Intermediate logics and their modal companions

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Modal logic and intuitionistic logic

Modal logic is an **expansion** of classical logic.

Additional modal operators have different meanings:

- alethic modalities (possibility, necessity),
- temporal modalities (since, until),
- deontic modalities (obligation, permission),
- epistemic modalities (knowledge),
- doxastic modalities (belief), etc.

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Surprisingly: intuitionistic and modal logic are

closely connected!

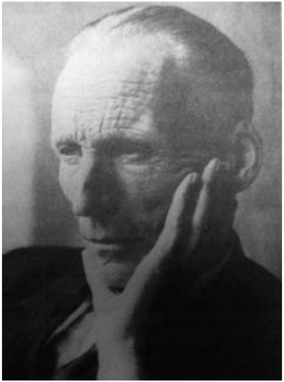
Overview of today's lecture

1. Intuitionistic logic and its Kripke semantics
2. Intermediate logics
3. Gödel translation
4. Modal companions of intermediate logics
5. Least and greatest modal companions
6. Blok-Esakia theorem: an overview

Intuitionistic logic

One of the cornerstones of classical reasoning is the law of excluded middle $p \vee \neg p$.

On the grounds that the only accepted reasoning should be constructive, the Dutch mathematician L. E. J. Brouwer rejected this law, and hence classical reasoning.



Luitzen Egbertus Jan Brouwer (1881 - 1966)

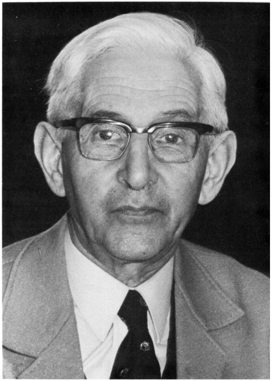
Intuitionistic logic

This resulted in serious debates between Hilbert and Brouwer. Other leading mathematicians of the time were also involved in this debate.

David Hilbert (1862 - 1943)

Intuitionistic logic

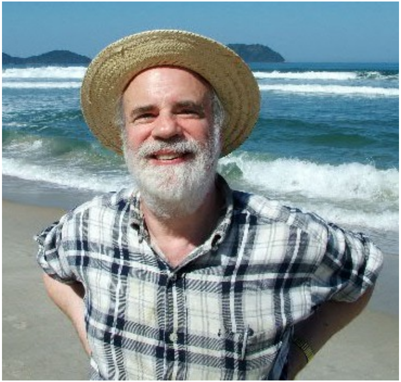
In 30's Brouwer's ideas led his student Heyting to axiomatize intuitionistic logic.



Arend Heyting (1898 - 1980)

Kripke semantics

In 50's and 60's Kripke discovered a relational (Kripke) semantics for intuitionistic and modal logic and proved completeness of intuitionistic logic wrt this semantics.



Saul Kripke

Kripke semantics

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An intuitionistic Kripke model is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ such that $\mathfrak{F}$ is an intuitionistic Kripke frame and V is an intuitionistic valuation; that is, a map $V : \textsc{Prop} \to \mathcal{P}(W)$ such that:

$$w \in V(p) \text{ and } wRv \text{ implies } v \in V(p).$$

Persistence: Information is never lost.

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We will consider only rooted frames.

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\mathfrak{M}, w \models \varphi \vee \psi & \text{if} & \mathfrak{M}, w \models \varphi \text{ or } \mathfrak{M}, w \models \psi; \\
\mathfrak{M}, w \models \varphi \to \psi & \text{if} & \forall v, \text{ if } (wRv \text{ and } \mathfrak{M}, v \models \varphi) \text{ then } \mathfrak{M}, v \models \psi;
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The satisfaction clause for intuitionistic $\varphi \to \psi$ resembles the satisfaction clause for modal $\Box(\varphi \to \psi)$.

