On the Endogenous Order of Play in Sequential Games

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Motivation

In many strategic interactions the order of moves is endogenously determined by the players' decisions, in pursuit of their interest.

- Agenda formation
- Nomination of candidates for an office
- Electoral campaigns
- Political debates
- Firm competition, R&D
- ...

Yet the usual game theoretic modelling takes the order as given by a protocol or by chance, in both cases exogenously.

This is not innocuous as the predicted equilibrium outcome can be very sensitive to the assumed order of moves.

Our Contribution

We consider strategic interactions where players take their actions sequentially and there is no a priori given order of moves:

- At each state any player can take an action from a feasible set of actions.
- The game ends if no player wants to take any additional action or if no player has a feasible action anymore.

Players' payoffs may depend not only on the actions taken by the players but also on their order.

We call these games games of addition.

We propose a protocol-free equilibrium notion where players' actions and the order of moves are determined simultaneously.

The equilibrium notion extends a concept that was introduced by Dutta, Jackson and Le Breton (2004)[1] and was also applied in Barberà and Gerber (2022, 2023)[2].

[1] Dutta, B., M. O. Jackson, and M. Le Breton (2004): "Equilibrium Agenda Formation" *Social Choice and Welfare*, 23, 21-57.

[2] Barberà, S., and A. Gerber (2022): "Deciding on what to Decide," *International Economic Review*, 63, 37-61.

Barberà, S., and A. Gerber (2023): "(Not) Addressing Issues in Electoral Campaigns," *Journal of Theoretical Politics*, 35, 259-291.

Related Literature

Timing of actions in duopoly games

- Hamilton and Slutsky (1990)
- Deneckere and Kovenock (1992)
- Van Damme and Hurkens (2004)

Agenda setting

- Dutta, Jackson and LeBreton (2004)
- Vartiainen (2014)
- Barberà and Gerber (2022)

Electoral campaigns

- Kamada and Sugaya (2020)
- Barberà and Gerber (2023)

The Model

Definition

A **(finite) game of addition** is given by $(N, A, \Sigma, (\succsim_i)_{i \in N})$ that satisfies the following conditions:

(i) $N = \{1, \ldots, n\}$ with $n \geq 2$ is the player set.

(ii) A is a nonempty finite action set.

(iii) Σ is a set of states that has the property that there exists some $M \geq 1$ such that each state $\sigma \in \Sigma$ is either empty ($\sigma = \varnothing$) or is a sequence $\sigma = (s_1, \ldots, s_m)$ with $m \leq M$ where $s_k = (i_k, a_k)$ with $i_k \in N$ and $a_k \in A$ for all $k = 1, \ldots, m$.

(iv) $\succsim_i$ is a complete and transitive preference relation on Σ for all $i = 1, \ldots, n$.

For $\sigma \in \Sigma$ let

$$A^i(\sigma) = \{a | \, (\sigma, (i, a)) \in \Sigma\}.$$

Special case: Each action can be taken at most once.

- Common action sets (Dutta et al., 2004; Barberà and Gerber, 2022): For all i,

$$\begin{aligned} A^i(\varnothing) &= A \\ \text{and } A^i(\sigma) &= A \setminus \{a | \, (j, a) \in \sigma \text{ for some player } j\} \text{ for all } \sigma \neq \varnothing. \end{aligned}$$

- Individual action sets (Barberà and Gerber, 2023): For all i,

$$\begin{aligned} A^i(\varnothing) &= A^i \\ \text{and } A^i(\sigma) &= A^i \setminus \{a | \, (i, a) \in \sigma\} \text{ for all } \sigma \neq \varnothing, \end{aligned}$$

  where $A^i \subseteq A$ is a set of possible actions of player i.

Equilibrium

$\sigma' \in \Sigma$ is a continuation state at σ if $\sigma = \varnothing$ or if $\sigma = (s_1 \ldots, s_m) \in \Sigma$ and $\sigma' = (\sigma, \ldots)$. By definition σ is a continuation state at σ.

Let $C(\sigma)$ be the set of all continuation states at $\sigma \in \Sigma$.

