Game Theory: Spring 2020

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Plan for Today

Today we are going to focus on the special case of zero-sum games and discuss two positive results that do not hold for games in general.

- new solution concepts: maximin and minimax solutions
- Minimax Theorem: maximin = minimax = NE for zero-sum games
- fictitious play: basic model for learning in games
- convergence result for the case of zero-sum games

The first part of this is also covered in Chapter 3 of the Essentials.

Zero-Sum Games

Today we focus on two-player games \( \langle N, A, u \rangle \) with \( N = \{1, 2\} \).

Notation: Given player \( i \in \{1, 2\} \), we refer to her opponent as \( -i \).

Recall: A zero-sum game is a two-player normal-form game \( \langle N, A, u \rangle \) for which \( u_i(a) + u_{-i}(a) = 0 \) for all action profiles \( a \in A \).

Examples include (but are not restricted to) games in which you can win \((+1)\), lose \((-1)\), or draw \((0)\), such as matching pennies:

\[
\begin{array}{cc}
H & T \\
\hline
H & -1 & 1 \\
T & 1 & -1 \\
\end{array}
\quad
\begin{array}{cc}
L & R \\
\hline
L & -5 & -3 \\
R & -3 & 3 \\
\end{array}
\quad
\begin{array}{cc}
T & B \\
\hline
T & 5 & 0 \\
B & 0 & -2 \\
\end{array}
\quad
\begin{array}{cc}
H & T \\
\hline
H & -1 & 1 \\
T & 1 & -1 \\
\end{array}
\]
A constant-sum game is a two-player normal-form game \( \langle N, A, u \rangle \) for which there exists a \( c \in \mathbb{R} \) such that \( u_i(a) + u_{-i}(a) = c \) for all \( a \in A \).

Thus: A zero-sum game is a constant-sum game with constant \( c = 0 \).

Everything about zero-sum games to be discussed today also applies to constant-sum games, but for simplicity we only talk about the former.

**Fun Fact:** Football is *not* a constant-sum game, as you get 3 points for a win, 0 for a loss, and 1 for a draw. But prior to 1994, when the “three-points-for-a-win” rule was introduced, World Cup games were constant-sum (with 2, 0, 1 points, for win, loss, draw, respectively).
Maximin Strategies

The definitions on this slide apply to arbitrary normal-form games . . .

Suppose player $i$ wants to maximise her worst-case expected utility (e.g., if all others conspire against her). Then she should play:

$$ s_i^* \in \arg\max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) $$

Any such $s_i^*$ is called a maximin strategy (usually there is just one).

Solution concept: assume each player will play a maximin strategy.

Call $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$ player $i$’s maximin value (or security level).
Exercise: Maximin and Nash

Consider the following two-player game:

```
<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>8</td>
</tr>
</tbody>
</table>
```

What is the maximin solution?

How does this relate to Nash equilibria?

Note: This is neither a zero-sum nor a constant-sum game.
Exercise: Maximin and Nash Again

Now consider this very similar game, which is zero-sum:

\[
\begin{array}{ccc}
T & L & R \\
B & -8 & 0 \\
8 & 0 & -8 \\
0 & 8 & 0 \\
\end{array}
\]

What is the maximin solution?

How does this relate to Nash equilibria?
Minimax Strategies

Now focus on two-player games only, with players $i$ and $-i$.

Suppose player $i$ wants to minimise $-i$’s best-case expected utility (e.g., to punish her). Then $i$ should play:

$$s^*_i \in \arg\min_{s_i \in S_i} \max_{s_{-i} \in S_{-i}} u_{-i}(s_i, s_{-i})$$

Remark: For a zero-sum game, an alternative interpretation is that player $i$ has to play first and her opponent $-i$ can respond.

Any such $s^*_i$ is called a minimax strategy (usually there is just one).

Call $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$ player $-i$’s minimax value.

So $i$’s minimax value is $\min_{s_{-i}} \max_{s_i} u_i(s_{-i}, s_i) = \min_{s_{-i}} \max_{s_i} u_i(s_{-i}, s_i)$. 

Equivalence of Maximin and Minimax Values

Recall: For two-player games, we have seen the following definitions.

- Player $i$’s **maximin value** is $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$.
- Player $i$’s **minimax value** is $\min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$.

**Lemma 1** In a two-player game, maximin and minimax value **coincide**:

$$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$$

We omit the proof. For the case of two actions per player, there is a helpful visualisation in the *Essentials*. Note that one direction is easy:

$(\leqslant)$ LHS is what $i$ can achieve when she has to move first, while RHS is what $i$ can achieve when she can move second. ✔

**Remark:** The lemma does *not* hold if we quantify over actions rather than strategies (counterexample: Matching Pennies).
The Minimax Theorem

Recall: A zero-sum game is a two-player game with $u_i(a) + u_{-i}(a) = 0$.

**Theorem 2 (Von Neumann, 1928)** In a zero-sum game, a strategy profile is a NE iff each player’s expected utility equals her minimax value.

**Proof:** Let $v_i$ be the minimax/maximin value of player $i$ (and $v_{-i} = -v_i$ that of player $-i$).

1. Suppose $u_i(s_i, s_{-i}) \neq v_i$. Then one player does worse than she could (note that here we use the zero-sum property!). So $(s_i, s_{-i})$ is not a NE. ✓

2. Suppose $u_i(s_i, s_{-i}) = v_i$. Then each player already defends optimally against this worst of all possible attacks. So $(s_i, s_{-i})$ is a NE. ✓

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Learning in Games

Suppose you keep playing the same game against the same opponents. You might try to learn their strategies.

