"A Possibility Theorem on Majority Decisions" by Amartya K. Sen

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OVERVIEW

- Sen's theorem identifies a group of conditions which are sufficient to guarantee that the social preference relation R generated by majority decisions is a weak social ordering, (*i.e.*, reflexive, connected, transitive)
- Majority decision makes an Arrovian social welfare function when every triple is value-restricted and every triple has an odd number of concerned voters
- Relationship between this and earlier results: Sen's proof generalizes work from Arrow, Black, Inada, and Ward.

Sen's Background until 1966

- His friend (Sukhamoy Chakravarty, at Presidency College) introduced him to Arrow's impossibility theorem in 1952
- The intellectual climate at Cambridge included debates between the Keynesians and neo-classicists
- After winning the Prize Fellowship from Trinity, he took four years to study philosophy
- During 1966, he was professing economics at the Delhi School of Economics and the University of Delhi

MAJORITY DECISIONS

Definition

The method of majority decisions means that xRy if and only if the number of individuals i such that xR_iy is at least as great as the number of individuals i such that yR_ix.

Important Note:

The key to this proof is that when majority votes are taken, the social ordering satisfies reflexivity[1] and connectedness.[2] Thus, for a weak social ordering, Sen only has to show that under Value-Restriction, transitivity is assured.

Definition

Forward circles are intransitive triples: xRy, yRz, and zRx.
Backward circles are intransitive triples: yRx, xRz, and zRy.

[1] $\forall x(xRx)$
[2] $\forall x, y(xRy \vee yRx)$

ASSUMPTION OF VALUE-RESTRICTED PREFERENCES

Assumption of Value-Restriction

A set of individual preferences is value-restricted if for every triple and some alternative in that triple, for every individual that alternative is not best, or for every individual that alternative is not worst, or for every individual that alternative is not medium.

STATEMENT OF POSSIBILITY THEOREM

Theorem 1 (Possibility Theorem for Value-Restricted Preferences)

The method of majority decision is a social welfare function satisfying Arrow's Conditions 2-5[3]*, and consistency for any number of alternatives, providing the preferences of concerned individuals over every triple of alternatives is Value-Restricted, and the number of concerned individuals for every triple is odd.*

By dropping Condition 1: that all "admissible" inputs are allowed; thus restricting inputs, there is transitivity (i.e. majority ensures a weak social order).

[3] Reminder: Positive Association, Independence of Irrelevant Alternatives, Citizens' Sovereignty, and Nondictatorship. *Not* Admissible Inputs.

PROOF OF POSSIBILITY THEOREM

- Lemma 1: that any inconsistency implies intransitivity in a social triple of alternatives. Thus, if no triple is intransitive, then majority maintains consistency. (Simple reductio.)
- Assume forward circle (*i.e.*, xRy, yRz, zRx). For each pair of conditions, we derive an equality.
- Three equalities for forward circles and three for backward circles. *E.g.*, assuming xRy and yRz, we get: (1.1) $N(x \geq y \geq z) + N(x > y > z) \geq N(z \geq y \geq x) + N(z > y > x)$
- Assume, for a contradiction, that for all $i \in N$, if $xR_iy \wedge yR_iz \Rightarrow i$ is indifferent between x, y, z
- Then $N(x \geq y \geq z) = N(x = y = z)$ and $N(x > y > z) = 0$, so: (1.1a) $N(x = y = z) \geq N(z \geq y \geq x) + N(z > y > x) \Rightarrow N(z > y > x) = 0$

