Tutorial #6

Exercise 1

Let $N = \{1, \ldots, 10\}$. For each of the following three games determine whether it is additive, whether it is convex, whether it is superadditive, and whether it is cohesive. Also determine which of these games has a nonempty core.

(a) $\langle N, v \rangle$ with $v : C \mapsto |C|$
(b) $\langle N, v \rangle$ with $v : C \mapsto |C|^2$
(c) $\langle N, v \rangle$ with $v : C \mapsto \min(|C|, 7)$

Exercise 2

Recall that a *voting game* is a tuple $\langle N, \boldsymbol{w}, q \rangle$, where $N = \{1, \ldots, n\}$ is a the set of *voters*, $\boldsymbol{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n_{\geqslant 0}$ is a vector of the *weights* of these voters, and $q \in \mathbb{R}_{>0}$ is the *quota*. The idea is that the voters need to decide on whether a given proposal should be accepted. If the sum of weights of the voters in favour meets the quota q, then the proposal is accepted. Any coalition $C \subseteq N$ is called a *winning coalition* in case $\sum_{i \in C} w_i \geqslant q$. We require $\sum_{i \in N} w_i \geqslant q$, as otherwise *no* coalition would be a winning coalition. Recall that every voting game is a simple game, and every simple game is a TU game. Specifically, the voting game $\langle N, \boldsymbol{w}, q \rangle$ is the TU game $\langle N, v \rangle$ with the characteristic function $v : C \mapsto \mathbb{1}_{\sum_{i \in C} w_i \geqslant q}$.

In this exercise we are going to explore how the choice of quota affects the nonemptiness of the core of a voting game. Recall that the core of a TU game $\langle N, v \rangle$ is the set of all imputations $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geqslant 0}$ that ensure that no coalition would want to break away from the grand coalition, i.e., that ensure that $\sum_{i \in C} x_i \geqslant v(C)$ for all $C \subseteq N$.

Note that we *want* our games to have nonempty cores: the imputations in the core, arguably, are the only 'reasonable' ways of distributing the surplus generated by the grand coalition.

(a) Recall that a TU game $\langle N, v \rangle$ is *cohesive* if $v(N) \geqslant v(C_1) + \cdots + v(C_K)$ for every partitioning $C_1 \uplus \cdots \uplus C_K$ of N into mutually disjoint coalitions. Also recall that in class we saw that cohesiveness is a necessary condition for nonemptiness of the core.

Give an example for a voting game that is not cohesive. What can you say about the quota of the voting game you found?

(b) Now let us focus on voting games with 'moderately high' quotas:

$$q \quad > \quad \frac{1}{2} \cdot (w_1 + \cdots + w_n)$$

Show that any such voting game is cohesive.

(c) So, intuitively speaking, it looks like increasing the quota makes it more likely that the core will be nonempty. So let us now consider voting games with 'very high' quotas:

$$q \quad > \quad (w_1 + \cdots + w_n) - \max\{w_1, \ldots, w_n\}$$

Recall that a TU game $\langle N, v\rangle$ is *convex* if $v(C \cup C') \geqslant v(C) + v(C') - v(C \cap C')$ holds for any two coalitions $C, C' \subseteq N$. Also recall that we saw in class that convexity is a sufficient condition for nonemptiness of the core.

So if we can show that voting games with 'very high' quotas are convex, then we have identified a sufficient condition on the quota for the voting game in question to have a nonempty core. But can we actually show that all such voting games are convex?

(d) Another route towards showing that a voting game must have a nonempty core is to use this characterisation result proved in class: a simple game has a nonempty core if and only if it has a veto player. Recall that $i \in N$ is a *veto player* of the TU game $\langle N, v\rangle$ if $v(C) = 0$ for all coalitions $C \subseteq N \setminus \{i\}$ that exclude that player.

Show that every voting game with a 'very high' quota must have a veto player.

(e) At this point, you have shown that every voting game with a 'very high' quota must have a nonempty core (by appealing to the characterisation result proved in class).

Is this is a positive result? And does the converse hold as well, i.e., does every voting game with a nonempty core have a 'very high' quota?

