Arguing about voting rules

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Introduction

Context

- Voting rule: a systematic way of aggregating different opinions and decide
- Multiple reasonable ways of doing this
- Different voting rules have different interesting properties
- None satisfy all desirable properties

Our goal

We want to easily communicate about strength and weaknesses of voting rules.

Voting rule

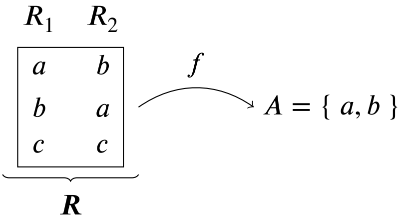
Alternatives $\mathscr{A} = \{ a, b, c, d, \dots \}$; $|\mathscr{A}| = m$

Possible voters $\mathscr{N} = \{ 1, 2, \dots \}$

Voters $\emptyset \subset N \subseteq \mathscr{N}$

Profile partial function $\boldsymbol{R}$ from $\mathscr{N}$ to linear orders on $\mathscr{A}$.

Voting rule function f mapping each $\boldsymbol{R}$ to winners $\emptyset \subset A \subseteq \mathscr{A}$.



Borda

- Jean-Charles de Borda, 1733–1799
- Given a profile $\boldsymbol{R}$:
- count the score of each alternative;
- the highest scores win.
- Score of $a \in \mathscr{A}$ for voter $i \in N$ is m minus its rank for that voter.

$$\boldsymbol{R} = \begin{matrix} a & b & b \\ d & c & a \\ c & a & c \\ b & d & d \end{matrix} .$$

- score a is…?

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$$\boldsymbol{R} = \begin{array}{ccc} a & b & b \\ d & c & a \\ c & a & c \\ b & d & d \end{array}.$$

- score a is…? $3 + 1 + 2 = 6$
- score b is $0 + 3 + 3 = 6$
- score c is $1 + 2 + 1 = 4$
- score d is $2 + 0 + 0 = 2$

Winners are $\{ a, b \}$.

Condorcet's principle

An idea from Marie Jean Antoine Nicolas de Caritat, marquis de Condorcet (1743–1794).

Condorcet's principle

We ought to take the Condorcet winner as sole winner if it exists.

- a *beats* b iff more than half the voters prefer a to b.
- a is a *Condorcet winner* iff a beats every other alternatives.

$$\boldsymbol{R} = \begin{matrix} a & b & b \\ d & c & a \\ c & a & c \\ b & d & d \end{matrix} .$$

How are voting rules analyzed?

- Examples featuring counter-intuitive results for some voting rules.
- Properties of voting rules, e.g. Borda does not satisfy Condorcet's principle.
- Axiomatization of a voting rule: accepting such principles lead to a unique voting rule.

Our objective

- Different voting rules
- Arguments in favor or against rules
- Dispersed in the literature
- Using mathematical formalism

We propose

- Common language
- Instantiate arguments on concrete examples

Goal: help understand strengths and weaknesses of given rules.

Outline

1 Context

2 Language

3 Arguing for Borda

4 Conclusion

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Example of axiom

- Dominance: if a dominates b in $\boldsymbol{R}$, then b may not win.
- We want a language to express this kind of axioms.

Language

We use propositional logic (with connectives $\neg, \vee, \wedge, \rightarrow$).

Atoms

- One atom for each $(\boldsymbol{R}, A)$, $\emptyset \subset A \subseteq \mathscr{A}$.
- An atom talks about assigning winners A to $\boldsymbol{R}$.
- Written $[\boldsymbol{R} \longmapsto A]$.

Semantics

Semantics v_f, given a voting rule f:

$$v_f([\boldsymbol{R} \longmapsto A]) = \mathrm{T} \text{ iff } f(\boldsymbol{R}) = A.$$

Shortcut notations

$\mathscr{P}_{\emptyset}(\mathscr{A})$ the set of subsets of $\mathscr{A}$, excluding the empty set.

Let $\alpha \subseteq \mathscr{P}_{\emptyset}(\mathscr{A})$ be a set of possible winning alternatives.