PROOF OF POSSIBILITY THEOREM

- By assuming indifference for i s.t. $xR_iy \wedge yR_iz$, also indifference for i s.t. $zR_iy \wedge yR_ix$
- So all are either unconcerned or peakedly concerned: $N = N(x = y = z) + N(x > y, y < z) + N(x < y, y > z)$
- But by assumption, xRy, yRz in social preferences. Thus: $N(x > y, y < z) \geq N(x < y, y > z)$, $N(x < y, y > z) \geq N(x > y, y < z)$
- Thus, $N(x > y, y < z) = N(x < y, y > z)$, i.e. number of concerned individuals is even. Contradiction.
- $N(x \geq y \geq z) = N(x = y = z)$ *inconsistent* with forward circle.
- Similar claims: three each for forward circles, backward.
- Each triple restriction: best, medium, or worst, corresponds to both (a) a forward restriction, (b) a backward restriction; prevents either intransitivity. □

COMPARING SEN'S THEOREM WITH OTHERS

- Arrow and Black's Single-Peaked Preferences: Counterexample to Black's formulation with indifference
- Inada shows that Arrow only needs the weaker condition of Single-Peaked Preferences on triples, not over all alternatives

Counterexample to Black

- In "On the Rationale of Group Decision-Making," (Black 1948) takes individual preference orderings but disallows complete indifference. However, in *The Theory of Committees and Elections*, (Black 1958) allows general indifference (4).
- Black claims the total number of voters is odd, rather than concerned voters being odd.
- The counterexample has to be single-peaked, but the majority of voters take xRy, yRz and $\neg xRz$.

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Counterexample

Let $N := \{1, 2, 3\}$ and $bP_1a \wedge aP_1c, aP_2c \wedge cP_2b, aI_3b, bI_3c$.
Majority gives you: $aRb^*, bRa, bRc^*, cRb, cPa, \neg aRc^*$.

Inada's Generalization of Arrow

- (Inada 1964) simple majority rule satisfies any number of alternatives when triple single-peakedness holds and odd individuals (528).
- (Inada 1964) shows that, like single-peakedness, single-cavedness is sufficient for possibility (529-30).
- Sen generalizes by saying that:
  1. The number of *concerned individuals* is odd for a triple, allowing for unconcerned individuals.
  2. Further, the number of individuals is even, but concerned individuals may be odd.
  3. Different value restrictions for differing triples.
- Essentially, (Indada 1964), (Arrow 1950) and (Black 1948) are all concerned with the concerned voters, and do not consider the non-impact of unconcerned voters.

DIAGRAM CONNECTING VALUE RESTRICTION

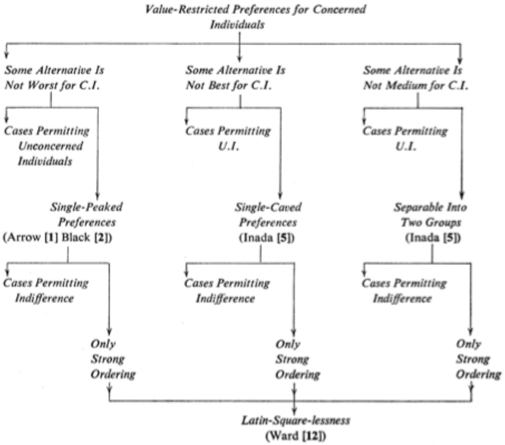

FIGURE 1.—Restriction on preferences for each triple.

Conclusion

- The primary value of Sen's Possibility Theorem is in showing that the (fairly intuitive) ideas of Black and Arrow can be further generalized.
- Major difference between Sen's treatment and others is the distinction between concerned and unconcerned voters.

Possible discussion questions:

- Clearly there may be unconcerned voters in any election. But in which applications might unconcerned voters actually submit unconcerned votes? For instance, as opposed to spoiled ballots (or simple abstentions).
- As (Inada 1964) pointed out, inconsistency is derivable from intransitive triples. Are there any intuitive ideas about why triples are sufficient?

SELECTED WORKS

- Arrow, Kenneth J. (1950): "A difficulty in the concept of social welfare," **J. Poli. Econ.** 58(4):328–46.
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- Inada, Ken-ichi (1964): "A note on the simple majority decision rule," **Econometrica** 32(4):525–31.
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