Exercise 3

The purpose of this exercise is to help you understand the proof of the characterisation theorem for the Shapley value presented in class. The theorem states that assigning payoffs to players in line with the Shapley value is the only way of assigning payoffs in a manner that satisfies four natural axioms: efficiency, symmetry, the dummy player axiom, and additivity.

One direction of the theorem is straightforward to prove: the Shapley value really does satisfy the four axioms (but do make sure you understand this!). The tricky part is to show that it is *the only way* of assigning payoffs that works.

But observe that if we can show that the axioms determine *some* unique payoff vector, then it can only be the vector of payoffs suggested by the Shapley value. To prove that, indeed, the axioms determine a unique solution, complete the following steps:

(a) Fix any set of players $N = \{1, \ldots, n\}$. Now let us define a family of simple games for this fixed set of players. For any given set $S \subseteq N$, let $\langle N, v_S\rangle$ be the TU game with the following characteristic function:

$$v_S(C) \quad = \quad \begin{cases} 1 & \text{if } C \supseteq S \\ 0 & \text{otherwise} \end{cases}$$

Suppose we want to find functions $x_1, \ldots, x_n$, with each x_i mapping any given game $\langle N, v_S\rangle$ to the payoff player i should receive when the grand coalition forms, such that (at least) the first three of our four axioms will be respected (efficiency, symmetry, and the dummy player axiom). Prove the following three claims (in that order):

(i) To satisfy the dummy player axiom, we must choose $x_i(N, v_S) = 0$ for all $i \notin S$. That is, players not belonging to S (defining the game) will get no payoff at all.

(*ii*) To satisfy efficiency, we must ensure $\sum_{i \in N} x_i(N, v_S) = 1$.

(*iii*) To satisfy the dummy player axiom, efficiency, and symmetry together, we must choose $x_i(N, v_s) = \frac{1}{|S|}$ for all $i \in S$. That is, the surplus of 1 generated by the grand coalition must be divided evenly between the players in S.

As an aside, observe that the Shapley value would recommend precisely the payoffs as stated in the three statements above. But, importantly, we inferred these payments from the axioms, *not* from the definition of the Shapley value. Also, note that this observation is not in fact required for our proof to go through.

(b) Next, let us consider a larger family of TU games with players in the fixed set N, namely the family of games $\langle N, \alpha_S \cdot v_S \rangle$ defined by some set $S \subseteq N$ together with a coefficient $\alpha_S \in \mathbb{R}_{\geqslant 0}$. A game in this family has the following characteristic function:

$$(\alpha_s \cdot v_S)(C) \quad = \quad \begin{cases} \alpha_S & \text{if } C \supseteq S \\ 0 & \text{otherwise} \end{cases}$$

Show that, in order to satisfy the first three axioms, for any such game we must choose $x_i(N, \alpha_S \cdot v_S) = \frac{\alpha_S}{|S|}$ for $i \in S$ and $x_i(N, \alpha_S \cdot v_S) = 0$ for $i \notin S$.

Observe that everything we have done up to this point will still go through if we extend the definition of TU games to allow for characterisation functions $v : 2^N \to \mathbb{R}$ with possibly negative values, if we extend the notion of payoff vector accordingly to allow for negative payoffs, and if we also consider games of the form $\langle N, \alpha_S \cdot v_S \rangle$ with $\alpha_S < 0$.

(c) Now consider an arbitrary TU game $\langle N, v \rangle$. We want to prove that we can find a collection of games (in the extended sense, where negative values are permitted) that are of the form $\langle N, \alpha_S \cdot v_S \rangle$, one for each $S \in 2^N \setminus \{\emptyset\}$, such that we obtain our given game as the sum of all of those games we found. That is, we want to show:

$$v(C) \quad = \sum_{S \in 2^N \setminus \{\emptyset\}} \alpha_S \cdot v_S(C) \quad \text{for all } C \subseteq N$$

Give this proof by showing that the following coefficients (and only those!) do the job:

$$\alpha_S \quad := \quad v(S) \; - \sum_{S' \in 2^S \setminus \{\emptyset, S\}} \alpha_{S'}$$

(d) Explain, using the additivity axiom, how this completes the proof.