Game Theory 2025

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Plan for Today

This and the next lecture are going to be about extensive games, where we model individual actions being played in sequence.

Today we focus on the basic model for this kind of scenario:

- modelling *extensive games* of perfect information
- *translation* from the extensive into the normal form
- *Zermelo's Theorem:* existence of pure Nash equilibria
- new solution concept: *subgame-perfect equilibria*
- famous examples: *ultimatum game* and *centipede game*

This material is also covered in Chapter 4 of the *Essentials*.

K. Leyton-Brown and Y. Shoham. *Essentials of Game Theory: A Concise, Multidisciplinary Introduction*. Morgan & Claypool Publishers, 2008. Chapter 4.

Strategic Games in Extensive Form

An *extensive-form game* is a tuple $\langle N, A, H, Z, \underline{i}, \underline{A}, \sigma, \boldsymbol{u} \rangle$, where

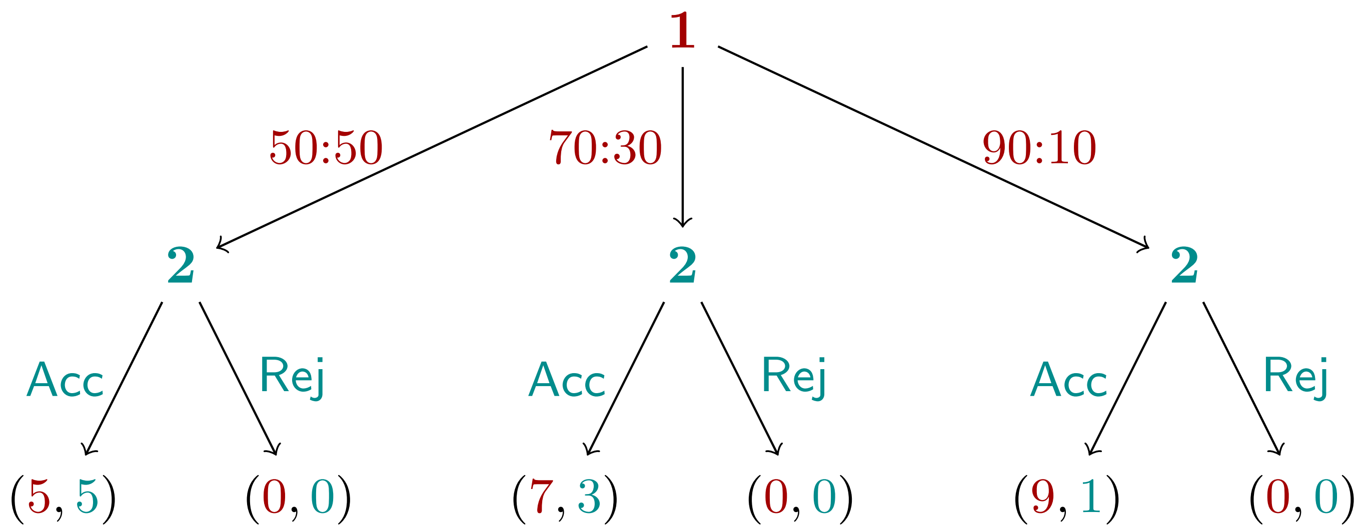
- $N = \{1, \ldots, n\}$ is a finite set of *players*;
- A is a (single) set of *actions*;
- H is a set of *choice nodes* (non-leaf nodes of the tree);
- Z is a set of *outcome nodes* (leaf nodes of the tree);
- $\underline{i} : H \to N$ is the *turn function*, fixing whose turn it is when;
- $\underline{A} : H \to 2^A$ is the *action function*, fixing the playable actions;
- $\sigma : H \times A \to H \cup Z$ is the (injective) *successor function*; and
- $\boldsymbol{u} = (u_1, \ldots, u_n)$ is a profile of *utility functions* $u_i : Z \to \mathbb{R}$.