A collection of sets of continuation states is a family of subsets of $C(\sigma)$ for each $\sigma \in \Sigma$.

Definition (Equilibrium Collections)

A collection of sets of continuation states $(CE(\sigma))_{\sigma \in \Sigma}$ is an equilibrium collection of sets of continuation states if the following three conditions are satisfied:

(E1) For all states σ, $CE(\sigma)$ is nonempty and any equilibrium continuation in $CE(\sigma)$ involves either stopping at σ or taking some action and then following an equilibrium path from there. Formal condition

(E2) For all states σ, stopping at σ is an equilibrium if and only if no player can improve by taking an action if one follows an equilibrium path from there. Formal condition

Definition (Contd.)

(E3) For all states σ, all equilibrium continuations at σ must be rationalizable and all rationalizable states are equilibrium continuations. Moreover, if all equilibrium continuations at σ are initiated by the same player then this player must weakly prefer the continuation over stopping at σ.

Formal condition

σ^* is an equilibrium state if there exists an equilibrium collection of sets of continuation states $(CE(\sigma))_{\sigma \in \Sigma}$ with $\sigma^* \in CE(\varnothing)$.

Examples: Two Players, One Action for Each Player

- Two players: player 1 and player 2.
- Each player has one action which can only be taken once:

$$A^1 = \{a\},\ A^2 = \{b\}.$$

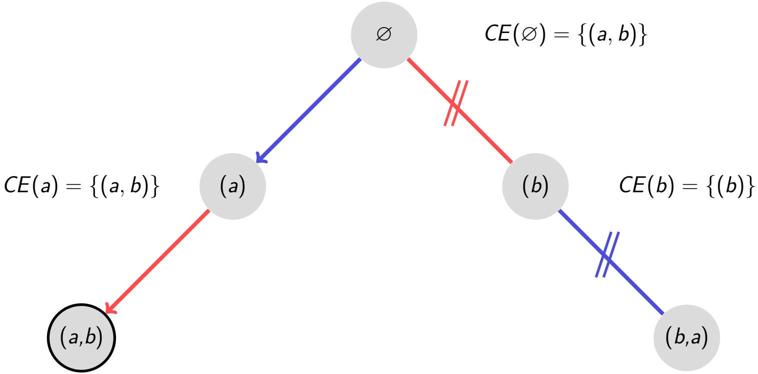
- Simplified notation for all states:

$$\varnothing,\ (a),\ (b),\ (a, b),\ (b, a).$$

Unique Equilibrium State

Player 1 $\quad (a) \succ_1 (b) \succ_1 (a,b) \sim_1 (b,a) \succ_1 \varnothing$

Player 2 $\quad \varnothing \succ_2 (a,b) \sim_2 (b,a) \succ_2 (b) \succ_2 (a).$



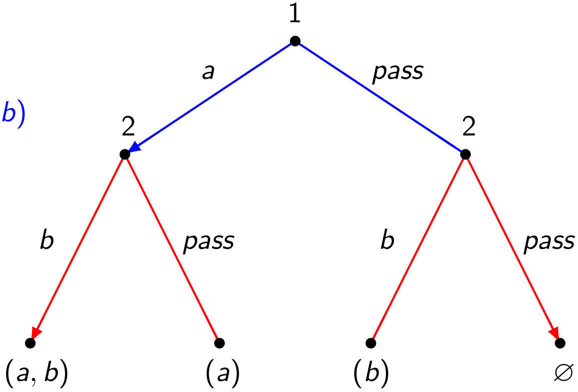
Difference to SPNE in Extensive Form Games

Player 1 $\quad (a) \succ_1 (b) \succ_1 (a,b) \sim_1 (b,a) \succ_1 \varnothing$

Player 2 $\quad \varnothing \succ_2 (a,b) \sim_2 (b,a) \succ_2 (b) \succ_2 (a).$

Player 1 is first mover

$\Rightarrow$ SPNE outcome (a, b)