Uni-profile clause

$[\boldsymbol{R} \overset{\in}{\longmapsto} \alpha]$ shortcut for:

$$\bigvee_{A \in \alpha} [\boldsymbol{R} \longmapsto A].$$

- Intuitive content.
- Called a uni-profile clause.

Domain knowledge

- We need some formulæ encoding the voting rule concept.
- Define κ as the set of all those formulæ.

Domain knowledge κ

1. a voting rule can't select more than one set of winners: for all $\boldsymbol{R}$ and all $\emptyset \subset A \neq B \subseteq \mathscr{A}$,
$$[\boldsymbol{R} \longmapsto A] \wedge [\boldsymbol{R} \longmapsto B] \rightarrow \bot.$$
2. a voting rule must select at least one set of winners: for all $\boldsymbol{R}$,
$$[\boldsymbol{R} \overset{\in}{\longmapsto} \mathscr{P}_{\emptyset}(\mathscr{A})].$$

L-axioms

- Now: “translate” axioms into language-axioms.
- An *l-axiom* is a set of formulæ.

Example (Dom)

L-axiom Dom: for each $\boldsymbol{R}$,

$$[\boldsymbol{R} \xmapsto{\in} \mathscr{P}_{\emptyset}(U_{\boldsymbol{R}})],$$

with $U_{\boldsymbol{R}}$ the set of alternatives in $\boldsymbol{R}$ that are not dominated.

Symmetric cancellation l-axiom

Example (Sym)

For each $\boldsymbol{R}$ consisting of a linear order and its inverse,

$$[\boldsymbol{R} \longmapsto \mathscr{A}].$$

Reinforcement l-axiom

Classical reinforcement axiom: consider

- $\boldsymbol{R}_1$, $\boldsymbol{R}_2$,
- having winners A_1, A_2,
- with $A_1 \cap A_2 \neq \emptyset$;

then winners in $\boldsymbol{R}_1 + \boldsymbol{R}_2$ must be $A_1 \cap A_2$.

Definition (Reinf)

For each $\boldsymbol{R}_1, \boldsymbol{R}_2, A_1, A_2 \subseteq \mathscr{A}, A_1 \cap A_2 \neq \emptyset$:

$$([\boldsymbol{R}_1 \longmapsto A_1] \wedge [\boldsymbol{R}_2 \longmapsto A_2]) \rightarrow [\boldsymbol{R}_1 + \boldsymbol{R}_2 \longmapsto A_1 \cap A_2].$$

Fishburn-against-Condorcet argument

Fishburn (1974, p. 544) argument against the Condorcet principle (see also `http://rangevoting.org/FishburnAntiC.html`).

Condorcet winner

$$w \text{ VS } ?, ? \in \{a, \ldots, h\} : \frac{51}{101}.$$

	nb voters					
	31	19	10	10	10	21
1	a	a	f	g	h	h
2	b	b	w	w	w	g
3	c	c	a	a	a	f
4	d	d	h	h	f	w
5	e	e	g	f	g	a
6	w	f	e	e	e	e
7	g	g	d	d	d	d
8	h	h	c	c	c	c
9	f	w	b	b	b	b

	ranks								
	1	2	3	4	5	6	7	8	9
w	0	30	0	21	0	31	0	0	19
a	50	0	30	0	21	0	0	0	0

Fishburn-versus-Condorcet l-axiom

Define $\boldsymbol{R}_F$ the profile shown in the previous slide.

Definition (Fishburn-versus-Condorcet)

The Fishburn-versus-Condorcet l-axiom is defined as:

$$[\boldsymbol{R}_F \xmapsto{\in} \mathscr{P}_{\emptyset}(A_{\boldsymbol{R}_F} \setminus \{ w \})].$$

L-axiomatization

An l-axiomatization is a set of l-axioms.

Definition (Conforming to J)

The rule f conforms to the l-axiomatization J iff v_f assigns the value $\mathtt{T}$ to all formulæ in j, for all $j \in J$.

