# Axiomatic characterization of the knapsack and greedy budget allocation rules

Federica Ceron, Stéphane Gonzalez, and Adriana Navarro-Ramos
(GATE Lyon Saint-Étienne)

Amsterdam/Saint-Etienne Workshop on Social Choice

# Purpose: A General Decision Problem Under Constraint

- We examine a class of decision problems where a finite set of costly and desirable items $\mathbb{P}$ must be allocated under a budget constraint denoted by $\ell > 0$.

# Purpose: A General Decision Problem Under Constraint

- We examine a class of decision problems where a finite set of costly and desirable items $\mathbb{P}$ must be allocated under a budget constraint denoted by $\ell > 0$.
- A budgeting instance $I = (m, c)$ of this problem consists of:
  1. A menu $m \in \mathbb{N}^{(\mathbb{P})} \setminus \{0\}$ representing the number of units of items available. The support, $supp(m) := \{p \in \mathbb{P} | m(p) > 0\}$, is finite.
  2. Each item $p \in supp(m)$ has a specific unitary cost $c(p) > 0$.

# Purpose: A General Decision Problem Under Constraint

- We examine a class of decision problems where a finite set of costly and desirable items $\mathbb{P}$ must be allocated under a budget constraint denoted by $\ell > 0$.
- A budgeting instance $I = (m, c)$ of this problem consists of:
  1. A menu $m \in \mathbb{N}^{(\mathbb{P})} \setminus \{0\}$ representing the number of units of items available. The support, $supp(m) := \{p \in \mathbb{P} | m(p) > 0\}$, is finite.
  2. Each item $p \in supp(m)$ has a specific unitary cost $c(p) > 0$.

**Objective:** The goal is to design a budget allocation rule $f$ which, for each instance $I = (m, c)$, identifies a menu $x \leq m$ such that

$$\sum_{p \in supp(x)} x(p)c(p) \leq \ell$$

ensuring the total cost of the selected items does not exceed the budget limit.

# Numerical Example: Budget Allocation Problem

- Consider a budget constraint $\ell = 100$ (e.g., \$100).
- Let's assume our universe of items $\mathbb{P}$ consists of three items: $p_1, p_2, p_3$.

# Numerical Example: Budget Allocation Problem

- Consider a budget constraint $\ell = 100$ (e.g., \$100).
- Let's assume our universe of items $\mathbb{P}$ consists of three items: $p_1, p_2, p_3$.
- The menu $m \in \mathbb{N}^{(\mathbb{P})} \setminus \{0\}$, representing the availability of items, and their costs are given as follows:

| Item | Units available ($m(p)$) | Cost ($c(p)$) |
|---|---|---|
| $p_1$ | 10 | \$2 |
| $p_2$ | 5 | \$10 |
| $p_3$ | 8 | \$5 |

# Numerical Example: Budget Allocation Problem

- Consider a budget constraint $\ell = 100$ (e.g., \$100).
- Let's assume our universe of items $\mathbb{P}$ consists of three items: $p_1, p_2, p_3$.
- The menu $m \in \mathbb{N}^{(\mathbb{P})} \setminus \{0\}$, representing the availability of items, and their costs are given as follows:

| Item | Units available ($m(p)$) | Cost ($c(p)$) |
|---|---|---|
| $p_1$ | 10 | \$2 |
| $p_2$ | 5 | \$10 |
| $p_3$ | 8 | \$5 |

- **Objective:** Design a budget allocation rule $f$ to identify a affordable allocation $x \leq m$ such that the total cost $\sum_{p \in supp(x)} x(p)c(p) \leq \ell$.

