Tutorial #5

Exercise 1

Consider the combinatorial auction instance shown below, with four bidders and three goods. Each bidder reports a value for every possible bundle of goods. For notational simplicity, set brackets have been omitted, i.e., ab stands for $\{a, b\}$ and so forth.

Bidder 1: $(\emptyset, 0),\ (a, 1),\ (b, 1),\ (c, 1),\ (ab, 2),\ (ac, 2),\ (bc, 2),\ (abc, 3)$
Bidder 2: $(\emptyset, 0),\ (a, 2),\ (b, 0),\ (c, 0),\ (ab, 2),\ (ac, 2),\ (bc, 0),\ (abc, 2)$
Bidder 3: $(\emptyset, 0),\ (a, 0),\ (b, 2),\ (c, 0),\ (ab, 2),\ (ac, 0),\ (bc, 2),\ (abc, 2)$
Bidder 4: $(\emptyset, 0),\ (a, 0),\ (b, 0),\ (c, 1),\ (ab, 8),\ (ac, 1),\ (bc, 1),\ (abc, 9)$

Answer the following questions:

(a) Describe the valuations of each of the four bidders in plain English.

(b) How many different allocations of goods to bidders are there?

(c) What is the allocation selected by the VCG mechanism?

(d) We know that the VCG mechanism is incentive-compatible. Thus, if the auction instance shown reflects the true valuations of the bidders, then none of them can do better by misrepresenting their own valuation. But can you find a possible way for two of the bidders to collude and obtain an outcome that is better for both of them?

Exercise 2

Suppose you want to run an auction to sell two identical goods to 100 bidders, each of which will buy at most one of the goods. You ask each of them for a sealed bid in which they indicate the valuation they supposedly assign to obtaining one of the goods. (For the sake of simplicity, let us assume that no two bidders will submit the exact same bid.) You decide to assign the goods to the two highest bidders and you are considering three pricing rules:

(a) Charge the highest bidder the price of the second-highest bid and charge the second-highest bidder the price of the third-highest bid.

(b) Charge both of them the price of the second-highest bid.

(c) Charge both of them the price of the third-highest bid.

Which of these pricing rules, if any, result in an incentive-compatible mechanism? Which of them, if any, correspond to the pricing rule of the VCG mechanism?

Exercise 3

To help you understand the result on the incentive-compatibility of the VCG mechanism, review the following details of the proof presented in class:

(a) We had defined the price function $p_i : \boldsymbol{V} \to \mathbb{R}$ used to determine the price that agent i has to pay in case the reported valuations are $\hat{\boldsymbol{v}} = (\hat{v}_i, \hat{\boldsymbol{v}}_{-i}) = (\hat{v}_1, \ldots, \hat{v}_n)$ as follows:

$$p_i(\hat{\boldsymbol{v}}) \quad = \quad \hat{v}_i(f(\hat{\boldsymbol{v}})) - \left(\sum_{j=1}^{n} \hat{v}_j(f(\hat{\boldsymbol{v}})) - \sum_{j \neq i} \hat{v}_j(f(\hat{\boldsymbol{v}}_{-i})) \right)$$

For each of the three quantities making up the sum on the right, describe in plain English what is being measured by that quantity.

(b) Verify the following equation and describe what the two sums on the right represent:

$$p_i(\hat{\boldsymbol{v}}) \quad = \quad \sum_{j \neq i} \hat{v}_j(f(\hat{\boldsymbol{v}}_{-i})) - \sum_{j \neq i} \hat{v}_j(f(\hat{\boldsymbol{v}}))$$

(c) Recall that we are always assuming that agents have quasi-linear utilities, so the utility experienced by agent i when reporting $\hat{v}_i$ and everybody else is reporting $\hat{\boldsymbol{v}}_{-i}$ will be $u_i(\hat{v}_i, \hat{\boldsymbol{v}}_{-i}) = v_i(f(\hat{v}_i, \hat{\boldsymbol{v}}_{-i})) - p_i(\hat{v}_i, \hat{\boldsymbol{v}}_{-i})$, where v_i is the true valuation of agent i.

Now observe that, of course, the only way agent i can influence her own utility under the outcome and prices chosen by the mechanism is through her own reported valuation $\hat{v}_i$. With all of this in mind, explain why it is in the best interest of agent i to report a valuation $\hat{v}_i$ that maximises the following quantity:

$$v_i(f(\hat{v}_i, \hat{\boldsymbol{v}}_{-i})) + \sum_{j \neq i} \hat{v}_j(f(\hat{v}_i, \hat{\boldsymbol{v}}_{-i}))$$

(d) Compare this to the quantity maximised by the social choice function f for a given profile $(\hat{v}_i, \hat{\boldsymbol{v}}_{-i})$ of reported valuations when f is used to pick an outcome $\omega \in \Omega$. Explain how all of this proves the incentive compatibility of the VCG mechanism.