New fairness concepts for allocating indivisible items

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This talk …

- An overview of well-known results in fair division with indivisible items
- New technical results (C., Garg, Rathi, Sharma, & Varriccio, 2023)

Fair division: some indicative problems

- An **inheritance**, consisting of a jewellery collection, pieces of antique furniture, and estate property, is to be divided among heirs
- Food donated to a **food bank** has to be given to charities
- Access to **rainwater reservoirs** has to be granted to farmers
- A **territorial dispute** has to be resolved between neighbouring countries
- A **partnership is dissolved**, and the ex-partners have to split assets and liabilities
- Responsibility for the **protection of refugees** has to be shared among EU countries

The research agenda: conceptual and computational challenges in fair division

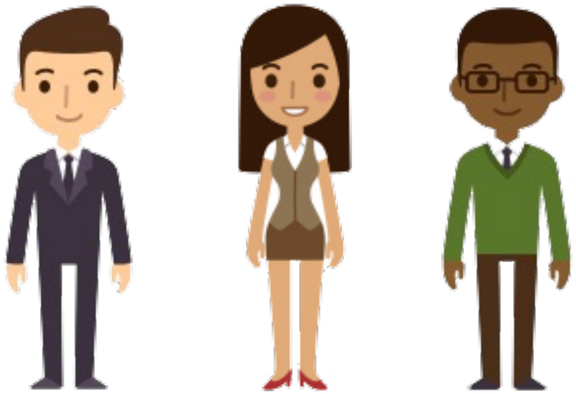
- **Computational questions**: How should fair division procedures for these scenarios work?
- Before that: need to define **fairness as a concept**

Allocating indivisible items

The basic setting

- **Indivisible** items

- **Agents** with **valuations** for the items (**additivity**)



- Goal: **divide** the items among the agents **in a fair manner**

An example

					
	$1000	$200	$600	$100	$100
	$700	$500	$100	$400	$300
	$500	$700	$400	$200	$200

An example

	**$1000**	$200	$600	$100	$100
	$700	$500	$100	**$400**	**$300**
	$500	**$700**	**$400**	$200	$200

What does “fairly” mean?

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- Two **interpretations of fairness**:
  - **Comparative**: to evaluate an allocation as fair, each agent compares the bundle of items allocated to her to the bundles allocated to other agents
  - **In absolute terms**: each agent defines a **threshold value** based on her view of the items to be allocated and evaluates as fair those allocations which give her value higher than the threshold

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- **Fairness notions**
  - **Envy freeness**: every agent prefers her bundle to that given to any other agent

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For every pair of agents i and j

Value of agent i for the bundle A_i allocated to her

Value of agent i for the bundle A_j allocated to agent j

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$$\forall i, j\colon\ v_i(A_i) \geq v_i(A_j)$$
  - **Proportionality**: every agent feels that she gets at least $1/n$-th of all items
$$\forall i\colon\ v_i(A_i) \geq \frac{1}{n} v_i(G)$$

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- **Fair…**
  - **E…** …gent prefe… …er bundle to … …gent

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Value of agent i for the bundle A_i allocated to her

For every agent i

Threshold value: $1/n$-th of the total value of agent i for all items

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- **Fairness notions**
  - Unfortunately, **envy free** and **proportional** allocations may not exist ☹

When is a fairness concept important/useful?

- Must be **fair** ☺
- Should always **exist**
- Must be **efficiently computable**

Relaxing envy-freeness

- **Envy freeness up to some item (EF1)**: every agent prefers her own bundle to the bundle of any other agent after eliminating some item from the latter

$$\forall i, j: \exists g \in A_j \text{ s.t. } v_i(A_i) \geq v_i(A_j \setminus \{g\})$$

- Proposed by Budish (2011)

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- EF1 always exist and can be computed in polynomial time
- Via the **draft mechanism** (folklore), **envy-cycle elimination** (Lipton, Markakis, Mossel, & Saberi, 2004), the **maximum Nash welfare allocation** (C., Kurokawa, Moulin, Procaccia, Shah, & Wang, 2019)

The draft mechanism

- Drafting order: 

$1200	$200	$300	$200	$100
$800	$500	$200	$300	$200
$800	$400	$400	$300	$100

The draft mechanism

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	$1200	$200	$300	$200	$100
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Envy-cycle elimination

- Allocate items **one by one**
- In each step j:
  - Allocate item j **to an agent that nobody envies**
  - If this creates a “cycle of envy”, **redistribute the bundles along the cycle**
- Crucial property:
  - Envy can be eliminated by removing just a **single good**
  - Implies **EF1**
- Lipton, Markakis, Mossel, & Saberi (2004)

