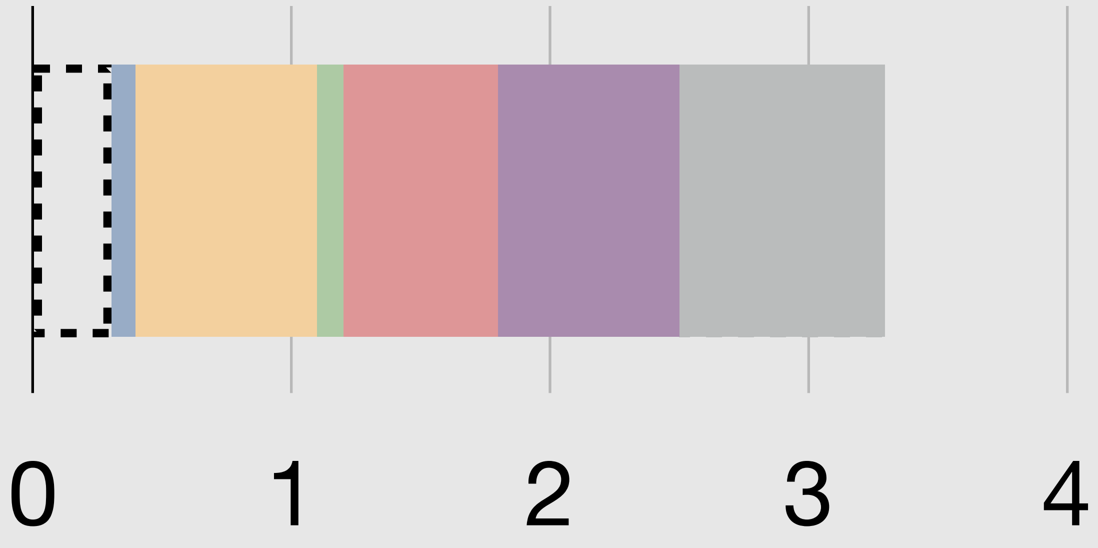
$$\mathbb{P}[S' = T] \geq \mathbb{P}[S = T] \, .$$



If a k-sized coalition gains residues, their ***joint selection probability*** *may not decrease.*

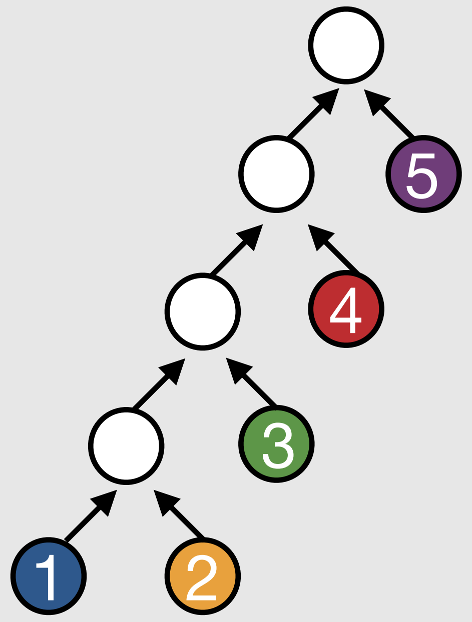
Rounding Rules Violating Selection Monotonicity

Grimmett's
(systematic rounding)



[Grimmett 04, Madow 49]

Pipage
(pivotal method)



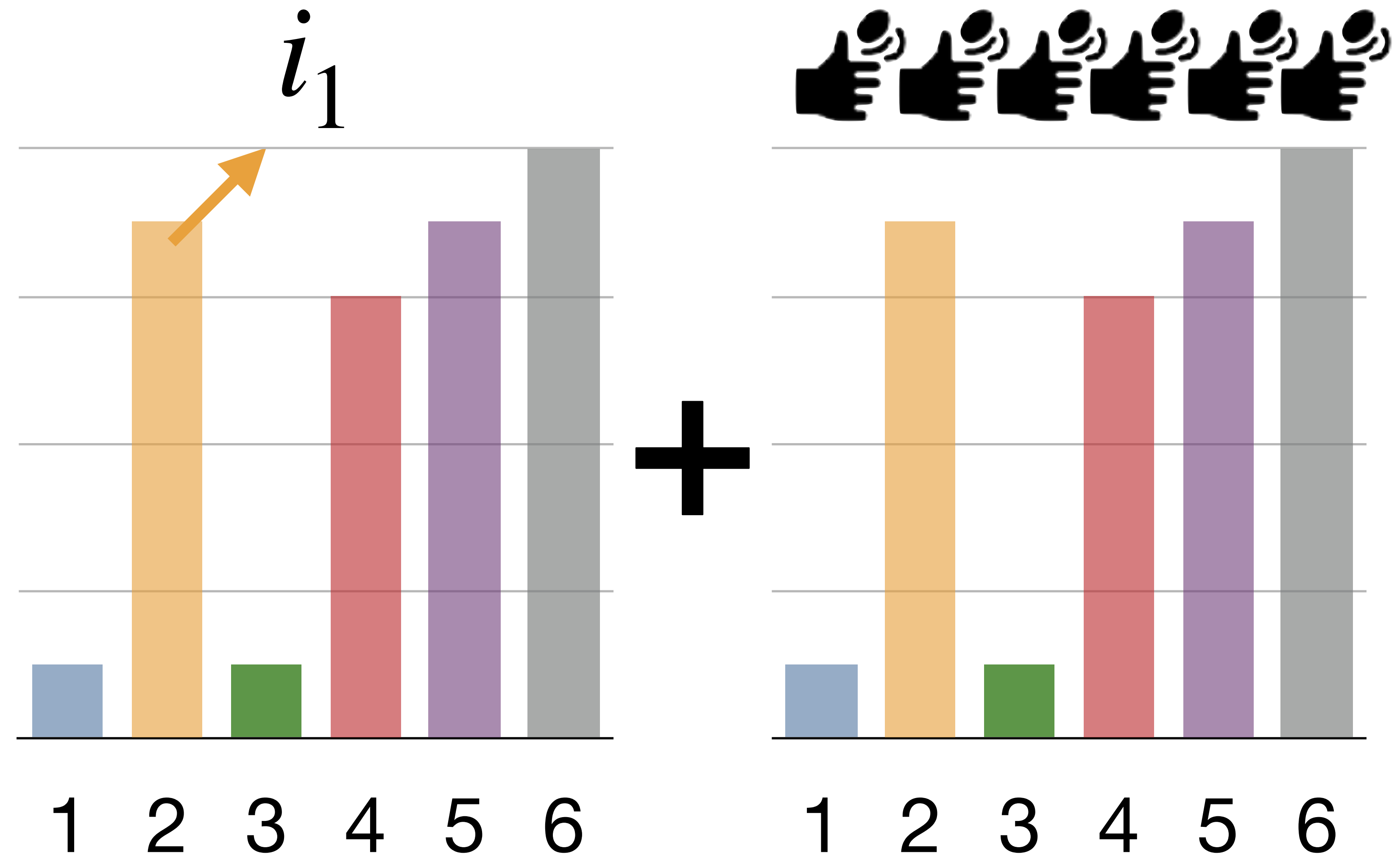
[Deville/Tilee 98, Srinivasan 01]