Must be *finite*. Must have exactly one *root* $h_0 \in H$ s.t. $h_0 \neq \sigma(h, a)$ for all $h \in H$ and $a \in A$. Must have $\underline{A}(h) \neq \emptyset$ for all nodes $h \in H$.

The successor function σ usually is *partial*. Requiring σ to be *injective* ensures every node has (at most) one parent (so we really get a tree).

Example: Ultimatum Game

Player 1 chooses a division of a given amount of money. Player 2 accepts this division or rejects it (in which case both get nothing).



What strategy would you adopt as Player 1? And as Player 2?

Exercise: *Describe this game using our formal definition of extensive games. Note that the picture is missing names for nodes in $H \cup Z$.*

Pure Strategies

Notation: Write $H_i := \{h \in H \mid \underline{i}(h) = i\}$ for the set of choice nodes in which it is player i's turn to choose an action.

A *pure strategy* for player i maps nodes $h \in H_i$ to actions in $\underline{A}(h)$. Thus, it is a function $\alpha_i : H_i \to A$ that respects $\alpha_i(h) \in \underline{A}(h)$.

Given a profile $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ of pure strategies, the *outcome* of the game is the outcome node computed by this program:

$h \leftarrow h_0$
$\textbf{while } h \notin Z \textbf{ do } h \leftarrow \sigma(h, \alpha_{\underline{i}(h)}(h))$
$\textbf{return } h$

Remark: A strategy describes what to do for *every* choice node where it would be your turn, even those you may never actually reach.

Translation to Normal Form

Every *extensive-form game* can be translated into a *normal-form game*.

Translating $\langle N, A, H, Z, \underline{i}, \underline{A}, \sigma, \boldsymbol{u} \rangle$ to normal-form game $\langle N^\star, \boldsymbol{A}^\star, \boldsymbol{u}^\star \rangle$:

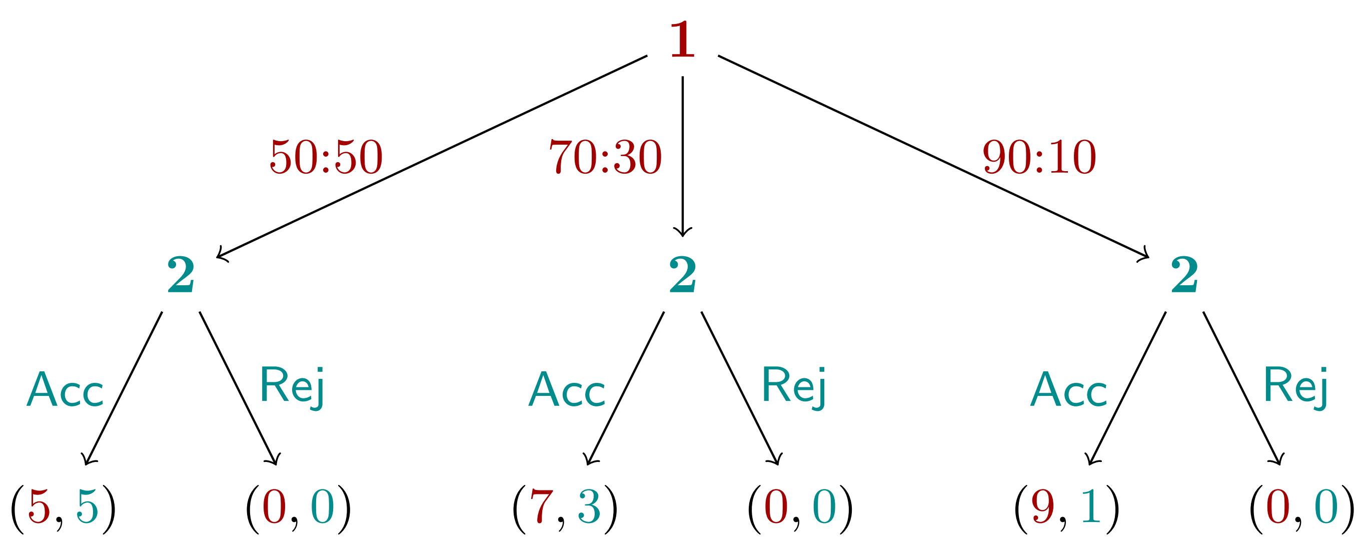
- $N^\star = N$, i.e., the set of players stays the same;
- $\boldsymbol{A}^\star = A_1^\star \times \cdots \times A_n^\star$, with $A_i^\star = \{\alpha_i : H_i \to A \mid \alpha_i(h) \in \underline{A}(h)\}$, i.e., the set of action profiles in the normal-form game is the set of pure-strategy profiles in the extensive game;
- $\boldsymbol{u}^\star = (u_1^\star, \ldots, u_n^\star)$, with $u_i^\star : \boldsymbol{\alpha} \mapsto u_i(\mathrm{out}(\boldsymbol{\alpha}))$, where $\mathrm{out}(\boldsymbol{\alpha})$ is the outcome of the extensive game under pure-strategy profile $\boldsymbol{\alpha}$.