Relaxing envy-freeness

- **Envy freeness up to any item (EFX)**: every agent prefers her own bundle to the bundle of any other agent after eliminating **any** item from the latter
$$\forall i, j, \forall g \in A_j:\ v_i(A_i) \geq v_i(A_j \setminus \{g\})$$
- Proposed by C., Kurokawa, Moulin, Procaccia, Shah, & Wang (2019), Gourves, Monnot, & Tilane (2014)

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- Proposed by C., Kurokawa, Moulin, Procaccia, Shah, & Wang (2019), Gourves, Monnot, & Tilane (2014)
- Not known whether it always exists for general instances
- Known results for agents with identical valuations, ordered valuations, three agents, and a few more
  - Plaut & Roughgarden (2020), Chaudhuri, Garg, & Mehlhorn (2020)
- Known results for relaxations of EFX (approximations, EFX with charity, etc.)
  - Amanatidis, Markakis, & Ntokos (2020), C., Gravin, & Huang (2019), Chaudhuri, Kavitha, Mehlhorn, & Sgouritsa (2021), Chaudhuri, Garg, Mehlhorn, Ruta, & Misra

Relaxing proportionality

- **Maximin share fairness (MMS)**: each agent's threshold is equal to the best guarantee when dividing the items into n bundles and getting the least valuable bundle

$$\forall i, v_i(A_i) \geq \theta_i = \max_B \min_j v_i(B_j)$$

- Proposed by Budish (2011)

For every agent i

Agent i's value is above the MMS threshold

MMS threshold = the maximum over all allocations B of the minimum value agent i has from B's bundles

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- Proposed by Budish (2011)
- Unfortunately, MMS allocations **may not exist**
  - Procaccia & Wang (2014), Kurokawa, Procaccia, & Wang (2018)
- Research has focused on **achieving MMS-approximations** in poly time
  - Amanatidis, Markakis, Nikzad, & Saberi (2017), Ghodsi, Hajiaghayi, Seddighin, Seddighin, & Yami (2018), Barman & Krishnamurthy (2020), Garg & Taki (2020)

Summarizing so far

- EF1: always exists, easy to achieve, **not fair**
- EFX: **not known whether it can be always satisfied**, fair
- MMS: **may not exist**, fair (if exists)

- See Bouveret & Lemaitre (2016), Aziz, Bouveret, C., Giagkousi, & Lang (2018) for taxonomies including more fairness concepts

Summarizing so far

- EF1: always exists, easy to achieve, **no**
- EFX: **not known whether it can be alw**
- MMS: **may not exist**, fair (if exists)

Still, EFX seems to be the most promising fairness property we have for indivisible items

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New fairness concepts

Fairness and knowledge

- What kind of **knowledge** do the agents need to have?
- Knowledge about the **items** and the **number of agents** only:
  - Proportionality, MMS
- Knowledge about the **whole allocation**:
  - EF, EFX, EF1

Epistemic envy-freeness (EEF)

- Informally: a **relaxation of EF** with a definition that uses only knowledge about items and number of agents
- Formal definition: the allocation $(A_1, A_2, \ldots, A_n)$ is EEF if, for every agent i, there is a **reallocation** $(B_1, \ldots, B_{i-1}, A_i, B_{i+1}, \ldots, B_n)$ of the items in which agent i is not envious, i.e., $v_i(A_i) \geq v_i(B_j)$ for every other agent j
- Aziz, C., Bouveret, Giagkousi, & Lang (2018)
- Unfortunately, EEF allocations may not exist

Epistemic envy-freeness up to any item (EEFX)

- Informally: a **relaxation of EFX** with a definition that uses only knowledge about items and number of agents
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$$\forall i, j \neq i, \forall g \in B_j \colon\ v_i(A_i) \geq v_i(A_j \setminus \{g\})$$

- C., Garg, Rathi, Sharma, & Varricchio (2023)

Minimum EFX value fairness (MXS)

- Informally: Each agent i gets a value that is at least as high as the minimum value agent i gets among all allocations where the EFX conditions for her are satisfied
- Formal definition: the allocation $(A_1, A_2, \dots, A_n)$ is **MXS** if
$$\forall i: v_i(A_i) \geq \theta_i = \min_{B \in EFX_i} v_i(B_i)$$
where the set EFX_i consists of those allocations $B = (B_1, B_2, \dots, B_n)$ such that
$$\forall j \neq i, g \in B_j: \; v_i(B_i) \geq v_i(B_j \setminus \{g\})$$
- C., Garg, Rathi, Sharma, & Varricchio (2023)