Conditional Poisson
(maximum entropy)

$$\Pr[S = T] = \prod_{i \in T} \pi_i$$

[Chen/Dempster/Liu 94]

Sampford Sampling



1. Sample i_1 from $[n]$ with probability $\propto p_i$
2. For every $i \in [n]$, perform Bernoulli trial with success probability p_i
3. If we observed $k-1$ **successes** and **a failure for** i_1 return. Else, start over.

$$f(A) = \sum_{i \in A} (1-p_i) \prod_{j \in A} p_j \prod_{j \notin A} (1-p_j) \qquad \Pr[S = T] = \frac{f(T)}{\sum_{A \in \binom{[n]}{k}} f(A)}$$

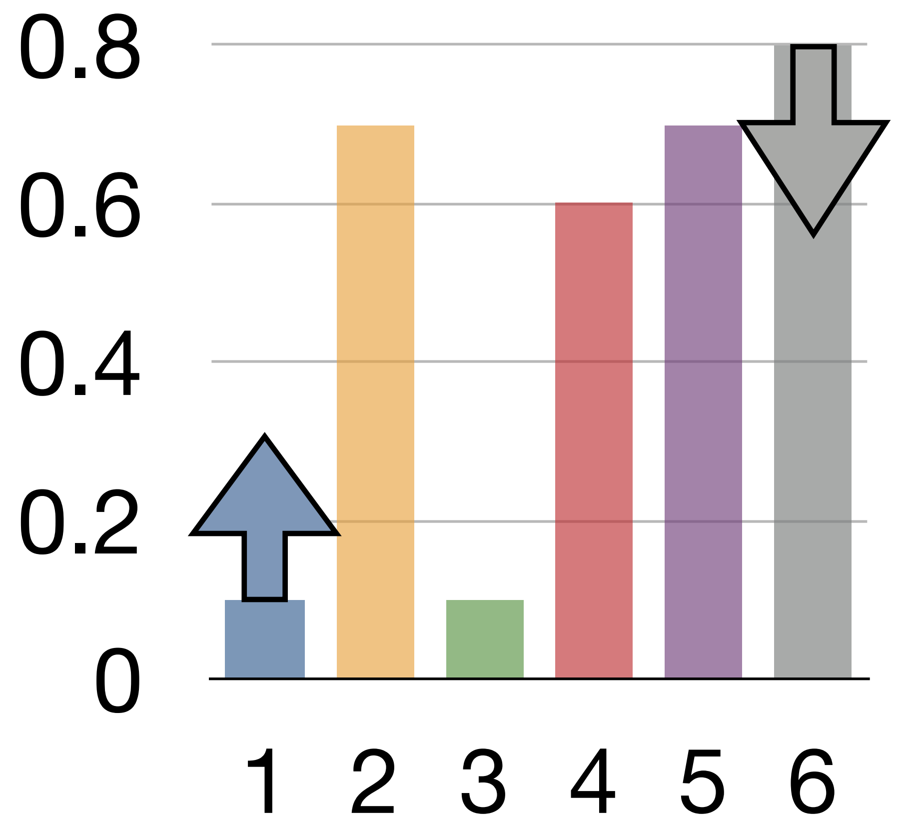
[Sampford 67]

Magic Lemma

$$\Pr[S = T] = \frac{f(T)}{\sum_{A \in \binom{[n]}{k}} f(A)}$$

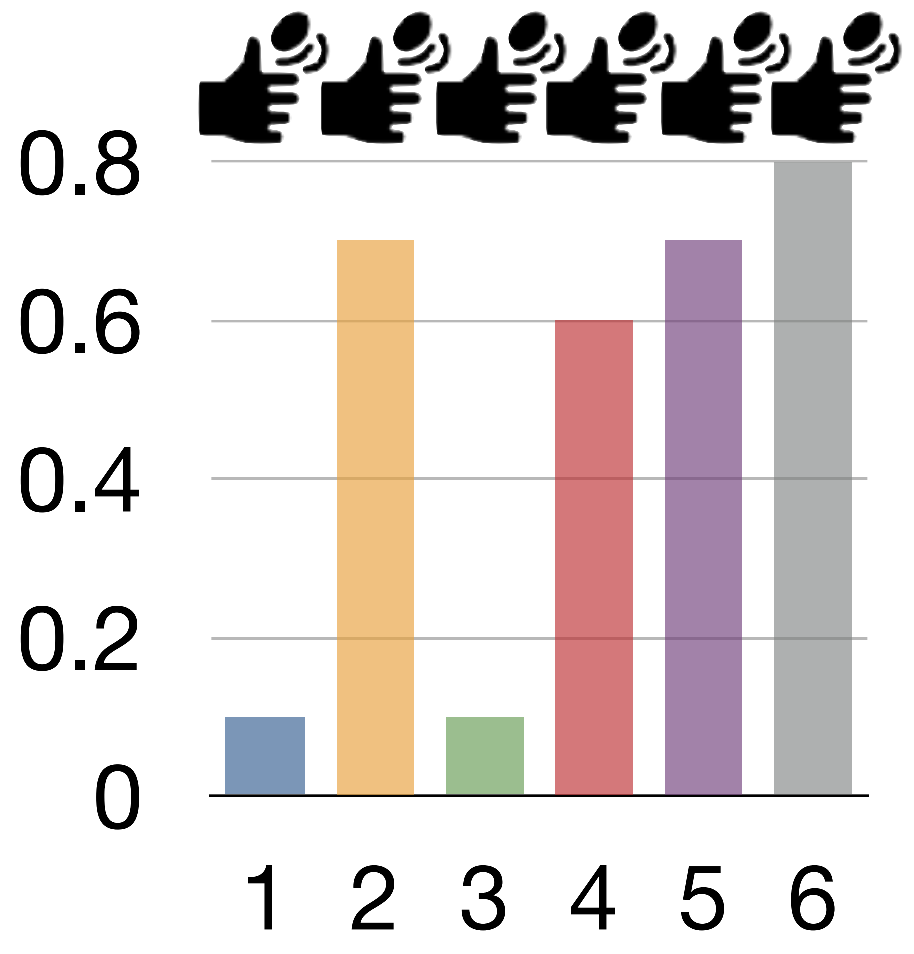
Observation: To prove selection monotonicity, it suffices to show that for any $\overrightarrow{p} \in [0,1)^n$ summing to k and $T = [k]$ it holds that:

$$\left(\frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_n}\right) P[S = T] \geq 0.$$



Magic Lemma: Let B be the random set containing each $i \in [n]$ independently with probability p_i. Then,

$$\sum_{A \in \binom{[n]}{k}} f(A) = \frac{1}{2}\,\mathbb{E}[\quad B \quad - k \quad].$$



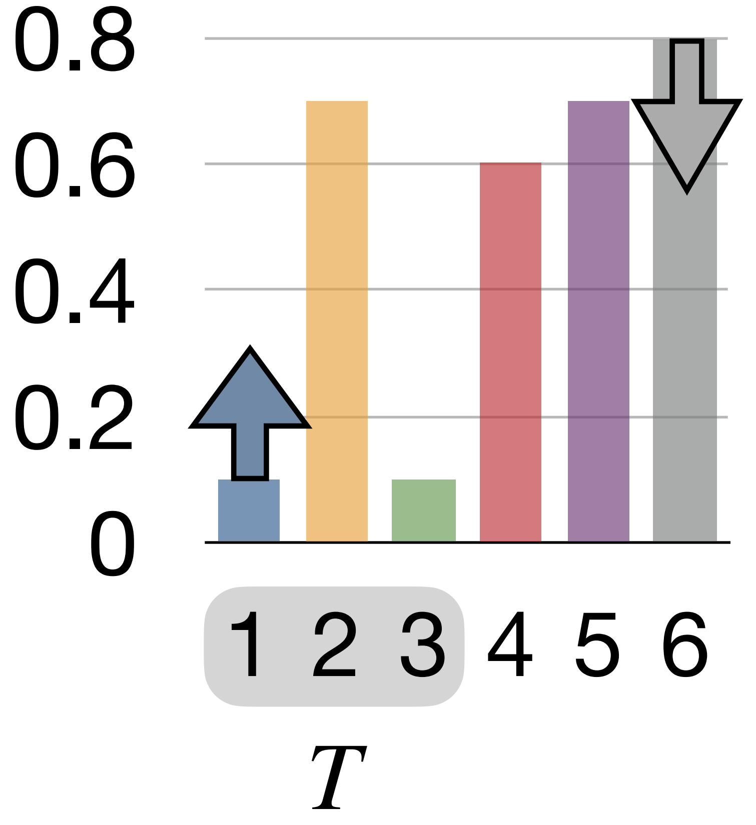
Sampford Sampling is Selection Monotone

Main Result: Sampford sampling is selection monotone.

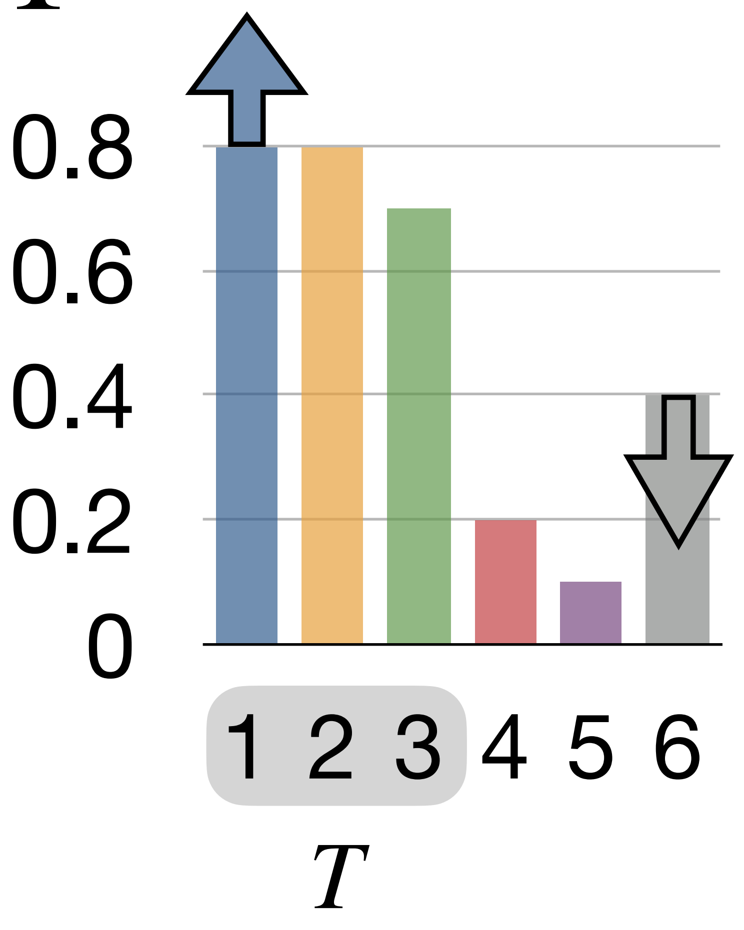
(Very rough) proof sketch: From the Magic Lemma we know that

