Strategic Manipulability without Resoluteness or Shared Beliefs: Gibbard-Satterthwaite Generalized

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The Generalisation of the GS-Theorem Allows Ties without Shared Beliefs

Recall the Gibbard-Satterthwaite Theorem:

Theorem (Gibbard-Satterthwaite, 1973, 1975)

If there are at least three alternatives to vote for, then there is no **surjective** *and* **strategy-proof** *voting procedure (mapping strict preferences for each individual to single winners among the alternatives), which is* **not dictatorial**.

- Three conditions are inconsistent:
  - Surjectivity (citizens' sovereignty)
  - Strategy-proofness (non-manipulability)
  - Non-dictatorship
- Actually another condition:
  - Resoluteness (single winners)
- Some authors generalized allowing ties, but
  - Shared beliefs (lottery is chosen together with winning set) or
  - Further restrictive assumptions on choice function or underlying social preference (neutrality, anonymity, acyclicity...)
- Duggan and Schwartz relaxed non-manipulability in a more general way than before
  - No shared beliefs about resolution of ties
  - Manipulability: only if an individual can profit regardless of the lottery
  - Need some remaining very weak resoluteness

Outline

1. The Authors: John Duggan, Thomas Schwartz
2. Setting, Definitions and Conditions
   - Citizens' sovereignty, non-dictatorship and residual resoluteness
3. Non-manipulability
   - $\neg\mathbf{M}$-Lemma and its proof
   - More intuitive definition
4. Impossibility Theorem
   - Proof outline
5. Relaxations of the conditions
6. Discussion

Duggan, J.; Schwartz, T.: *Strategic Manipulability without Resoluteness or Shared Beliefs: Gibbard-Satterthwaite Generalized*, Social Choice and Welfare, Vol. 17, 2000, pp. 85-93.

The Authors

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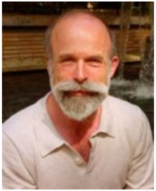
  - Writing book about Ronald Regan's international strategy during the Cold War



Setting, Notation and Basic Definitions

Notation

- A set of *alternatives* A
  - Elements denoted by x, y, z
  - Countable subsets denoted by X, Y
- A finite set of *individuals* $I = \{1, \ldots, n\}$
  - Elements denoted by i, j
- The set $\mathcal{P}$ of all strict linear orders on A (*preference orderings* P)
  - asymmetric, transitive, connected
- An *individual preference ordering* $P_i \in \mathcal{P}$ for each individual i, giving as the full picture a *(preference) profile* $\boldsymbol{P} = \langle P_1, P_2, \ldots, P_n \rangle \in \mathcal{P}^n$
- An *i-variant* of a profile $\boldsymbol{P}$ is another $\boldsymbol{P}'$ with $P_j = P'_j$ for all $j \neq i$
- An *X-lottery* is a function $\lambda : X \to (0, 1]$ with $\sum_{x \in X} \lambda(x) = 1$
- A *representative of* an individual preference ordering *P_i in X* is any function $u : X \to \mathbb{R}$ such that $u(x) > u(y) \iff xP_iy$

Definition

A *set choice function* $C : \mathcal{P}^n \to \text{Pow}(A) \setminus \emptyset$ is a function, which assigns a non-empty countable *winning set* $C(\boldsymbol{P}) \subseteq A$ to any profile $\boldsymbol{P} = \langle P_1, P_2, \ldots, P_n \rangle$.

Four Conditions

Definition (Citizen's Sovereignty ($\mathbf{CS}$))

A set choice function has the property of *Citizen's Sovereignty* if for all $x \in A$ there is a profile $\boldsymbol{P}$ that has a winning set $C(\boldsymbol{P})$ that includes x.

$$\forall x \, \exists \boldsymbol{P} \, [x \in C(\boldsymbol{P})]$$

Definition (Non-dictatorship ($\neg\mathbf{D}$))

A set choice function is *non-dictatorial* if there is no individual i such that, for all alternatives x and profiles $\boldsymbol{P}$, if $x = top(P_i)$, then $C(\boldsymbol{P}) = \{x\}$.

$$\neg\exists i \forall x, \boldsymbol{P} \, [x = top(P_i) \rightarrow C(\boldsymbol{P}) = \{x\}]$$

Definition (Residual Resoluteness ($\mathbf{RR}$))

A set choice function has *residual resoluteness* if $C(\boldsymbol{P})$ is a singleton in the case that all $P_{j \neq i}$ are the same, with x first and y second, and P_i is either the same as them or else the same but with y first and x second.

