Game Theory: Spring 2019

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Plan for Today

Today we are going to generalise our definition of “strategy” and allow players to randomise over several actions to play.

We are then going to generalise the notion of *Nash equilibrium* to this setting and discuss the following three topics:

- the (manual) *computation* of the Nash equilibria for small games
- the *existence* of Nash equilibria for arbitrary games
- (briefly) the computational *complexity* of finding Nash equilibria

Most of this is also covered in Chapters 1 and 2 of the *Essentials*.

Reminder

A *normal-form game* is a tuple $\langle N, A, u \rangle$, where

- $N = \{1, \ldots, n\}$ is a finite set of *players* (or *agents*);
- $A = A_1 \times \cdots \times A_n$ is a finite set of *action profiles* $a = (a_1, \ldots, a_n)$, with $A_i$ being the set of *actions* available to player $i$; and
- $u = (u_1, \ldots, u_n)$ is a profile of *utility functions* $u_i : A \to \mathbb{R}$.

An action profile $a$ is a *pure Nash equilibrium*, if no player $i$ wants to unilaterally deviate from her assigned action $a_i$: $u_i(a) \geq u_i(a'_i, a_{-i})$.

An action profile $a$ is *Pareto efficient*, if no other profile would be better for some player and no worse for any of the others.
Coordination Games

A (pure) coordination game is a normal-form game \( \langle N, A, u \rangle \) with \( u_i(a) = u_j(a) \) for all players \( i, j \in N \) and all action profiles \( a \in A \).

Example: A world with just two drivers. Which side of the road to use?

Remark: For this game, every pure NE is Pareto efficient. Nice.

Exercise: Is this the case for all coordination games?
Zero-Sum Games

A zero-sum game is a two-player normal-form game \( \langle N, A, u \rangle \) with 
\[ u_1(a) + u_2(a) = 0 \]
for all action profiles \( a \in A \). Example:

![Game Matrix]

What are the pure NE of this game? Intuitively, how should you play?
Mixed Strategies and Expected Utility

So far, the space of strategies available to player $i$ has simply been her set of actions $A_i$ (pure strategy = action). We now generalise and allow player $i$ to play any action in $A_i$ with a certain probability.

For any finite set $X$, let $\Pi(X) = \{p : X \rightarrow [0, 1] | \sum_{x \in X} p(x) = 1\}$ be the set of all probability distributions over $X$.

A mixed strategy $s_i$ for player $i$ is a probability distribution in $\Pi(A_i)$. The set of all her mixed strategies is $S_i = \Pi(A_i)$.

A mixed-strategy profile $s = (s_1, \ldots, s_n)$ is an element of $S_1 \times \cdots \times S_n$.

The expected utility of player $i$ for the mixed-strategy profile $s$ is:

$$u_i(s) = \sum_{\mathbf{a} \in A} \left[ u_i(\mathbf{a}) \cdot \prod_{j \in N} s_j(a_j) \right]$$

Remark: Note the overloading of the symbol $u_i$ (also denotes utility).
Types of Mixed Strategies

The *support* of strategy $s_i$ is the set of actions $\{a_i \in A_i \mid s_i(a_i) > 0\}$.

A mixed strategy $s_i$ is *pure* iff its support is a singleton.

A mixed strategy $s_i$ is *truly mixed* if it is not pure.

A mixed strategy $s_i$ is *fully mixed* if its support is the full set $A_i$. 
Example: Battle of the Sexes

Traditionally minded Rowena and Colin are planning a social activity. Worst of all would be not to agree on a joint activity; but if they do manage, Colin prefers auto racing and Rowena really prefers ballet.

\[
\begin{pmatrix}
A & B \\
A & 4 & 0 \\
B & 0 & 3 \\
\end{pmatrix}
\]

Suppose Rowena chooses to go to the ballet with 75% probability, while Colin chooses to go to the races with certainty (pure strategy):

\[s_1 = \left( \frac{1}{4}, \frac{3}{4} \right) \quad s_2 = (1, 0)\]

Then: \[u_1(s) = 2 \cdot \left( \frac{1}{4} \cdot 1 \right) + 0 \cdot \left( \frac{1}{4} \cdot 0 \right) + 0 \cdot \left( \frac{3}{4} \cdot 1 \right) + 8 \cdot \left( \frac{3}{4} \cdot 0 \right) = \frac{1}{2}\]
Mixed Nash Equilibria

Consider a game \( \langle N, A, u \rangle \) with associated (mixed) strategies \( s_i \in S_i \).

We say that strategy \( s_i^* \in S_i \) is a best response for player \( i \) to the (partial) strategy profile \( s_{-i} \) if \( u_i(s_i^*, s_{-i}) \geq u_i(s'_i, s_{-i}) \) for all \( s'_i \in S_i \).

We say that profile \( s = (s_1, \ldots, s_n) \) is a mixed Nash equilibrium, if \( s_i \) is a best response to \( s_{-i} \) for every player \( i \in N \).

Thus: no player has an incentive to unilaterally change her strategy.

Remark: Note how this definition mirrors that of pure Nash equilibria.