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CPC = classical propositional calculus
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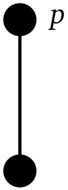

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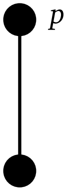
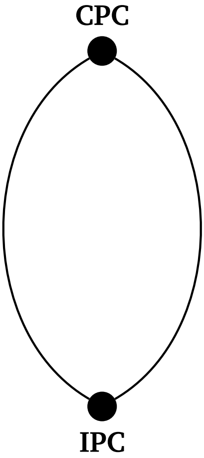


Assuming completeness, this shows that **IPC** $\subsetneq$ **CPC**.

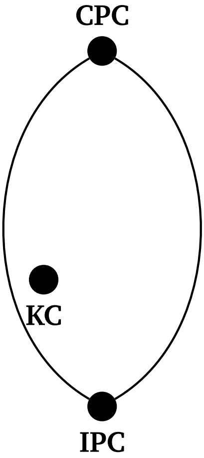
Intermediate logics



Intermediate logics

$\mathbf{KC} = \mathbf{IPC} + (\neg p \vee \neg\neg p)$

weak law of excluded middle



Intermediate logics

$\mathbf{LC} = \mathbf{IPC} + (p \to q) \vee (q \to p)$

Gödel-Dummett calculus

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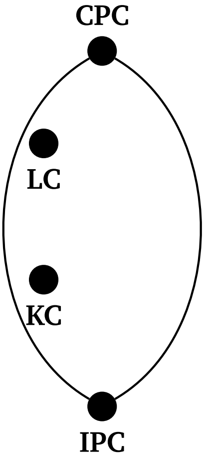

Intermediate logics

Logics in between **IPC** and **CPC** are called intermediate logics.

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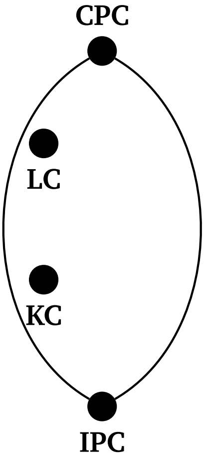

Intermediate logics

Theorem.

1. **IPC** is the logic of all intuitionistic frames.
2. **CPC** is the logic of a one-point frame.
3. **KC** is the logic of directed intuitionistic frames.
4. **LC** is the logic of linear intuitionistic frames.

Gödel translation

In the 30's Gödel defined a translation of intuitionistic logic into the modal logic **S4**.

Kurt Gödel (1906 - 1978)

Gödel translation

$$(\bot)^* = \bot,$$
$$(p)^* = \Box p, \text{ where } p \in \mathsf{Prop},$$
$$(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*,$$
$$(\varphi \vee \psi)^* = \varphi^* \vee \psi^*,$$
$$(\varphi \to \psi)^* = \Box(\varphi^* \to \psi^*).$$

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S4 is the modal logic of reflexive and transitive frames.

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Theorem (Gödel-McKinsey-Tarski) For each formula φ in the propositional language we have

$$\mathbf{IPC} \vdash \varphi \quad \text{iff} \quad \mathbf{S4} \vdash \varphi^*.$$

Topological semantics

They also defined topological semantics for modal and intuitonistic logic and proved that **S4** and **IPC** are complete wrt the real line $\mathbb{R}$.

Alfred Tarski (1901 - 1983)

Generalized Gödel embedding

Dummett and Lemmon in the 50's lifted the Gödel translation to intermediate logics and extensions of **S4**.

Michael Dummett (1925 - 2011)

Modal companions

A modal logic $M \supseteq \mathbf{S4}$ is a modal companion of an intermediate logic $L \supseteq \mathbf{IPC}$ if for any propositional formula φ we have

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Examples.

1. **S4** is a modal companion of **IPC**.
2. **S5** is a modal companion of **CPC**.
3. **S4.2** is a modal companion of **KC**.
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Recall that

- $\mathbf{S4.2} = \mathbf{S4} + \Diamond\Box p \to \Box\Diamond p$ is the logic of directed **S4**-frames.
- $\mathbf{S4.3} = \mathbf{S4} + \Box(\Box p \to \Box q) \vee \Box(\Box q \to \Box p)$ is the logic of linear **S4**-frames.