Four Conditions (continued)

Definition (Non-manipulability ($\neg$M))

A set choice function is called *non-manipulable* if there are no i-variant profiles $\boldsymbol{P}$, $\boldsymbol{P}'$ such that for all $C(\boldsymbol{P})$-lotteries λ and $C(\boldsymbol{P}')$-lotteries λ', some representative u of P_i in $C(\boldsymbol{P}) \cup C(\boldsymbol{P}')$ exists with $\sum_{x \in C(\boldsymbol{P}')} \lambda'(x)u(x) > \sum_{x \in C(\boldsymbol{P})} \lambda(x)u(x)$.

$$\neg \exists \boldsymbol{P}, \boldsymbol{P}' \left[\forall \lambda, \lambda' \exists u \left(\sum_{x \in C(\boldsymbol{P}')} \lambda'(x)u(x) > \sum_{x \in C(\boldsymbol{P})} \lambda(x)u(x) \right) \right]$$

Lemma ($\neg$M-Lemma)

If $\boldsymbol{P}'$ is an i-variant of $\boldsymbol{P}$ and $x \in C(\boldsymbol{P}')$, then

1. *there is $y \in C(\boldsymbol{P})$ with $y = x$ or $xP_i'y$, and*
2. *there is $y \in C(\boldsymbol{P})$ with $y = x$ or yP_ix.*

$$\forall \boldsymbol{P}', \boldsymbol{P} \, \forall x \in C(\boldsymbol{P}') \, \exists y \in C(\boldsymbol{P}) \, [x \geq_i' y] \tag{1}$$

$$\forall \boldsymbol{P}', \boldsymbol{P} \, \forall x \in C(\boldsymbol{P}') \, \exists y \in C(\boldsymbol{P}) \, [y \geq_i x] \tag{2}$$

Proof of $\neg\mathbf{M}$-Lemma

$$\neg\mathbf{M}: \quad \neg\exists \boldsymbol{P}, \boldsymbol{P}' \left[\forall\lambda,\lambda'\exists u\left(\sum_{x\in C(\boldsymbol{P})}\lambda(x)u(x) > \sum_{x\in C(\boldsymbol{P}')}\lambda'(x)u(x)\right)\right]$$

$$\neg\mathbf{M}\text{-Lemma}: \quad \forall \boldsymbol{P}', \boldsymbol{P}\,\forall x\in C(\boldsymbol{P}')[\underbrace{\exists y\in C(\boldsymbol{P})\,(x\succeq_i' y)}_{(1)} \wedge \underbrace{\exists y\in C(\boldsymbol{P})\,(y\succeq_i x)}_{(2)}]$$

Proof (of $\neg\mathbf{M}$-Lemma).

Pick $\boldsymbol{P}, \boldsymbol{P}'$ i-variants, $x\in C(\boldsymbol{P}')$. Suppose (1) false, then $y \succ_i' x$ for all $y\in C(\boldsymbol{P})$. Now let λ,λ' be a $C(\boldsymbol{P})$- and $C(\boldsymbol{P}')$-lottery, respectively, and define representative $u^*: C(\boldsymbol{P})\cup C(\boldsymbol{P}')\to\mathbb{R}$ of P_i': Set $u^*(x) := 1$ and define $u^*(z) := \frac{1}{d+1}$ for alternatives z ranked d steps lower in P_i'; and similarly $u^*(z) := 2 - \frac{1}{d+1}$ for alternatives z ranked d steps higher in P_i'. Then (since $0 < u^* < 2$) we have guaranteed convergence of $0 \leq \sum_{y\in C(\boldsymbol{P})}\lambda(y)u^*(y) \leq 2$ and $0 \leq \sum_{z\in C(\boldsymbol{P}')\setminus\{x\}}\lambda'(z)u^*(z) \leq 2$. Hence, can define new representative $u: C(\boldsymbol{P})\cup C(\boldsymbol{P}')\to\mathbb{R}$ of P_i' by setting

$$u(z) = \begin{cases} \min\left(u^*(x), \frac{\sum_{y\in C(\boldsymbol{P})}\lambda(y)u^*(y) - \sum_{z\in C(\boldsymbol{P}')\setminus\{x\}}\lambda'(z)u^*(z) - 1}{\lambda'(x)}\right) & \text{if } z = x \\ u^*(z) - (u^*(x) - u(x)) & \text{if } xP_i'z \\ u^*(z) & \text{else.} \end{cases}$$

