CHAPTER 7

An Introduction to Belief Merging and its Links with Judgment Aggregation

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Caveat: This is a working version. The text of the introduction and the exact content of the paper will still evolve.

7.1 Introduction

The main objective of this chapter is to present propositional belief merging and to exhibit some of its links and differences with judgment aggregation. Making this precise is important: though propositional belief merging and judgment aggregation are two logically-founded theories of belief aggregation, they differ in many aspects. Studying what makes them similar and what makes them dissimilar is useful for a better understanding of the pros and the cons of the two theories.

Some works linking the two settings already exist. In (Pigozzi, 2006), merging operators are used to aggregate judgments, with the strong assumption that the beliefs of the agents consist of a unique interpretation. In (Lang et al., 2011; Everaere et al., 2014b), judgment operators satisfying some of the most important properties of aggregation judgment are defined.

In this chapter, the focus is laid on the translation of merging postulates into the aggregation judgment setting and the translation of properties of aggregation judgment into the propositional merging settings. These translations highlight the intrinsic discrepancies between the two frameworks.

In propositional belief merging, a set of postulates characterizes the expected behavior of the operators, and a corresponding representation theorem exists. In judgment aggregation, some properties (often issued from vote theory) have been pointed out. However, these properties do not lead to a representation theorem, but instead several impossibility theorems show that these properties are not jointly compatible. Furthermore, the study of the translations of the properties in the other framework shows that some of them, while very natural in one setting, are unexpected in the other setting.

Another important difference concerns the nature of the input and the nature
of the output considered in the two settings. Propositional belief merging framework considers the beliefs of individual agents from a group. Beliefs are encoded by sets of propositional formulas, or, equivalently, by sets of interpretations. Interpretations are independent and mutually conflicting views of the same world, and each agent believes that the true world is one of the interpretations in her beliefs. In propositional merging, the notion of interpretation is thus fundamental, since the interpretations are the “candidates” of the decision process (in a nutshell, belief merging can be defined as a process which aims at finding the most plausible interpretations given the beliefs of the agents of the group).

Contrastingly, in aggregation judgment, the input is a set of judgments (0/reject or 1/accept) of a group of agents on a finite set of propositional formulas, the so-called agenda. No assumption is made on the way decisions/judgments are made by each agent. Whether the questions of the agenda are related to the beliefs or to the goals of the agents may have a huge impact on their judgment sets. As a matter of example, suppose that decisions are taken from the goals of agents. In this case, if the goal of an agent is \( a \), she will probably vote in favor of \( a \land b \), because \( a \land b \) guarantees \( a \). But if the agent believes \( a \), she is not able to vote in favor of \( a \land b \), because she does not know anything about \( b \).

In the following, after an introduction to belief merging and to judgment aggregation (which aims at introducing a number of key concepts, postulates and notations), we focus on the judgment generation issue: given the belief base of an agent and an agenda, how to define the corresponding judgment set of the agent? On this ground, we then show that in the general case, the results produced by a belief merging operator and those produced by a judgment operator can easily be incompatible, even if the two operators are rational ones (i.e., they satisfy some expected postulates). Interestingly, in the restricted case when the two approaches are equally informed (i.e., when the agenda is the set of all interpretations), every merging operator can be associated with an aggregation judgment correspondence, and vice-versa. We show that some close connections can be established, linking the satisfaction of expected postulates by the pairs of operators which correspond one another.

### 7.2 On Belief Merging

We consider a propositional language \( \mathcal{L} \) defined from a finite set \( \mathcal{P} \) of propositional symbols and the usual connectives, including \( \top \) and \( \bot \).

An interpretation (or state of the world) \( \omega \) is a total function from \( \mathcal{P} \) to \( \{0,1\} \). The set of all interpretations is noted \( \mathcal{W} \). An interpretation is usually denoted by a bit vector whenever a strict total order on \( \mathcal{P} \) is specified. It can also be viewed as the formula \( \bigwedge_{p \in \mathcal{P}} |\omega(p) = 1 \rangle p \land \bigwedge_{p \in \mathcal{P}} |\omega(p) = 0 \rangle \neg p \).

\( \omega \) is a model of a formula \( \phi \in \mathcal{L} \) if and only if it makes it true in the usual truth functional way. \( \models \) denotes logical entailment and \( \equiv \) denotes logical equivalence. \([\phi]\) denotes the set of models of formula \( \phi \), i.e., \([\phi] = \{\omega \in \mathcal{W} \mid \omega \models \phi\}\).

A belief base \( K \) is a finite set of propositional formulas \( \{\varphi_1, \ldots, \varphi_n\} \). We denote by \( \bigwedge K \) the conjunction of formulae of \( K \), i.e., \( \bigwedge K = \varphi_1 \land \ldots \land \varphi_n \). Often, in order
to simplify the notations, we will identify the base $K$ with the formula $\varphi = \bigwedge K$ which is the conjunction of the formulae of $K$. We suppose that each belief base is non-trivial, i.e., it is consistent but not valid and note $\mathcal{K}$ the set of bases.

A profile $E$ represents the beliefs of a group of $n$ agents involved in the merging process; formally $E$ is given by a vector $(K_1, \ldots, K_n)$ of belief bases, where $K_i$ is the belief base of agent $i$ (different agents are allowed to exhibit identical bases). $\bigwedge E$ denotes the conjunction of all elements of $E$, i.e., $\bigwedge E = \bigwedge K_1 \land \ldots \land \bigwedge K_n$ and $\sqcup$ denotes the vector union. $\mathcal{E}$ represents the set of all profiles. A profile $E$ is said to be consistent if and only if $\bigwedge E$ is consistent.