Thus, the full machinery developed for normal-form games (such as mixed strategies, Nash equilibria, other solution concepts) is available.

So why use the extensive form at all? Because it (often) is a more *compact* as well as *intuitive* form of representation.

Exercise: Translation to Normal Form

Recall the Ultimatum Game:



Sketch the normal form. How many matrix cells?

Translation from Normal Form

Can we also translate from *normal-form* to *extensive-form* games? No! At least not in all cases. So the normal form is more general.

Exercise: *Explain why it doesn't work for the Prisoner's Dilemma*.

	C	D
C	-10, -10	-25, 0
D	0, -25	-20, -20

Existence of Pure Nash Equilibria

Theorem 1 (Zermelo, 1913) *Every (finite) extensive-form game has at least one pure Nash equilibrium.*

Proof: Work your way up, from the 'lowest' choice nodes to the root. Label each $h \in H$ with an action $a^\star \in \underline{A}(h)$ and a vector $(u_1^h, \ldots, u_n^h)$:

- Find (one of) the best action(s) for the selected player $i^\star = \underline{i}(h)$:

$$a^\star \in \operatorname*{argmax}_{a \in \underline{A}(h)} u_{i^\star}^{\sigma(h,a)}$$

- Compute the utility labels u_i^h for node h for all agents $i \in N$:

$$u_i^h := u_i^{\sigma(h,a^\star)} \quad \text{(where } u_i^z := u_i(z) \text{ for any } z \in Z\text{)}$$

This process is well-defined and terminates. And by construction, the resulting assignment $\{h \mapsto a^\star\}$ of nodes to pure strategies is a NE. ✓

This method for solving a game is called *backward induction*.

E. Zermelo. Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels. Proc. 5th International Congress of Mathematicians, 1913.

Historical Note: Relevance to Chess

Of course, Zermelo did not phrase his result quite like that: extensive games and Nash equilibria were introduced much later than 1913.

The title of Zermelo's paper mentions *chess* (*das Schachspiel*) . . .

- Using essentially the same argument we have (backward induction) it is easy to see that chess must be determined: either *White* has a *winning strategy*, or *Black* has, or both players can force a draw.

