Game Theory 2021

Ulle Endriss
Institute for Logic, Language and Computation
University of Amsterdam
Plan for Today

Today we are going to focus on the special case of zero-sum games and discuss two positive results that do not hold for games in general.

- new solution concepts: maximin and minimax solutions
- Minimax Theorem: maximin = minimax = NE for zero-sum games
- fictitious play: basic model for learning in games
- convergence result for the case of zero-sum games

The first part of this is also covered in Chapter 3 of the Essentials.

Zero-Sum Games

Today we focus on two-player games \( \langle N, A, u \rangle \) with \( N = \{1, 2\} \).

**Notation:** Given player \( i \in \{1, 2\} \), we refer to her opponent as \( -i \).

Recall: A zero-sum game is a two-player normal-form game \( \langle N, A, u \rangle \) for which \( u_i(a) + u_{-i}(a) = 0 \) for all action profiles \( a \in A \).

Examples include (but are not restricted to) games in which you can win \((+1)\), lose \((-1)\), or draw \((0)\), such as matching pennies:
Constant-Sum Games

A constant-sum game is a two-player normal-form game $\langle N, A, u \rangle$ for which there exists a $c \in \mathbb{R}$ such that $u_i(a) + u_{-i}(a) = c$ for all $a \in A$.

Thus: A zero-sum game is a constant-sum game with constant $c = 0$.

Everything about zero-sum games to be discussed today also applies to constant-sum games, but for simplicity we only talk about the former.

Fun Fact: Football is not a constant-sum game, as you get 3 points for a win, 0 for a loss, and 1 for a draw. But prior to 1994, when the “three-points-for-a-win” rule was introduced, World Cup games were constant-sum (with 2, 0, 1 points, for win, loss, draw, respectively).
Maximin Strategies

The definitions on this slide apply to arbitrary normal-form games . . .

Suppose player $i$ wants to maximise her worst-case expected utility (e.g., if all others conspire against her). Then she should play:

$$s_i^* \in \arg\max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

Any such $s_i^*$ is called a maximin strategy (usually there is just one).

Solution concept: assume each player will play a maximin strategy.

Call $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$ player $i$’s maximin value (or security level).
Exercise: Maximin and Nash

Consider the following two-player game:

```
   L  R
T  2  0
8  0  0
B  0  4
```

What is the maximin solution?

How does this relate to Nash equilibria?

Note: This is neither a zero-sum nor a constant-sum game.
Exercise: Maximin and Nash Again

Now consider this fairly similar game, which is zero-sum:

```
  L    R
T  -8   0
  8    0
B  0   -8
  0    8
```

What is the maximin solution?

How does this relate to Nash equilibria?
Minimax Strategies

Now focus on two-player games only, with players $i$ and $-i$ . . .

Suppose player $i$ wants to minimise $-i$’s best-case expected utility (e.g., to punish her). Then $i$ should play:

$$s_i^* \in \arg\min_{s_i \in S_i} \max_{s_{-i} \in S_{-i}} u_{-i}(s_i, s_{-i})$$

Remark: For a zero-sum game, an alternative interpretation is that player $i$ has to play first and her opponent $-i$ can respond.

Any such $s_i^*$ is called a minimax strategy (usually there is just one).

Call $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$ player $-i$’s minimax value.

So $i$’s minimax value is $\min_{s_{-i}} \max_{s_i} u_i(s_{-i}, s_i) = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$. 
Equivalence of Maximin and Minimax Values

Recall: For two-player games, we have seen the following definitions.

- Player $i$’s **maximin value** is $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$.
- Player $i$’s **minimax value** is $\min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$.

**Lemma 1** In a two-player game, **maximin and minimax value coincide**:

$$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$$

We omit the proof. For the case of two actions per player, there is a helpful visualisation in the *Essentials*. Note that one direction is easy:

$(\leq)$ LHS is what $i$ can achieve when she *has to* move first, while
RHS is what $i$ can achieve when she *can* move second. ✓

**Remark:** The lemma does *not* hold if we quantify over actions rather than strategies (counterexample: Matching Pennies).
The Minimax Theorem

Recall: A zero-sum game is a two-player game with $u_i(a) + u_{-i}(a) = 0$.

**Theorem 2 (Von Neumann, 1928)** *In a zero-sum game, a strategy profile is a NE iff each player’s expected utility equals her minimax value.*

**Proof:** Let $v_i$ be the minimax/maximin value of player $i$ (and $v_{-i} = -v_i$ that of player $-i$).

1. Suppose $u_i(s_i, s_{-i}) \neq v_i$. Then one player does worse than she could (note that here we use the zero-sum property!). So $(s_i, s_{-i})$ is not a NE. ✓

2. Suppose $u_i(s_i, s_{-i}) = v_i$. Then one having a better response would mean the other’s security level is actually lower. So $(s_i, s_{-i})$ is a NE. ✓

Learning in Games

Suppose you keep playing the same game against the same opponents. You might try to learn their strategies.

A good hypothesis might be that the frequency with which player $i$ plays action $a_i$ is approximately her probability of playing $a_i$.

Now suppose you always best-respond to those hypothesised strategies. And suppose everyone else does the same. What will happen?