Exercise: Can you think of a game with infinitely many Nash equilibria?
Example: Driving Game

Recall: A world with just two drivers. Which side of the road to use?

For this game, it is easy to guess what the Nash equilibria are:

1. pure NE: both pick left with certainty: \(((1, 0), (1, 0))\) [optimal!]
2. pure NE: both pick right with certainty: \(((0, 1), (0, 1))\) [optimal!]
3. both choose fifty-fifty: \(((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))\) [anything is best response!]

There is no other NE: Suppose I pick \((\frac{1}{2} + \epsilon, \frac{1}{2} - \epsilon)\), e.g., \((0.51, 0.49)\). Then your best response is \((1, 0)\), to which my best response is \((1, 0)\).
Computing Nash Equilibria

Suppose we have guessed (correctly) that this game has exactly one NE \((s_1, s_2)\) and that it is fully mixed. How to compute it?

\[
\begin{array}{ccc}
& L & R \\
T & 6 & 5 \\
& 4 & 7 \\
B & 2 & 1 \\
& 3 & 8 \\
\end{array}
\]

Let \(s_1 = (p, 1-p)\) and \(s_2 = (q, 1-q)\). If you use a mixed strategy, you must be indifferent between your two actions. Thus:

- **Player 2** is indifferent: \(4p + 2(1-p) = 5p + 1(1-p) \Rightarrow p = \frac{1}{2}\)
- **Player 1** is indifferent: \(6q + 7(1-q) = 3q + 8(1-q) \Rightarrow q = \frac{1}{4}\)
**Exercise: Game of Chicken**

To establish their relative levels of bravery, Rowena and Colin race their cars towards a cliff at full speed. Each can *jump out* or *wait*. If both wait, they die. If both jump, nothing happens. Otherwise, whoever jumps faces humiliation, while the other one wins.

\[
\begin{array}{c|c|c|c}
 & J & W \\
\hline
J & 0 & 10 & -5 \\
\hline
W & -5 & -20 & 10 \\
\end{array}
\]

What are the Nash equilibria of this game?
Nash’s Theorem

Recall that some games do not have pure Nash equilibria. Good news:

**Theorem 1 (Nash, 1951)** Every (finite) normal-form game has at least one (truly mixed or pure) Nash equilibrium.

We are now going to prove this seminal result. **Plan:**

- Definition of function $f$ from strategy profiles to strategy profiles, simulating the updates players might use to try to improve their lot.
- Lemma showing that $s$ is a NE iff $s$ is a fixed point of $f$.
- Presentation of *Brouwer’s Fixed-Point Theorem* (but no proof), giving sufficient conditions for a function to have a fixed point.
- Lemma showing that $f$ meets the conditions of Brouwer’s Theorem.

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Heuristic Improvement Dynamics

Let $S = S_1 \times \cdots \times S_n$ be the space of all mixed-strategy profiles.

In any given profile $s \in S$, player $i$ will look for a new strategy that improves her payoff. Let $f_i : S \to S_i$ describe how she updates.

Player $i$ might use a heuristic. First, for every pure strategy $a \in A_i$, she computes her gain if she switches to $a$ (and nobody else moves):

$$g_i(s, a) = \max\{u_i(a, s_{-i}) - u_i(s), 0\}$$

Then, she updates the probability of every action $a \in A_i$, in line with these expected gains (ensuring they still add up to 1), from $s_i(a)$ to:

$$f_i(s)(a) = \frac{s_i(a) + g_i(s, a)}{\sum_{a' \in A_i} s_i(a') + g_i(s, a')} = \frac{s_i(a) + g_i(s, a)}{1 + \sum_{a' \in A_i} g_i(s, a')}$$

If everyone does this, we get a global update function $f : S \to S$ with $f(s) = (f_1(s), \ldots, f_n(s))$. Exercise: Suppose $s$ is a NE. What is $f(s)$?
Nash Equilibria and Fixed Points

Recall: \( f : S \rightarrow S \) with \( f(s) = (f_1(s), \ldots, f_n(s)) \), where:

\[
f_i(s)(a) = \frac{s_i(a) + g_i(s, a)}{1 + \sum_{a' \in A_i} g_i(s, a')}
\]

\[
\begin{bmatrix}
g_i(s, a) = \max\{u_i(a, s_{-i}) - u_i(s), 0\}
\end{bmatrix}
\]

**Lemma 2** A strategy profile \( s \in S \) is a Nash equilibrium iff \( s \) is a fixed point of the update function \( f \) (meaning that \( f(s) = s \)).