MXS: an example

			ROLLS ROYCE		
	$500	$600	$200	$400	$300
	$700	$700	$300	$200	$100
	$900	$600	$200	$200	$100

MXS: an example

Let’s compute the MXS threhsolds first

			ROLLS ROYCE			θ_i
	\$500	\$600	\$200	\$400	\$300	
	\$700	\$700	\$300	\$200	\$100	
	\$900	\$600	\$200	\$200	\$100	

MXS: an example

Let's compute the MXS threhsolds first

						θ_i
	\$500	\$600	\$200	\$400	\$300	\$500
	\$700	\$700	\$300	\$200	\$100	\$600
	\$900	\$600	\$200	\$200	\$100	\$500

MXS: an example

Now let's compute the allocation

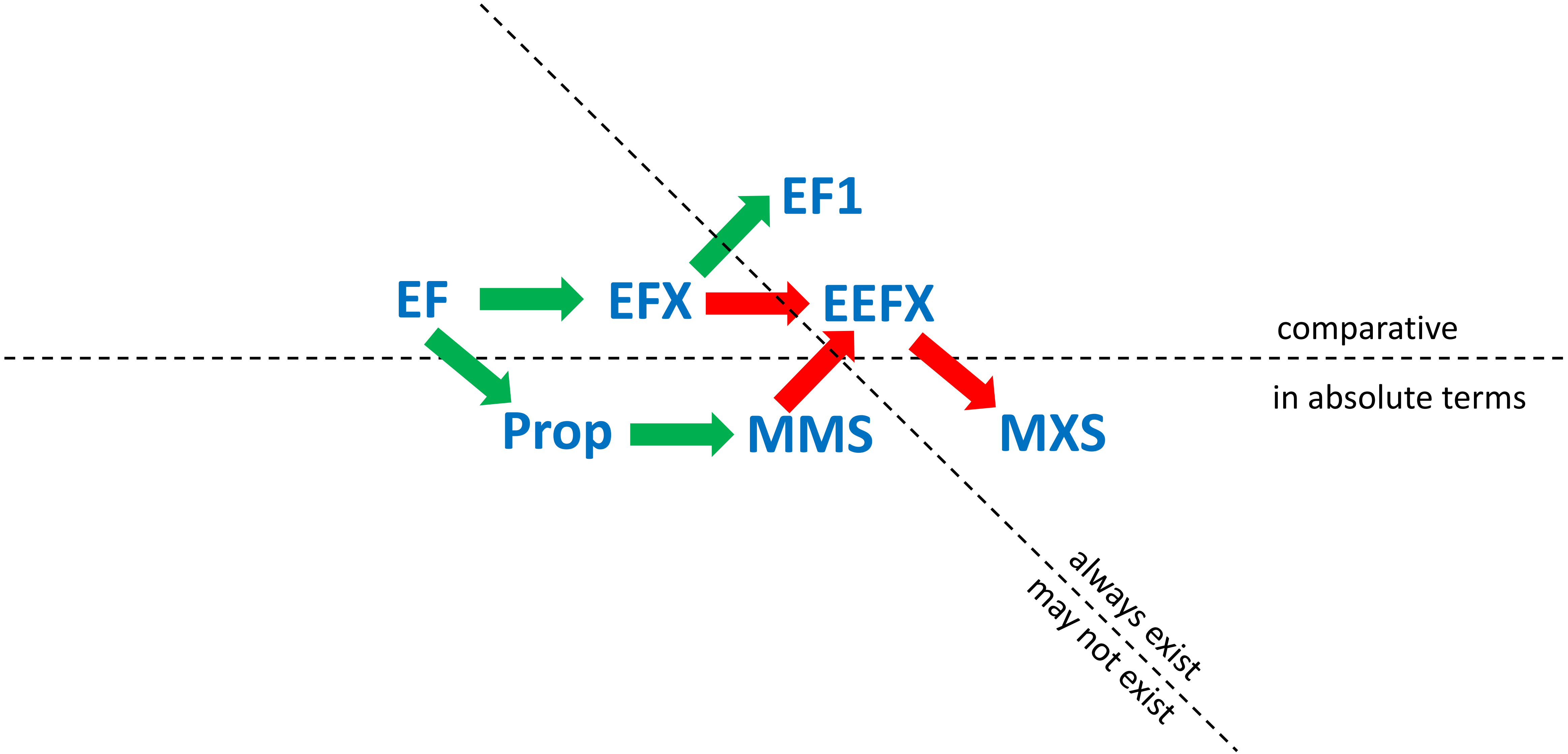
			ROLLS ROYCE			θ_i
	\$500	\$600	\$200	\$400	\$300	\$500
	\$700	\$700	\$300	\$200	\$100	\$600
	\$900	\$600	\$200	\$200	\$100	\$500