An l-axiomatization is consistent iff there exists a voting rule conformant to it.

Arguments

Definition (Argument)

An argument grounded on J is a pair (*claim*, *proof*),

- J an I-axiomatization,
- *claim* a uni-profile clause (thus of the form $[\boldsymbol{R} \xmapsto{\in} \alpha]$),
- *proof* a natural deduction proof of the claim grounded on J.

- The argument shows that for all voting rules f conformant to J, $f(\boldsymbol{R})$ selects a set of winners among α.
- The argument claims that it is only reasonable to choose the winners among α for $\boldsymbol{R}$ (provided J is accepted).
- *Consistent* arguments require a consistent I-axiomatization.

A simple argument

Claim

Consider:

- $R = \begin{matrix} a & b & a & c \\ b & c & b & b \\ c & a & c & a \end{matrix}$,
- $J = \left\{ \textsc{Dom}, \textsc{Sym}, \textsc{Reinf} \right\}$.

We can prove that for f compliant with J:

$$[R \xmapsto{\in} \{ \{ a \}, \{ b \}, \{ a, b \} \}].$$

See how?

A simple argument

Claim

Consider:

- $\boldsymbol{R} = \begin{matrix} a & b & a & c \\ b & c & b & b \\ c & a & c & a \end{matrix}$,
- $J = \{ \textsc{Dom}, \textsc{Sym}, \textsc{Reinf} \}$.

We can prove that for f compliant with J:

$$[\boldsymbol{R} \stackrel{\in}{\longmapsto} \{ \{ a \}, \{ b \}, \{ a, b \} \}].$$

See how? Consider $\boldsymbol{R}_D = \begin{matrix} a & b \\ b & c \\ c & a \end{matrix}$, $\boldsymbol{R}_S = \begin{matrix} a & c \\ b & b \\ c & a \end{matrix}$, $\boldsymbol{R} = \boldsymbol{R}_D + \boldsymbol{R}_S$.

Example proof

$$\boldsymbol{R}_D = \begin{matrix} a & b \\ b & c \\ c & a \end{matrix}, \boldsymbol{R}_S = \begin{matrix} a & c \\ b & b \\ c & a \end{matrix}, \boldsymbol{R} = \boldsymbol{R}_D + \boldsymbol{R}_S = \begin{matrix} a & b & a & c \\ b & c & b & b \\ c & a & c & a \end{matrix}.$$

1. $[\boldsymbol{R}_D \overset{\in}{\longmapsto} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$ (Dom)
2. $[\boldsymbol{R}_S \longmapsto \{ a, b, c \}]$ (Sym)
3. $([\boldsymbol{R}_D \longmapsto \{ a \}] \wedge [\boldsymbol{R}_S \longmapsto \{ a, b, c \}]) \rightarrow [\boldsymbol{R} \longmapsto \{ a \}]$ (Reinf)
4. $([\boldsymbol{R}_D \longmapsto \{ b \}] \wedge [\boldsymbol{R}_S \longmapsto \{ a, b, c \}]) \rightarrow [\boldsymbol{R} \longmapsto \{ b \}]$ (Reinf)
5. $([\boldsymbol{R}_D \longmapsto \{ a, b \}] \wedge [\boldsymbol{R}_S \longmapsto \{ a, b, c \}]) \rightarrow [\boldsymbol{R} \longmapsto \{ a, b \}]$ (Reinf)
6. $[\boldsymbol{R}_D \longmapsto \{ a \}] \rightarrow [\boldsymbol{R} \longmapsto \{ a \}]$ (PR from 2 & 3)
7. $[\boldsymbol{R}_D \longmapsto \{ b \}] \rightarrow [\boldsymbol{R} \longmapsto \{ b \}]$ (PR from 2 & 4)
8. $[\boldsymbol{R}_D \longmapsto \{ a, b \}] \rightarrow [\boldsymbol{R} \longmapsto \{ a, b \}]$ (PR from 2 & 5)
9. $[\boldsymbol{R}_D \longmapsto \{ a \}] \vee [\boldsymbol{R}_D \longmapsto \{ b \}] \vee [\boldsymbol{R}_D \longmapsto \{ a, b \}]$ (rewrite 1)
10. $[\boldsymbol{R} \longmapsto \{ a \}] \vee [\boldsymbol{R} \longmapsto \{ b \}] \vee [\boldsymbol{R} \longmapsto \{ a, b \}]$ (PR from 6–9)
11. $[\boldsymbol{R} \overset{\in}{\longmapsto} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$ (rewrite 10)

Example shortened

It may be useful to tweak l-axioms in order to skip steps which will seem intuitive to humans.