$$P[S = T] = \frac{2\,f(T)}{\mathbb{E}[\;B\;-k\;]}.$$

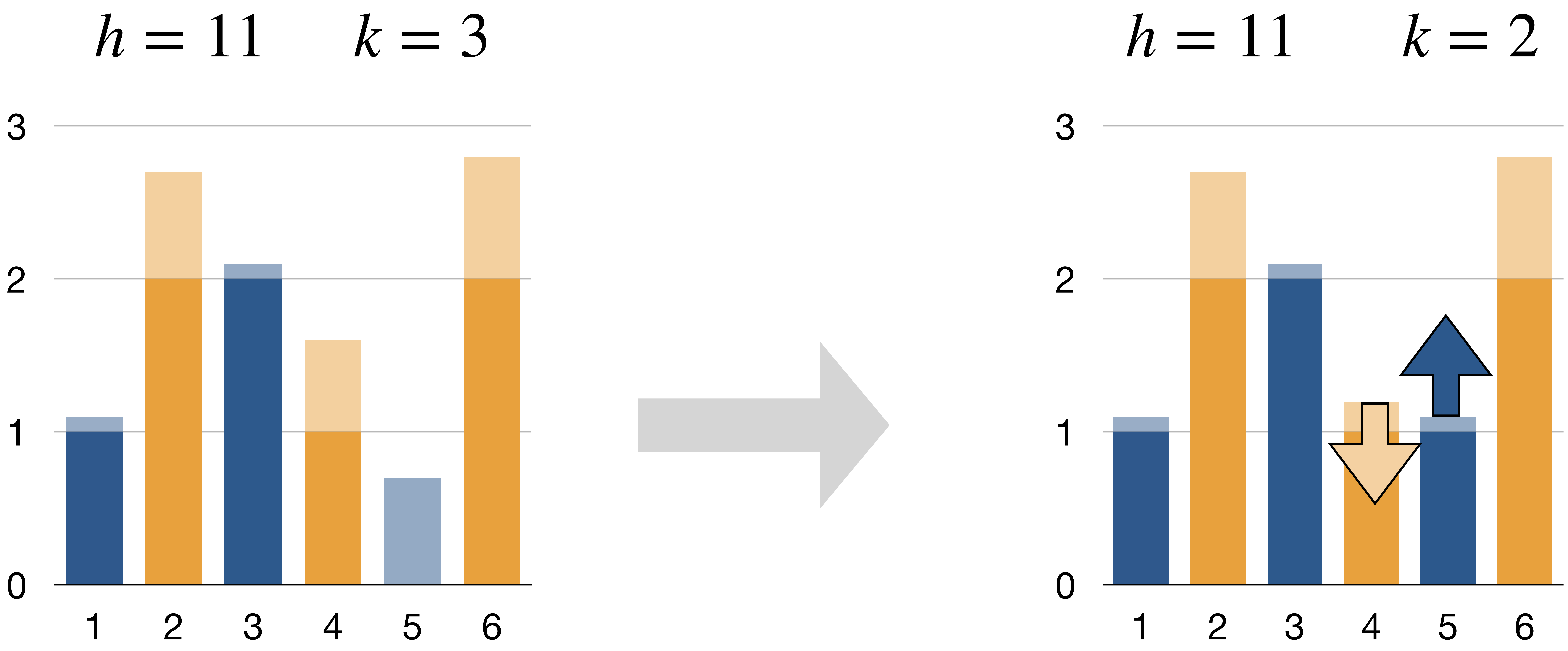
Easy Case: $\sum_{i \in T} p_i \leq k - 1$



Hard Case: $\sum_{i \in T} p_i > k - 1$



Selection Monotonicity Revisited



What if lower quotas change?