We denote by $E^n$ the profile in which $E$ appears $n$ times, more precisely $E^n = \bigcup_{i=1}^{n} E$. Two profiles $E = (K_1, \ldots, K_n)$ and $E = (K'_1, \ldots, K'_n)$ are equivalent, noted $E \equiv E'$, iff there exists a permutation $\pi$ over $\{1, \ldots, n\}$ s.t. for each $i \in 1, \ldots, n$, we have $K_i \equiv K'_{\pi(i)}$.

A base (formula) $K$ is complete if it has only one model. A profile $E$ is complete if all the bases of $E$ are complete. If $\leq$ denotes a pre-order on $\mathcal{W}$ (i.e. a reflexive and transitive relation), then $<$ denotes the associated strict order defined by $\omega < \omega'$ if and only if $\omega \leq \omega'$ and $\omega' \not< \omega$. A pre-order is total if $\forall \omega, \omega' \in \mathcal{W}$, $\omega \leq \omega'$ or $\omega' \leq \omega$. A pre-order that is not total is called partial. Let $\leq$ be a pre-order on $A$, $B \subseteq A$, then $\min(B, \leq) = \{b \in B \mid \forall a \in B \; a \not< b\}$.

If $A$ is a set, we denote $|A|$ the cardinal of $A$. The symbol $\subseteq$ will denote set containment and $\subset$ strict set containment, i.e., $A \subset B$ if and only if $A \subseteq B$ and $A \not\equiv B$.

An integrity constraint $\mu$ is a consistent formula restricting the possible results of the merging process.

Merging operators we will consider are functions from the set of profiles and the set of propositional formulae (that will represent integrity constraints) to the set of bases, i.e. $\Delta : \mathcal{E} \times \mathcal{L} \mapsto \mathcal{K}$. We will use the notation $\Delta_\mu(E)$ instead of $\Delta(E, \mu)$. $\Delta(E)$ is a short for $\Delta_\top(E)$.

We first study the logical properties of propositional merging operators and state a representation theorem for these operators in terms of pre-orders on interpretations.

The logical properties given in (Konieczny and Pino Pérez, 2002) for characterizing IC belief merging operators are:

**Definition 7.1.** A merging operator $\Delta$ is an IC merging operator iff it satisfies the following properties:

1. **(IC0)** $\Delta_\mu(E) \models \mu$
2. **(IC1)** If $\mu$ is consistent, then $\Delta_\mu(E)$ is consistent
3. **(IC2)** If $\bigwedge E$ is consistent with $\mu$, then $\Delta_\mu(E) \equiv \bigwedge E \land \mu$
4. **(IC3)** If $E_1 \equiv E_2$ and $\mu_1 \equiv \mu_2$, then $\Delta_{\mu_1}(E_1) \equiv \Delta_{\mu_2}(E_2)$
5. **(IC4)** If $K_1 \models \mu$ and $K_2 \models \mu$, then $\Delta_\mu((K_1, K_2)) \land K_1$ is consistent if and only if $\Delta_{\mu_1}((K_1, K_2)) \land K_2$ is consistent

$^1$This identification will be done when the approach is not sensitive to syntactical representation. When the approach is sensitive to syntactical representation, it will be important to distinguish between $K$ and the conjunction of its formulae (see e.g. (Konieczny et al., 2004)).
\textbf{(IC5)} \( \Delta_\mu(E_1) \land \Delta_\mu(E_2) \models \Delta_\mu(E_1 \cup E_2) \)

\textbf{(IC6)} If \( \Delta_\mu(E_1) \land \Delta_\mu(E_2) \) is consistent, then \( \Delta_\mu(E_1 \cup E_2) \models \Delta_\mu(E_1) \land \Delta_\mu(E_2) \)

\textbf{(IC7)} \( \Delta_{\mu_1}(E) \land \mu_2 \models \Delta_{\mu_1 \land \mu_2}(E) \)

\textbf{(IC8)} If \( \Delta_{\mu_1}(E) \land \mu_2 \) is consistent, then \( \Delta_{\mu_1 \land \mu_2}(E) \models \Delta_{\mu_1}(E) \)

See (Konieczny and Pino Pérez, 2002) for some explanations of these properties.

Let us now give some examples of IC merging operators from the family of distance-based merging operators (Konieczny et al., 2004):

\textbf{Definition 7.2.} A (pseudo-)distance between interpretations is a function \( d: \mathcal{W} \times \mathcal{W} \rightarrow \mathbb{R}^+ \) such that for any \( \omega_1, \omega_2 \in \mathcal{W} \):

- \( d(\omega_1, \omega_2) = d(\omega_2, \omega_1) \)
- \( d(\omega_1, \omega_2) = 0 \) iff \( \omega_1 = \omega_2 \)

\textbf{Definition 7.3.} An aggregation function is a mapping\(^2\) \( f \) from \( \mathbb{R}^m \) to \( \mathbb{R} \), which satisfies:

- If \( x_i \geq x_i' \), then \( f(x_1, ..., x_i, ..., x_m) \geq f(x_1, ..., x_i', ..., x_m) \) \hspace{1cm} \text{(non-decreasingness)}
- \( f(x_1, ..., x_m) = 0 \) if \( \forall i, x_i = 0 \) \hspace{1cm} \text{(minimality)}
- \( f(x) = x \) \hspace{1cm} \text{(identity)}
- If \( \sigma \) is a permutation over \( \{1, ..., m\} \), then \( f(x_1, ..., x_m) = f(x_{\sigma(1)}, ..., x_{\sigma(m)}) \) \hspace{1cm} \text{(symmetry)}

Some additional properties can also be considered for \( f \), especially:

- If \( x_i > x_i' \), then \( f(x_1, ..., x_i, ..., x_m) > f(x_1, ..., x_i', ..., x_m) \) \hspace{1cm} \text{(strict non-decreasingness)}
- If \( f(x_1, ..., x_n, z) \leq f(y_1, ..., y_n, z) \), then \( f(x_1, ..., x_n) \leq f(y_1, ..., y_n) \) \hspace{1cm} \text{(decomposition)}
- If \( \forall i, z > y_i \), then \( f(z, x_1, ..., x_n) > f(y_1, ..., y_{n+1}) \) \hspace{1cm} \text{(strict preference)}

\textbf{Definition 7.4.} Let \( d \) and \( f \) be respectively a distance between interpretations and an aggregation function. The distance-based merging operator \( \Delta_{d,f} \) is defined by

\[ [\Delta_{d,f}^\mu](E) = \min([\mu], \leq_E) \]

where the total pre-order \( \leq_E \) on \( \mathcal{W} \) is defined in the following way (with \( E = (K_1, ..., K_n) \)):

- \( \omega \leq_E \omega' \) iff \( d(\omega, E) \leq d(\omega', E) \)
- \( d(\omega, E) = f(d(\omega, K_1), ..., d(\omega, K_n)) \)

\(^2\)Strictly speaking, it is a family of mappings, one for each \( m \).
• \( d(\omega, K) = \min_{\omega' \supseteq K} d(\omega, \omega') \)

For usual aggregation functions, and whatever the chosen distance, the corresponding distance-based operators exhibit good logical properties:

**Proposition 7.1 (Konieczny and Pino Pérez (2002)).** For any distance \( d \), if \( f \) is equal to \( \Sigma \), leximax\(^3\), leximin, or \( \Sigma^n \) (sum of the \( n^{th} \) powers), then \( \Delta^{d,f} \) is an IC merging operator.

Some formula-based merging operators have also been defined (Baral et al., 1991, 1992). The general principle underlying them is to find maximal consistent subsets of formulas from the initial profile. One important limitation of these approaches is to possibly forget some important pieces of information, as the number of bases supporting each formula. Furthermore, these operators exhibit much less expected postulates than the model-based ones (Konieczny, 2000).

Other approaches to merge propositional belief bases are \( DA^2 \) operators (Konieczny et al., 2004), which use two aggregation functions (a first one to extract pieces of information from inconsistent bases and a second one to aggregate the resulting pieces of information); conflict-based merging operators (Everaere et al., 2008) and default-based merging operators (Delgrande and Schaub, 2007) have also been defined.

Additional merging postulates have been also designed. Among them, the **Unanimity** postulate (on formulas or on models) or **Disjunction** has been studied in (Everaere et al., 2010). Other properties inspired by similar conditions in social choice literature have been translated into the propositional merging framework. This led to the introduction of different notions: truth tracking (Everaere et al., 2007), rationalization (Konieczny et al., 2011), egalitarism (Everaere et al., 2014a) and a study of voting properties in the context of merging (Haret et al., 2016).

### 7.3 On Judgment Aggregation

We briefly present some definitions and notations used in the following.\(^4\)

An **agenda** \( X = \{\varphi_1, \ldots, \varphi_m\} \) is a finite, non-empty and totally ordered set of non-trivial (i.e., consistent but not valid) propositional formulas.

A **judgment** on a formula \( \varphi_k \) of \( X \) is an element of \( D = \{1, 0, \star\} \), where 1 means that \( \varphi_k \) is supported, 0 that \( \neg \varphi_k \) is supported, \( \star \) that neither \( \varphi_k \) nor \( \neg \varphi_k \) are supported. A **judgment set** on \( X \) is a mapping \( \gamma \) from \( X \) to \( D \), also viewed as a \( m \)-vector over \( D \), when the cardinality of \( X \) is \( m \). For each \( \varphi_k \) of \( X \), \( \gamma \) is supposed to satisfy \( \gamma(\neg \varphi_k) = \neg \gamma(\varphi_k) \), where \( \neg \gamma \) is given by \( \neg \gamma(\varphi_k) = \star \) iff \( \gamma(\varphi_k) = \star \), \( \neg \gamma(\varphi_k) = 1 \) iff \( \gamma(\varphi_k) = 0 \), and \( \neg \gamma(\varphi_k) = 0 \) iff \( \gamma(\varphi_k) = 1 \).

Judgment sets are often asked to be consistent and resolute: A judgment set is **resolute** iff \( \forall \varphi_k \in X, \gamma(\varphi_k) = 0 \) or \( \gamma(\varphi_k) = 1 \). A judgment set \( \gamma \) on \( X \) is **consistent** iff the associated formula (judgment) \( \hat{\gamma} = \bigwedge_{\varphi_k \in X, \gamma(\varphi_k) = 1} \varphi_k \land \bigwedge_{\varphi_k \in X, \gamma(\varphi_k) = 0} \neg \varphi_k \) is consistent.

\(^3\)Also referred to as \( G_{max} \).

\(^4\)Most of these notations depart from (but are equivalent to) the ones usually used in judgment aggregation papers.
Aggregating judgments consists in associating a set of collective judgment sets with a profile of individual judgment sets: a profile $\Gamma = (\gamma_1, \ldots, \gamma_n)$ of judgment sets on $X$ is a non-empty vector of judgments sets on $X$. $\Gamma$ is consistent (resp. resolute) when each judgment set in it is consistent (resp. resolute).