Proof of $\neg\mathbf{M}$-Lemma (continued)

$$\neg\mathbf{M}: \quad \neg\exists \boldsymbol{P}, \boldsymbol{P}' \left[\forall \lambda, \lambda' \exists u \left(\sum_{x\in C(\boldsymbol{P})} \lambda(x)u(x) > \sum_{x\in C(\boldsymbol{P}')} \lambda'(x)u(x) \right)\right]$$

$$\neg\mathbf{M}\text{-Lemma:} \quad \forall \boldsymbol{P}', \boldsymbol{P}\, \forall x \in C(\boldsymbol{P}')[\underbrace{\exists y \in C(\boldsymbol{P})\,(x \geq_i' y)}_{(1)} \wedge \underbrace{\exists y \in C(\boldsymbol{P})\,(y \geq_i x)}_{(2)}]$$

$$u(z) = \begin{cases} \min\left(u^*(x), \frac{\sum_{y\in C(\boldsymbol{P})} \lambda(y)u^*(y) - \sum_{z\in C(\boldsymbol{P}')\setminus\{x\}} \lambda'(z)u^*(z) - 1}{\lambda'(x)}\right) & \text{if } z = x \\ u^*(z) - (u^*(x) - u(x)) & \text{if } xP_i'z \\ u^*(z) & \text{else.} \end{cases}$$

Proof (of $\neg\mathbf{M}$-Lemma) continued.

From first line of case distinction we get

$$\sum_{y\in C(\boldsymbol{P})} \lambda(y)u^*(y) - \sum_{z\in C(\boldsymbol{P}')\setminus\{x\}} \lambda'(z)u^*(z) > u(x)\lambda'(x)$$

and hence

$$\sum_{y\in C(\boldsymbol{P})} \lambda(y)u(y) > \sum_{z\in C(\boldsymbol{P}')\setminus\{x\}} \lambda'(z)u(z) + u(x)\lambda'(x) = \sum_{z\in C(\boldsymbol{P}')} \lambda'(z)u(z).$$

Contradiction to $\neg\mathbf{M}$. (Proof for (2) is analogous.) □

¬M-Lemma Yields New Intuitive Understanding of ¬M-condition

¬M-Lemma: $$\forall \boldsymbol{P}', \boldsymbol{P}\, \forall x \in C(\boldsymbol{P}')[\underbrace{\exists y \in C(\boldsymbol{P})\,(x \geq_i' y)}_{(1)} \wedge \underbrace{\exists y \in C(\boldsymbol{P})\,(y \geq_i x)}_{(2)}]$$

$$\iff \quad \neg\exists \boldsymbol{P}', \boldsymbol{P}\, \exists x \in C(\boldsymbol{P}')[\forall y \in C(\boldsymbol{P})\,(x <_i' y) \vee \forall y \in C(\boldsymbol{P})\,(y <_i x)]$$

Definition

1. A set choice function C is *manipulable by a pessimist* if there are i-variant profiles $\boldsymbol{P}$, $\boldsymbol{P}'$ and an $x \in C(\boldsymbol{P}')$ among the winners of the "truthful" profile $\boldsymbol{P}'$ such that all winners $C(\boldsymbol{P})$ of the "manipulated" profile $\boldsymbol{P}$ are ranked higher than x by the "truthful" ordering P_i'.
2. A set choice function C is *manipulable by an optimist* if there are i-variant profiles $\boldsymbol{P}$, $\boldsymbol{P}'$ and an $x \in C(\boldsymbol{P}')$ among the winners of the "manipulated" profile $\boldsymbol{P}'$ such that all winners $C(\boldsymbol{P})$ of the "truthful" profile $\boldsymbol{P}$ are ranked lower than x by by the "truthful" ordering P_i.

- A set choice function C is *non-manipulable** if it is neither manipulable by a pessimist nor by an optimist. ($\iff$ ¬M-Lemma)

Remark

Under the assumption of countable choice sets, ¬M-Lemma is equivalent to ¬M.

The Impossibility Theorem and its Proof

Theorem (Duggan, Schwarz (2000))

If $|A| \geq 3$ then there is no set choice function that can simultaneously satisfy Conditions $\neg\mathbf{M}$, $\mathbf{CS}$, $\neg\mathbf{D}$ and $\mathbf{RR}$.