Monotone
Apportionment Rules

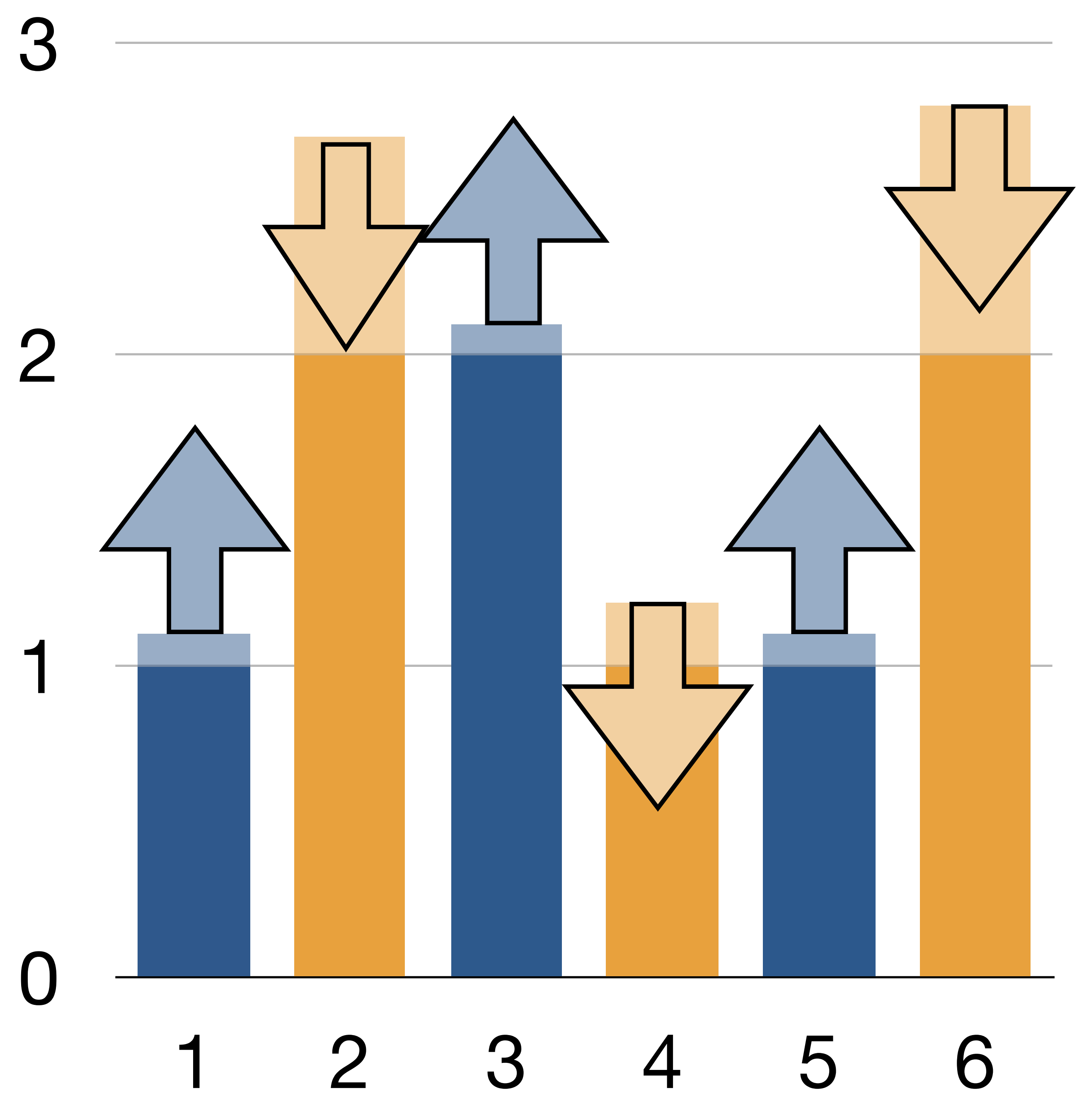
Threshold Monotonicity

Let $\overrightarrow{q}$ and $\overrightarrow{q}'$ be two vote share (or quota) vectors and T be a coalition such that

- $q_i' \geq q_i$ for $i \in T$, and
- $q_i' \leq q_i$ for $i \notin T$.

An apportionment rule is **threshold-monotone** if for all $\ell \in [h]$ it holds that:

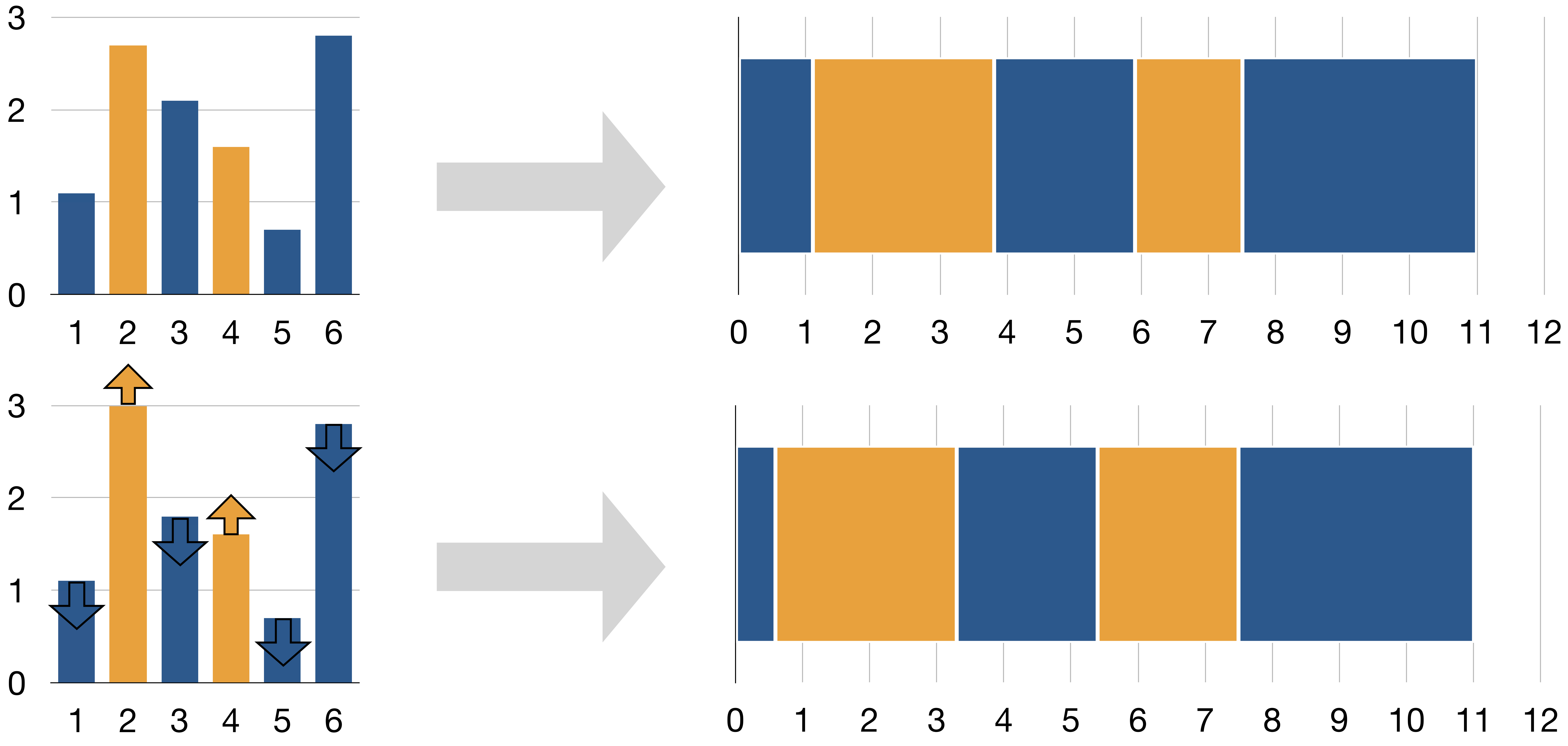
$$\mathbb{P}\left[\sum_{i \in T} a_i' \geq \ell\right] \geq \mathbb{P}\left[\sum_{i \in T} a_i \geq \ell\right].$$



If a coalition gains vote share, their probability of receiving ***any threshold*** *of seats may not decrease.*