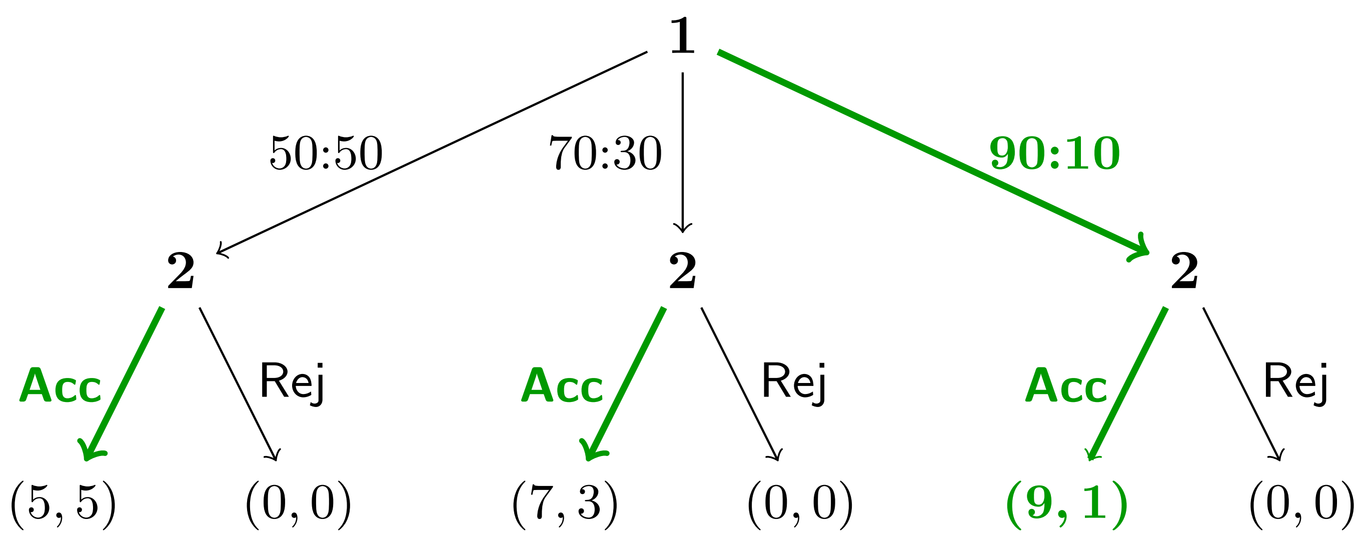
  Remark: True if there is a rule saying that loops result in draws.

- Of course, the *existence* of such a strategy does not mean that anyone *knows* what it actually looks like (the game tree is *too big*).

- Still, the basic idea of backward induction is at the bottom of any *chess-playing program* (and the same is true for similar games).

Example: Backward Induction

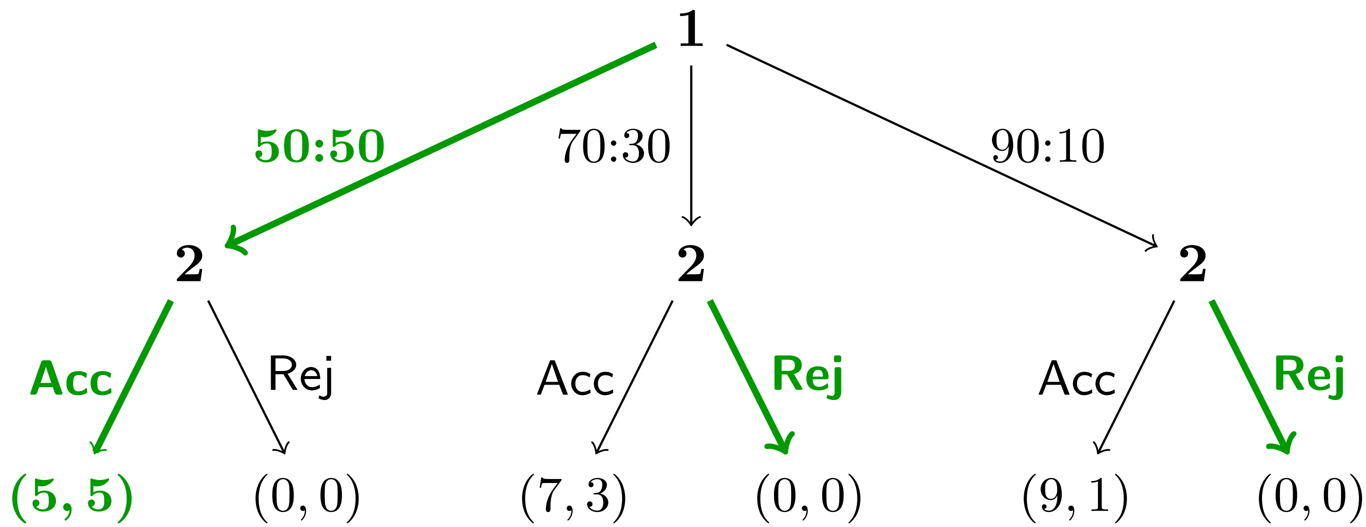
Here is the (only!) *Nash equilibrium* ($90{:}10$, Acc-Acc-Acc) you will find by applying *backward induction* to the *Ultimatum Game:*