For each agenda $X$, a judgment aggregation method $Ag$ associates with a consistent profile $\Gamma$ on $X$ a non-empty set $Ag(\Gamma)$ of collective judgment sets on $X$, also viewed as a formula (the collective judgment) $\hat{Ag}(\Gamma) = \bigvee_{\gamma \in Ag(\Gamma)} \hat{\gamma}$. For $\varphi_k \in X$, we note $Ag(\Gamma)(\varphi_k) = 1$ (resp. $Ag(\Gamma)(\varphi_k) = 0$) if and only if $\forall \gamma \in Ag(\Gamma)$, $\gamma(\varphi_k) = 1$ (resp. $\forall \gamma \in Ag(\Gamma)$, $\gamma(\varphi_k) = 0$), and $Ag(\Gamma)(\varphi_k) = *$ in the remaining case. When $Ag(\Gamma)$ is a singleton for each $\Gamma$, the judgment aggregation operator is called a (deterministic) judgment aggregation rule, and it is called a judgment aggregation correspondence otherwise (Lang et al., 2011).

Usual rationality properties pointed out so far for judgment aggregation (JA) operators are:

**Universal domain.** The domain of $Ag$ is the set of all consistent profiles.

**Collective rationality.** For any profile $\Gamma$ in the domain of $Ag$, $Ag(\Gamma)$ is a set of consistent collective judgment sets.

**Collective resoluteness.** For any profile $\Gamma$ in the domain of $Ag$, $Ag(\Gamma)$ is a set of resolute collective judgment sets.

It is usually expected that agents play symmetric roles:

**Anonymity.** For any two profiles $\Gamma = (\gamma_1, \ldots, \gamma_n)$ and $\Gamma' = (\gamma'_1, \ldots, \gamma'_n)$ in the domain of $Ag$ which are permutations one another, we have $Ag(\Gamma) = Ag(\Gamma')$.

**Neutrality.** For any $\varphi_p, \varphi_q$ in the agenda $X$ and profile $\Gamma$ in the domain of $Ag$, if $\forall i \gamma_i(\varphi_p) = \gamma_i(\varphi_q)$, then $Ag(\Gamma)(\varphi_p) = Ag(\Gamma)(\varphi_q)$.

A more demanding property is independence:

**Independence.** For any $\varphi$ in the agenda $X$ and profiles $\Gamma$ and $\Gamma'$ in the domain of $Ag$, if $\forall i \gamma_i(\varphi) = \gamma'_i(\varphi)$, then $Ag(\Gamma)(\varphi) = Ag(\Gamma')(\varphi)$.

**Systematicity.** For any $\varphi_p, \varphi_q$ in the agenda $X$ and profiles $\Gamma$ and $\Gamma'$ in the domain of $Ag$, if $\forall i \gamma_i(\varphi_p) = \gamma'_i(\varphi_q)$, then $Ag(\Gamma)(\varphi_p) = Ag(\Gamma')(\varphi_q)$.

It is clear that Systematicity is equivalent to Independence and Neutrality.

The above properties are quite standard (List, 2012), but are incompatible:

**Proposition 7.2 (List and Pettit, 2002).** There exists no judgment aggregation rule that satisfy $R$-universal domain, collective rationality, collective resoluteness, systematicity and anonymity.

This theorem is quite negative, but it relies on some strong assumptions. First is the resoluteness (completeness) assumptions of the individuals (R-universal domain), that can be criticized, since one cannot expect all agents to have an opinion on all possible issues; this is also the case of the collective completeness property, that is helpful for making decisions, but forces to make some choices even when there is no evidence enough to do so. Thus the collective completeness requirement imposes sometimes to discriminate further some judgment sets, using additional information not given in the input profile, as such, it conflicts with
the anonymity and neutrality conditions. Suppose for instance a perfect tie (say, about a unique issue $\varphi$ in the agenda, with 4 votes for and 4 votes against it), why and how to make a distinction between $\varphi$ and $\neg\varphi$? See (Gärdenfors, 2006) for criticisms on the collective completeness property.

The systematicity property is also highly criticizable (Everaere et al., 2014b). Indeed, it prevents from considering judgment aggregation as an optimization process trying to achieve a best compromise, which is often expected. The following example illustrates it:

**Example 7.1.** Let us consider an agenda $X$ composed of the following six formulas: $\varphi_1 = (\neg a \lor \neg b \lor \neg c \lor \neg d \lor \neg e)$, $\varphi_2 = a$, $\varphi_3 = b$, $\varphi_4 = c$, $\varphi_5 = d$, $\varphi_6 = e$. Let us consider the profiles $\Gamma$ and $\Gamma'$ on this agenda, as given by Table 7.1. In the (resolute) profile $\Gamma$, every formula has a majority of votes, so using simple majority vote all the formulas have to be selected, which would lead to an inconsistent collective judgment set. So (at least) one of the six formulas has to be rejected by the judgment aggregation correspondence. There is a unanimity for $\varphi_1$, so it seems sensible to select $\varphi_1$, in the result. All the other formulae except $\varphi_2$ are quasi-unanimous (they get all votes but one). The less supported formula is $\varphi_2$, so the expected result is $\gamma_\mathcal{P} = \{\varphi_1, \neg\varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6\}$.

Consider now the profile $\Gamma'$. The simple majority vote leads to a consistent collective judgment set $\gamma_{\mathcal{P}'} = \{\neg\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6\}$, which thus appears as the expected result. So, though the individual judgments are the same ones in both profiles for $\varphi_2$, in the expected result for $\mathcal{P}$ $\neg\varphi_2$ is selected, whereas for $\mathcal{P}'$ $\varphi_2$ is selected. Since $\varphi_2$ gets the same votes pros and cons in the two profiles, no judgment aggregation method satisfying systematicity is not allowed to make such a distinction.

This example illustrates clearly that the individual judgments on an issue cannot be considered independently from those for the other issues.