Grimmett's Satisfies Threshold Monotonicity

For Coalitions of Size 2



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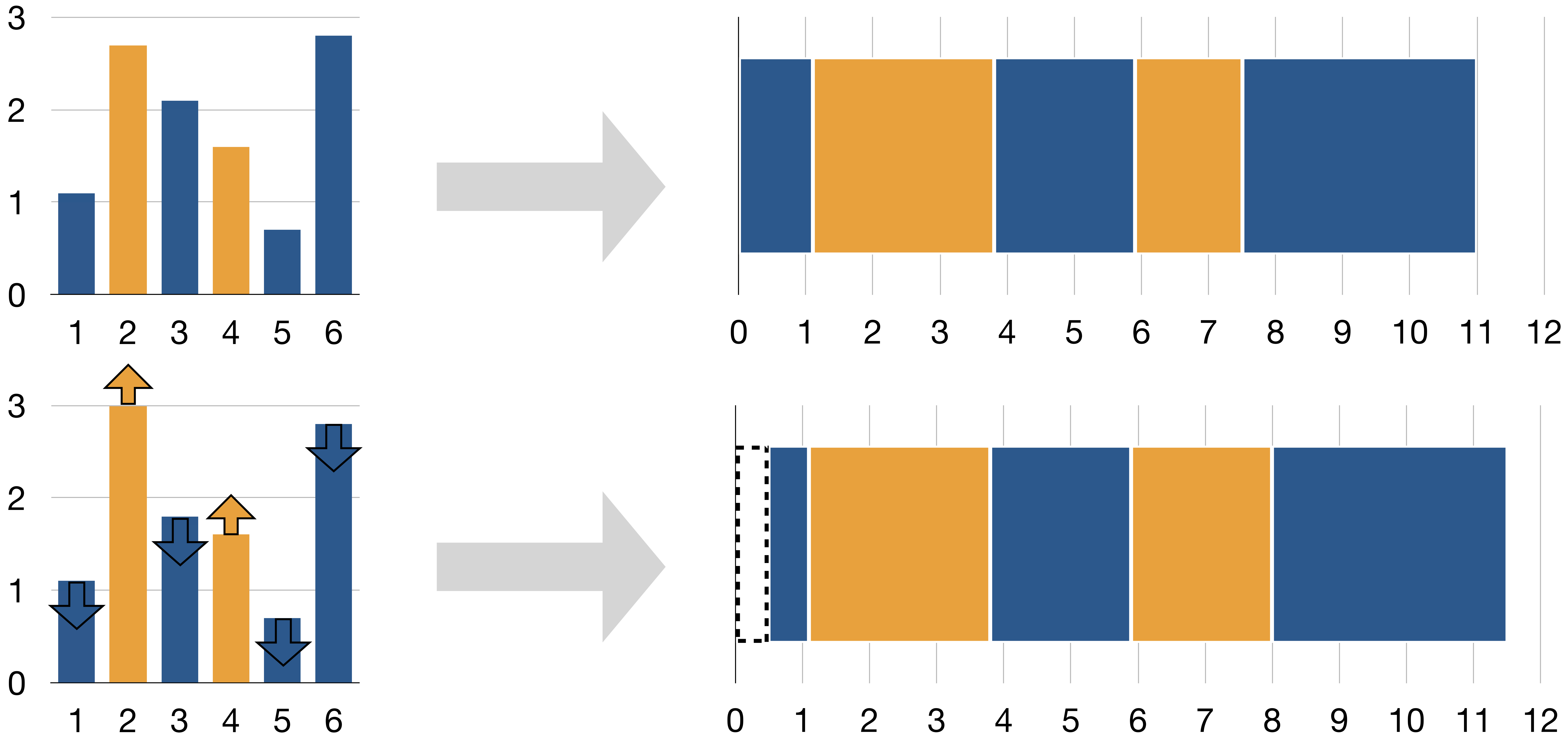

Conjecture: **Sampford's** apportionment method satisfied **threshold monotonicity**.

Vote-Count Threshold Monotonicity

Let $\vec{v}$ and $\vec{v}'$ be two **vote vectors** and T be a coalition such that

- $v_i' \geq v_i$ for $i \in T$, and
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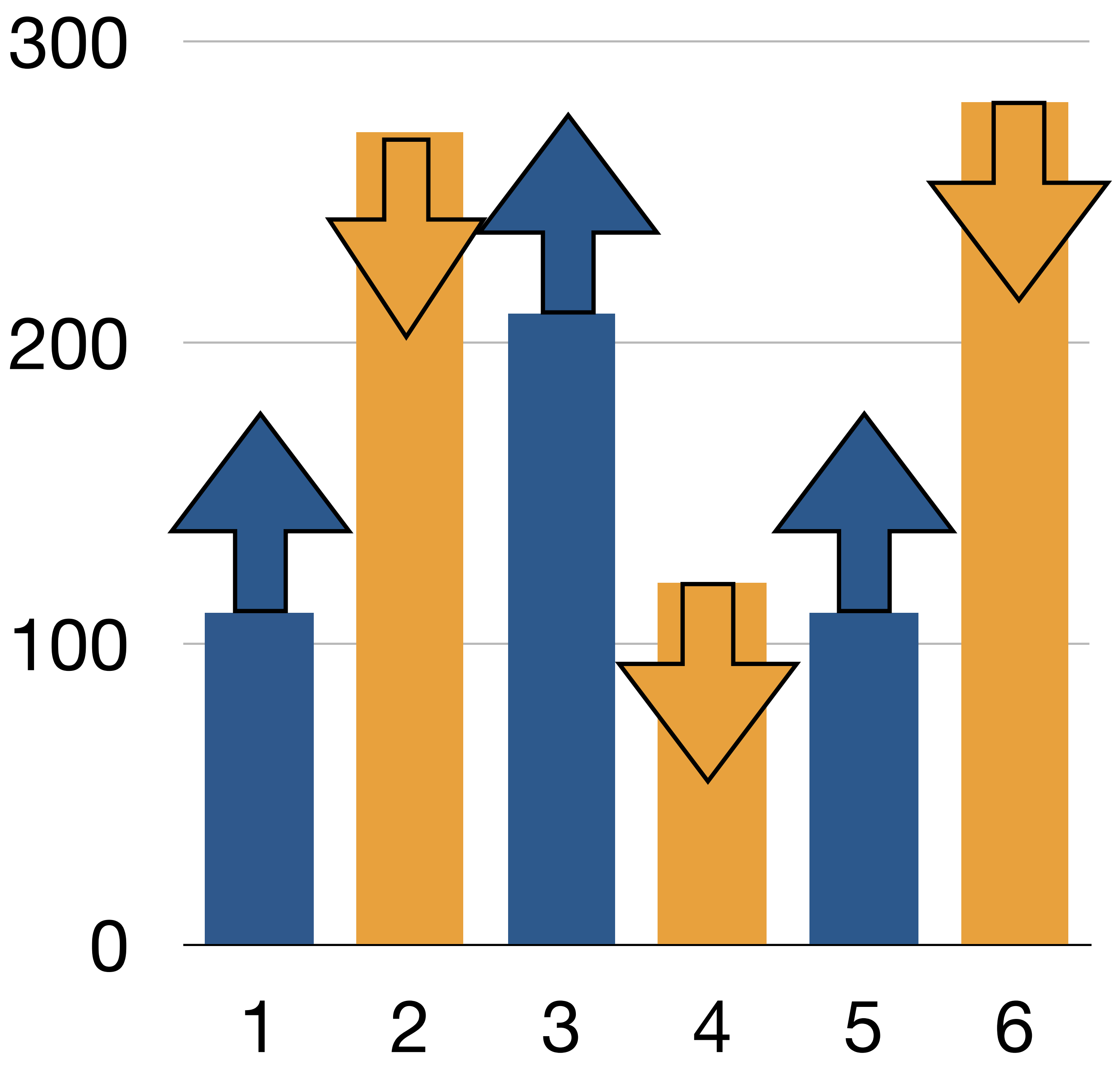
An apportionment rule is **vote-count threshold-monotone** if for all $\ell \in [h]$ it holds that:

$$\mathbb{P}\left[\sum_{i \in T} a_i' \geq \ell\right] \geq \mathbb{P}\left[\sum_{i \in T} a_i \geq \ell\right].$$



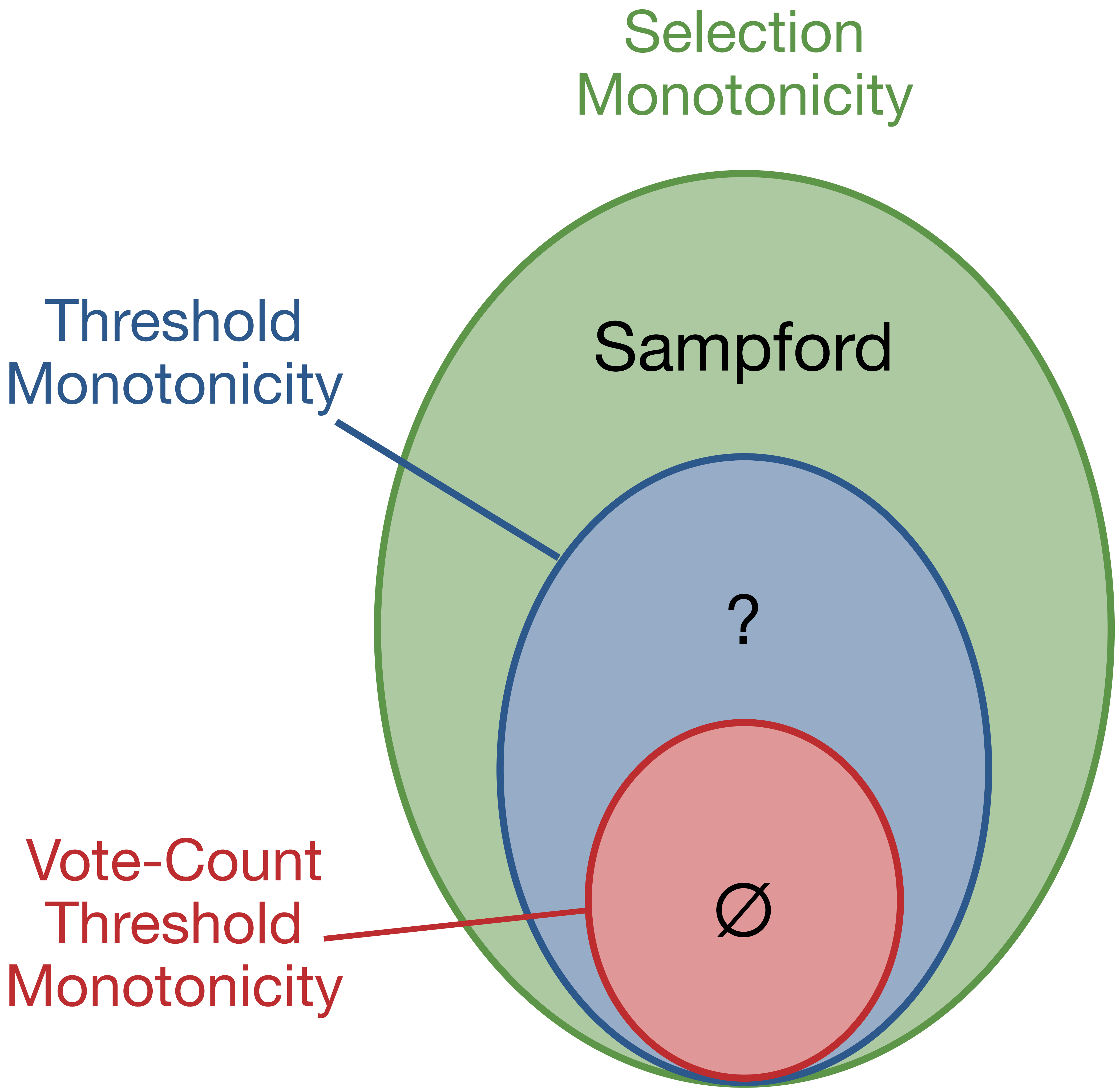
*If a coalition gains **votes**, their probability of receiving any threshold of seats may not decrease.*