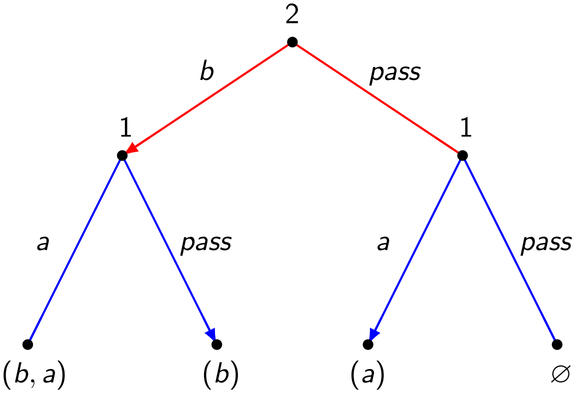


Player 1 $(a) \succ_1 (b) \succ_1 (a,b) \sim_1 (b,a) \succ_1 \varnothing$

Player 2 $\varnothing \succ_2 (a,b) \sim_2 (b,a) \succ_2 (b) \succ_2 (a)$.

Player 2 is first mover

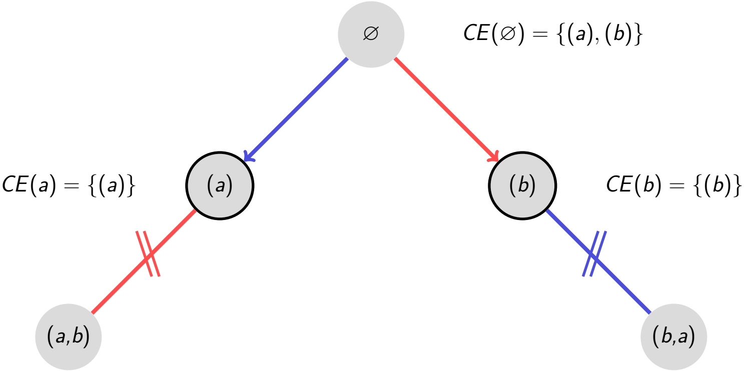
$\Rightarrow$ SPNE outcome (b)



Multiple Equilibrium States

Player 1 $(a) \succ_1 (b) \succ_1 \varnothing \succ_1 (a,b) \sim_1 (b,a)$

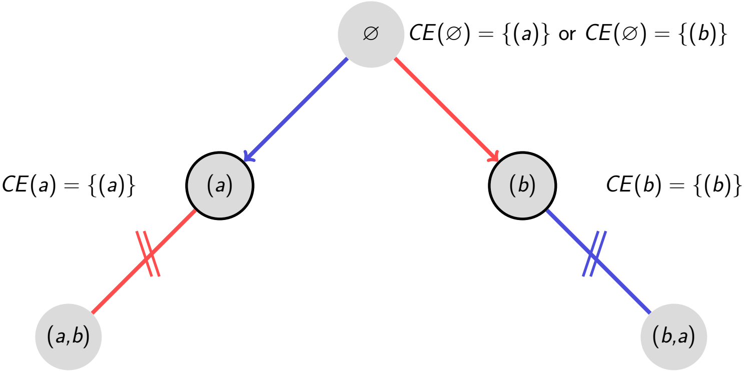
Player 2 $(b) \succ_2 (a) \succ_2 \varnothing \succ_2 (a,b) \sim_2 (b,a)$.



Multiple Equilibrium Collections

Player 1 $\quad (b) \succ_1 (a) \succ_1 \varnothing \succ_1 (a,b) \sim_1 (b,a)$

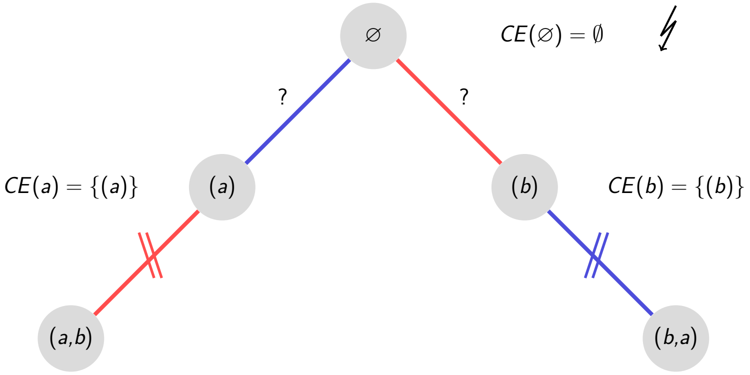
Player 2 $\quad (a) \succ_2 (b) \succ_2 \varnothing \succ_2 (a,b) \sim_2 (b,a).$