Other properties are also very attractive for JA operators, such as unanimity (Everaere et al., 2014b) and majority preservation (Lang et al., 2011).

**Unanimity.** For any $\varphi_k \in X$, for any profile $\Gamma$ in the domain of $Ag$, if $\exists x \in \{0,1\}$ s.t. $\forall \gamma_i \in \Gamma$, $\gamma_i(\varphi_k) = x$, then for every $\gamma_\mathcal{P} \in Ag_\mathcal{P}$, we have $\gamma_\mathcal{P}(\varphi_k) = x$. Note that unanimity is not required when $x = \star$, since in this case it makes sense to let the acceptance of $\varphi_k$ depends on the acceptance of other (logically related) formulas.

**Majority preservation.** If the judgment set obtained using the majority rule is consistent and resolute, then $Ag_\mathcal{P}$ is a singleton which consists of this set.

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Several definitions are possible for the majority rule when abstention is allowed. Here, one con-
Majority preservation (Lang et al., 2011; Slavkovic, 2012) is a very natural property, stating that if the simple majority vote on each issue leads to a consistent judgment set, then the judgment aggregation correspondence must output precisely this set. Indeed, it is sensible to stick to the result furnished by a simple majority vote when no doctrinal paradox occurs.

Let us now review some of the judgment aggregation operators that have been put forward in the literature. Usual judgment aggregation operators are majority, supermajority, premise-based, conclusion-based, sequential priority (List, 2012). In (Pigozzi, 2006) distance-based (merging-based) operators are pointed out. In (Lang et al., 2011) several families of judgment aggregation operators based on minimization, inspired from operators considered in voting theory and in AI, are defined. Majority preservation is presented as a natural requirement for such operators. In (Everaere et al., 2014b) another family of operators, called ranked majority operators, based on the number of votes received by each formula, has been introduced.

### 7.4 Projecting a belief base on an agenda

As explained previously, belief merging and judgment aggregation consider different inputs. In belief merging, an input profile consists of a profile of belief bases, representing the beliefs of a group of agents. In judgment aggregation, agents answer "yes" (1), "no" (0) or "undetermined" (⋆) to a set of questions (the agenda), and the input profile is a vector of such answers (alias judgment sets). Of course agents might use their beliefs to answer the questions, but it is out of the scope of judgment aggregation methods to specify how.

Imagine that the beliefs $K_i$ of an agent $i$ are known, given a question $\varphi_k$, what could be the opinion of the agent on the question? Suppose that an agent only believes that $a \land b$ is true, and questions her about $a$: she will probably answer "yes" to the question because she necessarily believes that $a$ is true. If the question is $\neg b$, she will probably answer "no" because $b$ being false is incompatible with her beliefs. Suppose now that the agent just believes that $a$ is true, and that the question is $a \land b$. In this case the agent probably has no opinion on the question (the question is contingent given her beliefs), thus she will probably answer "undetermined".

What we need to define to make it formal is a notion of projection function, which characterizes the answers (i.e., the judgment set) an agent can give to the questions of the agenda, depending on her current belief base. We call such projection functions decision policies, and our purpose is first to characterize axiomatically the rational ones:

**Definition 7.5.** A decision policy $p : \mathcal{L} \times \mathcal{L} \to \{0, 1, \star\}$ is a mapping associating an element of $\{0, 1, \star\}$ with any pair of non-trivial formulas $(K, \varphi)$ and satisfying:

1. if $K_1 \equiv K_2$, then $\forall \varphi, p(K_1, \varphi) = p(K_2, \varphi)$
2. if $\varphi_1 \equiv \varphi_2$, then $\forall K, p(K, \varphi_1) = p(K, \varphi_2)$

Considered that the majority rule gives 1 (resp. 0) when the number of agents reporting 1 (resp. 0) is strictly greater than the number of agents reporting 0 (resp. 1), and it gives $\star$ otherwise.

$^6$Called strong majority preservation in (Slavkovic, 2012).
3. \( p(\varphi, \varphi) = 1 \)

Conditions 1 and 2 can be viewed as a formal counterpart, respectively, of a neutrality condition and of an anonymity condition for decision policies, but we will refrain from using such a terminology here because of a possible confusion with the corresponding rationality conditions on judgment aggregation methods (in particular, the "neutrality" and "anonymity" conditions here do not entail respectively the neutrality property or the anonymity property of a judgment aggregation correspondence as defined previously).

Given an agenda \( X = \{\varphi_1, \ldots, \varphi_m\} \) and a belief base \( K \) (respectively a profile \( E = (K_1, \ldots, K_n) \) of belief bases), every decision policy \( p \) induces a judgment set \( p_X(K) = (p(K, \varphi_1), \ldots, p(K, \varphi_m)) \) (resp. a profile of judgment sets \( p_X(E) = (p_X(K_1), \ldots, p_X(K_n)) \)).

Examples of decision policies are the following ones:

- \( p_B(K, \varphi) = \begin{cases} 
1 & \text{if } K \models \varphi \\
0 & \text{if } K \models \neg \varphi \\
⋆ & \text{otherwise}
\end{cases} \)

- \( p_C(K, \varphi) = \begin{cases} 
1 & \text{if } K \land \varphi \not\models \bot \\
0 & \text{otherwise}
\end{cases} \)

The belief decision policy \( p_B \) makes sense when beliefs are considered. According to it an agent answers "yes" (resp. "no") to a given question precisely when it (resp. its negation) is a logical consequence of her belief base; in the remaining case she answers "undetermined".

