Symmetric mechanisms for two-sided matching problems

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One-to-one two-sided matching problem

- two sets of agents **having equal size**
- each agent of a set has preferences on the agents of the other set (linear orders)

Problem

Find a procedure that "properly" matches any agent of the first set with **one and only one** agent of the other set.

Gale-Shapley algorithm

- agents interpreted as **individuals**
- the first group is denoted by W
  the members of W are called **women**
- the second group is denoted by M
  the members of M are called **men**

$|W| = |M| = n \geqslant 2$

$W = \{1, \dots, n\} \quad M = \{n+1, \dots, 2n\}$

$I = W \cup M$

every woman has preferences in $\mathbf{L}(M)$ (linear orders on M)

every man has preferences in $\mathbf{L}(W)$ (linear orders on W)

A preference profile is a function

$$p : I \to \mathbf{L}(W) \cup \mathbf{L}(M)$$

such that, for every $z \in I$, $\begin{cases} \text{if } z \in W, \text{ then } p(z) \in \mathbf{L}(M) \\ \text{if } z \in M, \text{ then } p(z) \in \mathbf{L}(W) \end{cases}$

$$\mathbf{P} \ = \ \text{set of preference profiles}$$

$W = \{1, 2, 3\} \quad M = \{4, 5, 6\}$

$$p = \begin{array}{|ccc||ccc|} \hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline\hline 4 & 5 & 6 & 3 & 1 & 2 \\ 5 & 6 & 4 & 1 & 2 & 3 \\ 6 & 4 & 5 & 2 & 3 & 1 \\ \hline \end{array}$$

Definition

A matching is an element μ of $Sym(I)$ such that

- for every $x \in W$, $\mu(x) \in M$
- for every $y \in M$, $\mu(y) \in W$
- for every $z \in I$, $\mu(\mu(z)) = z$

$\mathbf{M}$ = set of matchings

$W = \{1, 2, 3\} \quad M = \{4, 5, 6\}$

The elements of **M** are

$$(14)(25)(36) \qquad (14)(26)(35) \qquad (16)(25)(34)$$

$$(15)(24)(36) \qquad (15)(26)(34) \qquad (16)(24)(35)$$

Let $p \in \mathbf{P}$ and $\mu \in \mathbf{M}$

- μ is stable if there is no $(x, y) \in W \times M$ such that $y \succ_{p(x)} \mu(x)$ and $x \succ_{p(y)} \mu(y)$.

- μ is minimally optimal if there is an individual who is not matched by μ to her/his worst choice.

$$\text{Stability} \quad \Rightarrow \quad \text{Minimal optimality}$$

Theorem (Gale-Shapley, 1962)

For every $p \in \mathbf{P}$ there exists a stable matching.

woman-oriented Gale-Shapley algorithm

$\Downarrow$

woman-optimal stable matching

man-oriented Gale-Shapley algorithm

$\Downarrow$

man-optimal stable matching

1	2	3	4	5	6

$p =$

4	5	6	3	1	2
5	6	4	1	2	3
6	4	5	2	3	1

- woman-optimal stable: (14)(25)(36)
- man-optimal stable: (15)(26)(34)
- stable: (14)(25)(36), (15)(26)(34)
- minimally optimal: all the matchings but (16)(24)(35)

Definition

A matching mechanism (MM) is a nonempty-valued correspondence from **P** to **M**.

Problem

Find a MM satisfying "good" properties.

Some matching mechanisms

$$GS_w(p) = \Big\{\text{the woman-optimal stable matching for } p\Big\}$$

$$GS_m(p) = \Big\{\text{the man-optimal stable matching for } p\Big\}$$

$$ST(p) = \Big\{\text{stable matchings for } p\Big\}$$

$$MO(p) = \Big\{\text{minimally optimal matchings for } p\Big\}$$

Definition

Let F be a MM.

F is resolute if, for every $p \in \mathbf{P}$, $|F(p)| = 1$.

- ST and MO are not resolute
- GS_w and GS_m are resolute

GS_w is not fair: it gives a systematic advantage to women.

GS_m is not fair: it gives a systematic advantage to men.

How to define and measure fairness?

- We identify the level of fairness of a matching mechanism with its level of symmetry
- We study symmetry via an algebraic approach developed by Bubboloni and Gori in SCT

$$G^* = \Big\{\varphi \in Sym(I) : \{\varphi(W), \varphi(M)\} = \{W, M\}\Big\} \qquad \mathbf{M} \subseteq G^*$$

$$G = \Big\{\varphi \in Sym(I) : \varphi(W) = W \text{ and } \varphi(M) = M\Big\} \trianglelefteq G^*$$

$$\varphi \in G \text{ exchanges } \begin{cases} \text{women's names among women} \\ \text{men's names among men} \end{cases}$$

$\varphi \in G^* \backslash G$ exchanges women's and men's names.

Given $p \in \mathbf{P}$ and $\varphi \in G^*$,

p^φ = preference profile obtained by p by exchanging individual names according to φ

$p =$

1	2	3	4	5	6
4	5	6	3	1	2
5	6	4	1	2	3
6	4	5	2	3	1

$\varphi = (123)(46) \in G$

$p^\varphi =$

2	3	1	6	5	4
6	5	4	1	2	3
5	4	6	2	3	1
4	6	5	3	1	2

$=$

1	2	3	4	5	6
4	6	5	3	2	1
6	5	4	1	3	2
5	4	6	2	1	3

.