# Numerical Example: Budget Allocation Problem

- Consider a budget constraint $\ell = 100$ (e.g., \$100).
- Let's assume our universe of items $\mathbb{P}$ consists of three items: $p_1, p_2, p_3$.
- The menu $m \in \mathbb{N}^{(\mathbb{P})} \setminus \{0\}$, representing the availability of items, and their costs are given as follows:

| Item | Units available ($m(p)$) | Cost ($c(p)$) |
|---|---|---|
| $p_1$ | 10 | \$2 |
| $p_2$ | 5 | \$10 |
| $p_3$ | 8 | \$5 |

- **Objective:** Design a budget allocation rule $f$ to identify a affordable allocation $x \leq m$ such that the total cost $\sum_{p \in supp(x)} x(p)c(p) \leq \ell$.
- **Example of Allocation:** Let's allocate $x(p_1) = 10$, $x(p_2) = 2$, and $x(p_3) = 3$.
- The total cost is $10(\$2) + 2(\$10) + 3(\$5) = \$20 + \$20 + \$15 = \$55$.
- This allocation satisfies the budget constraint $\ell = 100$.

# Applications

There are numerous applications of this kind of problems, such as:

- Consumer choice: where items represent goods or services for shopping, and $\ell$ is the consumer's budget.
- Time management: where $\ell$ represents the number of available hours, and items are tasks, each consuming a portion of this time.
- Carbon quota management: managing activities within a fixed carbon emission limit ($\ell$), where each activity (item) contributes to the total emissions.

# Application: Participatory Budgeting

1. The fixed budget $\ell$ represents the total public funds allocated for community projects.
2. Assume a set of proposed projects $\mathbb{P} = \{p_1, p_2, p_3, \ldots, p_k\}$, each with a specific cost $c(p_1), c(p_2), \ldots, c(p_k)$.
3. The goal is to design a budget allocation rule that takes into account scores or preferences assigned by voters to each project, thereby influencing the selection process.
4. **Objective:** Allocate funds to projects in a way that reflects voter preferences while adhering to the budget limit $\ell$.

# Budget Allocation Rule

- A budget allocation rule $f$ is a correspondence that, for each budgeting instance $I = (m, c)$, associates a set $f(I)$ of $I$-affordable allocations. These are menus $x \leq m$ such that the total cost of selected items does not exceed the budget limit $\ell$, i.e.,

$$f(I) \subseteq \mathcal{A}(I) := \left\{ x \in \mathbb{N}^{(\mathbb{P})} \setminus \{0\} \,\middle|\, x \leq m \text{ and } \sum_{p \in supp(x)} x(p)c(p) \leq \ell \right\}.$$

# Budget Allocation Rule: The Knapsack Method

- The **Knapsack Method** assigns a value $s(p) \in \mathbb{R}_+$ to each item $p \in \mathbb{P}$, indicating the utility or importance of the item.
- The objective is to select for each instance $(m, c)$ an affordable menu $x \in \mathcal{A}(I)$ that maximizes the total value:

$$\sum_{p \in supp(x)} s(p)x(p)$$

# Budget Allocation Rule: The Greedy Method

- The **Greedy Method** ranks items $p \in \mathbb{P}$ according to a well-order $\succ_{\mathbb{P}}$ based on their utility or importance per unit cost.
- Items are then selected in sequence, from highest to lowest ranked, until adding another item would exceed the budget limit $\ell$.

# Knapsack Method: A Numerical Example

Consider the following setup:

- A fixed budget $\ell = 9$.
- A universe of items $\mathbb{P} = \{p_1, p_2, p_3, p_4\}$.
- $m(p) = 1$ for each $p \in \mathbb{P}$ (exactly one of each item is available).

The costs and values (utilities) of the items are given in the table below:

| Item | Cost $c(p)$ | Value $s(p)$ |
|---|---|---|
| $p_1$ | 4 | 5 |
| $p_2$ | 6 | 6 |
| $p_3$ | 3 | 3 |
| $p_4$ | 2 | 2 |

Using the Knapsack Method, we aim to maximize the total value of selected items, i.e., the sum of $s(p)$ values, without exceeding the budget $\ell$.