Observe that with the consistency decision policy \( p_C \) it is possible to have together \( p_C(K_i, \varphi_k) = 1 \) and \( p_C(K_i, \neg \varphi_k) = 1 \) (for instance a belief base equivalent to \( a \) is consistent with \( b \) and with \( \neg b \)). In order to avoid this problem, some additional conditions must be ensured:

**Definition 7.6.** Let \( p : L \times L \to \{0, 1, ⋆\} \) be a decision policy. It is a rational decision policy if it satisfies the two following conditions:

4. If \( p(K, \varphi) = 1 \), then \( p(K, \neg \varphi) = 0 \)

5. If \( K_1 \land K_2 \) is consistent and if \( p(K_1, \varphi) = 1 \)
   then \( p(K_1 \land K_2, \varphi) = 1 \)

It turns out that these two additional conditions fully characterize the belief decision policy:

**Proposition 7.3.** \( p \) is a rational decision policy iff \( p = p_B \).

We also have the following expected property when \( p_B \) is used:

**Proposition 7.4.** \( p_B \) guarantees individual consistency: whatever the belief base \( K \) and the agenda \( X \), if \( \hat{\gamma} \) is the judgment set on \( X \) induced by \( p_B \) given \( K \), then the associated judgment \( \hat{\gamma} \) is consistent.

The last two propositions justify to focus on the \( p_B \) policy, and this is what we do in the following.
Our objective is first to determine whether some logical connections between the formulas $\varphi_\Delta$ and $\varphi_{Ag}$ exist whenever $\Delta$ and $Ag$ are "rational". Especially, we focus on the unanimity condition and the majority preservation condition on $Ag$ which are natural ones.

One first needs to give a couple of notations:

**Definition 7.7.** Let $E = (K_1, \ldots, K_n)$ be a profile of belief bases and let $p$ be a decision policy. Let $\Delta$ be a belief merging operator and $Ag$ be a judgment aggregation correspondence. Let $X = \{\varphi_1, \ldots, \varphi_m\}$ be an agenda. There are two ways to define a collective judgment on $X$ (see Figure ??):

- if the project-then-aggregate path $Ag \circ p$ is followed, then the output is $\varphi_{Ag} = \hat{Ag}_{pX}(E)$,
- if the merge-then-project path $p \circ \Delta$ is followed, then the output is $\varphi_{\Delta} = pX(\Delta(E))$.

Each of $Ag \circ p$ and $p \circ \Delta$ can be viewed as an aggregation operator associating with a profile $E$ of belief bases and an agenda $X$ a (set of) collective judgment set(s), interpreted as a propositional formula (a collective judgment). The point is that the two resulting collective judgments are not necessarily compatible, even when $\Delta$ and $Ag$ are rational operators. More precisely:

**Proposition 7.5.** There exist an IC merging operator $\Delta$, a judgment aggregation operator $Ag$ satisfying unanimity such that for a profile $E$ of belief bases and a singleton agenda $X$, $\varphi_{Ag} \land \varphi_{\Delta}$ is inconsistent.

Let $\text{diff}(\omega, \omega')$ be the set of propositional variables on which $\omega$ and $\omega'$ differ:

**Definition 7.8.** A distance $d$ is normal iff $\forall \omega_1, \omega_2, \omega_3, \omega_4 \in W, d(\omega_1, \omega_2) \leq d(\omega_3, \omega_4)$ whenever $\text{diff}(\omega_1, \omega_2) \subseteq \text{diff}(\omega_3, \omega_4)$.

This normality property expresses a very natural idea: if $\omega_1$ and $\omega_2$ differ on a given subset $D$ of variables and $\omega_3$ and $\omega_4$ differ on a superset of $D$, then $\omega_3$ should not be considered closer to $\omega_4$ than $\omega_1$ is to $\omega_2$. In particular, all usual distances are normal, in particular the Hamming distance and the Drastic distance (Konieczny and Pino Pérez, 2002) are normal distances.

**Proposition 7.6.** Let $\Delta_{d,f}$ be a distance-based merging operator with $d$ any normal distance and $f$ any strictly non-decreasing function, and let $Ag$ satisfies majority preservation, one can find a profile $E$ of belief bases and a (singleton) agenda $X$ such that $\varphi_{Ag} \land \varphi_{\Delta}$ is inconsistent.

This result is quite important, because most reasonable distances between interpretations (Hamming distance, Drastic distance) are normal ones and most reasonable aggregation functions ($\Sigma$, leximax, leximin, $\Sigma''$, ...) satisfy strict non-decreasingness. Thus in the general case the results obtained by using rational IC merging methods can be inconsistent with the results obtained by using rational JA methods.
7.6 BM vs JA: the case of full agendas

Let us now investigate the connections between belief merging and judgment aggregation in the case when the two approaches are equally informed, i.e., when the agenda $X$ gathers all interpretations of $W$.

In the following, in order to simplify the notations, since $X$ is fixed, we write $p(K_i)$ instead of $p_X(K_i)$ and $p(E)$ instead of $p_X(E)$.

For any belief base $K_i$ of $E$ and any $\omega \in X$, we have $p(K_i,\omega) = 0$ iff $K_i \models \neg \omega$, i.e., $p(K_i,\omega) = 0$ iff $\omega \not\in K_i$. So $p(K_i,\omega) \neq 0$ iff $\omega \models K_i$. Observe that when questions are interpretations, a belief base $K_i$ that is not complete (i.e., with more than one model) cannot lead to answer 1, but only to $*$ or to 0. Whatever the case, $p(K_i)$ contains necessarily at most one 1 and at least one 0 (since $K_i$ is supposed to be non-trivial).