MXS: an example

Now let's compute the allocation

			ROLLS ROYCE			θ_i
	**$500**	$600	$200	$400	$300	$500
	$700	**$700**	$300	$200	$100	$600
	$900	$600	**$200**	**$200**	**$100**	$500

A geometry of fairness properties



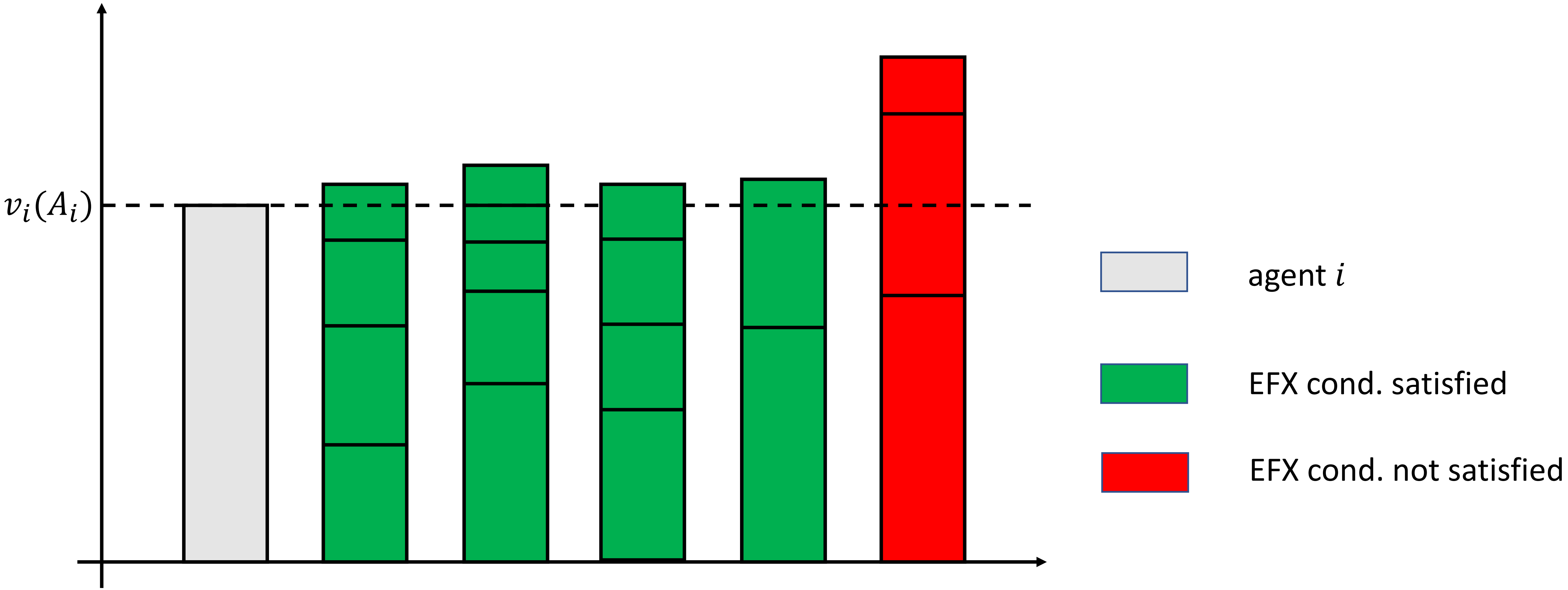
EEFX ➡ MXS

- Proof: Let $A = (A_1, \ldots, A_n)$ be **EEFX**. Then, for every agent i, there exists a reallocation $B = (B_1, \ldots, B_{i-1}, A_i, B_{i+1}, \ldots, B_n)$ so that the EFX conditions are satisfied for agent i, $B \in EFX_i$
- Hence,