The optimal selection with the maximum total value under the budget constraint would be $\{p_1, p_3, p_4\}$, with a total cost of 9 and a total value of 10.

# Greedy Method: A Numerical Example

Consider the same setup as before:

- A fixed budget $\ell = 9$.
- A universe of items $\mathbb{P} = \{p_1, p_2, p_3, p_4\}$.
- $m(p) = 1$ for each $p \in \mathbb{P}$ (exactly one of each item is available).

The costs of the items and their ranking according to their utility per unit cost are given in the table below:

| Item | Cost $c(p)$ | Rank |
|---|---|---|
| $p_1$ | 4 | $2^{nd}$ |
| $p_2$ | 6 | $1^{st}$ |
| $p_3$ | 3 | $3^{rd}$ |
| $p_4$ | 2 | $4^{th}$ |

Using the Greedy Method, we select items in the order of the ranking until the budget limit is reached.

The selection of items in order of their rank under the budget constraint would be $\{p_2, p_3\}$ with a total cost of $9$.

# LexicHahn Rules

- Introducing a new class of budget allocation rules.
- A budget allocation rule $f$ is a LexicHahn rule if there exist:
  1. A set $Z$,
  2. A well-order $\succ_Z$ on $Z$, and
  3. A mapping $s : \mathbb{P} \to \mathbb{R}^Z_{>0}$,

  such that, for every budgeting instance $I = (m, c)$,

$$f(I) := \underset{\geq^{lex}_{\succ_Z} : x \in \mathcal{A}(I)}{\operatorname{argmax}} \sum_{p \in supp(x)} s(p) \cdot x(p) \ . \tag{1}$$

- Here, $\mathcal{A}(I)$ represents the set of all $I$-affordable allocations, and the decision rule selects the allocation $x$ that maximizes the lexicographically ordered sum of values assigned to each item in $x$, considering their quantities and the mapping $s$.

# LexicHahn Rules: A Numerical Example (1)

Consider the following setup:

- A fixed budget $\ell = 9$.
- A universe of items $\mathbb{P} = \{p_1, p_2, p_3, p_4, p_5\}$.
- For simplicity, let's assume $m(p) = 1$ for each $p \in \mathbb{P}$ (indicating exactly one of each item is available), forming our menu $m$.

The costs and values of the items are given by:

| Item | Cost $c(p)$ | $s_1(p)$ | $s_2(p)$ |
|---|---|---|---|
| $p_1$ | 6 | 2 | 0 |
| $p_2$ | 1 | 0 | 15 |
| $p_3$ | 3 | 1 | 4 |
| $p_4$ | 4 | 1 | 5 |
| $p_5$ | 1 | 1 | 5 |

# LexicHahn Rules: A Numerical Example (2)

$\ell = 9$

| Item | Cost $c(p)$ | $s_1(x(p))$ | $s_2(x(p))$ |
|---|---|---|---|
| $p_1$ | 6 | 2 | 0 |
| $p_2$ | 1 | 0 | 15 |
| $p_3$ | 3 | 1 | 4 |
| $p_4$ | 4 | 1 | 5 |
| $p_5$ | 1 | 1 | 5 |

- Our goal using the LexicHahn rule is to maximize the total value of selected items. In other words, we seek to maximize the sum of $s(x(p))$ values, while not exceeding the budget $\ell$.
- The optimal selection offering maximum total value under this budget constraint is $\{p_2, p_3, p_4, p_5\}$. This selection totals to a cost of 9, resulting in a total opinion value of (3,29).

# Knapsack and Greedy as Special Cases of LexicHahn Rules

The Knapsack and Greedy budget allocation rules are specific instances of LexicHahn rules:

- **Knapsack Rule:** A LexicHahn rule with $|Z| = 1$, simplifying $\geq^{lex}$ to the natural order $\geq$ of real numbers. It focuses on maximizing the sum of utilities within the budget $\ell$.
- **Greedy Rule:** A LexicHahn rule where $Z = \mathbb{P}$, making lexicographical comparisons based on the occurrence of each item within the menu's support. This approach prioritizes items based on predefined criteria, selecting them sequentially within the budget limit $\ell$.