**Proof:** \((\Rightarrow)\) If \( s \) is a NE, no strategy, certainly no pure strategy \( a \), can increase \( i \)'s payoff, i.e., \( g_i(s, a) = 0 \). Thus: \( f_i(s)(a) = s_i(a) \) ✓

\((\Leftarrow)\) Suppose \( s \) is a fixed point of \( f \), i.e., \( f_i(s)(a) = s_i(a) \). **Cases:**

- \( g_i(s, a) = 0 \) for all \( i \in N \) and \( a \in A_i \): If impossible to improve by pure strategy, then also by mixed strategy. Thus, \( s \) is a NE. ✓

- **Otherwise**, use definition of \( f_i \) to infer \( s_i(a) = g(s, a) / \sum_{a' \in A_i} g(s, a') \).
  Thus \( s_i(a) > 0 \iff g(s, a) > 0 \), meaning \( s_i(a) > 0 \iff u_i(a, s_{-i}) > u_i(s) \).
  But this contradicts \( u_i(s) = \sum_{a \in A_i} u_i(a, s_{-i}) \cdot s_i(a) \). ✓
Brouwer’s Fixed-Point Theorem

Theorem 3 (Brouwer, 1911) Let $X \subseteq \mathbb{R}^m$ ($X \neq \emptyset$) be compact and convex. Then every continuous function $f : X \to X$ has a fixed point.

Recall that this means that there exists an $x \in X$ such that $f(x) = x$.

Explanation of the terminology used:

- **Compactness.** $X$ is compact if it is closed (contains its limit points) and bounded (any two elements have distance $\leq K$, for some $K$).
- **Convexity.** $X$ is convex if any point “between” $x$ and $y$ in $X$ is also in $X$, i.e., if $\lambda \cdot x + (1 - \lambda) \cdot y \in X$ for any $\lambda \in [0, 1]$.
- **Continuity.** You know this one (“no sudden jumps”).

Examples

On $X = [0, 1]$, the function $f : x \mapsto x^2$ has the fixed points 0 and 1.

On $X = \{ (x, y) \in [0, 1]^2 \mid x^2 + y^2 \leq 1 \}$, the “mirroring” function $f : (x, y) \mapsto (-x, -y)$ has the fixed point $(0, 0)$.

Put a map of Amsterdam on a table somewhere in A’dam. Then some point on the map will be directly above the location it represents. ($X$ is the set of locations in A’dam; $f$ is the projection to the map.)

On $X = [0, 1)$, the function $f : x \mapsto \frac{x+1}{2}$ has no fixed point, because $X$ is not closed. But for $X = [0, 1]$ we get the fixed point 1.

On $X = \mathbb{R}$, the function $f : x \mapsto x + 1$ has no fixed point, because $X$ is not bounded (even though it is closed).

On $X = \{-1, 1\}$, the function $f : x \mapsto -x$ has no fixed point, because $X$ is not convex.

On $X = [0, 9]$, the function $f : x \mapsto ([x] + 1) \mod 10$ has no fixed point, because $f$ is not continuous.
Applying Brouwer’s Fixed-Point Theorem

Recall: $f : S \to S$ with $f(s) = (f_1(s), \ldots, f_n(s))$, where:

$$f_i(s)(a) = \frac{s_i(a) + g_i(s, a)}{1 + \sum_{a' \in A_i} g_i(s, a')}$$

$$g_i(s, a) = \max\{u_i(a, s_{-i}) - u_i(s), 0\}$$

The set $S$ of strategy profiles and our update function $f : S \to S$ satisfy the conditions in Brouwer’s Fixed-Point Theorem:

- **Compactness** of $S$: Clear when we think of $s_i$ as a vector of $|A_i|$ probabilities, i.e., $S_i \subseteq [0, 1]^{A_i}$ and $S \subseteq [0, 1]^m$ for $m = \prod_i |A_i|$.

- **Convexity** of $S$: Follows from fact that every linear combination of probabilities is itself a probability.

- **Continuity** of $f$: As all $u_i : S \to \mathbb{R}$ are continuous, so are all $g_i : S \times A_i \to \mathbb{R}_{\geq 0}$, all $f_i(s) : A_i \to [0, 1]$, and all $f_i : S \to S_i$.

This concludes the proof of Nash’s Theorem. ✓
Complexity of Computing Nash Equilibria

Note: The proof of Nash’s Theorem does not provide us with a method to actually compute Nash equilibria, because it is not constructive.

Both the design of algorithms for computing Nash equilibria and the analysis of the computational complexity of this task are important research topics in algorithmic game theory.

Regarding the complexity:

- Finding a NE is in $NP$: if you guess a NE, I can easily verify.
- But likely not not $NP$-hard: guaranteed existence would be atypical.
- Complete for $PPAD$ (“polynomial parity argument for directed graphs”), which lies “between” P and NP. Believed to be intractable.

Discussion: NE as model of rational behaviour vs. high complexity.

Summary

We have introduced the notion of a \textit{mixed strategy}, where a player randomises over several pure strategies (i.e., actions). And:

- \textit{Nash’s Theorem}: every normal-form game has a Nash equilibrium
- technique for \textit{computing} NE’s for small (two-player) games

The NE, although not perfect, is the most important \textit{solution concept} in game theory, and we’ll return to it frequently.

Points of concern regarding the notion of (mixed) NE:

- Is it reasonable to assume players are perfectly \textit{rational}?  
- Is it reasonable to assume players work with \textit{probabilities}?  
- Is it reasonable to assume players can handle the high \textit{complexity}?  
- What if there are \textit{many} Nash equilibria? How do you choose?

\textbf{What next?} Focus on a specific class of games with nice properties, the so-called \textit{congestion games}.  