No Equilibrium

Player 1 $\varnothing \succ_1 (a) \succ_1 (b) \succ_1 (a, b) \sim_1 (b, a)$

Player 2 $(a) \succ_2 (b) \succ_2 \varnothing \succ_2 (a, b) \sim_2 (b, a).$



Games with Common Action Sets

Assume that there is a common action set for all players and that each action can only be taken once:

(C1) For all $i \in N$,

$$A^i(\varnothing) = A$$

$$\text{and} \quad A^i(\sigma) = A \setminus \{a | \, (j, a) \in \sigma \text{ for some player } j\} \text{ for all } \sigma \in \Sigma \setminus \varnothing.$$

Moreover, assume that players' preferences over states only depend on the sequence of actions that have been taken but not on the identities of the players who have taken the actions:

(C2) If $\sigma, \sigma' \in \Sigma$ are such that $\sigma|_A = \sigma'_A$, where $\sigma|_A = (a_1, \ldots, a_m)$ if $\sigma = ((i_1, a_1), \ldots, (i_m, a_m))$, then

$$\sigma \sim_i \sigma' \quad \text{for all } i \in N.$$

Existence for Games with Common Action Sets

Theorem

Let $(N, A, \Sigma, (\succsim_i)_{i \in N})$ be a game of addition such that that (C1) and (C2) are satisfied. Then there exists an equilibrium collection of sets of continuation states $(CE(\sigma))_{\sigma \in \Sigma}$.

Existence for Two-Player Zero-Sum Games

Theorem

Let $n = 2$ and let players' preferences be zero-sum, i.e. for all $\sigma, \sigma' \in \Sigma$,

$$\sigma \succsim_1 \sigma' \iff \sigma' \succsim_2 \sigma.$$

Then there exists an equilibrium collection of sets of continuation states $(CE(\sigma))_{\sigma \in \Sigma}$.

Special Case: Order Independent Preferences

- Let $n = 2$ and let each player i have K^i actions such that each action can only be taken once.
- For each $\sigma \in \Sigma$ and $i = 1, 2$, let $z^i(\sigma) \in \{0,1\}^{K^i}$ be given by

$$z_k^i(\sigma) = \begin{cases} 1, \text{ if } (i, a_k^i) \in \sigma \\ 0, \text{ if } (i, a_k^i) \notin \sigma \end{cases}$$

  for $k = 1, \ldots, K^i$.
- There exists some payoff function $p : \{0,1\}^{K^1} \times \{0,1\}^{K^2} \to \mathbb{R}$ such that for all $\sigma, \sigma' \in \Sigma$,

$$\sigma \succsim_1 \sigma' \iff p(z^1(\sigma), z^2(\sigma)) \geq p(z^1(\sigma'), z^2(\sigma'))$$

  and

$$\sigma \succsim_2 \sigma' \iff p(z^1(\sigma), z^2(\sigma)) \leq p(z^1(\sigma'), z^2(\sigma')).$$

Theorem (Uniqueness for order independent preferences)

Assume that preferences are order-independent and $p(z) \neq p(z')$ for all $z \neq z'$.

(i) *For all states σ, all continuation equilibria in $CE(\sigma)$ and all equilibrium collections of sets of continuation states are outcome equivalent.*

(ii) *For all states σ, the continuation equilibria can be characterized by properties of the unique payoff $p^*(z^1(\sigma), z^2(\sigma))$ of player 1 in all continuation equilibria at σ which satisfies*

$$\begin{aligned} \min_{k: z^2_k(\sigma)=0} p^*\left(z^1(\sigma), z^2(\sigma) + e^k\right) &\geq p^*\left(z^1(\sigma), z^2(\sigma)\right) \\ &\geq \max_{k: z^1_k(\sigma)=0} p^*\left(z^1(\sigma) + e^k, z^2(\sigma)\right), \end{aligned}$$

where e^k is the kth unit vector in $\mathbb{R}^K$.