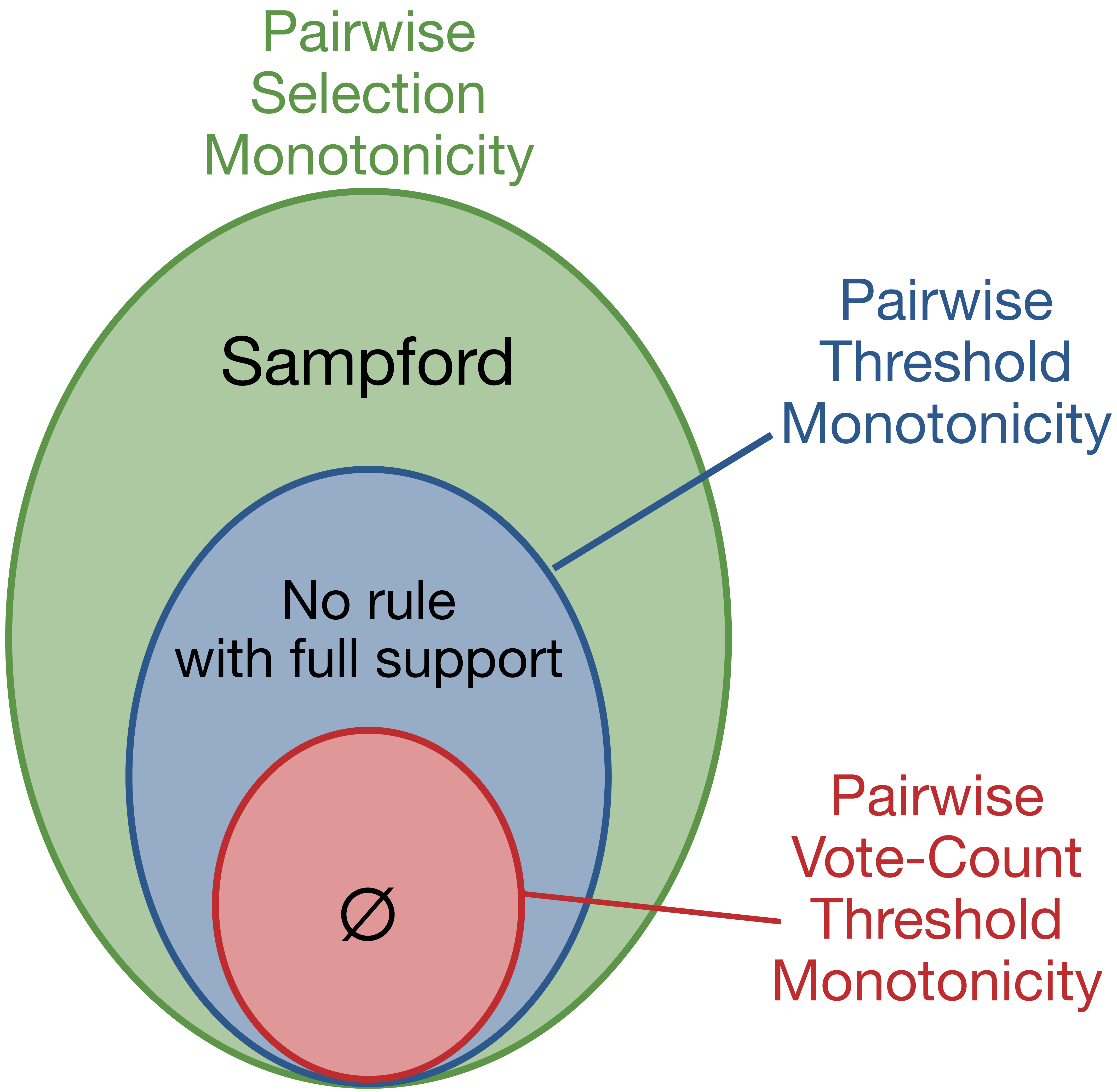
Impossibility: No apportionment method satisfying ex-ante proportionality and ex-post quota can satisfy **vote-count threshold monotonicity**.

Conclusion

The Axiomatic Landscape

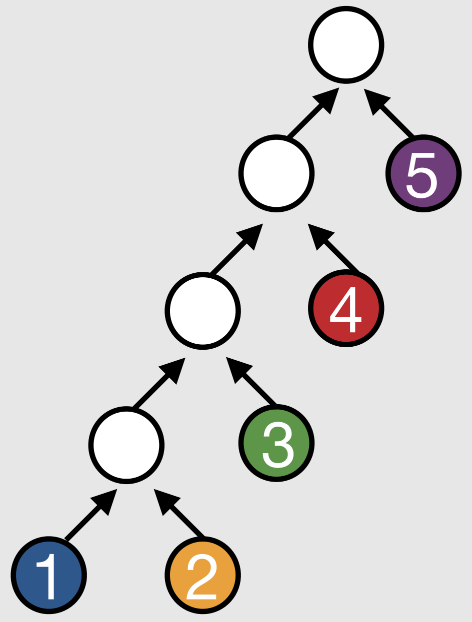
Of Monotone Randomized Apportionment





Applications Beyond Apportionment

Pipage
(pivotal method)



[Deville/Tilee 98, Srinivasan 01]

Pipage rounding yields approximation algorithms for **Steiner tree problems**, **k-median**, **committee selection** and **online algorithms**.

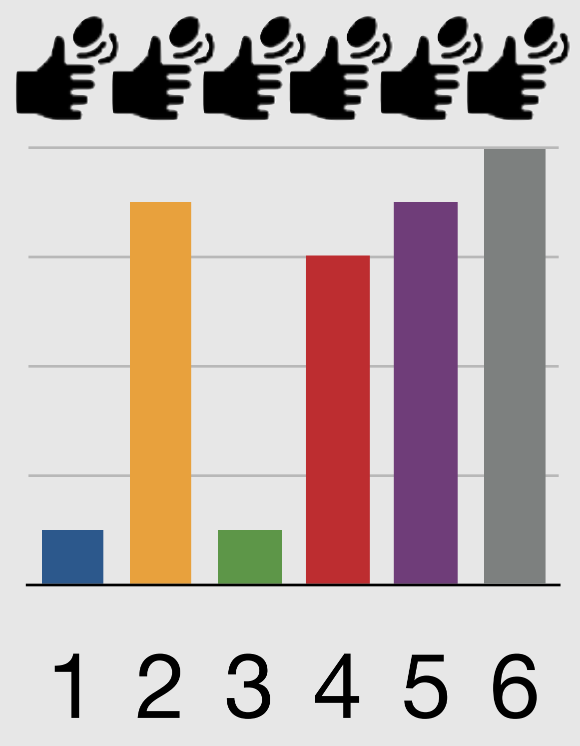
Important properties:

1. ex-ante proportionality
2. selection of k elements
3. negative correlation

Sampford sampling satisfies properties 1-3 plus **selection monotonicity**.

Applications Beyond Apportionment

Sampford



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Thank you!