We assume in what follows that the judgment aggregation correspondence $Ag$ under consideration satisfies both the collective resoluteness condition and the collective rationality condition. This is a harmless assumption when $X = W$ provided that $Ag$ outputs at least one consistent judgment set (which is not a very demanding condition). Indeed, let $Ag_T = \{\gamma_1, \ldots, \gamma_k\}$ be the set of collective judgment sets given by $Ag$ on the profile $\Gamma$ of individual judgment sets on $X$. For each $\gamma_i \in Ag_T$ such that $\gamma_i$ is consistent, let $\gamma_i^R$ be the set of resolute and consistent collective judgment sets obtained by replacing in $\gamma_i$ precisely one $*$ by 1 when $\gamma_i$ does not contain any 1, and the other $*$ by 0 (so that $\gamma_i^R$ contains $e$ elements whenever $\gamma_i$ contains $e$ $*$ but no 1). Let $Ag_T^R = \bigcup_{\gamma_i \in Ag_T} \gamma_i^R$. We have $\widehat{Ag}_T = \widetilde{Ag}_T^R$. So the collective judgments are the same ones for $Ag_T$ and $Ag_T^R$ and this explains why one can safely suppose that collective resoluteness holds. We call abstention-free correspondence associated with $Ag$ the judgment aggregation correspondence which associates $Ag_T^R$ with the input profile $\Gamma$.

Interestingly, when $X = W$, we can recover the belief base of any agent from her judgment set $\gamma$ (and not just deduce her judgment set from her belief base, unlike what happens in the general case). Thus the inverse mapping $p^{-1}$ of $p$ can be defined as follows (up to logical equivalence): $[p^{-1}(\gamma)] = \{\omega \in W \mid \gamma(\omega) \neq 0\}$. $[p^{-1}(\gamma)]$ is the set of models of the belief base of the agent reporting the judgment set $\gamma$.

On this ground, one can define a judgment aggregation correspondence $Ag = Ag^\Delta$ from a merging operator $\Delta$ and a merging operator $\Delta = \Delta^Ag$ from a judgment aggregation correspondence $Ag$. Given an interpretation $\omega$ (also viewed as a formula), let the induced judgment set $\gamma_\omega$ be equal to $p(\omega)$. By construction, $[\gamma_\omega] = \{\omega\}$, thus $\gamma_\omega$ is consistent.

**Definition 7.9.**

- Given a merging operator $\Delta$ and a profile $E = (K_1, \ldots, K_n)$, we define $Ag^\Delta_{p(E)} = \{\gamma_\omega \mid \omega \models \Delta(E)\} = \{\gamma_\omega \mid p(\Delta(E),\omega) \neq 0\}$.

- Given a judgment aggregation correspondence $Ag$ and a profile $\Gamma = (\gamma_1, \ldots, \gamma_n)$ of judgment sets, we note $p^{-1}(\Gamma) = (p^{-1}(\gamma_1), \ldots, p^{-1}(\gamma_n))$ and $\Delta^Ag(p^{-1}(\Gamma)) = \{\omega \in W \mid \gamma_\omega \in Ag_T\}$.

It is easy to prove that in the case $X = W$ the two aggregation paths corresponding respectively to $\Delta$ and to $Ag^\Delta$ lead to equivalent results:

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7Remember that $p$ denotes here the belief decision policy.
Proposition 7.7. Let $X$ be a complete agenda. Let $\varphi_{Ag} = \widehat{Ag}(\Delta_{p(E)})$ and $\varphi_{\Delta} = p(\Delta(E))$. We have $\varphi_{Ag} \equiv \varphi_{\Delta}$.

Furthermore, when $X$ is the complete agenda $\mathcal{W}$, every belief merging operator corresponds to one judgment aggregation correspondence, and vice-versa. More precisely, we have that:

Proposition 7.8. $\Delta = \Delta^{(Ag)}$ and $Ag = Ag^{(\Delta)}$

Thus, Definition 7.9 induces a one-to-one mapping between the merging operator and the corresponding judgment aggregation correspondence. This bijection will be used in the following to show how IC postulates and judgment aggregation properties are related.

Let us now parse the IC postulates and determine their counterparts in judgment aggregation (when they exist):

(IC0) By construction of $\Delta^{Ag}$ (IC0) is satisfied, so (IC0) does not correspond to any non-trivial condition on $Ag$.

(IC1) If $\mu$ is consistent, then $\Delta_{\mu}(E)$ is consistent

Proposition 7.9. $\Delta^{Ag}$ satisfies (IC1) iff $Ag$ satisfies universal domain.

(IC2) Let us define an additional property for JA methods:

Definition 7.10. Let $\Gamma = (\gamma_1, \ldots, \gamma_n)$ be a profile of judgment sets on an agenda $X$.

- $\varphi \in X$ is unanimous for $\Gamma$ iff $\forall i \in \{1, \ldots, n\}, \gamma_i(\varphi) \neq 0$.
- $\Gamma$ is consensual iff there exists $\varphi \in X$ which is unanimous for $\Gamma$.
- A judgment aggregation correspondence $Ag$ satisfies consensuality iff for every consensual profile $\Gamma$ of judgment sets on an agenda $X$, for every $\varphi \in X$, $Ag_{\Gamma}(\varphi) \neq 0$ iff $\varphi$ is unanimous for $\Gamma$.

Proposition 7.10. $\Delta^{Ag}$ satisfies (IC2) iff $Ag$ satisfies consensuality.

(IC3) If $E_1 \equiv E_2$, then $\Delta(E_1) \equiv \Delta(E_2)$

Proposition 7.11. $\Delta^{Ag}$ satisfies (IC3) iff $Ag$ satisfies anonymity.

(IC4) The neutrality condition on $Ag$ is not sufficient to ensure that $\Delta^{Ag}$ satisfies (IC4).