S4-frames and their skeletons

Let us look at an **S4**-frame $\mathfrak{G}$.

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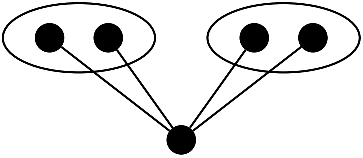

$\mathfrak{G}$

$\mathbf{S4}$-frames and their skeletons

Let us look at preordered (relfexive and transitive) frame $\mathfrak{G}$.

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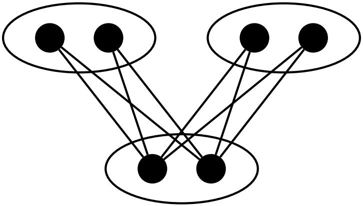

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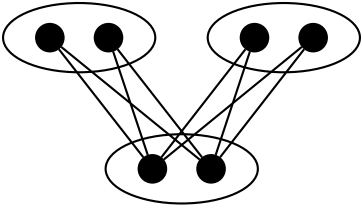

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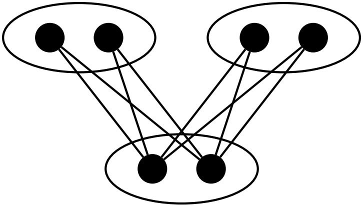

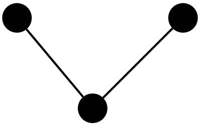

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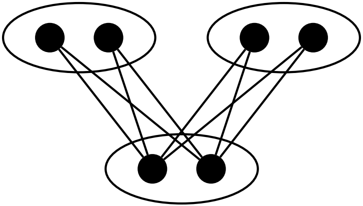

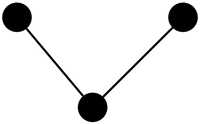

Thus we can think of an **S4**-frame as a poset of clusters.

$\mathbf{S4}$-frames and their skeletons

Lemma. Let $\mathfrak{G}$ be such that $\mathfrak{F}$ is its skeleton, then for any intuitionistic formula φ:

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Key idea: $\mathfrak{G}$ and $\mathfrak{F}$ have **matching** upward closed subsets.

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Let $Log(\mathfrak{F}) = \{\varphi : \mathfrak{F} \models \varphi\}$. We call it the intermediate logic of $\mathfrak{F}$.

Let $\mathfrak{F}$ be a finite intuitionistic frame. We let K denote a class of **S4**-frames that have $\mathfrak{F}$ as their skeleton.

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To prove an analogue of this result for all intermediate logics we need algebras and duality.

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Examples

Recall that **CPC** $= Log(\mathfrak{F}_1)$, where

$\mathfrak{F}_1$

Which modal logics are modal companions of **CPC**?

$\mathfrak{G}_1$ $\mathfrak{G}_2$ $\mathfrak{G}_3$

$$Log(\mathfrak{G}_1) \supsetneqq Log(\mathfrak{G}_2) \supsetneqq Log(\mathfrak{G}_3) \supsetneqq \ldots \supsetneqq \mathbf{S5}$$

Exercise: Verify these inclusions. Find formulas showing that the inclusions are strict.

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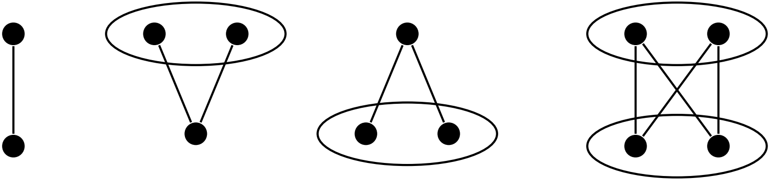

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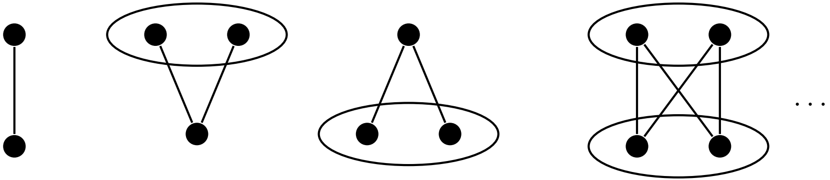

Exercise: Do these modal companions form a chain?