Extensions

The result can be extended to 2-player zero-sum games

- where there are restrictions on the number of actions each player can take,
- where players can take the same action several times,
- where payoffs are path dependent and satisfy an additional condition.

Conclusion

- Protocol-free equilibrium concept: Actions must be taken sequentially, but no assumption on order of moves.
- An endogenous order of moves can lead to fundamentally different predictions than subgame perfect Nash equilibrium in an extensive form game where the order of moves is fixed.
- We have general existence results for large classes of games.
- For two-player zero-sum games we have provided sufficient conditions for a unique equilibrium outcome.

THANKS!

Condition (E1)

For all $\sigma \in \Sigma$, $CE(\sigma)$ is a nonempty subset of

$$\bigcup_{i=1}^{n} \bigcup_{a \in A^i(\sigma)} CE(\sigma, (i, a)) \cup \{\sigma\}.$$

Back

Condition (E2)

For all $\sigma \in \Sigma$, $\sigma \in CE(\sigma)$ if and only if for all $i = 1, \ldots, n$,

$$\sigma \succsim_i \sigma' \text{ for all } \sigma' \in \bigcup_{a \in A^i(\sigma)} CE(\sigma, (i, a)).$$

Back

Rationalizability

Define $\sigma' = (\sigma, (i, a'), \ldots) \in \Sigma$ to be rationalizable relative to $\sigma \in \Sigma$ if $\sigma' \in CE(\sigma, (i, a'))$ and the following conditions are satisfied:

(i) If there exists some $j \neq i$ with $(\sigma, (j, a), \ldots) \in CE(\sigma)$, then there exists some $\sigma'' \in CE(\sigma)$ with $\sigma'' = (\sigma, (k, a''), \ldots)$ for some $k \neq i$ (k may but need not be equal to j) such that $\sigma' \succsim_i \sigma''$.

(ii) For all $a'' \neq a'$ there exists some $\sigma'' \in CE(\sigma, (i, a''))$ such that $\sigma' \succsim_i \sigma''$.

(iii) If $\sigma \in CE(\sigma)$ then also $\sigma' \succsim_i \sigma$.

Condition (E3)

Let $\sigma \in \Sigma$.

- If $\sigma' \in CE(\sigma, (i, a'))$ is rationalizable, then $\sigma' \in CE(\sigma)$.
- Conversely, if $\sigma' = (\sigma, (i, a'), \ldots) \in CE(\sigma)$ and either $\sigma \in CE(\sigma)$ or $\sigma'' = (\sigma, (j, a''), \ldots) \in CE(\sigma)$ for some $(j, a'') \neq (i, a')$, then σ' is rationalizable.
- If $\sigma' = (\sigma, (i, a'), \ldots) \in CE(\sigma)$ and $\nexists j \neq i$ such that $\sigma'' = (\sigma, (j, a''), \ldots) \in CE(\sigma)$ for some $a'' \in A^j(\sigma)$, then $\sigma' \succsim_i \sigma$.

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Definition

A **(finite) extensive form game with perfect information** is given by $(N, A, H, P, (\succsim_i)_{i \in N})$ that satisfies the following conditions:

(i) $N = \{1, \ldots, n\}$ with $n \geq 2$ is the player set.

(ii) A is a nonempty finite action set.

(iii) H is a set of histories that has the property that there exists some $M \geq 1$ such that each history $h \in H$ is either empty $(h = \varnothing)$ or is a sequence $h = (a_1, \ldots, a_m)$ with $m \leq M$ where $a_k \in A$ for all $k = 1, \ldots, m$. A history $h = (a_1, \ldots, a_m)$ is terminal if there exist no a_{m+1} such that $(a_1, \ldots, a_m, a_{m+1}) \in H$.

(iv) P is a player function that assigns a player $P(h) \in N$ to every nonterminal history h, i.e. $P(h)$ is the player who takes an action after the history h.

(v) $\succsim_i$ is a complete and transitive preference relation on the set of terminal histories for all $i = 1, \ldots, n$.

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