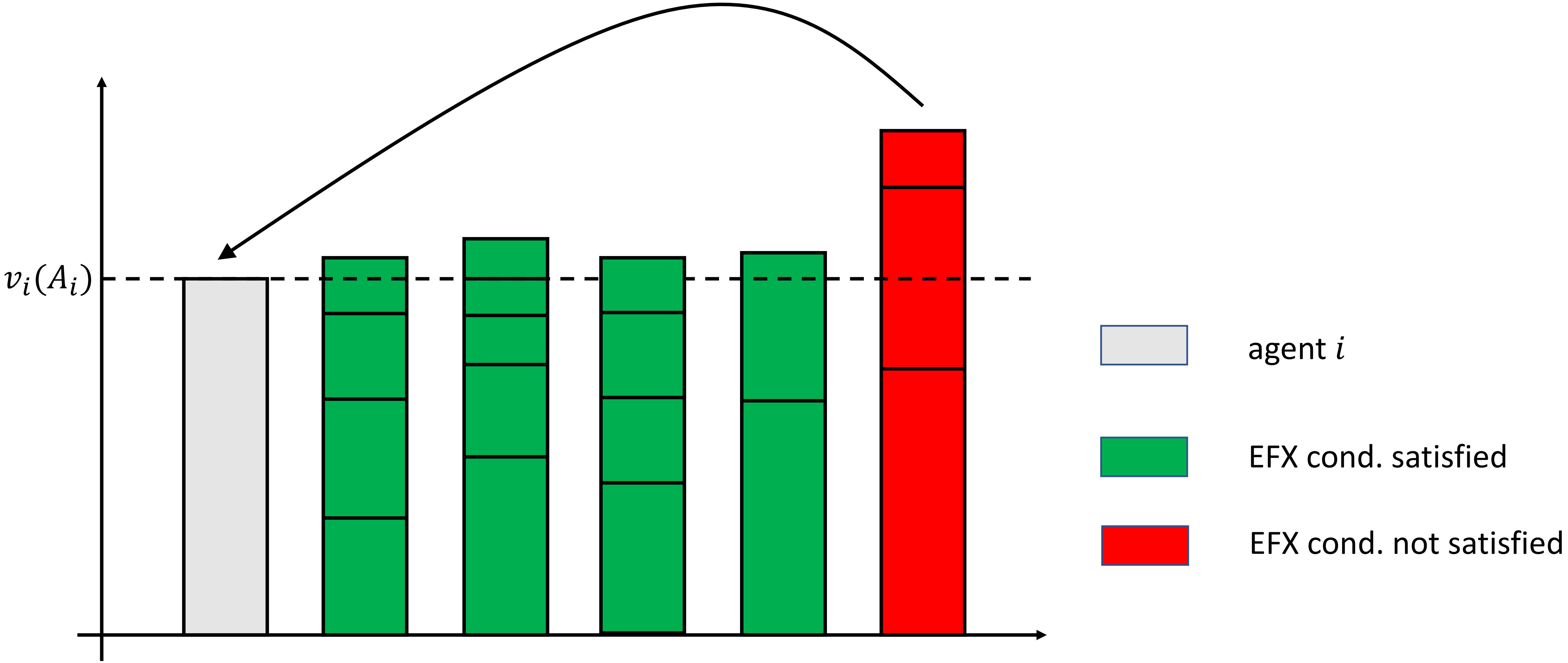
$$v_i(A_i) \geq \min_{B' \in EFX_i} v_i(B'_i) = MXS_i$$

- I.e., A is also **MXS**

MMS → EEFX



MMS ➡ EEFX (contd.)



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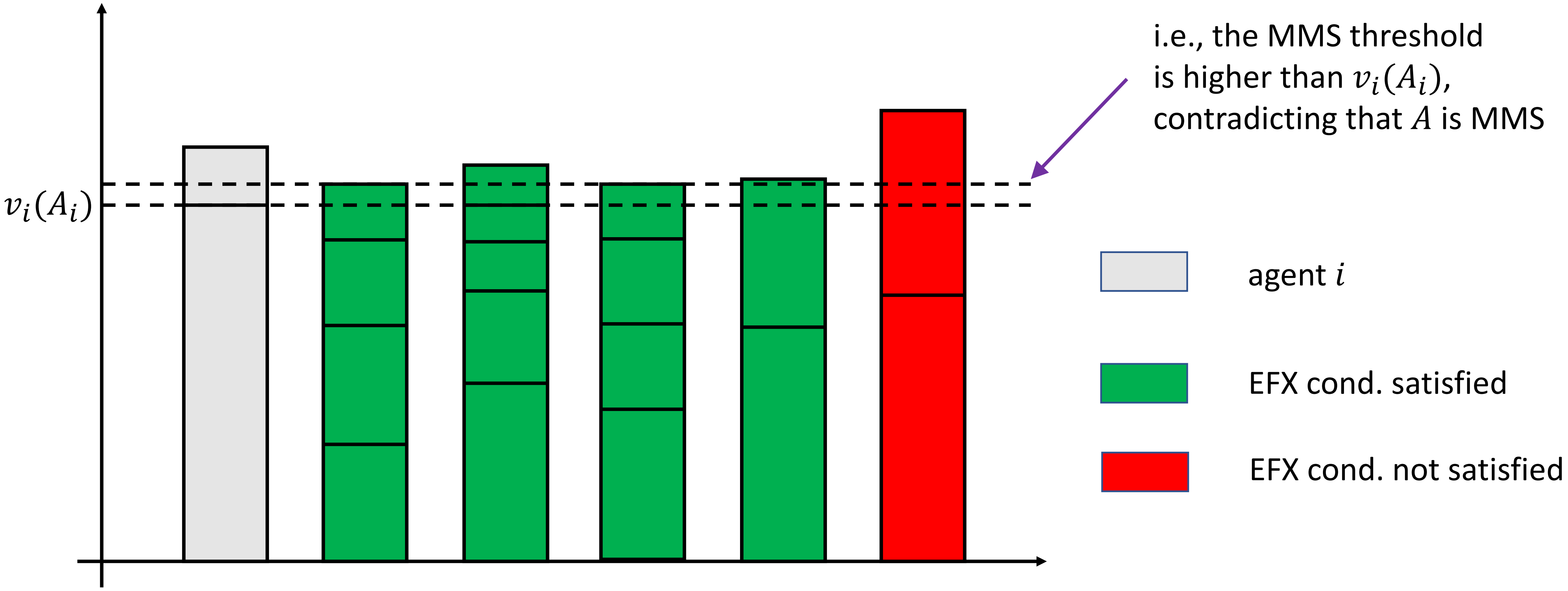