Exercise: *Is this the only pure Nash equilibrium for this game?*

Noncredible Threats

There are several other Nash equilibria, such as (50:50, Acc-Rej-Rej):



Indeed, no player has an incentive to unilaterally change her strategy. Nevertheless, this does not seem a reasonable solution for the game: Player 2's *threats* to reject are *not credible*.

Example: In the hypothetical situation where the righthand subgame is reached, to reject would be a strictly dominated strategy for Player 2.

Subgame-Perfect Equilibrium

Every internal node $h \in H$ induces a *subgame* in the natural manner.

A strategy profile s is a *subgame-perfect equilibrium* of an extensive game G_0 if, for every (not necessarily proper) subgame G of G_0, the restriction of s to G is a Nash equilibrium.

Theorem 2 (Selten, 1965) *Every (finite) extensive-form game has at least one subgame-perfect equilibrium.*

Proof: This is what we showed when we proved Zermelo's Theorem. ✓

Remark: Selten (1965) introduced the concept of SPE for a more specific family of games and did not quite state the theorem above, but these ideas are clearly implicit in that paper.

R. Selten. Spieltheoretische Behandlung eines Oligopolmodells mit Nachfrageträgheit. *Zeitschrift für die Gesamte Staatswissenschaft*, 121(2):301–324, 1965.

Let's Play: Centipede Game

We start in the choice node on the left. At each step, the player whose turn it is can choose between going right and going down:

$$\begin{tikzcd} \mathbf{1} \arrow[r] \arrow[d] & \mathbf{2} \arrow[r] \arrow[d] & \mathbf{1} \arrow[r] \arrow[d] & \mathbf{2} \arrow[r] \arrow[d] & (50, 50) \\ (10, 10) & (0, 40) & (30, 30) & (20, 60) & \end{tikzcd}$$

strategies:

- *down-down*
- *down-right*
- *right-down*
- *right-right*

Variant 1: Two volunteers play in full public view, step by step.

Variant 2: Everyone plays, specifying your full strategy in advance.

Specify strategies for both roles. We then run a round-robin tournament and pick one person from the set of top-scorers to receive ₵100.

Discussion

It appears that humans rarely play their SPE strategies.

And even when they do, this can result in counterintuitive effects:

> Suppose you play your SPE strategy, but your opponent doesn't.
>
> Then you are committed to continuing to play a strategy that you devised on the basis of an assumption (full rationality of your opponent) that just turned out to be wrong . . .

Summary

This has been an introduction to extensive games, where we model (for the first time) the sequential nature of most real games:

- definition of the formal model
- pure strategies as functions from choice nodes to actions
- translation into normal form is always possible
- translation from normal form into extensive form is not
- noncredible threats call for new solution concept: SPE
- subgame-perfect equilibrium = NE in every subgame
- backward induction shows: SPE and pure NE always exist

What next? Modelling imperfect (not incomplete) information, where you are not sure about the exact prior moves made by your opponents.