Definition (Reinforcement-sets)

For each $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, $\alpha_1, \alpha_2 \subseteq \mathscr{P}_{\emptyset}(\mathscr{A})$, ${}_{\cap Q \neq \emptyset, Q \in \alpha_1 \times \alpha_2}$:
$([\boldsymbol{R}_1 \stackrel{\in}{\longmapsto} \alpha_1] \land [\boldsymbol{R}_2 \stackrel{\in}{\longmapsto} \alpha_2]) \rightarrow [\boldsymbol{R}_1 + \boldsymbol{R}_2 \stackrel{\in}{\longmapsto} \bigcup_{A_1 \in \alpha_1, A_2 \in \alpha_2} \{ A_1 \cap A_2 \}]$.

1. $[\boldsymbol{R}_D \stackrel{\in}{\longmapsto} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$ (Dom)
2. $[\boldsymbol{R}_S \longmapsto \{ a, b, c \}]$ (Sym)
3. $((1) \land (2)) \rightarrow [\boldsymbol{R} \stackrel{\in}{\longmapsto} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$ (Reinf-sets)
4. $[\boldsymbol{R} \stackrel{\in}{\longmapsto} \{ \{ a \}, \{ b \}, \{ a, b \} \}]$

Soundness and completeness

Consider an l-axiomatization J and a claim $c = [\boldsymbol{R} \xmapsto{\in} \alpha]$.

Theorem (Soundness)

If there exists an argument (c, proof) grounded on J, the claim holds given J.

Theorem (Completeness)

If the claim holds given J, then there exists an argument (c, proof) grounded on J.

This is easily obtained from the soundness and completeness of natural deduction in propositional logic.

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Argument building for Borda

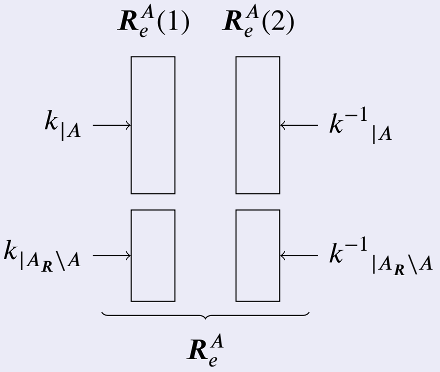
Write f_B for the Borda rule.

- We want to produce an argument justifying Borda's output.
- Given $\boldsymbol{R}$, we want an argument with claim $[\boldsymbol{R} \longmapsto f_B(\boldsymbol{R})]$.
- Basis: Young (1974)'s axiomatization of the Borda rule.
- Our l-axiomatization uses three simple profile types plus REINF.

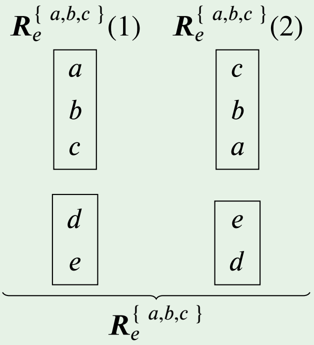
Elementary profile

Fix an arbitrary linear order k on $\mathscr{A}$.

Definition (Elementary profile)



Example



Cyclic profiles

Definition (Cyclic profile)

$\boldsymbol{R}_c^S$, where S is a complete cycle in $\mathscr{A}$, is the profile composed by all m possible cyclic offsets of S as preference orderings.