# Key Components of Building a Budget Allocation Rule

When constructing our budget allocation rule, there are two key elements to consider:

- **Evaluation:** This pertains to how we evaluate each item into a usable format. The evaluation could result in a real score, a qualitative evaluation, a simple acceptance or rejection, a multi-criteria evaluation, a ranking or ranking profile (a ranking per voter, often used in social choice), a song...
- **Processing:** This concerns how we utilize the evaluation of each item to arrive at a decision. In our case, the decision is the selection of a bundle. This involves taking into account the set of possible options, the budgetary constraints, and the priorities or values that we associate with the evaluations.

## Presentation of the axioms

# Exhaustiveness

EXHAUSTIVENESS. If any selected menu $x \in f(I)$ is such that no additional item $p$ can be added without exceeding the budget limit $\ell$.

# Completeness

COMPLETENESS. If for any two affordable allocations $x$ and $y$, there exists a budgeting instance $I$ where both $x, y$ are affordable, and at least one of them is in $f(I)$.

# Transitivity

TRANSITIVITY. If for an instance $I = (m, c)$, both allocations $x$ and $y$ are affordable and $x$ is chosen, and for another instance $I' = (m', c')$, both allocations $y$ and $z$ are affordable and $y$ is chosen, then there exists an instance $\hat{I} = (\hat{m}, \hat{c})$ where $x$ is chosen and $z$ is affordable.

# Weak Axiom of Revealed Preferences

WEAK AXIOM OF REVEALED PREFERENCES (WARP). If for an instance $I = (m, c)$, both allocations $x$ and $y$ are affordable ($x, y \in \mathcal{A}(I)$) and:

- $x$ is chosen ($x \in f(I)$),
- while $y$ is not ($y \notin f(I)$),

then, for any other instance $I' = (m', c')$ where both $x$ and $y$ are affordable ($x, y \in \mathcal{A}(I')$), it must still be that $y$ is not chosen ($y \notin f(I')$).

# Addition Consistency

ADDITION CONSISTENCY. If, in some instance $I = (m, c)$, two menus $x, y$ are affordable ($x, y \in \mathcal{A}(I)$), and:

- $x$ is chosen ($x \in f(I)$),
- while $y$ is not ($y \notin f(I)$),

then, for any menu $z$, there exists another instance $I' = (m', c')$ where both $x + z$ and $y + z$ are affordable, but only $x + z$ is chosen in $f(I')$.

# First Theorem

**Theorem (Characterization of LexicHahn Rules)**

*A budget allocation rule $f$ satisfies Completeness, Transitivity, Weak Axiom of Revealed Preferences, and Addition Consistency if and only if it is a LexicHahn rule.*

# Knapsack and Greedy

- Somewhat surprisingly, we can obtain a characterization of the knapsack and the greedy rules by adding two “opposite” axioms that deal with whether or nor quantity can outrun quality:

# Knapsack and Greedy

- Somewhat surprisingly, we can obtain a characterization of the knapsack and the greedy rules by adding two "opposite" axioms that deal with whether or nor quantity can outrun quality:

  1. Quantity Over Quality : A sufficiently large quantity of some qualitatively inferior items can outrun a bundle of higher quality.

# Knapsack and Greedy

- Somewhat surprisingly, we can obtain a characterization of the knapsack and the greedy rules by adding two "opposite" axioms that deal with whether or nor quantity can outrun quality:

  1. Quantity Over Quality : A sufficiently large quantity of some qualitatively inferior items can outrun a bundle of higher quality.
  2. Quality Over Quantity: A large quantity of a qualitatively inferior item cannot outrun a bundle of higher quality.