A good hypothesis might be that the frequency with which player $i$ plays action $a_i$ is approximately her probability of playing $a_i$.

Now suppose you always best-respond to those hypothesised strategies. And suppose everyone else does the same. What will happen?

We are going to see that for zero-sum games this process converges to a NE. This yields a method for computing a NE for the (non-repeated) game: just imagine players engage in such “fictitious play”.
Empirical Mixed Strategies

Given a history of actions $H_i^\ell = a_i^0, a_i^1, \ldots, a_i^{\ell-1}$ played by player $i$ in $\ell$ prior plays of game $\langle N, A, u \rangle$, fix her empirical mixed strategy $s_i^\ell \in S_i$:

$$s_i^\ell(a_i) = \frac{1}{\ell} \cdot \#\{k < \ell \mid a_i^k = a_i\} \quad \text{for all } a_i \in A_i$$

relative frequency of $a_i$ in $H_i^\ell$
Best Pure Responses

Recall: Strategy $s^*_i \in S_i$ is a best response for player $i$ to the (partial) strategy profile $s_{-i}$ if $u_i(s^*_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ for all $s'_i \in S_i$.

Due to the linearity of expected utilities we get:

**Observation 3** For any given (partial) strategy profile $s_{-i}$, the set of best responses for player $i$ must include at least one pure strategy.

So we can restrict attention to best pure responses for player $i$ to $s_{-i}$:

$$a^*_i \in \arg\max_{a_i \in A_i} u_i(a_i, s_{-i})$$
Fictitious Play

Take any action profile \( a^0 \in A \) for the normal-form game \( \langle N, A, u \rangle \).

Fictitious play of \( \langle N, A, u \rangle \), starting in \( a^0 \), is the following process:

- In round \( \ell = 0 \), each player \( i \in N \) plays action \( a^0_i \).
- In any round \( \ell > 0 \), each player \( i \in N \) plays a best pure response to her opponents' empirical mixed strategies:

\[
\begin{align*}
    a^\ell_i &\in \arg\max_{a_i \in A_i} u_i(a_i, s^\ell_{-i}), \quad \text{where} \\
    s^\ell_{i'}(a_{i'}) &= \frac{1}{\ell} \cdot \#\{k < \ell \mid a^k_{i'} = a_{i'}\} \quad \text{for all } i' \in N \text{ and } a_{i'} \in A_{i'}
\end{align*}
\]

Assume some deterministic way of breaking ties between maxima.

This yields a sequence \( a^0 \rightarrow a^1 \rightarrow a^2 \rightarrow \ldots \) with a corresponding sequence of empirical-mixed-strategy profiles \( s^0 \rightarrow s^1 \rightarrow s^2 \rightarrow \ldots \).

Question: Does \( \lim_{\ell \to \infty} s^\ell \) exist and is it a meaningful strategy profile?
**Example: Matching Pennies**

Let’s see what happens when we start in the upper lefthand corner $HH$ (and break ties between equally good responses in favour of H):

\[
\begin{array}{c|c|c}
 & H & T \\
\hline
H & -1 & 1 \\
\hline
T & 1 & -1 \\
\end{array}
\]

Any strategy can be represented by a single probability (of playing H).

\[
\begin{align*}
HH \left( \frac{1}{1}, \frac{1}{1} \right) & \to HT \left( \frac{2}{2}, \frac{1}{2} \right) \to HT \left( \frac{3}{3}, \frac{1}{3} \right) \to TT \left( \frac{3}{4}, \frac{1}{4} \right) \to TT \left( \frac{3}{5}, \frac{1}{5} \right) \\
& \to TT \left( \frac{3}{6}, \frac{1}{6} \right) \to TH \left( \frac{3}{7}, \frac{2}{7} \right) \to TH \left( \frac{3}{8}, \frac{3}{8} \right) \to TH \left( \frac{3}{9}, \frac{4}{9} \right) \\
& \to TH \left( \frac{3}{10}, \frac{5}{10} \right) \to HH \left( \frac{4}{11}, \frac{6}{11} \right) \to HH \left( \frac{5}{12}, \frac{7}{12} \right) \to \cdots
\end{align*}
\]

**Exercise:** Can you guess what this will converge to?
**Convergence Profiles are Nash Equilibria**

In general, \( \lim_{\ell \to \infty} s^\ell \) does not exist (no guaranteed convergence). But:

**Lemma 4** If fictitious play converges, then to a Nash equilibrium.

**Proof:** Suppose \( s^* = \lim_{\ell \to \infty} s^\ell \) exists. To see that \( s^* \) is a NE, note that \( s^*_i \) is the strategy that \( i \) seems to play when she best-responds to \( s^*_{-i} \), which she believes to be the profile of strategies of her opponents. ✓

**Remark:** This lemma is true for arbitrary (not just zero-sum) games.
Convergence for Zero-Sum Games

Good news:

**Theorem 5 (Robinson, 1951)** *For any zero-sum game and initial action profile, fictitious play will converge to a Nash equilibrium.*

We know that if FP converges, then to a NE. Thus, we still have to show that it will converge. The proof of this fact is difficult and we are not going to discuss it here.

Julia Robinson (1919–1985)

Summary

We have seen that zero-sum games are particularly well-behaved:

- **Minimax Theorem**: your expected utility in a Nash equilibrium will simply be your minimax/maximin value
- **Convergence of fictitious play**: if each player keeps responding to their opponent's estimated strategy based on observed frequencies, these estimates will converge to a Nash equilibrium

Both results give rise to alternative methods for computing a NE.

**What next?** Players who have incomplete information (are uncertain) about certain aspects of the game, such as their opponents’ utilities.