Definition

Let F be a MM and $U \subseteq G^*$. F is U-symmetric if, for every $p \in \mathbf{P}$ and $\varphi \in U$,

$$F(p^{\varphi}) = \left\{ \varphi\mu\varphi^{-1} \in Sym(I) : \mu \in F(p) \right\}$$

U-symmetry is interpreted as a measure of fairness

If $V \subseteq U$, then U-symmetry implies V-symmetry

$$W = \{1,2,3\} \quad M = \{4,5,6\} \qquad F \text{ a } G^*\text{-symmetric } \mathrm{MM}$$

$$p \in \mathbf{P} \quad \varphi = (1426)(35) \in G^*$$

$$F(p) = \Big\{(14)(25)(36), (15)(26)(34)\Big\}$$

$$\Downarrow$$

$$F(p^{\varphi}) = \Big\{\varphi(14)(25)(36)\varphi^{-1}, \varphi(15)(26)(34)\varphi^{-1}\Big\}$$

$$= \Big\{(42)(63)(51), (43)(61)(52)\Big\} = \Big\{(15)(24)(36), (16)(25)(34)\Big\}$$

anonimity or peer indifference $=$ G-symmetry
Masarani-Gokturk (1989), Sasaki-Toda (1992)

gender indifference $=$ $\{\mu\}$-symmetry
Masarani-Gokturk (1989), Endriss (2020)

gender fairness $=$ $(G^{*}\backslash G)$-symmetry
Özkal-Sanver (2004)

Resoluteness and G^*-symmetry are highly desirable

- GS_w and GS_m are resolute but not G^*-symmetric
- ST and MO are G^*-symmetric but not resolute

Problem

Is there a resolute and G^*-symmetric $\textsc{mm}$?

Definition

Let F and H be MMs.

H is a refinement of F if, for every $p \in \mathbf{P}$, $H(p) \subseteq F(p)$.

General problem

Given a MM F and $U \subseteq G^*$, is there a U-symmetric resolute refinement of F?

Crucial fact

$$f : G^* \to Sym(\mathbf{P}),$$

$$\varphi \mapsto \left(f(\varphi) : \mathbf{P} \to \mathbf{P}, \quad p \to p^{\varphi} \right)$$

is a group homomorphism (an action of G^* on $\mathbf{P}$)

for every $\varphi_1, \varphi_2 \in G^*$ and $p \in \mathbf{P}$, $\quad p^{\varphi_1 \varphi_2} = (p^{\varphi_2})^{\varphi_1}$

Proposition

Let F be a MM and $U \subseteq G^*$. Then

$$F \text{ is } U\text{-symmetric} \quad \Leftrightarrow \quad F \text{ is } \langle U \rangle\text{-symmetric}$$

Only subgroups of G^* matter

$$(G^* \backslash G)\text{-symmetry} \; = \; G^*\text{-symmetry}$$

Definition

Let $U \leqslant G^*$ and $p \in \mathbf{P}$. Then

$$Stab_U(p) = \left\{ \varphi \in U : p^{\varphi} = p \right\} \leqslant U$$

Definition

Let $U \leqslant G^*$ and $p \in \mathbf{P}$. Then

$$C^U(p) = \left\{ \mu \in \mathbf{M} : \forall\, \varphi \in Stab_U(p) \;\; \varphi\mu\varphi^{-1} = \mu \right\}$$

General existence result

Theorem

Let F be a $\textsc{mm}$ and $U \leqslant G^*$. Assume that F is U-symmetric. The following facts are equivalent:

- F admits a U-symmetric and resolute refinement
- for every $p \in \mathbf{P}$, $C^U(p) \cap F(p) \neq \varnothing$

Corollary

Let $U \leqslant G^*$. The following facts are equivalent:

- there exists a U-symmetric and resolute $\textsc{mm}$
- for every $p \in \mathbf{P}$, $C^U(p) \neq \varnothing$

G^*-symmetry

Theorem

If n is even, there exists no resolute and G^*-symmetric MM.

Proof. We find $p \in \mathbf{P}$ such that $C^{G^*}(p) = \varnothing$. □

Theorem

If n is odd, there exists a resolute and G^*-symmetric MM.

Proof. We consider $p \in \mathbf{P}$ and prove that $C^{G^*}(p) \neq \varnothing$.

Step 1. We prove that $Stab_{G*}(p)$ is a semi-regular group.

Step 2. We use semi-regularity to build an element in $C^{G^*}(p)$. □

If $n = 3$ there are $2^{298} \cdot 3^{362}$ resolute and G^*-symmetric MMs

Theorem

There exists no resolute, G^*-symmetric and minimally optimal MM.

Proof. If n is even, there are no resolute and G^*-symmetric MM. If n is odd, we find $p \in \mathbf{P}$ such that $C^{G^*}(p) \cap MO(p) = \varnothing$. □

Corollary (Endriss, 2020, $n \geqslant 3$)

There exists no resolute, G^*-symmetric and stable MM.

Proof. Simply note that ST is a refinement of MO. □

The End

Thank you for your attention