Definition

- $X \subseteq A$ is called a *top set* in a profile $\boldsymbol{P}$ if xP_iy for all $x \in X$, $i \in I$ and $y \notin X$.
- A profile $\boldsymbol{P}'$ is an *xy-twin* of another profile $\boldsymbol{P}$ if $xP'_iy \leftrightarrow xP_iy$ for all $i \in I$.

Proof.

- Define a "social preference" function $F : \mathcal{P}^n \rightarrow A^2$ from a set choice function C by

$$xF(\boldsymbol{P})y \iff (x \neq y) \wedge (\forall \boldsymbol{P}' \; xy\text{-twin of } \boldsymbol{P} \text{ with top set } \{x,y\})[C(\boldsymbol{P}') = \{x\}]$$

- Under the assumption of $\neg\mathbf{M}$, $\mathbf{CS}$, $\neg\mathbf{D}$ and $\mathbf{RR}$ show properties of F, which are known to be inconsistent

□

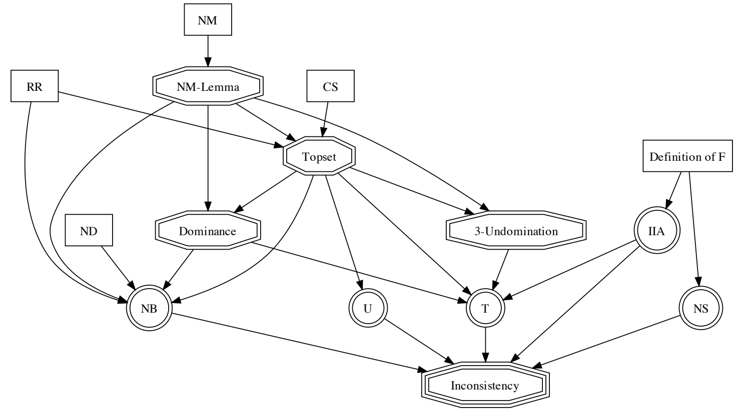
The Impossibility Theorem and its Proof (continued)

Proof.

- Define a "social preference" function $F : \mathcal{P}^n \to A^2$ from a set choice function by

$$xF(\boldsymbol{P})y \iff (x \neq y) \wedge (\forall \boldsymbol{P}' \ xy\text{-twin of } \boldsymbol{P} \text{ with top set } \{x,y\})[C(\boldsymbol{P}') = \{x\}]$$

- Under the assumption of $\neg\mathbf{M}$, $\mathbf{CS}$, $\neg\mathbf{D}$ and $\mathbf{RR}$ show properties of F, which are known to be inconsistent



Relaxation of $\mathbf{RR}$

Definition (Residual Resoluteness ($\mathbf{RR}$))

A set choice function has *residual resoluteness* if $C(\boldsymbol{P})$ is a singleton in the case that all $P_{j \neq i}$ are the same, with x first and y second, and P_i is either the same as them or else the same but with y first and x second.

- Avoid $\mathbf{RR}$ by strengthening $\mathbf{CS}$ to $\mathbf{CS+}$, and $\neg\mathbf{D}$ to $\neg\mathbf{D+}$:

Definition

- $\mathbf{CS+}$: For all alternatives $x \in A$, some profile $\boldsymbol{P}$ has $C(P) = \{x\}$.
  - Compare $\mathbf{CS}$: $\forall x \, \exists \boldsymbol{P} \, [x \in C(\boldsymbol{P})]$
- $\neg\mathbf{D+}$: No individual i is such that, for all alternatives x and profiles $\boldsymbol{P}$, $x = top(P_i)$ implies $x \in C(X)$.
  - Compare $\neg\mathbf{D}$: $\neg\exists i \forall x, \boldsymbol{P} \, [x = top(P_i) \rightarrow C(\boldsymbol{P}) = \{x\}]$

- Strengthening only one of them is not enough ($\rightarrow$ dual dictators)
- Both (strengthened) conditions carry implicit resoluteness
  - $\mathbf{CS+}$: Each outcome can be chosen as a singleton
  - $\neg\mathbf{D+}$: Bans procedures that pick all alternatives ranked first by someone ($\rightarrow$ example from Gibbard)
- Weakening $\mathbf{RR}$?
  - Two-member choice sets ($\rightarrow$ dual dictators)
  - Only to case when everyone agrees ($\rightarrow$ dual dictators)