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In order to regain the intuition of the relational semantics we use a duality between algebras and general frames.

Greatest and least modal companions

Esakia and independently Maksimova in the 70's developed the theory of Heyting and closure algebras. Esakia also developed an order-topological duality for closure and Heyting algebras.

Leo Esakia (1934 - 2010)

Larisa Maksimova

Grzegorczyk's logic

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This gives a purely syntactic characterization of the least and greatest modal companions of an intermediate logic.

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Andrzej Grzegorczyk (1922 – 2014)

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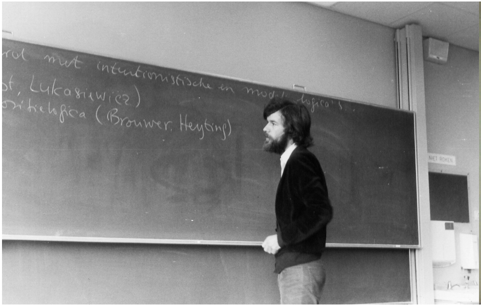
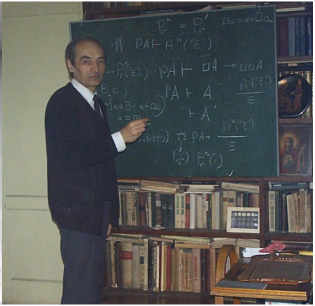
3. (Blok-Esakia) $\sigma : \Lambda(\mathbf{IPC}) \to \Lambda(\mathbf{Grz})$ is an isomorphism from the lattice of intermediate logics onto the lattice of extensions of **Grz**.

Blok-Esakia theorem



Wim Blok (1947 - 2003)



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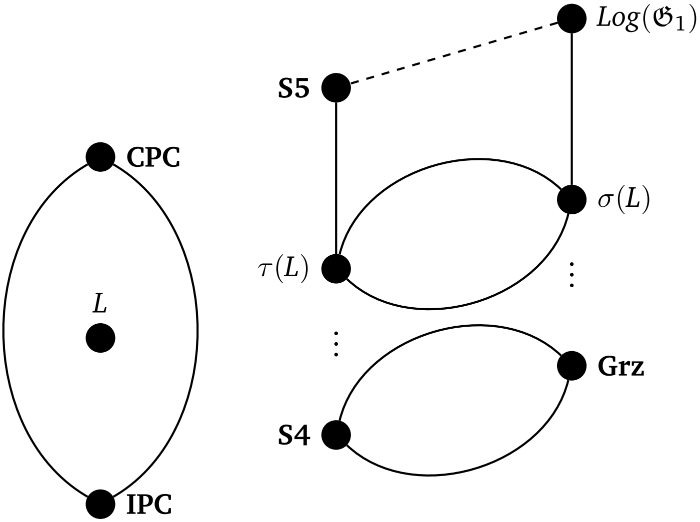
This method is very complex.

Nowadays we can provide a simplified algebraic approach to this method.

Picture of $\Lambda(\mathbf{IPC})$ and $\Lambda(\mathbf{S4})$



Exercises

1. Describe the intermediate logic whose modal companion is $\mathbf{S4.1} = \mathbf{S4} + (\Box\Diamond p \to \Diamond\Box p)$?

2. Is there a modal logic M with $\mathbf{S4} \subseteq M \subseteq \mathbf{S5}$ such that for no intermediate logic L we have $\tau(L) = M$? Justify your answer.

3. How many modal companions does the intermediate logic of the two element chain have? Justify your answer.

4. Is there an intermediate logic that has a finite number of modal companions? Justify your answer.

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Thank you!