Example

$$\boldsymbol{R}_c^{\langle a,b,c,d \rangle} = \begin{matrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{matrix}.$$

Borda l-axiomatization

ELEM for all A: $[\boldsymbol{R}_e^A \longmapsto A]$.

CYCL for all S: $[\boldsymbol{R}_c^S \longmapsto \mathscr{A}]$.

REINF as previously but generalized to any number of summed profiles.

CANC cancellation: when all pairs of alternatives (a, b) in a profile are such that a is preferred to b as many times as b to a, then the set of winners must be $\mathscr{A}$.

An example

Consider $\mathscr{A} = \{ a, b, c, d \}$ and a profile $\boldsymbol{R}$ defined as:

$$\boldsymbol{R} = \begin{matrix} a & c \\ b & b \\ d & a \\ c & d \end{matrix} .$$

An example

Consider $\mathscr{A} = \{ a, b, c, d \}$ and a profile $\boldsymbol{R}$ defined as:

$$\boldsymbol{R} = \begin{matrix} a & c \\ b & b \\ d & a \\ c & d \end{matrix} .$$

We want to justify that $f_B(\boldsymbol{R}) = \{ a, b \}$.

Sketch

- Consider any $\boldsymbol{R}' = q_1 \boldsymbol{R}_e^{a,b} + q_2 \boldsymbol{R}_e^{a,b,c} + \sum_{S \in \mathcal{S}} q_S \boldsymbol{R}_c^S$, $q_1, q_2, q_S \in \mathbb{N}$, $\mathcal{S}$ some set of cycles.
- In $\boldsymbol{R}'$, $W = \{ a, b \}$ must win.
- Assume that for some $k \in \mathbb{N}$, $\overline{k\boldsymbol{R}} + \boldsymbol{R}'$ cancel.
- Then $k\boldsymbol{R}$ has winners W. (Skipping details.)
- Then $\boldsymbol{R}$ has winners W.

Our task: find $\boldsymbol{R}'$ a combination of elementary and cyclic profiles such that $\overline{k\boldsymbol{R}} + \boldsymbol{R}'$ cancel.
Good news: this is always possible.

Application on the example

Define $\boldsymbol{R}' = \boldsymbol{R}_e^{a,b} + 2\boldsymbol{R}_e^{a,b,c} + \boldsymbol{R}_c^{\langle c,b,a,d \rangle} + \boldsymbol{R}_c^{\langle b,d,c,a \rangle}$.

1. $[\boldsymbol{R}_e^{a,b} \longmapsto \{ a, b \}]$ (ELEM)
2. $[\boldsymbol{R}_e^{a,b,c} \longmapsto \{ a, b, c \}]$ (ELEM)
3. $[\boldsymbol{R}_c^{\langle c,b,a,d \rangle} \longmapsto \mathscr{A}]$ (CYCL)
4. $[\boldsymbol{R}_c^{\langle b,d,c,a \rangle} \longmapsto \mathscr{A}]$ (CYCL)
5. $[\boldsymbol{R}' \longmapsto \{ a, b \}]$ (REINF, 1, 2, 3, 4)
6. $[4\boldsymbol{R} + \overline{4\boldsymbol{R}} \longmapsto \mathscr{A}]$ (CANC)
7. $[4\boldsymbol{R} + \overline{4\boldsymbol{R}} + \boldsymbol{R}' \longmapsto \{ a, b \}]$ (REINF, 5, 6)
8. $[\overline{4\boldsymbol{R}} + \boldsymbol{R}' \longmapsto \mathscr{A}]$ (CANC)
9. $[4\boldsymbol{R} \longmapsto \{ a, b \}]$ (REINF, 7, 8)
10. $[\boldsymbol{R} \longmapsto \{ a, b \}]$ (REINF, 9)

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2. Language
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Conclusion

- A language to express desirable properties of voting rules.
- We can then instanciate concrete arguments (example-based).
- May render some arguments in the specialized literature accessible to non experts.
- Other logics may be better suited!
- Extensions may permit to *debate* about voting rules.
- Provides a way to study appreciation of arguments.

Thank you for your attention!

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