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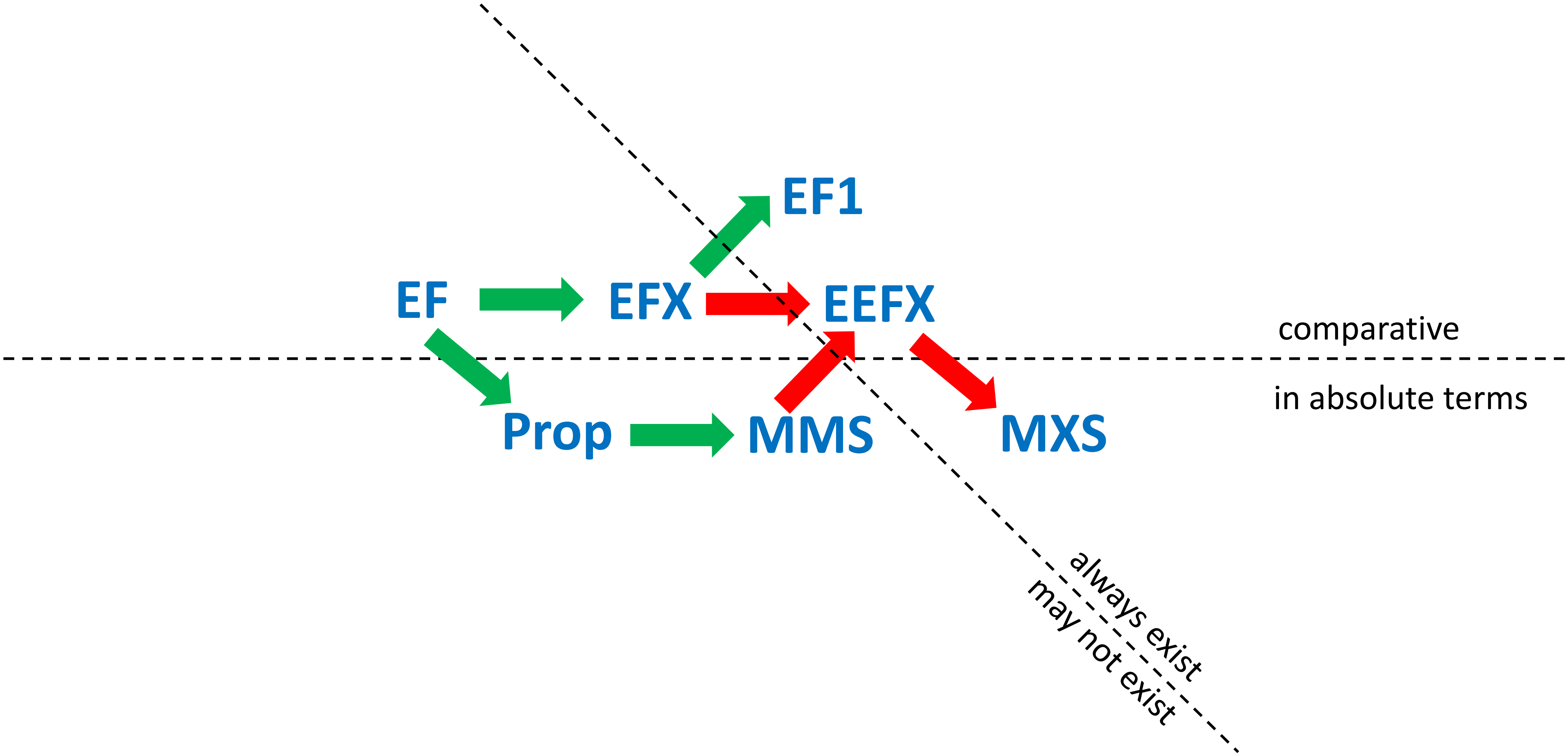

Main result: EEFX and MXS are awesome!