# Quantity Over Quality

QUANTITY OVER QUALITY. Considering a situation where two menus $x$ and $y$ are affordable in some instance $I = (m, c)$, and $x$ is chosen over $y$:

- There should exist an instance $I' = (m', c')$ where, with the addition of a certain menu $z$, both $x$ and $y + z$ are affordable, and both are chosen.
- Additionally, there exists an integer $k$ such that in a scenario where $k$ replicas of $y$ (denoted $k \cdot y$) are affordable along with $x$, then $k \cdot y$ along with $x$ are chosen, indicating a preference for accumulating more of $y$ under certain conditions.

# Quantity Over Quality

QUANTITY OVER QUALITY. Considering a situation where two menus $x$ and $y$ are affordable in some instance $I = (m, c)$, and $x$ is chosen over $y$:

- There should exist an instance $I' = (m', c')$ where, with the addition of a certain menu $z$, both $x$ and $y + z$ are affordable, and both are chosen.

- Additionally, there exists an integer $k$ such that in a scenario where $k$ replicas of $y$ (denoted $k \cdot y$) are affordable along with $x$, then $k \cdot y$ along with $x$ are chosen, indicating a preference for accumulating more of $y$ under certain conditions.

$\longrightarrow$ Roughly, one can always improve upon an inferior bundle by either adding sufficiently many additional items or by replicating the inferior menu by a factor $k$.

# Quality Over Quantity

QUALITY OVER QUANTITY. Given two menus $x$ and $y$ that do not share items, if in an instance $I = (m, c)$:

- Both $x$ and $y$ are affordable,
- $x$ is chosen ($x \in f(I)$) and $y$ is not ($y \notin f(I)$),

then, for any number of copies $n \in \mathbb{N}$ of $y$, there exists another instance $I' = (m', c')$ where:

- $x$ and $n \cdot y$ (multiple copies of $y$) are both affordable,
- $x$ is chosen ($x \in f(I')$) and $n \cdot y$ is not ($n \cdot y \notin f(I')$).

$\longrightarrow$ This axiom illustrates that the preference for a higher quality menu $x$ remains even when the quantity of a lesser preferred menu $y$ is increased, emphasizing the importance of quality over mere quantity.

# Result

**Theorem**

1. *The Knapsack rule*[a] *is the only budget allocation rule that satisfies Completeness, Transitivity, Weak Axiom of Revealed Preferences, Addition Consistency, and Quantity Over Quality.*
2. *The Greedy rule is the only budget allocation rule that satisfies Completeness, Transitivity, Weak Axiom of Revealed Preferences, Addition Consistency, and Quality Over Quantity.*

[a] Under few regularity assumptions

# Introducing the Comparative Worth Axiom

If we claim that menu $x$ is "really better" than menu $y$, then we can consider that $x$ is giving us a certain surplus in quality. This surplus should be able to compensate for having a larger number of additional copies of $y$. If no such compensation is possible, regardless of the quantity of $x$, this implies that, in fact, $x$ and $y$ have the same value.

# Formal Presentation of the Axiom

Comparative Worth.

- Consider two menus $x$ and $y$, within an instance $I = (m, c)$ where both are affordable, and $x$ is chosen ($x \in f(I)$).
- If for all $n_1, n_2 \in \mathbb{N}$ with $n_1 < n_2$, there exists an instance $I' = (m', c')$ where both $n_1 \cdot x$ and $n_2 \cdot y$ are affordable, but only $n_2 \cdot y$ is chosen ($n_2 \cdot y \in f(I')$ and $n_1 \cdot x \notin f(I')$),
- then, $y$ should also be chosen in the original instance ($y \in f(I)$).

# Alternative axiomatization

**Theorem**

*The Knapsack rule is the only budget allocation rule that satisfies Completeness, Transitivity, Weak Axiom of Revealed Preferences, Addition Consistency, and Comparative Worth.*

# Thank you!