(IC5) Let us now define two additional properties for JA operators, based on the consistency condition for voting methods (Young, 1975; Alcantud and Laruelle, 2013). These two properties correspond respectively to (IC5) and (IC6).

Weak consistency. Let $\Gamma = (\gamma_1, \ldots, \gamma_n)$ and $\Gamma' = (\gamma'_1, \ldots, \gamma'_p)$ be two profiles of judgment sets on an agenda $X$ and in the domain of $Ag$. For any $\varphi \in X$, if $Ag_{\Gamma}(\varphi) = 1$ and $Ag_{\Gamma'}(\varphi) = 1$, then $Ag_{\Gamma \sqcup \Gamma'}(\varphi) = 1$.

This property states that if a formula is not accepted by a profile $\Gamma$, and by a profile $\Gamma'$, then it must be accepted by the union of the profiles.

Consistency. Let $\Gamma = (\gamma_1, \ldots, \gamma_n)$ and $\Gamma' = (\gamma'_1, \ldots, \gamma'_p)$ be two profiles of judgment sets on an agenda $X$ and in the domain of $Ag$. If there is $\varphi \in X$ s.t. $Ag_{\Gamma}(\varphi) = 1$ and
\[ Ag_{\Gamma'}(\varphi) = 1, \text{ then for every } \psi \in X, \text{ if } Ag_{\Gamma \sqcup \Gamma'}(\psi) = 1 \text{ then } Ag_{\Gamma}(\psi) = 1 \text{ and } Ag_{\Gamma'}(\psi) = 1. \]

This property states that if there is at least a formula that is accepted by two subprofiles \( \Gamma \) and \( \Gamma' \), then each formula that is accepted by the whole profile \( \Gamma \sqcup \Gamma' \) should be accepted by each of the two subprofiles \( \Gamma \) and \( \Gamma' \).

Quite surprisingly these conditions have not been considered as standard ones for judgment aggregation methods (we are only aware of (Lang et al., 2011; Slavkovic, 2012) which point out the consistency condition, under the name "separability").

**Proposition 7.12.** \( \Delta^{Ag} \) satisfies (IC5) iff \( Ag \) satisfies weak consistency

(IC6) If \( \Delta(E_1) \wedge \Delta(E_2) \) is consistent, then \( \Delta(E_1 \sqcup E_2) \models \Delta(E_1) \wedge \Delta(E_2) \)

**Proposition 7.13.** \( \Delta^{Ag} \) satisfies (IC6) iff \( Ag \) satisfies consistency

We do not consider (IC7) and (IC8) since they are obviously satisfied by any merging operator when the integrity constraints \( \mu_1 \) and \( \mu_2 \) are valid.

The following proposition sums up the results:

**Proposition 7.14.**

- If \( Ag \) satisfies collective rationality, consensuality, anonymity, neutrality, weak consistency, consistency, then \( \Delta^{Ag} \) satisfies (IC0) to (IC3) and (IC5), (IC6).

- If \( \Delta \) is an IC merging operator, then \( Ag^\Delta \) satisfies collective rationality, consensuality, anonymity, weak consistency, and consistency.

### 7.7 On the compatibility of BM and JA

In Proposition 7.14 we pointed out a list of properties required for a JA operator to correspond to an IC merging operator in the complete agenda case. A key question is whether these properties can be satisfied by some judgment aggregation operator.

We give a positive answer to this issue, considering some JA correspondences \( \delta^{RM_\oplus} \) defined in (Everaere et al., 2014b). Roughly, each \( \delta^{RM_\oplus} \) correspondence consists in selecting in the set of all consistent and resolute judgment sets the "best score" ones, where the score of each judgment set is defined as the \( \oplus \)-aggregation of an \( m \)-vector of values (one value per question in the agenda \( X \), reflecting the number of agents supporting the question in the input profile \( \Gamma \)). Note that, by construction, the sets of collective judgment sets computed using \( \delta^{RM_\oplus} \) contain only consistent and resolute judgment sets (thus, \( \delta^{RM_\oplus} \) coincides with the abstention-free correspondence associated with it). Finally, when \( X = W \), the set of all consistent and resolute judgment sets coincide with \( \{ \gamma_\omega \mid \omega \in W \} \).

**Proposition 7.15.** When the agenda is complete, for any \( \oplus \) satisfying strict non-decreasingness, the ranked majority judgment aggregation correspondence \( \delta^{RM_\oplus} \) satisfies universal domain, collective rationality, collective resoluteness, anonymity, neutrality, unanimity, consensuality, and majority preservation. It does not satisfy independence. For \( \oplus = \Sigma \), weak consistency and consistency are also satisfied.
Let us now step back to the general case, when the agenda $X$ is not complete. First, let us observe that in this case, no JA correspondence can satisfy both consensuality and majority preservation.

**Proposition 7.16.** The consensuality property and the majority preservation property cannot be satisfied together in the general case.

Unsurprisingly, unanimity and consensuality are connected:

**Proposition 7.17.** Consensuality implies unanimity.

Unfortunately, the quite good behaviour of $\delta^{RM_\oplus}$ in the complete agenda case does not lift to the general case:

**Proposition 7.18.** In the general case, $\delta^{RM_\oplus}$ satisfies universal domain, collective rationality, collective resoluteness, anonymity, neutrality. For any $\oplus$ satisfying strict non-decreasingness, $\delta^{RM_\oplus}$ satisfies majority preservation, but does not satisfy weak consistency, consistency, or consensuality. If $\oplus$ satisfies strict preference and decomposition, then $\delta^{RM_\oplus}$ satisfies unanimity. Finally, $\delta^{RM_\oplus}$ does not satisfy independence.

### 7.8 Conclusion and discussion

### Bibliography