- Theorem: EEFX and MXS allocations **always exist** and **can be computed in polynomial time**

An algorithm for EEFX (and MXS)

- Step 1: Enumerate the items as $g_1, g_2, \ldots, g_m$ and **redistribute the values** so that each agent has her j-th highest value for item g_j
- Step 2: Run **envy-cycle elimination** on this **ordered instance**
- Step 3: Redistribute the items to the bundles. For $j = 1, \ldots, m$, agent who currently has item g_j is asked to **pick her best available item**

Envy-cycle elimination (implementation of step 2)

- Lipton, Markakis, Mossel, & Saberi (2004)
- Allocate items **one by one** (**ordered** from the most to the least valued one)
- In each step j:
  - Allocate item j **to an agent that nobody envies**
  - If this creates a “cycle of envy”, **redistribute the bundles along the cycle**
- Crucial property:
  - Envy can be eliminated by removing a **single item** (the last one inserted in a bundle)
  - Implies **EF1** (actually, **EFX**)
- Barman & Krishnamourthy (2020)

An example

	$500	$600	$200	$400	$300
	$700	$700	$300	$200	$100
	$900	$600	$200	$200	$100

Step 1: redistributing the values

	1	2	3	4	5
	$600	$500	$400	$300	$200
	$700	$700	$300	$200	$100
	$900	$600	$200	$200	$100

Step 2: envy-cycle elimination

	1	**2**	**3**	**4**	**5**
	$600	$500	$400	$300	$200
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Step 3: redistribute items to the bundles

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picking sequence

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An algorithm for EEFX (and MXS)

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  - Bouveret & Lemaitre (2016)

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- Step 2: Run envy-cycle elimination to this ordered instance
  - Yields an EFX allocation for the ordered instance (Barman & Krishnamourthy, 2020, Plaut & Roughgarden, 2020)

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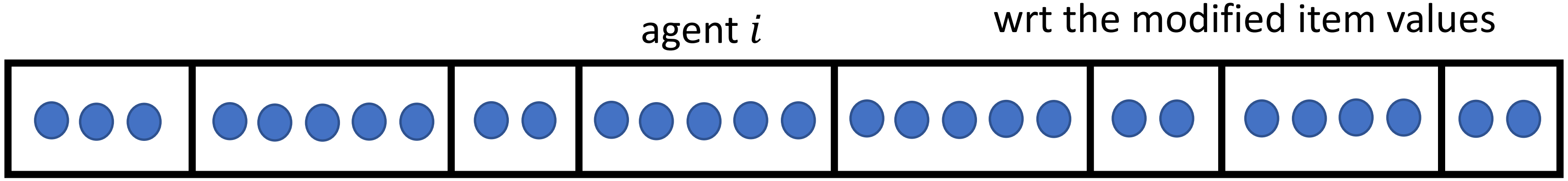
What happens at step 3?

at the end of step 2, agent i is EFX-happy wrt the modified item values



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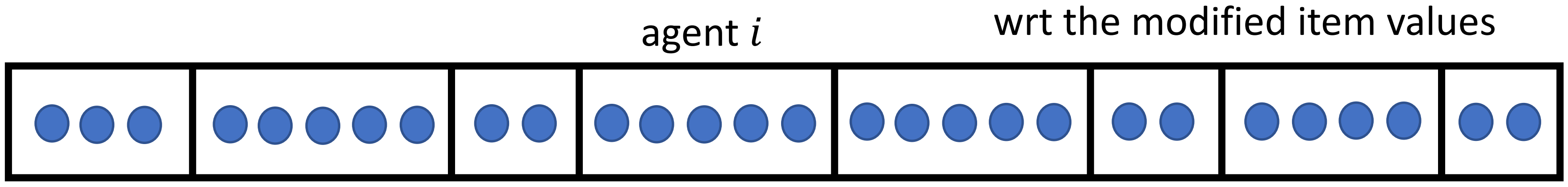
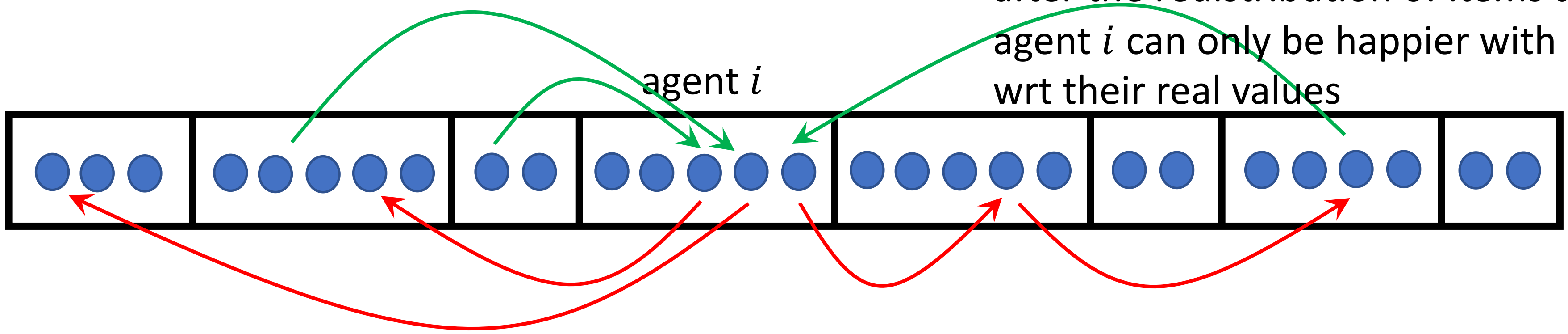