Relaxation of $\mathbf{CS}$, $\neg\mathbf{D}$

Definition (Citizen's Sovereignty ($\mathbf{CS}$))

A set choice function has the property of *Citizen's Sovereignty* if for all $x \in A$ there is a profile $\boldsymbol{P}$ that has a winning set $C(\boldsymbol{P})$ that includes x.

$$\forall x\, \exists \boldsymbol{P}\, [x \in C(\boldsymbol{P})]$$

- $\mathbf{CS}$ implies that any alternative is feasible
- Can avoid this by defining profiles on a larger set $B \supseteq A$ instead
- Then C can depend on infeasible alternatives, too
  - e.g. indicating strengths of preferences
- $\neg\mathbf{M}$ is defined to consider feasible alternatives only
  - $\mathcal{C}(\boldsymbol{P})$-lotteries, representative of P_i on $\mathcal{C}(\boldsymbol{P}) \cup \mathcal{C}(\boldsymbol{P}')$

Definition (Non-dictatorship ($\neg\mathbf{D}$))

A set choice function is *non-dictatorial* if there is no individual i such that, for all alternatives x and profiles $\boldsymbol{P}$, if $x = top(P_i)$, then $C(\boldsymbol{P}) = \{x\}$.

$$\neg\exists i \forall x, \boldsymbol{P}\, [x = top(P_i) \rightarrow C(\boldsymbol{P}) = \{x\}]$$

- (Almost) only matters for resolute choice functions
  - $\exists \boldsymbol{P}[|C(\boldsymbol{P})| > 1 \wedge \forall i \exists x (x = top(P_i))] \implies \neg\mathbf{D}$

Relaxation of $\neg\mathbf{M}$

Definition (Non-manipulability ($\neg\mathbf{M}$))

A set choice function is called *non-manipulable* if there are no i-variant profiles $\boldsymbol{P}$, $\boldsymbol{P}'$ such that for all $C(\boldsymbol{P})$-lotteries λ and $C(\boldsymbol{P}')$-lotteries λ', some representative u of P_i in $C(\boldsymbol{P}) \cup C(\boldsymbol{P}')$ exists with $\sum_{x \in C(\boldsymbol{P}')} \lambda'(x)u(x) > \sum_{x \in C(\boldsymbol{P})} \lambda(x)u(x)$.

$$\neg\exists \boldsymbol{P}, \boldsymbol{P}' \left[\forall\lambda, \lambda' \exists u \left(\sum_{x \in C(\boldsymbol{P}')} \lambda'(x)u(x) > \sum_{x \in C(\boldsymbol{P})} \lambda(x)u(x) \right) \right]$$

- Strengthen $\forall\lambda\forall\lambda'\exists u$ to $\exists u\forall\lambda\forall\lambda'$ or $\forall\lambda\forall\lambda'\forall u$
  - Weakens $\neg\mathbf{M}$ $\rightarrow$ strengthens theorem
  - Counterexample: pick, if exists, Condorcet, else all
- Relaxation of support set
  - Condition taylor-made for proof and weak
  - Usefulness? ($\rightarrow$ discussion)
- Shift to $\neg\mathbf{M}$-Lemma (non-manipulability*) instead of $\neg\mathbf{M}$-condition
  - Allows uncountable choice sets
  - Equivalent if we assume countable choice sets
- Allow "contracting" manipulations
  - Proof breaks down
  - Potentially stronger version allows "contracting" manipulations only if following manipulations are not even profitable with respect to the original "honest" ordering

Conclusion

- Generalisation of GS-Theorem allowing ties
  - More general than before
  - No shared beliefs about resolution of ties
  - Manipulability: only if an individual can profit regardless of the lottery
  - Need some remaining very weak resoluteness
  - Proof via result on "social preference" functions

Conditions:

- Non-manipulability ($\neg\mathbf{M}$)
  - $\neg\mathbf{M}$-Lemma, its proof and intuition (optimist, pessimist), better taken as definition?
  - Infinitely many alternatives (convergence, Riemann Rearrangement Theorem, practical relevance?)
  - Relaxation of support set useful?
- Citizen's Sovereignty ($\mathbf{CS}$)
  - any alternative feasible
  - relaxable
- Non-dictatorship ($\neg\mathbf{D}$)
  - Nearly irrelevant for non-resolute set choice functions
  - Mistake in paper
- Residual Resoluteness ($\mathbf{RR}$)
  - Avoidable at cost
  - But replacement has implicit resoluteness