We are going to see that for zero-sum games this process converges to a NE. This yields a method for computing a NE for the (non-repeated) game: just imagine players engage in such “fictitious play”.
Empirical Mixed Strategies

Given a history of actions $H_i^\ell = a_i^0, a_i^1, \ldots, a_i^{\ell-1}$ played by player $i$ in $\ell$ prior plays of game $\langle N, A, u \rangle$, fix her empirical mixed strategy $s_i^\ell \in S_i$:

$$s_i^\ell(a_i) = \frac{1}{\ell} \cdot \# \{ k < \ell \mid a_i^k = a_i \} \quad \text{for all } a_i \in A_i$$

relative frequency of $a_i$ in $H_i^\ell$
Best Pure Responses

Recall: Strategy $s_i^* \in S_i$ is a best response for player $i$ to the (partial) strategy profile $s_{-i}$ if $u_i(s_i^*, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_i' \in S_i$.

Due to expected utilities being convex combinations of plain utilities:

**Observation 3** For any given (partial) strategy profile $s_{-i}$, the set of best responses for player $i$ must include at least one pure strategy.

So we can restrict attention to best pure responses for player $i$ to $s_{-i}$:

$$a_i^* \in \arg\max_{a_i \in A_i} u_i(a_i, s_{-i})$$
Fictitious Play

Take any action profile $a^0 \in A$ for the normal-form game $\langle N, A, u \rangle$.

Fictitious play of $\langle N, A, u \rangle$, starting in $a^0$, is the following process:

- In round $\ell = 0$, each player $i \in N$ plays action $a^0_i$.
- In any round $\ell > 0$, each player $i \in N$ plays a best pure response to her opponents’ empirical mixed strategies:

  $$a^\ell_i \in \arg\max_{a_i \in A_i} u_i(a_i, s^\ell_{-i}),$$

  where

  $$s^\ell_{i'}(a_{i'}) = \frac{1}{\ell} \cdot \#\{k < \ell \mid a^k_{i'} = a_{i'}\} \text{ for all } i' \in N \text{ and } a_{i'} \in A_{i'}$$

Assume some deterministic way of breaking ties between maxima.

This yields a sequence $a^0 \rightarrow a^1 \rightarrow a^2 \rightarrow \ldots$ with a corresponding sequence of empirical-mixed-strategy profiles $s^0 \rightarrow s^1 \rightarrow s^2 \rightarrow \ldots$

**Question:** Does $\lim_{\ell \to \infty} s^\ell$ exist and is it a meaningful strategy profile?
Example: Matching Pennies

Let's see what happens when we start in the upper lefthand corner HH (and break ties between equally good responses in favour of H):

Any strategy can be represented by a single probability (of playing H).

\[ HH \left( \frac{1}{1}, \frac{1}{1} \right) \rightarrow HT \left( \frac{2}{2}, \frac{1}{2} \right) \rightarrow HT \left( \frac{3}{3}, \frac{1}{3} \right) \rightarrow TT \left( \frac{3}{4}, \frac{1}{4} \right) \rightarrow TT \left( \frac{3}{5}, \frac{1}{5} \right) \]
\[ \rightarrow TT \left( \frac{3}{6}, \frac{1}{6} \right) \rightarrow TH \left( \frac{3}{7}, \frac{2}{7} \right) \rightarrow TH \left( \frac{3}{8}, \frac{3}{8} \right) \rightarrow TH \left( \frac{3}{9}, \frac{4}{9} \right) \]
\[ \rightarrow TH \left( \frac{3}{10}, \frac{5}{10} \right) \rightarrow HH \left( \frac{4}{11}, \frac{6}{11} \right) \rightarrow HH \left( \frac{5}{12}, \frac{7}{12} \right) \rightarrow \cdots \]

Exercise: Can you guess what this will converge to?
Convergence Profiles are Nash Equilibria

In general, $\lim_{\ell \to \infty} s^\ell$ does not exist (no guaranteed convergence). But:

**Lemma 4** If fictitious play converges, then to a Nash equilibrium.

**Proof:** Suppose $s^* = \lim_{\ell \to \infty} s^\ell$ exists. To see that $s^*$ is a NE, note that $s^*_i$ is the strategy that $i$ seems to play when she best-responds to $s^*_{-i}$, which she believes to be the profile of strategies of her opponents. ✓

**Remark:** This lemma is true for arbitrary (not just zero-sum) games.
Convergence for Zero-Sum Games

Good news:

**Theorem 5 (Robinson, 1951)** *For any zero-sum game and initial action profile, fictitious play will converge to a Nash equilibrium.*

We know that if FP converges, then to a NE. Thus, we still have to show that it will converge. The proof of this fact is difficult and we are not going to discuss it here.

Summary

We have seen that zero-sum games are particularly well-behaved:

- **Minimax Theorem**: your expected utility in a Nash equilibrium will simply be your minimax/maximin value
- **Convergence of fictitious play**: if each player keeps responding to their opponent’s estimated strategy based on observed frequencies, these estimates will converge to a Nash equilibrium

Both results give rise to alternative methods for computing a NE.

**What next?** Players who have incomplete information (are uncertain) about certain aspects of the game, such as their opponents’ utilities.