after the redistribution of items at step 3 agent i can only be happier with her items wrt their real values



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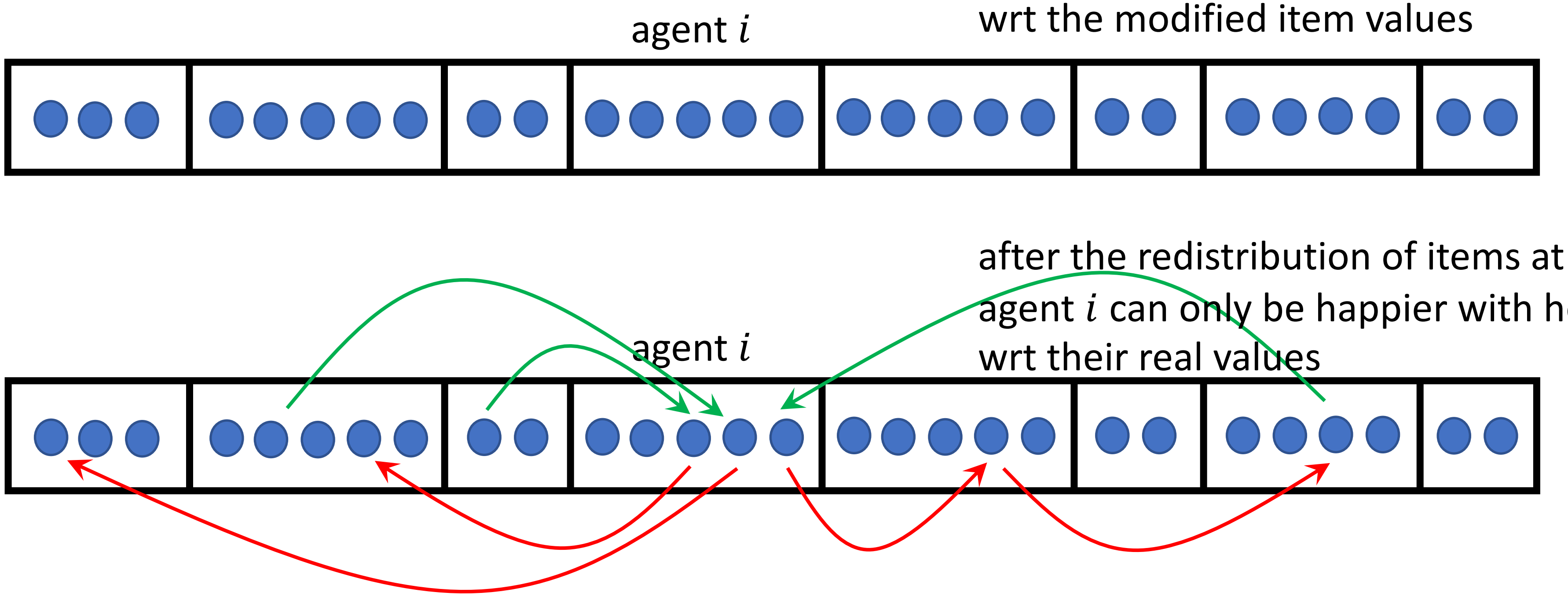

after the redistribution of items at step 3 agent i can only be happier with her items wrt their real values

… and there is a redistribution of the items (i.e., their assignment at the end of step 2) which makes agent i EFX-happy

Takeaway message

- **EFX is still an important property** and we should further explore it
- But why not focusing on **alternative fairness concepts** in parallel?
- In particular, on concepts that are related to it, like EEFX and MXS
- **Reconsider existing algorithms** (they may do more than we think)

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Thank you!